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Fractional differentiation in the self-affine case I – Random functions

N. Patzschke and M. Zähle

Friedrich-Schiller-Universität, Jena, Germany

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The invariance structure of self-affine functions and measures leads to the concept of fractional Cesáro derivatives and densities, respectively. In the present paper the case of random functions from $\mathbb{R}^p$ into $\mathbb{R}^q$ is considered. It is shown that the corresponding derivatives exist a.s. and equal a constant in the ergodic case. Part II will deal with the class of self-similar extremal processes and certain extensions. In Part III the fractional density of the Cantor measure will be evaluated, and arbitrary self-similar random measures will be treated in Part IV. There exists a deeper connection to fractional differentiation in the theory of function spaces which will be established elsewhere.

Introduction

A well-known theorem of Paley, Wiener and Zygmund (1933) states that one-dimensional Brownian motion $B(t)$ is nowhere differentiable with probability 1. The sample paths and the level sets are fractals of Hausdorff dimensions $\frac{3}{2}$ and $\frac{1}{2}$, respectively. Moreover, their exact Hausdorff measures which agree with the occupation measures and the local time measures, respectively, are explicitly determined by the local behaviour of $B(t)$ (for references cf. Taylor, 1986; Kahane, 1985). The latter, i.e., the law of iterated logarithm

$$\limsup_{t\to 0} \frac{B(t)}{t^{1/2}(2\ln|\ln t|)^{1/2}} = 1 \quad \text{a.s.},$$

results essentially from the following two characterizing properties of Brownian motion: $B(t)$ has stationary independent increments with $EB(1)^2 = 1$ and it is $\frac{1}{2}$-self-similar, i.e.,

$$\rho^{-1/2}B(\rho(\cdot)) \overset{\mathrm{d}}{=} B(\cdot), \quad \rho > 0.$$

Although this scale invariance suggests fractional differentiation with exponent $\frac{1}{2}$, the law of iterated logarithm shows that it cannot hold in the usual sense of function theory. However, we show in this paper that the fractional derivatives in the mean

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \frac{B(t+\mathrm{e}^{-r}) - B(t)}{\mathrm{e}^{-r/2}}\,\mathrm{d}r$$

Correspondence to: Dr. M. Zähle, Mathematische Fakultät, Friedrich-Schiller-Universität, Universitätshochhaus, 17.0G, O-6900 Jena, Germany.

exist and (by symmetry) are equal to zero at almost all t with probability 1. For the absolute derivatives of B the corresponding statement looks more interesting:

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \frac{|B(t+\mathrm{e}^{-r})-B(t)|}{\mathrm{e}^{-r/2}}\,\mathrm{d}r = E|B(1)|$$

for almost all t with probability 1.

We first prove the a.s. existence of the derivative at $t=0$. As an immediate consequence of stationary increments and measurability we obtain the assertion for almost all t with respect to Lebesgue measure. Moreover, the statement remains valid for almost all t from the zero set of Brownian motion with respect to its exact Hausdorff measure.

Similarly, if $B^*(t)=\sup_{0\le s\le t} B(s)$ one can prove (cf. Part II) that

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \frac{B^*(t+\mathrm{e}^{-r})-B^*(t)}{\mathrm{e}^{-r/2}}\,\mathrm{d}r = EB^*(1)$$

at almost all record instants t where $B^*(t)=B(t)$ with probability 1.

Our approach was inspired by a talk of *T.* Bedford on fractional densities of the middle-third Cantor set and of the zero set of Brownian motion held at the 18th Winter School on Abstract Analysis in Srni (1990) as well as by a paper of U. Zähle (1991).

More generally, let $X(t)$ be a measurable random function from $\mathbb{R}^p$ into $\mathbb{R}^q$ with finite expectations which is D-scale invariant (self-similar), i.e., it has stationary increments and satisfies

$$\rho^{-D}X(\rho(\cdot)) \overset{\mathrm{d}}{=} X(\cdot), \quad 0<\rho<1,$$

for some $D>0$. Then the random fractional derivative in direction $v\in\mathbb{R}^p$ at almost all $t\in\mathbb{R}^p$ is determined by the average

$$d_D X(t)v = \lim_{R\to\infty}\frac{1}{R}\int_0^R \frac{X(t+\mathrm{e}^{-r}v)-X(t)}{\mathrm{e}^{-rD}}\,\mathrm{d}r,$$

with probability 1. $d_D X(\cdot)v$ is explicitly calculated and under a certain ergodicity assumption it agrees with the constant $EX(v)$.

In full generality our result is as follows (cf. Theorems 1 and 2). Let U, V be linear contractions in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, whose eigenvalues are all positive. Then the powers U^r, V^r for arbitrary real r are meaningful. Let X be a measurable random function from $\mathbb{R}^p$ into $\mathbb{R}^q$ with finite expectations which is (U, V)-self-affine, i.e., it has stationary increments and satisfies

$$V^{-1}X(U(\cdot)) \overset{\mathrm{d}}{=} X(\cdot).$$

In this case there exists at almost all $t\in\mathbb{R}^p$ the corresponding affine directional derivative

$$d_{U,V}X(t)v = \lim_{R\to\infty}\frac{1}{R}\int_0^R V^{-r}(X(t+U^r v)-X(t))\,\mathrm{d}r,$$

with probability 1. $d_{U,V}X(\cdot)v$ is explicitly computed and equals a constant for ergodic distributions, in particular, for all X as above with independent increments. The phrase 'almost all t' may be understood in the Lebesgue sense as well as in terms of local time measures if the latter exist.

Further examples of D-self-similar random processes are fractional Brownian motion and certain stable processes. For references and constructions see Vervaat (1985). Recall that we additionally assume finite expectations so that the example of O'Brien and Vervaat (1985) does not fit our model. For α-stable $(1/\alpha)$-self-similar processes this assumption means that $1<\alpha<2$. Self-affine random functions arise, e.g., from linear transformations of vector functions whose coordinates are independent D-scale invariant random functions with different D.

1. Palm relationships for random functions with stationary increments

The aim of this section is to make available tools for extending the existence of the fractional derivatives at the point zero to almost all points from the zero level sets of self-affine random local time functions. Thereby we will show that the distributions of arbitrary random functions with stationary increments are the Palm distributions of their stationary flows with respect to the kernels defined by the occupation measure as well as by the local time measure (if it exists). This interpretation allows the transition mentioned above from the point zero to almost all points of the sets carrying these random measures without any further assumptions such as strong Markov property used in Bedford and Fisher (1990). The same approach will be applied in Part II of this paper to the case of random functions which have stationary increments with respect to random time measures, in particular, to the case of extremal processes. The idea of Palm distributions also plays an essential role in the case of fractional densities of self-similar random measures (cf. Part IV).

We use here concepts and notations from U. Zähle (1991). Let $\mathscr{R}^n$ be the Borel σ-algebra and $\mathscr{L}^n$ be Lebesgue measure in Euclidean space $\mathbb{R}^n$. $M(n)$ is the family of Radon measures on $[\mathbb{R}^n, \mathscr{R}^n]$. map$(p,q)$ denotes the space of functions from $\mathbb{R}^p$ to $\mathbb{R}^q$, mble(p,q) the subspace of $(\mathscr{R}^p, \mathscr{R}^q)$-measurable functions, and *map*(p,q) the Daniel–Kolmogorov σ-algebra $(\mathscr{R}^q)^{\mathbb{R}^p}$. Define the groups $(T_{s,z})_{(s,z)\in\mathbb{R}^p\times\mathbb{R}^q}$, $(\Theta_s)_{s\in\mathbb{R}^p}$ of measurable transformations of map(p,q) by

$$(T_{s,z}f)(t)=f(t+s)-z, \quad t\in\mathbb{R}^p,$$

and

$$\Theta_s f = T_{s,f(s)}f,$$

respectively.

For any $\mathscr{L}^p$-measurable $f\in$ map(p,q) the *occupation measure* $\nu_f\in M(p+q)$ may be determined by

$$\nu_f(A\times B)=\mathscr{L}^p\{t\in A: f(t)\in B\}, \quad A\in\mathscr{R}^p, \ B\in\mathscr{R}^q.$$

It is concentrated on graph(f). Note that

$$\nu_{T_{s,z}f}(\cdot)=\nu_f((\cdot)+(s,z)),\quad (s,z)\in\mathbb{R}^p\times\mathbb{R}^q.$$

We call f as before a *local time function* (LT) if for $\mathscr{L}^q$-almost all y and for $y=0$ there exist the vague measure limits

$$\tau_f(y,\cdot)=\lim_{\varepsilon\to 0}\mathscr{L}^q(B(0,\varepsilon))^{-1}\nu_f((\cdot)\times B(y,\varepsilon)),$$

where $B(y,\varepsilon)$ is the ball with centre y and radius ε. $\tau_f(y,\cdot)$ is said to be the *local time measure* of f at level y. Obviously, we have for those y,

$$\tau_f(y,\cdot)=\tau_{T_{0,y}f}(0,\cdot).$$

For brevity denote the local time measure of f at level zero $\tau_f(0,\cdot)$ by τ_f. The well-known relation between the occupation measure and the local time measure of an LT function f may then be formulated as follows:

$$\nu_f=\int\int_{\mathbb{R}^q}1_{(\cdot)}(t,y)\tau_{T_{0,y}f}(\mathrm{d}t)\mathscr{L}^q(\mathrm{d}y).\tag{1}$$

We now turn to stochastic versions. By a *measurable* ($\mathscr{L}^p$*-measurable*) *random function* X with values in $\mathbb{R}^q$ we mean a mapping from a basic probability space $[\Omega,\mathscr{F},P]$ into map(p,q) such that the map $(\omega,t)\to X_\omega(t)$ is measurable with respect to $\mathscr{F}\otimes\mathscr{R}^p$ (to the $P\times\mathscr{L}^p$-completion of $\mathscr{F}\otimes\mathscr{R}^p$). The distribution of X will be denoted by P_X. Note that in this case P_X-almost all $f\in$ map(p,q) are measurable ($\mathscr{L}^p$-measurable) and ν_X is a random measure. An $\mathscr{L}^p$-measurable X is said to be of *local time* (LT) if P_X-almost all $\mathscr{L}^p$-measurable f are LT. For such X, τ_X is a random measure. (In this paper we are only interested in invariance properties of LT functions with stationary increments. We will not deal with the difficult problem of checking the LT condition. Note that for LT the relation $p\geq q$ is necessary.)
Now suppose that the $\mathscr{L}^p$-measurable random function X has *stationary increments*, i.e., the distribution of Θ_tX does not depend on $t\in\mathbb{R}^p$, and that $X(0)=0$ with probability 1. Then there exists a biunique relationship between the distribution P_X and its *flow* H_X, i.e., the (non-finite) $(T_{s,z})$-invariant quasi-distribution on [map(p,q), *map*(p,q)] defined by

$$H_X=\int\int 1_{(\cdot)}(T_{0,y}f)L^q(\mathrm{d}y)P_X(\mathrm{d}f).$$

(Note that H_X is σ-finite and is also concentrated on $\mathscr{L}^p$-measurable functions.) P_X may be determined by H_X through a 'random shift' according to the corresponding occupation measure, i.e., P_X is the *Palm distribution* of H_X with respect to the *kernel* ν:

Proposition 1.

$$P_X = \mathscr{L}^{p+q}(C)^{-1} \int\int_C 1_{(\cdot)}(T_{s,z}f)\nu_f(\mathrm{d}(s,z))H_X(\mathrm{d}f)$$

for arbitrary $C \in \mathscr{R}^p \otimes \mathscr{R}^q$ *with* $0 < \mathscr{L}^{p+q}(C) < \infty$.

Proof. By definition of H_X the right-hand side of the assertion equals

$$\mathscr{L}^{p+q}(C)^{-1} \int\int\int 1_C(s,z)1_{(\cdot)}(T_{s,z+y}f)\nu_{T_{0,y}f}(\mathrm{d}(s,z))\mathscr{L}^q(\mathrm{d}y)P_X(\mathrm{d}f)$$

$$= \mathscr{L}^{p+q}(C)^{-1} \int\int\int 1_C(s,z-y)1_{(\cdot)}(T_{s,z}f)\nu_f(\mathrm{d}(s,z)\mathscr{L}^q(\mathrm{d}y)P_X(\mathrm{d}f)$$

$$\left(\text{since} \int g(s,z)\nu_{T_{t,y}f}(\mathrm{d}(s,z)) = \int g(s-t,z-y)\nu_f(\mathrm{d}(s,z))\right)$$

$$= \mathscr{L}^{p+q}(C)^{-1} \int\int\int 1_C(s,f(s)-y)1_{(\cdot)}(\Theta_s f)\mathscr{L}^p(\mathrm{d}s)\mathscr{L}^q(\mathrm{d}y)P_X(\mathrm{d}f)$$

(in view of the definition of ν_f)

$$= \mathscr{L}^{p+q}(C)^{-1} \int\int\int 1_C(s,y)\mathscr{L}^q(\mathrm{d}y)1_{(\cdot)}(\Theta_s f)\mathscr{L}^p(\mathrm{d}s)P_X(\mathrm{d}f)$$

(by Fubini and invariance of $\mathscr{L}^q$)

$$= \mathscr{L}^{p+q}(C)^{-1} \int\int\int 1_{(\cdot)}(f)P_X(\mathrm{d}f)1_C(s,y)\mathscr{L}^p(\mathrm{d}s)\mathscr{L}^q(\mathrm{d}y)$$

(by Fubini and stationary increments of X)

$$= P_X. \qquad \square$$

An extension of this relationship is the key for the results of Part II. It may be completed by a version for the local time measure τ_X if it exists. Suppose that X is LT. Then in view of the defintion of the flow for H_X-almost all $f \in \mathrm{map}(p,q)$ the local time measure τ_f at level zero exists. Moreover, P_X may also be interpreted as the *Palm distribution* of H_X with respect to the *kernel* τ:

Proposition 2. *Under the above conditions we have*

$$P_X = \mathscr{L}^p(A)^{-1} \int\int_A 1_{(\cdot)}(T_{t,0}f)\tau_f(\mathrm{d}t)H_X(\mathrm{d}f)$$

for arbitrary $A \in \mathscr{R}^p$ *with* $0 < \mathscr{L}^p(A) < \infty$.

Proof. By definition of H_X the right-hand side equals

$$\mathscr{L}^p(A)^{-1} \int\int\int_A 1_{(\cdot)}(T_{t,y}f)\tau_{T_{0,y}f}(\mathrm{d}t)\mathscr{L}^q(\mathrm{d}y)P_X(\mathrm{d}f)$$

$$= \mathscr{L}^p(A)^{-1} \int\int 1_A(s)1_{(\cdot)}(T_{s,z}f)\nu_f(\mathrm{d}(s, z))P_X(\mathrm{d}f)$$

$$= \mathscr{L}^p(A)^{-1} \int\int 1_A(s)1_{(\cdot)}(\Theta_s f)\mathscr{L}^p(\mathrm{d}s)P_X(\mathrm{d}f)$$

$$= P_X,$$

following (1), definition of ν, and (Θ_s)-invariance of P_X. □

Remark. Special cases of Propositions 1 and 2 (for the distributions of the random measures ν_X and τ_X, respectively,) are proved in U. Zähle (1991). In the next section we need only Proposition 2, since ν_X is closely related to stationary increments.

2. Self-affine random functions

Let U, V be linear contractions in $\mathbb{R}^p$and $\mathbb{R}^q$, respectively, with strictly positive eigenvalues. Then for any real r the rth powers U^r and V^r make sense. For $r>0$ they are also contractions. Recall that a random function from $\mathbb{R}^p$ into $\mathbb{R}^q$ is said to be (U, V)-*self-affine* if it has stationary increments and if

$$V^{-1}X(U(\cdot)) \stackrel{\mathrm{d}}{=} X$$

(cf. U. Zähle (1991), $\stackrel{\mathrm{d}}{=}$ means equality of the distributions).

Proposition 3. *If X is self-affine then we have $X(0)=0$ with probability* 1.

Proof. The proof is obvious. □

In the special case of *D-scale invariant* X, i.e., when

$$\rho^{-D}X(\rho(\cdot)) \stackrel{\mathrm{d}}{=} X, \quad 0<\rho<1,$$

we may put $U=\mathrm{e}^{-1}\mathrm{id}$, $V=\mathrm{e}^{-D}\mathrm{id}$, $r=-\ln\rho$, and then X is (U^r, V^r)-self-affine for any $r>0$.

Definition. For any $f\in \mathrm{mble}(p, q)$ the mean (U, V)-*fractional derivative* of f at $t\in\mathbb{R}^p$ in direction $v\in\mathbb{R}^p$ is determined by

$$d_{U,V}f(t)v = \lim_{R\to\infty} \frac{1}{R}\int_0^R V^{-r}(f(t+U^r v)-f(t))\,\mathrm{d}r$$

if this limit exists.

We are interested in the random version $d_{U,V}X$ for measureable X as above. Let $S_{U,V}:\mathrm{map}(p,q)\to\mathrm{map}(p,q)$ be given by

$$S_{U,V}f=V^{-1}f(U(\cdot)),$$

and denote the pullback of the σ-algebra of $S_{U,V}$-invariant sets from $map(p,q)$ under the mapping X by $\mathscr{I}_{U,V}$. $\mathscr{I}^t_{U,V}$ is the corresponding σ-algebra if X is replaced by Θ_tX. It may be interpreted as the σ-algebra of events for X which are $S_{U,V}$-invariant in the shifted coordinate system with $(t, X(t))$ as origin.

Theorem 1. *Let X be a measurable (U, V)-self-affine random function with finite expectations. Then for any $v\in\mathbb{R}^p$ with probability 1 there exist at $\mathscr{L}^p$-almost all t including $t=0$ the fractional derivatives of X in direction v and may be computed by*

$$d_{U,V}X(t)v=E\left[\int_0^1 V^{-r}(X(t+U^rv)-X(t))\,\mathrm{d}r\,\middle|\,\mathscr{I}^t_{U,V}\right].$$

Moreover, if X is (U^r, V^r)-self-affine for all $r>0$ then we have with probability 1 that

$$d_{U,V}X(t)v=E\left[X(t+v)-X(t)\,\middle|\,\bigcap_{r>0}\mathscr{I}^t_{U^r,V^r}\right]$$

for $\mathscr{L}^p$-almost all t including $t=0$.

Proof. We first consider the case $t=0$ and show that

$$d_{U,V}X(0)v=E\left[\int_0^1 V^{-r}(X(U^rv))\,\mathrm{d}r\,\middle|\,\mathscr{I}_{U,V}\right]$$

with probability 1.

By definition of the map $S_{U,V}$ we get for any natural N,

$$\begin{aligned}\frac{1}{N}\int_0^N V^{-r}X(U^rv)\,\mathrm{d}r&=\frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 V^{-(n+r)}X(U^{n+r}v)\,\mathrm{d}r\\&=\frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 V^{-r}S^n_{U,V}X(U^rv)\,\mathrm{d}r.\end{aligned}$$

The distribution P_X is invariant under $S_{U,V}$ and therefore according to Birkhoff's ergodic theorem the last expression tends to

$$E\left[\int_0^1 V^{-r}X(U^rv)\,\mathrm{d}r\,\middle|\,\mathscr{I}_{U,V}\right]$$

as $N\to\infty$ with probability 1. In order to prove convergence of the Cesáro averages for arbitrary $R\to\infty$ it suffices to consider the positive and the negative parts of the coordinates of the integrands separately. For them we make use of the inequality

$$\frac{[R]}{R}\frac{1}{[R]}\int_0^{[R]}\cdots\leqslant\frac{1}{R}\int_0^R\cdots\leqslant\frac{[R]+1}{R}\frac{1}{[R]+1}\int_0^{[R]+1}\cdots.$$

By the above convergence which holds for the $S_{U,V}$-transformations of arbitrary integrable functions of X the limits of the right- and the left-hand sides as $R\to\infty$ exist and coincide. Thus, the assertion is true.

Similarly, if X is (U^r, V^r)-self-affine for all $r>0$ then we may apply the individual ergodic theorem to the transformation group $(S_{U^r,V^r})_{r\in\mathbb{R}^1}$ (which is measurable on the space of measurable functions) in order to obtain that

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R V^{-r}X(U^r v)\,\mathrm{d}r = E\left[X(v)\,\middle|\,\bigcap_r \mathscr{I}_{U^r,V^r}\right]$$

with probability 1, where $\bigcap_r \mathscr{I}_{U^r,V^r}$ agrees with the σ-algebra of events for X which are S_{U^r,V^r}-invariant for all r.

Finally, let D be the set of those $(\omega, t)\in\Omega\times\mathbb{R}^p$ where $d_{U,V}X_\omega(t)v$ exists and equals the expression in the assertion. Note that it is $\mathscr{F}\otimes\mathscr{R}^p$-measurable and may be represented by $D=\{(\omega,t): (X_\omega, t)\in\tilde{D}\}$ for some $\tilde{D}\in map(p,q)\otimes\mathscr{R}^p$. Put $D_0=\{f\in \mathrm{map}(p,q): (f,0)\in\tilde{D}\}$. Then we get for the complement sets

$$\int\!\!\int 1_{D^c}(\omega,t)\mathscr{L}^p(\mathrm{d}t)P(\mathrm{d}\omega)=\int\!\!\int 1_{D^c}(\omega,t)P(\mathrm{d}\omega)\mathscr{L}^p(\mathrm{d}t)$$
$$=\int E1_{D_0^c}(\Theta_t X)\mathscr{L}^p(\mathrm{d}t).$$

In view of the (Θ_t)-invariance of P_X and the above result for $t=0$ the function under the last integral is identically zero. Consequently, $\int 1_{D^c}(\omega,t)\mathscr{L}^p(\mathrm{d}t)=0$ for P-almost all ω, i.e., for these ω we have $(\omega,t)\in D$ for $\mathscr{L}^p$-almost all t. $\square$

Remark. By similar arguments as in the last step of the preceding proof for fixed $t\in\mathbb{R}^p$, $d_{U,V}X(t)v$ is determined for $\mathscr{L}^p$-almost all $v\in\mathbb{R}^p$ with probability 1. Note that the differential $d_{U,V}X(t)$ is a non-linear function which is S_{U^r,V^r}-invariant in the $\mathscr{L}^p$-sense for all $r>0$.

Theorem 2. *If X is as in Theorem 1 and possesses the LT property then the statements remain valid for τ_X-almost all t (instead of $\mathscr{L}^p$-almost all t).*

Proof. Let D_0 be as in the proof of Theorem 1. (It is the set of those functions where the derivative at zero exists and equals the corresponding expression.) Theorem 1 implies $P_X(D_0^c)=0$. From Proposition 2 we now infer

$$0=P_X(D_0^c)=\mathscr{L}^p(A)^{-1}\int\!\!\int_A 1_{D_0^c}(T_{t,0}f)\tau_f(\mathrm{d}t)H_X(\mathrm{d}f).$$

Hence, for H_X-almost all f,

$$\int_A 1_{D_0^c}(T_{t,0}f)\tau_f(\mathrm{d}t)=0.$$

Since A is arbitrary we conclude that

$$\int 1_{D_0^c}(T_{t,0}f)\tau_f(\mathrm{d}t)=0$$

for H_X-almost all f. Using that for any $s\in\mathbb{R}^p$,

$$\int 1_{D_0^c}(T_{t+s,0}f)\tau_{T_{s,0}f}(\mathrm{d}t)=\int 1_{D_0^c}(T_{t,0}f)\tau_f(\mathrm{d}t),$$

we obtain for f as before

$$\int_A\int 1_{D_0^c}(T_{t+s},{}_0f)\tau_{T_{s,0}f}(\mathrm{d}t)\tau_f(\mathrm{d}s)=0.$$

Consequently, Proposition 2 implies

$$\begin{aligned}0&=\mathscr{L}^p(A)^{-1}\int\int_A\int 1_{D_0^c}(T_{t+s,0}f)\tau_{T_{s,0}f}(\mathrm{d}t)\tau_f(\mathrm{d}x)H_X(\mathrm{d}f)\\&=\int\int 1_{D_0^c}(T_{t,0}f)\tau_f(\mathrm{d}t)P_X(\mathrm{d}f).\end{aligned}$$

Therefore we get with probability 1,

$$\int\int 1_{D_0^c}(T_{t,0}X)\tau_X(\mathrm{d}t)=0.$$

Since τ_X is concentrated on the zero level set of X this yields the assertion. □

3. Ergodicity conditions

Recall that the proof of Theorem 1 is based on the ergodic theorem for the mappings $S_{U,V}$ and $(S_{U^r,V^r})_{r\in\mathbb{R}^1}$, respectively. If P_X is ergodic with respect to these transformations then the limits in the ergodic theorem are equal to the expectations of the corresponding functions of X. Thus we obtain the following.

Corollary. *If X is as in Theorem 2 and ergodic under the corresponding affinities then the fractional derivatives are constant:*

$$d_{U,V}X(t)v=E\int_0^1 V^{-r}X(U^rv)\,\mathrm{d}r$$

in the (U,V)-self-affine case and

$$d_{U,V}X(t)v=EX(v)$$

in the (U^r,V^r)-self-affine case for $\mathscr{L}^p$-almost all t inclusive $t=0$ with probability 1. *The same result is true in the LT case with respect to the local time measure τ_X.* □

We now will derive a natural sufficient condition for ergodicity. Recall that X has stationary increments, i.e., P_X is invariant under the flow $(\Theta_t)_{t\in\mathbb{R}^p}$. For $a>0$ let $map^a(p,q)$ be the σ-algebra on map(p,q) generated by the increments of the functions outside the ball in $\mathbb{R}^p$ with centre 0 and radius a. As usual, P_X is said to be *uniformly* (Θ_t)*-mixing* if for any $A\in map(p,q)$,

$$\lim_{a\to\infty}\sup_{B\in map^a(p,q)}|P_X(A\cap B)-P_X(A)P_X(B)|=0.$$

Proposition 4. *Any uniformly (Θ_t)-mixing (U,V)-self-affine measurable random function X is ergodic with respect to the mapping $S_{U,V}$. (The analogous statement holds for the flow (S_{U^r,V^r}).)*

Proof. We will show the sharper mixing relation

$$\lim_{n\to\infty}P_X(A\cap S^n_{U,V}B)=P_X(A)P_X(B)$$

for any $A,B\in map(p,q)$. By the structure of the σ-algebra $map(p,q)$ and since $X(0)=0$ with probability 1 there exist some $B^\alpha\in map^\alpha(p,q)$ such that $B^\alpha\supset B$ and

$$\lim_{\alpha\to 0}P_X(B^\alpha\backslash B)=0.$$

From the estimation

$$\begin{aligned}&|P_X(A\cap S^n_{U,V}B)-P_X(A)P_X(B)|\\&\quad=|P_X(A\cap S^n_{U,V}B^\alpha)-P_X(A\cap S^n_{U,V}(B^\alpha\backslash B))+P_X(A)P_X(B^\alpha\backslash B)-P_X(A)P_X(B^\alpha)|\\&\quad\leqslant|P_X(A\cap S^n_{U,V}B^\alpha)-P_X(A)P_X(B^\alpha)|+2P_X(B^\alpha\backslash B),\end{aligned}$$

we conclude that it suffices to prove the asserted mixing relation for A and B^α when α is fixed. Note that

$$S^n_{U,V}B^\alpha=\{V^{-n}f(U^n(\cdot)):f\in B^\alpha\},$$

and U is a non-singular linear contraction.

Hence, for any $a>0$ there exists an $n(a)$ such that for all $n>n(a)$,

$$S^n_{U,V}B^\alpha\in map^a(p,q).$$

Thus, the uniform (Θ_t)-mixing condition implies the convergence

$$\lim_{n\to\infty}P_X(A\cap S^n_{U,V}B^\alpha)=P_X(A)P_X(B^\alpha).\qquad\square$$

Remark. It is easy to verify that any measurable random function with stationary *independent increments* is uniformly (Θ_t)-mixing. Therefore self-affine functions of this type possess constant fractional derivatives (cf. the corollary).

Note that all our results remain valid if we replace the fractional derivatives by their absolute variants

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R |V^{-r}(f(t+U^r v)-f(t))|\,\mathrm{d}r.$$

(Then in the explicit expressions of Theorem 1 the corresponding absolute value signs have to be inserted.)

The ideas of this paper may also be applied to the case of fractional densities of self-affine random measures. Moreover, our appoach carries over to fractional differentiation of deterministic 'self-similar' functions by means of a suitable randomization. This will be demonstrated in a forthcoming part. In particular, we will extend results of Bedford and Fisher (1990).

References

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