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On Representations of Integers by Indefinite Ternary Quadratic Forms

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integral solutions of the equation $f(x)=q$ where x lies in the ball of radius T centered at the origin. We are interested in the asymptotic behavior of $N(T, f, q)$ as $T \to \infty$. We deduce from the results of our joint paper with Z. Rudnick that $N(T, f, q) \sim c E_{HL}(T, f, q)$ as $T \to \infty$, where $E_{HL}(T, f, q)$ is the Hardy–Littlewood expectation (the product of local densities) and $0 \leqslant c \leqslant 2$. We give examples of f and q such that c takes the values 0, 1, 2. © 2001 Academic Press

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0. INTRODUCTION

Let f be a nondegenerate indefinite integral-matrix quadratic form of n variables:

$$f(x_1, \ldots, x_n) = \sum_{i,\, j=1}^{n} a_{ij} x_i x_j, \qquad a_{ij} \in \mathbf{Z}, \qquad a_{ij} = a_{ji}.$$

Let $q \in \mathbf{Z}$, $q \neq 0$. Let $W = \mathbf{Q}^n$. Consider the affine quadric X in W defined by the equation

$$f(x_1, \ldots, x_n) = q.$$

We wish to count the representations of q by the quadratic form f, that is the integer points of X.

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Since f is indefinite, the set $X(\mathbf{Z})$ can be infinite. We fix a Euclidean norm $|\cdot|$ on $\mathbf{R}^n$. Consider the counting function

$$N(T, X) = \#\{x \in X(\mathbf{Z}) : |x| \leqslant T\}$$

where $T \in \mathbf{R}$, $T > 0$. We are interested in the asymptotic behavior of $N(T, X)$ as $T \to \infty$.

When $n \geqslant 4$, the counting function $N(T, X)$ can be approximated by the product of local densities. For a prime p set

$$\mu_p(X) = \lim_{k \to \infty} \frac{\# X(\mathbf{Z}/p^k\mathbf{Z})}{(p^k)^{n-1}}.$$

For almost all p it suffices to take $k = 1$:

$$\mu_p(X) = \frac{\# X(\mathbf{F}_p)}{p^{n-1}}.$$

Set $\mathfrak{S}(X) = \prod_p \mu_p(X)$; this product converges absolutely (for $n \geqslant 4$) and is called the singular series. Set

$$\mu_\infty(T, X) = \lim_{\varepsilon \to 0} \frac{\mathrm{Vol}\{x \in \mathbf{R}^n : |x| \leqslant T, |f(x) - q| < \varepsilon/2\}}{\varepsilon},$$

which is called the singular integral. For $n \geqslant 4$ the following asymptotic formula holds:

$$N(T, X) \sim \mathfrak{S}(X)\, \mu_\infty(T, X) \qquad \text{as} \quad T \to \infty.$$

This follows from results of [2, 6.4] (which are based on analytical results of [6, 7, 8]). For certain non-Euclidean norms the similar result was earlier proved by the Hardy–Littlewood circle method, cf. [5] in the case $n \geqslant 5$ and [9] in the more difficult case $n = 4$.

We are interested here in the case $n = 3$, a ternary quadratic form. This case is beyond the range of the Hardy–Littlewood circle method. Set $D = \det(a_{ij})$. We assume that $-qD$ is not a square. Then the product $\mathfrak{S}(X) = \prod \mu_p(X)$ conditionally converges (see Sect. 1 below), but in general $N(T, X)$ is not asymptotically $\mathfrak{S}(X)\, \mu_\infty(T, X)$. From results of [2] it follows that

$$N(T, X) \sim c_X \mathfrak{S}(X)\, \mu_\infty(T, X) \qquad \text{as} \quad T \to \infty$$

with $0 \leqslant c_X \leqslant 2$, see details in Section 1.5 below. We wish to know what values can c_X take.

A case when $c_X = 0$ was already known to Siegel, see also [2, 6.4.1]. Consider the quadratic form

$$f_1(x_1, x_2, x_3) = -9x_1^2 + 2x_1x_2 + 7x_2^2 + 2x_3^2,$$

and take $q = 1$. Let X be defined by $f_1(x) = q$. Then f_1 does not represent 1 over $\mathbf{Z}$, so $N(T, X) = 0$ for all T. On the other hand, f_1 represents 1 over $\mathbf{R}$ and over $\mathbf{Z}_p$ for all p, and $\mathfrak{S}(X)\, \mu_\infty(T, X) \to \infty$ as $T \to \infty$. Thus $c_X = 0$ (see details in Sect. 2).

We show that c_X can take the value 2. Recall that two integral quadratic forms f, f' are in the same genus if they are equivalent over $\mathbf{R}$ and over $\mathbf{Z}_p$ for every prime p, cf. e.g. [3].

Theorem 0.1. *Let f be an indefinite integral-matrix ternary quadratic form, $q \in \mathbf{Z}$, $q \neq 0$, and let X be the affine quadric defined by the equation $f(x) = q$. Assume that f represents q over $\mathbf{Z}$ and that there exists a quadratic form f' in the genus of f such that f' does not represent q over $\mathbf{Z}$. Then $c_X = 2$:*

$$N(T, X) \sim 2\mathfrak{S}(X)\, \mu_\infty(T, X) \qquad \textit{as} \quad T \to \infty.$$

Theorem 0.1 will be proved in Section 3.

Example 0.1.1. Let $f_2(x_1, x_2, x_3) = -x_1^2 + 64x_2^2 + 2x_3^2$, $q = 1$. Then f_2 represents 1 ($f_2(1, 0, 1) = 1$) and the quadratic form f_1 considered above is in the genus of f_2 (cf. [4, 15.6]). The form f_1 does not represent 1. Take $|x| = (x_1^2 + 64x_2^2 + 2x_3^2)^{1/2}$. By Theorem 0.1 $c_X = 2$ for the variety X: $f_2(x) = 1$. Analytic and numeric calculations give $2\mathfrak{S}(X)\, \mu_\infty(T, X) \sim 0.794T$. On the other hand, numeric calculations give for $T = 10{,}000$ the value $N(T, X)/T = 0.8024$.

We also show that c_X can take the value 1.

Theorem 0.2. *Let f be an indefinite integral-matrix ternary quadratic form, $q \in \mathbf{Z}$, $q \neq 0$, and let X be the affine quadric defined by the equation $f(x) = q$. Assume that $X(\mathbf{R})$ is two-sheeted (has two connected components). Then $c_X = 1$:*

$$N(T, X) \sim \mathfrak{S}(X)\, \mu_\infty(T, X) \qquad \textit{as} \quad T \to \infty.$$

Theorem 0.2 will be proved in Section 4.

EXAMPLE 0.2.1. Let f_2 and $|x|$ be as in Example 0.1.1, $q=-1$, $X\colon f_2(x)=q$. Then $X(\mathbf{R})$ has two connected components, and by Theorem 0.2 $c_X=1$. Analytic and numeric calculations give $\mathfrak{S}(X)\,\mu_\infty(T,X)\sim 0.7065T$. On the other hand, numeric calculations give for $T=10,000$ the value $N(T,X)/T=0.7048$.

Question 0.3. Can c_X take values other than 0, 1, 2?

The plan of the paper is the following. In Section 1 we describe results of [2] in the case of 2-dimensional affine quadrics. In Section 2 we treat in detail the example of $c_X=0$. In Section 3 we prove Theorem 0.1. In Section 4 we prove Theorem 0.2.

1. RESULTS OF [2] IN THE CASE OF TERNARY QUADRATIC FORMS

Let f be an indefinite ternary integral-matrix quadratic form

$$f(x_1,x_2,x_3)=\sum_{i,j=1}^{3} a_{ij}x_ix_j,\qquad a_{ij}\in\mathbf{Z},\qquad a_{ij}=a_{ji}.$$

Let $q\in\mathbf{Z}$, $q\neq 0$. Let $D=\det(a_{ij})$. We assume that $-qD$ is not a square.

Let $W=\mathbf{Q}^3$ and let X denote the affine variety in W defined by the equation $f(x)=q$, where $x=(x_1,x_2,x_3)$. We assume that X has a $\mathbf{Q}$-point x^0. Set $G=\mathrm{Spin}(W,f)$, the spinor group of f. Then G acts on W on the left, and X is an orbit (a homogeneous space) of G.

1.1. *Rational Points in Adelic Orbits*

Let $\mathbf{A}$ denote the adèle ring of $\mathbf{Q}$. The group $G(\mathbf{A})$ acts on $X(\mathbf{A})$; let $\mathcal{O}_{\mathbf{A}}$ be an orbit. We would like to know whether $\mathcal{O}_{\mathbf{A}}$ has a $\mathbf{Q}$-rational point.

Let W' denote the orthogonal complement of x^0 in W, and let f' denote the restriction of f to W'. Let H be the stabilizer of x^0 in G, then $H=\mathrm{Spin}(W',f')$. Since $\dim W'=2$, the group H is a one-dimensional torus.

We have $\det f'=D/q$, so up to multiplication by a square $\det f'=qD$. It follows that up to multiplication by a scalar, f' is equivalent to the quadratic form u^2+qDv^2. Set $K=\mathbf{Q}(\sqrt{-qD})$, then K is a quadratic extension of $\mathbf{Q}$, because $-qD$ is not a square. The torus H is anisotropic over $\mathbf{Q}$ (because $-qD$ is not a square), and H splits over K. Let $\mathbf{X}_*(H_K)$ denote the cocharacter group of H_K, $\mathbf{X}_*(H_K)=\mathrm{Hom}(\mathbb{G}_{m,K},H_K)$; then $\mathbf{X}_*(H_K)\simeq\mathbf{Z}$. The non-neutral element of $\mathrm{Gal}(K/\mathbf{Q})$ acts on $\mathbf{X}_*(H_K)$ by multiplication by -1.

Let $\mathcal{O}_{\mathbf{A}}$ be an orbit of $G(\mathbf{A})$ in $X(\mathbf{A})$, $\mathcal{O}_{\mathbf{A}} = \prod \mathcal{O}_v$ where $\mathcal{O}_v$ is an orbit of $G(\mathbf{Q}_v)$ in $X(\mathbf{Q}_v)$, v runs over the places of $\mathbf{Q}$, and $\mathbf{Q}_v$ denotes the completion of $\mathbf{Q}$ at v. We define local invariants $v_v(\mathcal{O}_v) = \pm 1$. If $\mathcal{O}_v = G(\mathbf{Q}_v) \cdot x^0$, then we set $v_v(\mathcal{O}_v) = +1$, if not, we set $v_v(\mathcal{O}_v) = -1$. Then $v_v(\mathcal{O}_v) = +1$ for almost all v. We define $v(\mathcal{O}_{\mathbf{A}}) = \prod v_v(\mathcal{O}_v)$ where $\mathcal{O}_{\mathbf{A}} = \prod \mathcal{O}_v$. Note that the local invariants $v_v(\mathcal{O}_v)$ depend on the choice of the rational point $x^0 \in X(\mathbf{Q})$; one can prove, however, that their product $v(\mathcal{O}_{\mathbf{A}})$ does not depend on x^0.

Let $x \in X(\mathbf{A})$. We set $v(x) = v(G(\mathbf{A}) \cdot x)$. Then $v(x)$ takes values ± 1; it is a locally constant function on $X(\mathbf{A})$, because the orbits of $G(\mathbf{A})$ are open in $X(\mathbf{A})$.

For $x \in X(\mathbf{A})$ define $\delta(x) = v(x) + 1$. In other words, if $v(x) = -1$ then $\delta(x) = 0$, and if $v(x) = +1$ then $\delta(x) = 2$. Then δ is a locally constant function on $X(\mathbf{A})$.

THEOREM 1.1. *An orbit $\mathcal{O}_{\mathbf{A}}$ of $G(\mathbf{A})$ in $X(\mathbf{A})$ has a $\mathbf{Q}$-rational point if and only if $v(\mathcal{O}_{\mathbf{A}}) = +1$.*

Below we will deduce Theorem 1.1 from [2, Theorem 3.6].

1.2. *Proof of Theorem* 1.1

For a torus T over a field k of characteristic 0 we define a finite abelian group $C(T)$ as follows

$$C(T) = (\mathbf{X}_*(T_{\bar{k}})_{\mathrm{Gal}(\bar{k}/k)})_{\mathrm{tors}}$$

where $\bar{k}$ is a fixed algebraic closure of k, $\mathbf{X}_*(T_{\bar{k}})_{\mathrm{Gal}(\bar{k}/k)}$ denotes the group of coinvariants, and $(\cdot)_{\mathrm{tors}}$ denotes the torsion subgroup. If k is a number field and k_v is the completion of k at a place v, then we define $C_v(T) = C(T_{k_v})$. There is a canonical map $i_v : C_v(T) \to C(T)$ induced by an inclusion $\mathrm{Gal}(\bar{k}_v/k_v) \to \mathrm{Gal}(\bar{k}/k)$. These definitions were given for connected reductive groups (not only for tori) by Kottwitz [10]; see also [2, 3.4]. Kottwitz writes $A(T)$ instead of $C(T)$.

We compute $C(H)$ for our one-dimensional torus H over $\mathbf{Q}$. Clearly

$$C(H) = (\mathbf{X}_*(H_K)_{\mathrm{Gal}(K/\mathbf{Q})})_{\mathrm{tors}} = \mathbf{Z}/2\mathbf{Z}.$$

We have $C_v(H) = 1$ if $K \otimes \mathbf{Q}_v$ splits, and $C_v(H) \simeq \mathbf{Z}/2\mathbf{Z}$ if $K \otimes \mathbf{Q}_v$ is a field. The map i_v is injective for any v.

We now define the local invariants $\kappa_v(\mathcal{O}_v)$ as in [2], where $\mathcal{O}_v$ is an orbit of $G(\mathbf{Q}_v)$ in $X(\mathbf{Q}_v)$. The set of orbits of $G(\mathbf{Q}_v)$ in $X(\mathbf{Q}_v)$ is in canonical bijection with $\ker [H^1(\mathbf{Q}_v, H) \to H^1(\mathbf{Q}_v, G)]$, cf. [13, I-5.4, Corollary 1 of Proposition 36]. Hence $\mathcal{O}_v$ defines a cohomology class $\xi_v \in H^1(\mathbf{Q}_v, H)$. The local Tate–Nakayama duality for tori defines a canonical homomorphism $\beta_v : H^1(\mathbf{Q}_v, H) \to C_v(H)$, see Kottwitz [10, Theorem 1.2]. (Kottwitz defines the map β_v in a more general setting, when H is any connected reductive group over a number field.) The homomorphism β_v is an isomorphism for any v. We set $\kappa_v(\mathcal{O}_v)=\beta_v(\xi_v)$. Note that if $\mathcal{O}_v=G(\mathbf{Q}_v)\cdot x^0$, then $\xi_v=0$ and $\kappa_v(\mathcal{O}_v)=0$; if $\mathcal{O}_v\neq G(\mathbf{Q}_v)\cdot x^0$, then $\xi_v\neq 0$ and $\kappa_v(\mathcal{O}_v)=1$.

We define the Kottwitz invariant $\kappa(\mathcal{O}_{\mathbf{A}})$ of an orbit $\mathcal{O}_{\mathbf{A}}=\prod \mathcal{O}_v$ of $G(\mathbf{A})$ in $X(\mathbf{A})$ by $\kappa(\mathcal{O}_{\mathbf{A}})=\sum_v i_v(\kappa_v(\mathcal{O}_v))$. We identify $C(H)$ with $\mathbf{Z}/2\mathbf{Z}$, and $C_v(H)$ with a subgroup of $\mathbf{Z}/2\mathbf{Z}$. With this identifications $\kappa(\mathcal{O}_{\mathbf{A}})=\sum \kappa_v(\mathcal{O}_v)$.

We prefer the multiplicative rather than additive notation. Instead of $\mathbf{Z}/2\mathbf{Z}$ we consider the group $\{+1,-1\}$, and set

$$\nu_v(\mathcal{O}_v)=(-1)^{\kappa_v(\mathcal{O}_v)},\qquad \nu(\mathcal{O}_{\mathbf{A}})=(-1)^{\kappa(\mathcal{O}_{\mathbf{A}})}.$$

Here $\nu_v(\mathcal{O}_v)$ and $\nu(\mathcal{O}_{\mathbf{A}})$ take the values ± 1. We have $\nu(\mathcal{O}_{\mathbf{A}})=\prod \nu_v(\mathcal{O}_v)$. Since $\kappa_v(\mathcal{O}_v)=0$ if and only if $\mathcal{O}_v=G(\mathbf{Q}_v)\cdot x^0$, we see that $\nu_v(\mathcal{O}_v)=+1$ if and only if $\mathcal{O}_v=G(\mathbf{Q}_v)\cdot x^0$. Hence our $\nu_v(\mathcal{O}_v)$ and $\nu(\mathcal{O}_{\mathbf{A}})$ coincide with $\nu_v(\mathcal{O}_v)$ and $\nu(\mathcal{O}_{\mathbf{A}})$, resp., introduced in Section 1.1.

By Theorem 3.6 of [2] an adelic orbit $\mathcal{O}_{\mathbf{A}}$ contains $\mathbf{Q}$-rational points if and only if $\kappa(\mathcal{O}_{\mathbf{A}})=0$. With our multiplicative notation $\kappa(\mathcal{O}_{\mathbf{A}})=0$ if and only if $\nu(\mathcal{O}_{\mathbf{A}})=+1$. Thus $\mathcal{O}_{\mathbf{A}}$ contains $\mathbf{Q}$-points if and only if $\nu(\mathcal{O}_{\mathbf{A}})=+1$. We have deduced Theorem 1.1 from [2, Theorem 3.6]. ∎

1.3. *Tamagawa Measure*

We define a gauge form on X, i.e. a regular differential form $\omega\in\Lambda^2(X)$ without zeroes. Recall that X is defined by the equation $f(x)=q$. Choose a differential form μ of degree 2 on W such that $\mu\wedge df=dx_1\wedge dx_2\wedge dx_3$, where x_1, x_2, x_3 are the coordinates in $W=\mathbf{Q}^3$. Let $\omega=\mu|_X$, the restriction of μ to X. Then ω is a gauge form on X, cf. [2, 1.3], and it does not depend on the choice of μ. The gauge form ω is G-invariant, because there exists a G-invariant gauge form on X, cf. [2, 1.4], and a gauge form on X is unique up to a scalar multiple, cf. [2, Corollary 1.5.4].

For any place v of $\mathbf{Q}$ one associates with ω a local measure m_v on $X(\mathbf{Q}_v)$, cf. [14, 2.2]. We show how to define a Tamagawa measure on $X(\mathbf{A})$, following [2, 1.6.2].

We have by [2, 1.8.1], $\mu_p(X)=m_p(X(\mathbf{Z}_p))$, where $\mu_p(X)$ is defined in the Introduction. By [14, Theorem 2.2.5], for almost all p we have $m_p(X(\mathbf{Z}_p))=\#X(\mathbf{F}_p)$.

We compute $\#X(\mathbf{F}_p)$. The group $\mathrm{SO}(f)(\mathbf{F}_p)$ acts on $X(\mathbf{F}_p)$ with stabilizer $\mathrm{SO}(f')(\mathbf{F}_p)$, where $\mathrm{SO}(f')(\mathbf{F}_p)$ is defined for almost all p. This action is transitive by Witt's theorem. Thus we obtain that $\#X(\mathbf{F}_p)=\#\mathrm{SO}(f)(\mathbf{F}_p)/\#\mathrm{SO}(f')(\mathbf{F}_p)$. By [1, III-6],

$$\#\mathrm{SO}(f)(\mathbf{F}_p)=p(p^2-1),\qquad \#\mathrm{SO}(f')(\mathbf{F}_p)=p-\chi(p),$$

where $\chi(p) = -1$ if $f' \bmod p$ does not represent 0, and $\chi(p) = +1$ if $f' \bmod p$ represents 0. We have $\chi(p) = (\frac{-qD}{p})$. We obtain for $p \nmid qD$

$$\# X(\mathbf{F}_p) = \frac{p(p^2-1)}{p-\chi(p)}, \qquad \mu_p(X) = \frac{\# X(\mathbf{F}_p)}{p^2} = \frac{1-1/p^2}{1-\chi(p)/p}.$$

For $p \mid qD$ set $\chi(p) = 0$. We define

$$L_p(s, \chi) = (1 - \chi(p)\, p^{-s})^{-1}, \qquad L(s, \chi) = \prod_p L_p(s, \chi),$$

where s is a complex variable. We set

$$\lambda_p = L_p(1, \chi)^{-1} = 1 - \frac{\chi(p)}{p}, \qquad r = L(1, \chi)^{-1}.$$

Then the product $\prod_p (\lambda_p^{-1} \mu_p)$ converges absolutely, hence the family (λ_p) is a family of convergence factors in the sense of [14, 2.3]. We define, as in [2, 1.6.2], the measures

$$m_f = r^{-1} \prod_p (\lambda_p^{-1} m_p), \qquad m = m_\infty m_f,$$

then m_f is a measure on $X(\mathbf{A}_f)$ (where $\mathbf{A}_f$ is the ring of finite adèles) and m is a measure on $X(\mathbf{A})$. We call m the Tamagawa measure on $X(\mathbf{A})$.

1.4. *Counting Integer Points*

For $T > 0$ set $X(\mathbf{R})^T = \{x \in X(\mathbf{R}) : |x| \leqslant T\}$.

THEOREM 1.2.

$$N(T, X) \sim \int_{X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})} \delta(x)\, dm.$$

In other words,

$$N(T, X) \sim 2m(\{x \in X(\mathbf{R})^T \times X(\hat{\mathbf{Z}}) : \nu(x) = +1\}). \tag{1}$$

Theorem 1.2 follows from [2, Theorem 5.3] (cf. [2, 6.4] and [2, Definition 2.3]).

For comparison note that

$$m(X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})) = m_\infty(X(\mathbf{R})^T)\, m_f(X(\hat{\mathbf{Z}})) = \mu_\infty(T, X)\, \mathfrak{S}(X), \tag{2}$$

cf. [2, 1.8].

The following lemma will be used in the proof of Theorem 0.1.

LEMMA 1.3. *Assume that there exists* $y \in X(\mathbf{R} \times \hat{\mathbf{Z}})$ *such that* $\nu(y) = +1$. *Then the set* $X(\mathbf{Z})$ *is infinite.*

Proof. Since ν is a locally constant function on $X(\mathbf{A})$, there exists a nonempty open subset $\mathcal{U}_f \in X(\hat{\mathbf{Z}})$ and an orbit $\mathcal{U}_\infty$ of $G(\mathbf{R})$ in $X(\mathbf{R})$ such that $\nu(x) = +1$ for all $x \in \mathcal{U}_\infty \times \mathcal{U}_f$. Set $\mathcal{U}_\infty^T = \{x \in \mathcal{U}_\infty : |x| \leqslant T\}$, then $m_\infty(\mathcal{U}_\infty^T) \to \infty$ as $T \to \infty$. We have

$$\int_{X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})} \delta(x)\, dm \geqslant \int_{\mathcal{U}_\infty^T \times \mathcal{U}_f} \delta(x)\, dm = 2m_\infty(\mathcal{U}_\infty^T)\, m_f(\mathcal{U}_f)\ .$$

Since $2m_\infty(\mathcal{U}_\infty^T)\, m_f(\mathcal{U}_f) \to \infty$ as $T \to \infty$, we see that

$$\int_{X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})} \delta(x)\, dm \to \infty \qquad \text{as} \quad T \to \infty,$$

and by Theorem 1.2 $N(T, X) \to \infty$. Hence $X(\mathbf{Z})$ is infinite.

1.5. *The Constant* c_X

Here we prove the following result:

PROPOSITION 1.4.

$$N(T, X) \sim c_X \mathfrak{S}(X)\, \mu_\infty(T, X) \qquad \textit{as} \quad T \to \infty$$

with some constant c_X, $0 \leqslant c_X \leqslant 2$.

Proof. If $X(\mathbf{R})$ has two connected components, then by Theorem 0.2 (which we will prove in Section 4 below), $N(T, X) \sim \mathfrak{S}(X)\, \mu_\infty(T, X)$, so the proposition holds with $c_X = 1$.

If $X(\mathbf{R})$ has one connected component, then $X(\mathbf{R})$ consists of one $G(\mathbf{R})$-orbit and $\nu_\infty(X(\mathbf{R})) = +1$. For an orbit $\mathcal{O}_f = \prod \mathcal{O}_p$ of $G(\mathbf{A}_f)$ in $X(\mathbf{A}_f)$ we set $\nu_f(\mathcal{O}_f) = \prod_p \nu_p(\mathcal{O}_p)$. We regard ν_f as a locally constant function on $X(\mathbf{A}_f)$ taking the values ± 1. Define $X(\hat{\mathbf{Z}})_+ = \{x_f \in X(\hat{\mathbf{Z}}) : \nu_f(x_f) = +1\}$. We have

$$\int_{X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})} \delta(x)\, dm = 2m_\infty(X(\mathbf{R})^T)\, m_f(X(\hat{\mathbf{Z}})_+).$$

Set $c_X = 2m_f(X(\hat{\mathbf{Z}})_+)/m_f(X(\hat{\mathbf{Z}}))$, then $0 \leqslant c_X \leqslant 2$ and

$$\int_{X(\mathbf{R})^T \times X(\hat{\mathbf{Z}})} \delta(x)\, dm = c_X m_\infty(X(\mathbf{R})^T)\, m_f(X(\hat{\mathbf{Z}})) = c_X \mu_\infty(T, X)\, \mathfrak{S}(X).$$

Using Theorem 1.2, we see that

$$N(T, X) \sim c_X \mu_\infty(T, X)\, \mathfrak{S}(X) \qquad \text{as} \quad T \to \infty.$$

2. AN EXAMPLE OF $c_X = 0$

Let

$$f_1(x_1, x_2, x_3) = -9x_1^2 + 2x_1x_2 + 7x_2^2 + 2x_3^2, \qquad q = 1.$$

This example was mentioned in [2, 6.4.1]. Here we provide a detailed exposition.

Consider the variety X defined by the equation $f_1(x) = q$. We have $f_1(-\frac{1}{2}, \frac{1}{2}, 1) = 1$. It follows that f_1 represents 1 over $\mathbf{R}$ and over $\mathbf{Z}_p$ for $p > 2$.

We have $f_1(4, 1, 1) = -127 \equiv 1 \pmod{2^7}$. We prove that f_1 represents 1 over $\mathbf{Z}_2$. Define a polynomial of one variable $F(Y) = f_1(4, 1, Y) - 1$, $F \in \mathbf{Z}_2[Y]$. Then $F(1) = -2^7$, $|F(1)|_2 = 2^{-7}$, $F'(Y) = 4Y$, $|F'(1)^2|_2 = 2^{-4}$, $|F(1)|_2 < |F'(1)^2|_2$. By Hensel's lemma (cf. [11, II-§2, Proposition 2]) F has a root in $\mathbf{Z}_2$. Thus f_1 represents 1 over $\mathbf{Z}_2$.

Now we prove that f_1 does not represent 1 over $\mathbf{Z}$. I know the following elementary proof from D. Zagier.

We prove the assertion by contradiction. Assume on the contrary that

$$-9x_1^2 + 2x_1x_2 + 7x_2^2 + 2x_3^2 = 1 \qquad \text{for some} \quad x_1, x_2, x_3 \in \mathbf{Z}.$$

We may write this equation as follows:

$$2x_3^2 - 1 = (x_1 - x_2)^2 + 8(x_1 - x_2)(x_1 + x_2).$$

The left hand side is odd, hence $x_1 - x_2$ is odd and therefore $x_1 + x_2$ is odd. We have $(x_1 - x_2)^2 \equiv 1 \pmod 8$. Hence the right hand side is congruent to $1 \pmod 8$. We see that x_3 is odd, hence $2x_3^2 - 1 \equiv 1 \pmod{16}$. But

$$8(x_1 - x_2)(x_1 + x_2) \equiv 8 \qquad \pmod{16}.$$

It follows that

$$(x_1 - x_2)^2 \equiv 9 \qquad \pmod{16}$$

$$x_1 - x_2 \equiv \pm 3 \qquad \pmod 8.$$

Therefore $x_1 - x_2$ must have a prime factor $p \equiv \pm 3 \pmod 8$. Hence $2x_3^2 - 1$ has a prime factor $p \equiv \pm 3 \pmod 8$. On the other hand, if $p \mid (2x_3^2 - 1)$, then

$$2x_3^2 \equiv 1 \qquad (\text{mod } p)$$

and 2 is a square modulo p, $(\frac{2}{p}) = 1$. By the quadratic reciprocity law $p \equiv \pm 1 \pmod 8$. Contradiction. We have proved that f_1 does not represent 1 over $\mathbf{Z}$, hence $N(T, X) = 0$ for all T.

On the other hand,

$$\mathfrak{S}(X)\, \mu_\infty(T, X) = m_f(X(\hat{\mathbf{Z}}))\, m_\infty(X(\mathbf{R})^T).$$

Since $X(\hat{\mathbf{Z}})$ is a nonempty open subset in $X(\mathbf{A}_f)$, $m_f(X(\hat{\mathbf{Z}})) > 0$. Now $m_\infty(X(\mathbf{R})^T) \to \infty$ as $T \to \infty$. Hence $\mathfrak{S}(X)\, \mu_\infty(T, X) \to \infty$ as $T \to \infty$, and thus $c_X = 0$.

3. PROOF OF THEOREM 0.1

LEMMA 3.1. *Let k be a field of characteristic different from 2, and let V be a finite-dimensional vector space over k. Let f be a non-degenerate quadratic form on V. Let $u \in \mathrm{GL}(V)(k)$, $f' = u^* f$. Then the map $y \mapsto uy : V \to V$ takes the orbits of* $\mathrm{Spin}(f)(k)$ *in V to the orbits of* $\mathrm{Spin}(f')(k)$.

Proof. Let $x \in V$, $f(x) \neq 0$. The reflection (symmetry) $r_x = r_{f,x} : V \to V$ is defined by

$$r_x(y) = y - \frac{2B(x, y)}{f(x)}\, x, \qquad y \in V,$$

where B is the symmetric bilinear form on V associated with f. Every $s \in \mathrm{SO}(f)(k)$ can be written as

$$s = r_{x_1} \cdots r_{x_l} \tag{3}$$

cf. [12, Theorem 43:3]. The spinor norm $\theta(s)$ of s is defined by

$$\theta(s) = f(x_1) \cdots f(x_l) \qquad (\text{mod } k^{*2}) \in k^*/k^{*2}$$

and it does not depend on the choice of the representation given by (3), cf. [12, §55]. Let $\Theta(f)$ denote the image of $\mathrm{Spin}(f)(k)$ in $\mathrm{SO}(f)(k)$. Then $s \in \mathrm{SO}(f)(k)$ is contained in $\Theta(f)$ if and only if $\theta(s) = 1$, cf. [13, III-3.2] or [3, Chap. 10, Theorem 3.3].

Now let u, f' be as above. Then $r_{f',\,ux} = u r_{f,\,x} u^{-1}$, $f'(ux) = f(x)$, and so $\theta_{f'}(usu^{-1}) = \theta_f(s)$. We conclude that $u\Theta(f)\,u^{-1} = \Theta(f')$ and that the map $y \mapsto uy$ takes the orbits of $\Theta(f)$ in V to the orbits of $\Theta(f')$. ∎

Let f, f' be integral-matrix quadratic forms on $\mathbf{Z}^n$ and assume that f' is in the genus of f. Then there exists $u \in \mathrm{GL}_n(\mathbf{R} \times \hat{\mathbf{Z}})$ such that $f'(x) = f(u^{-1}x)$ for $x \in \mathbf{A}^n$. Let $q \in \mathbf{Z}$, $q \neq 0$. Let X denote the affine quadric $f(x) = q$, and X' denote the quadric $f'(x) = q$.

LEMMA 3.2. *The map $x \mapsto ux : \mathbf{A}^n \to \mathbf{A}^n$ takes $X(\mathbf{R} \times \hat{\mathbf{Z}})$ to $X'(\mathbf{R} \times \hat{\mathbf{Z}})$ and takes orbits of* $\mathrm{Spin}(f)(\mathbf{A})$ *in $X(\mathbf{A})$ to orbits of* $\mathrm{Spin}(f')(\mathbf{A})$ *in $X'(\mathbf{A})$.*

Proof. Let A denote the matrix of f, and A' denote the matrix of f'. We have

$$(u^{-1})^t\,Au^{-1} = A', \qquad A = u^tA'u.$$

The variety X is defined by the equation $x^tAx = q$, and X' is defined by $x^tA'x = q$. One can easily check that the map $x \mapsto ux$ takes $X(\mathbf{R} \times \hat{\mathbf{Z}})$ to $X'(\mathbf{R} \times \hat{\mathbf{Z}})$ and $X(\mathbf{A})$ to $X'(\mathbf{A})$.

In order to prove that the map $x \mapsto ux : X(\mathbf{A}) \to X'(\mathbf{A})$ takes the orbits of $\mathrm{Spin}(f)(\mathbf{A})$ to the orbits of $\mathrm{Spin}(f')(\mathbf{A})$, it suffices to prove that the map $x \mapsto u_v x : X(\mathbf{Q}_v) \to X'(\mathbf{Q}_v)$ takes the orbits of $\mathrm{Spin}(f)(\mathbf{Q}_v)$ to the orbits of $\mathrm{Spin}(f')(\mathbf{Q}_v)$ for every v, where u_v is the v-component of u. This last assertion follows from Lemma 3.1. ∎

PROPOSITION 3.3. *Let f' and q be as in Theorem* 0.1, *in particular f' represents q over $\mathbf{Z}_v$ for any v (we set $\mathbf{Z}_\infty = \mathbf{R}$), but not over $\mathbf{Z}$. Let X' be the quadric defined by $f'(x) = qv$. Then $X'(\mathbf{R} \times \hat{\mathbf{Z}})$ is contained in one orbit of* $\mathrm{Spin}(f')(\mathbf{A})$.

Proof. Set $G' = \mathrm{Spin}(f')$. We prove that $X'(\mathbf{Z}_v)$ is contained in one orbit of $G'(\mathbf{Q}_v)$ for every v by contradiction. Assume on the contrary that for some v the set $X'(\mathbf{Z}_v)$ has nontrivial intersection with two orbits of $G'(\mathbf{Q}_v)$. Then ν_v takes both values $+1$ and -1 on $X'(\mathbf{Z}_v)$. It follows that ν takes both values $+1$ and -1 on $X'(\mathbf{R} \times \hat{\mathbf{Z}})$. Hence by Lemma 1.3 X' has infinitely many $\mathbf{Z}$-points. This contradicts to the assumption that f' does not represent q over $\mathbf{Z}$. ∎

Proof of Theorem 0.1. Let $u \in \mathrm{GL}_3(\mathbf{R} \times \hat{\mathbf{Z}})$ be such that $f'(x) = f(u^{-1}x)$. Let X, X' be as above, in particular X' has no $\mathbf{Z}$-points. By Proposition 3.3 $X'(\mathbf{R} \times \hat{\mathbf{Z}})$ is contained in one orbit of $\mathrm{Spin}(f')(\mathbf{A})$. It follows from Lemma 3.2 that $X(\mathbf{R} \times \hat{\mathbf{Z}})$ is contained in one orbit of $\mathrm{Spin}(f)(\mathbf{A})$. Since f represents q over $\mathbf{Z}$, this orbit has $\mathbf{Q}$-rational points, and ν equals $+1$ on

$X(\mathbf{R}\times\hat{\mathbf{Z}})$. Thus δ equals 2 on $X(\mathbf{R}\times\hat{\mathbf{Z}})$, and by Formulas (1) and (2) of Section 1.4 $N(T,X)\sim 2\mathfrak{S}(X)\,\mu_\infty(T,X)$. ▮

4. PROOF OF THEOREM 0.2

We prove Theorem 0.2. We define an involution τ_∞ of $X(\mathbf{R})$ by $\tau_\infty(x)=-x$, $x\in X(\mathbf{R})\subset\mathbf{R}^3$. Since $f(x)=f(-x)$, τ_∞ is well defined, i.e takes $X(\mathbf{R})$ to itself. Since $|-x|=|x|$, τ_∞ takes $X(\mathbf{R})^T$ to itself. We define an involution τ of $X(\mathbf{A})$ by defining τ as τ_∞ on $X(\mathbf{R})$ and as 1 on $X(\mathbf{Q}_p)$ for all prime p. Then τ respects the Tamagawa measure m on $X(\mathbf{A})$.

By assumption $X(\mathbf{R})$ has two connected components. These are the two orbits of $\mathrm{Spin}(f)(\mathbf{R})$. The involution τ_∞ of $X(\mathbf{R})$ interchanges these two orbits. Thus we have

$$\nu_\infty(\tau_\infty(x_\infty))=-\nu_\infty(x_\infty)\qquad\text{for all}\quad x_\infty\in X(\mathbf{R})\tag{4}$$

$$\nu(\tau(x))=-\nu(x)\qquad\text{for all}\quad x\in X(\mathbf{A}).\tag{5}$$

Let $X(\mathbf{R})_1$ and $X(\mathbf{R})_2$ be the two connected components of $X(\mathbf{R})$. Set

$$X(\mathbf{R})_1^T=X(\mathbf{R})_1\cap X(\mathbf{R})^T,\qquad X(\mathbf{R})_2^T=X(\mathbf{R})_2\cap X(\mathbf{R})^T$$

Then τ interchanges $X(\mathbf{R})_1^T\times X(\hat{\mathbf{Z}})$ and $X(\mathbf{R})_2^T\times X(\hat{\mathbf{Z}})$. From Formula (5) in this section we have

$$\int_{X(\mathbf{R})_1^T\times X(\hat{\mathbf{Z}})}\nu(x)\,dm=-\int_{X(\mathbf{R})_2^T\times X(\hat{\mathbf{Z}})}\nu(x)\,dm,$$

hence

$$\int_{X(\mathbf{R})^T\times X(\hat{\mathbf{Z}})}\nu(x)\,dm=0.$$

Since $\delta(x)=\nu(x)+1$, we obtain

$$\int_{X(\mathbf{R})^T\times X(\hat{\mathbf{Z}})}\delta(x)\,dm=\int_{X(\mathbf{R})^T\times X(\hat{\mathbf{Z}})}dm=m(X(\mathbf{R})^T\times X(\hat{\mathbf{Z}})),$$

and $m(X(\mathbf{R})^T\times X(\hat{\mathbf{Z}}))=\mathfrak{S}(X)\,\mu_\infty(T,X)$. By Theorem 1.2

$$N(T,X)\sim\int_{X(\mathbf{R})^T\times X(\hat{\mathbf{Z}})}\delta(x)\,dm.$$

Thus $N(T,X)\sim\mathfrak{S}(X)\,\mu_\infty(T,X)$ as $T\to\infty$, i.e. $c_X=1$. ▮

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