Computers and Mathematics with Applications 60 (2010) 1214–1219

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The contraction principle for set valued mappings on a metric space with a graph

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ARTICLE INFO

Article history:
Received 18 November 2009
Received in revised form 31 May 2010
Accepted 1 June 2010

Keywords:
Fixed point
Directed graph
Metric space
Set valued mapping

ABSTRACT

Let (X, d) be a metric space and $F : X \rightsquigarrow X$ be a set valued mapping. We obtain sufficient conditions for the existence of a fixed point of the mapping F in the metric space X endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X and the set of edges of G is $E(G) = \{(x, y) : (x, y) \in X \times X\}$.

1. Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: Banach's contraction principle [1, Theorem 2.1] and Tarski's fixed point theorem [2,3]. Generalizing the Banach contraction principle for set valued mapping to metric spaces, Nadler [4] obtained the following result:

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space and $F : X \rightsquigarrow X$ be a set valued mapping such that $F(x)$ is a nonempty closed bounded subset of X. If there exists a $\kappa \in (0, 1)$ such that*

$$D(F(x), F(y)) \le \kappa d(x, y), \quad \text{for all } x, y \in X,$$

where D is the Hausdorff metric on $CB(X)$, then F has a fixed point in X.

A number of extensions/generalizations of Nadler's theorem were obtained by different authors; see for instance [5–13] and references cited therein. The Tarski theorem was extended to set valued mapping by different authors; see [14–16].

Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [17] and they proved the following result:

Theorem 1.2 ([17]). *Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

1. *There exists a $\kappa \in (0, 1)$ with*

$$d(f(x), f(y)) \le \kappa d(x, y) \quad \text{for all } x \succeq y.$$

2. *There exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$.*

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doi:10.1016/j.camwa.2010.06.003

Then f is a Picard Operator (PO), that is f has a unique fixed point $x^* \in X$ and for each $x \in X$,

$$\lim_{n\to\infty} f^n(x) = x^*.$$

After this, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered sets; see [18–31] and references cited therein. Nieto and Rodríguez-López in [28], proved the following:

Theorem 1.3 ([28]). *Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ be an order preserving mapping such that there exists a $\kappa \in (0, 1)$ with*

$$d(f(x), f(y)) \leq \kappa d(x, y) \quad \text{for all } x \succeq y.$$

Assume that one of the following conditions holds:

1. *f is continuous and there exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$;*
2. *$(X, d, \preceq)$ is such that for any nondecreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \preceq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \preceq fx_0$;*
3. *$(X, d, \preceq)$ is such that for any nonincreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \succeq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \succeq fx_0$.*

Then f has a fixed point. Moreover if $(X, \preceq)$ is such that every pair of elements of X has an upper or a lower bound, then f is a PO.

Recently Jachymski et al. [32,33] established a result which generalized the results of [23,27–31] to single-valued mapping in metric spaces with a graph instead of partial ordering. They proved the following:

Theorem 1.4 ([32]). *Let (X, d) be a complete metric space, and let the triple (X, d, G) have the following property:*

For any $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n}, x) \in E(G)$ for $n \in N$.

Let $f : X \to X$ be a G-contraction, $X_f := \{x \in X : (x, f(x)) \in E(G)\}$. Then the following statements hold:

1. *$\mathrm{card}\,\mathrm{Fix} f = \mathrm{card}\{[x]_{\widetilde{G}} : x \in X_f\}$.*
2. *$\mathrm{Fix} f \neq \emptyset$ if and only if $X_f \neq \emptyset$.*
3. *f has a unique fixed point if and only if there exists an $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*
4. *For any $x \in X_f$, $f|_{[x]_{\widetilde{G}}}$ is a PO.*
5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*

The aim of this paper is to study the existence of fixed points for set valued mappings in metric spaces endowed with a graph G by defining the G-contraction.

2. Preliminaries

Let (X, d) be a complete metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let

$$D(A, B) := \max\left\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\right\},$$

where

$$d(a, B) := \inf_{b\in B} d(a, b).$$

Mapping D is said to be a *Hausdorff metric* induced by d.

Let $F : X \rightsquigarrow X$ be a set valued mapping i.e., $X \ni x \mapsto Fx$ is a subset of X.

Definition 2.1. A point $x \in X$ is said to be a *fixed point* of the set valued mapping F if $x \in F(x)$.

Let $\mathrm{Fix}\,F := \{x \in X : x \in F(x)\}$ denote the set of fixed points of the mapping F and $\Delta := \{(x, x) : x \in X\}$ denote the diagonal of the cartesian product $X \times X$.

Consider a directed graph G such that the set of its vertices coincides with X (i.e., $V(G) = X$) and where the set of its edges $E(G)$ is such that $\Delta \subseteq E(G)$. We assume that G has no parallel edges and obtain a weighted graph by assigning to each edge the distance between the vertices. We can identify G as $(V(G), E(G))$. G^{-1} denotes the conversion of a graph G, the graph obtained from G by reversing the direction of its edges. $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the directions of the edges of G. We consider G as a directed graph whose set of edges is symmetric; thus we have

$$E(\widetilde{G}) := E(G) \cup E(G^{-1}).$$

Definition 2.2. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

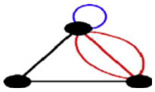

Fig. 1. A graph with parallel edges.

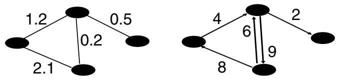

Fig. 2. A weighted graph and a digraph.

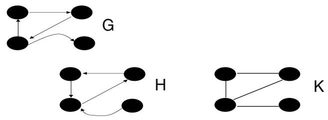

Fig. 3. H is the conversion and K is the undirected graph obtained from digraph G.

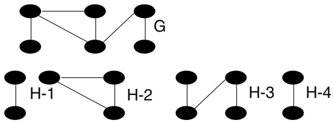

Fig. 4. H-1, H-2, H-3, H-4 are subgraphs of graph G.

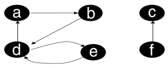

Fig. 5. ‘a’ to ‘e’ is a path of length 3.

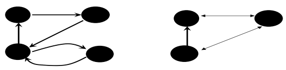

Fig. 6. Connected digraphs.

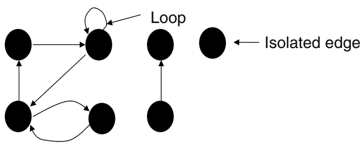

Fig. 7. A digraph having three components, each of which is a subgraph.

Definition 2.3. Let x and y be vertices in a graph G. A *path* in G from x to y of length n ($n \in N \cup \{0\}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ distinct vertices such that $x_0 = x, x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

Definition 2.4. The number of edges in G constituting the path is called the *length of the path*.

Definition 2.5. A graph G is *connected* if there is a path between any two vertices of G.

If a graph G is not connected then it is called disconnected and its different paths are called the components of G. Every component of G is a subgraph of it. Moreover G is weakly connected if $\widetilde{G}$ is connected.

Let G_x be the component of G, consisting of all edges and vertices which are contained in some path in G beginning at x. Assume that G is such that $E(G)$ is symmetric; then the equivalence class $[x]_G$ defined on $V(G)$ by the rule R (xRy if there is a path from x to y) is such that $V(G_x) = [x]_G$ (see Figs. 1–7).

For details regarding the above definitions from graph theory we refer the reader to Diestel [34].

Definition 2.6. Let $F : X \rightsquigarrow X$ be a set valued mapping with nonempty closed and bounded values. The mapping F is said to be a G-contraction if there exists a $\kappa \in (0, 1)$ such that

$$D(Fx, Fy) \leq \kappa d(x, y) \quad \text{for all } (x, y) \in E(G)$$

and if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u, v) \leq \kappa d(x, y) + \alpha, \quad \text{for each } \alpha > 0$$

then $(u, v) \in E(G)$.

Proposition 2.7. *If $F : X \rightsquigarrow X$ is a G-contraction then F is also a G^{-1}-contraction.*

Proof. It follows easily by the symmetry of D and d. □

Definition 2.8. A *partial order* relation is a binary relation $\preceq$ on X which satisfies the following conditions:

(i) $x \preceq x$ (reflexivity),
(ii) if $x \preceq y$ and $y \preceq x$ then $x = y$ (antisymmetry),
(iii) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitivity),

for all x, y and z in X.

A set with a partial order $\preceq$ is called a *partially ordered set*.

Definition 2.9. Let $(X, \preceq)$ be a partially ordered set and suppose $x, y \in X$. Points x and y are said to be *comparable elements* of X if either $x \preceq y$ or $x \succeq y$.

Lemma 2.10 ([35]). *If $A, B \in CB(X)$ and $a \in A$ then for each positive number α there exists a $b \in B$ such that $d(a, b) \leq D(A, B) + \alpha$.*

Lemma 2.11 ([35]). *Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} D(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

Property A ([32, Remark 3.1]). *For any sequence $(x_n)_{n\in N}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$, then $(x_n, x) \in E(G)$.*

3. Main results

We begin with the following theorem that gives the existence of a fixed point for set valued mappings (not necessarily unique) in metric spaces endowed with a graph.

Theorem 3.1. *Let (X, d) be a complete metric space and suppose that the triple (X, d, G) has the Property A. Let $F : X \rightsquigarrow X$ be a G-contraction and $X_F := \{x \in X : (x, u) \in E(G)$ for some $u \in F(x)\}$.*

Then the following statements hold:

1. For any $x \in X_F$, $F|_{[x]_{\widetilde{G}}}$ has a fixed point.
2. If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X.
3. If $X' := \cup\{[x]_{\widetilde{G}} : x \in X_F\}$, then $F|_{X'}$ has a fixed point.
4. If $F \subseteq E(G)$ then F has a fixed point.
5. *Fix* $F \neq \emptyset$ if and only if $X_F \neq \emptyset$.

Proof. 1. Let $x_0 \in X_F$; then there exists an $x_1 \in F(x_0)$ such that $(x_0, x_1) \in E(G)$. Since F is a G-contraction, we have

$$D(F(x_0), F(x_1)) \leq \kappa d(x_0, x_1).$$

Using Lemma 2.10, we have the existence of an $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) \leq D(F(x_0), F(x_1)) + \kappa \leq \kappa d(x_0, x_1) + \kappa. \tag{1}$$

Again because of F is a G-contraction $(x_1, x_2) \in E(G)$, we have

$$D(F(x_1), F(x_2)) \leq \kappa d(x_1, x_2),$$

and Lemma 2.10 gives the existence of an $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) \leq D(F(x_1), F(x_2)) + \kappa^2. \tag{2}$$

Using inequality (1) in (2) we have

$$d(x_2, x_3) \leq \kappa^2 d(x_0, x_1) + 2\kappa^2. \tag{3}$$

Continuing in this way we have $x_{n+1} \in F(x_n)$ such that $(x_n, x_{n+1}) \in E(G)$ and

$$d(x_n, x_{n+1}) \leq \kappa^n d(x_0, x_1) + n\kappa^n. \tag{4}$$

Next we will show that (x_n) is a Cauchy sequence in X.

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq d(x_0, x_1) \sum_{n=0}^{\infty} \kappa^n + \sum_{n=0}^{\infty} n\kappa^n < \infty.$$

Thus (x_n) is a Cauchy sequence and hence converges to some point (say) x in the complete metric space X.

Now we show that x is a fixed point of the mapping F. By using the Property A and the fact of F being a G-contraction, we have

$$D(F(x_n), F(x)) \leq \kappa d(x_n, x).$$

Since $x_{n+1} \in F(x_n)$ and $x_n \to x$, then by Lemma 2.11, $x \in F(x)$. Next, as $(x_n, x) \in E(G)$, for $n \in N$, we infer that $(x_0, x_1, \ldots .x_n, x)$ is a path in G and so $x \in [x_0]_{\widetilde{G}}$.

2. Since $X_F \neq \emptyset$, there exists an $x_0 \in X_F$, and since G is weakly connected, then $[x_0]_{\widetilde{G}} = X$ and by 1, mapping F has a fixed point.

3. It follows easily from 1 and 2.

4. $F \subseteq E(G)$ implies that all $x \in X$ are such that there exists some $u \in F(x)$ with $(x, u) \in E(G)$; so $X_F = X$ and by 2 and 3, F has a fixed point.

5. Let $Fix\,F \neq \emptyset$; this implies that there exists an $x \in Fix\,F$ such that $x \in F(x)$. $\Delta \subseteq E(G)$; therefore $(x, x) \in E(G)$ which implies that $x \in X_F$. So $X_F \neq \emptyset$. Conversely if $X_F \neq \emptyset$ then $Fix\,F \neq \emptyset$ follows from 2 and 3. □

Remark 3.2. If we assume G is such that $E(G) := X \times X$ then clearly G is connected and our Theorem 3.1 gives Nadler's theorem. Moreover if F is single valued then we get the Banach contraction theorem.

The following is a direct consequence of Theorem 3.1.

Corollary 3.3. *Let (X, d) be a complete metric space and the triple (X, d, G) have the Property A. If G is weakly connected then every G-contraction $F : X \rightsquigarrow X$ such that $(x_0, x_1) \in E(G)$ for some $x_1 \in F(x_0)$ has a fixed point.*

Remark 3.4. Let G be such that $E(G) := \{(x, y) : x \preceq y \vee x \succeq y\}$. In this case the G-contraction is defined as follows:

If there exists a $\kappa \in (0, 1)$ such that

$$D(Fx, Fy) \leq \kappa d(x, y) \quad \text{for all } (x, y) \in E(G) \text{ with } x \preceq y \text{ or } x \succeq y$$

and if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u, v) \leq \kappa d(x, y) + \alpha, \quad \text{for each } \alpha > 0$$

then $(u, v) \in E(G)$ with $u \preceq v$ or $v \preceq u$.

If F is a single-valued mapping then Theorem 3.1 partially generalizes the result of Ran and Reurings, Nieto and Rodríguez-López, and Jachymski [17,28,32].

Example 3.5. Let $X = \{(0, 0), (0, 0.1), (0.1, 0.1)\} := V(G)$ be a subset of R^2 and $E(G) := \{((0.1, 0.1), (0, 0)), ((0, 0.1), (0.1, 0.1))\}$ be such that $\Delta \subseteq E(G)$.

Let d be the Euclidean metric on X defined as

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

so that (X, d) is a complete metric space. Let

$$F : X \rightsquigarrow X$$

be a set valued mapping defined as

$$F(x) = \begin{cases} \{(0, 0)\} & \text{if } x = (0, 0) \\ X & \text{if } x = (0, 0.1) \\ \emptyset & \text{if } x = (0.1, 0.1). \end{cases}$$

Now for all $(x, y) \in E(G)$, F is a G-contraction. Also all other assumptions of Theorem 3.1 are satisfied and F has a fixed point.

Example 3.6. Consider $X = \{(0, 1), (1, 0)\} := V(G)$ as a subset of R^2 with the Euclidean metric defined as in the above example so that (X, d) is a complete metric space and $E(G) := \Delta$.

Let

$$F : X \rightsquigarrow X$$

be a set valued mapping defined as

$$F(x) = \begin{cases} \{(1, 0), (0, 1)\} & \text{if } x = (1, 0) \\ \{(1, 0)\} & \text{if } x = (0, 1). \end{cases}$$

Since $(1, 0) \in X$ is such that there exists $(1, 0) \in F(1, 0)$ with $((1, 0), (1, 0)) \in E(G)$, then $X_F \neq \emptyset$. Also F is a G-contraction and the other assumptions of Theorem 3.1 are satisfied, and F has a fixed point.

The following example shows a case where although Property A is satisfied, F has no fixed point. In fact F has a fixed point if in addition to the other assumptions of Theorem 3.1, F is a G-contraction.

Example 3.7. Consider $X = \{0, 0.5, 1\} := V(G)$ to be a subset of R with the usual metric defined as $d(x, y) =| x - y |$, so that (X, d) is a complete metric space and $E(G) := \{(1, 0.5), (0, 1)\}$ is such that $\Delta \subseteq E(G)$.

Define $F : X \rightsquigarrow X$ as

$$F(x) = \begin{cases} \{1, 0.5\} & \text{if } x = 0 \\ \{0, 1\} & \text{if } x = 0.5 \\ \{0\} & \text{if } x = 1. \end{cases}$$

Now since $(1, 0.5) \in E(G)$ and $D(F(1), F(0.5)) = 1, d((1, 0.5)) = 0.5$, then for all elements of $E(G)$ the contraction condition is not satisfied. Although $X_F \neq \emptyset$ and the other assumptions of Theorem 3.1 are satisfied, yet F has no fixed point.

Acknowledgements

The authors are grateful to the referees for precise remarks allowing us to improve the presentation of the paper. We also acknowledge with thanks the Higher Education Commission of Pakistan research grant 20-918/R&D/07.

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