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The conformal Killing equation on forms—prolongations and applications ☆

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Abstract

We construct a conformally invariant vector bundle connection such that its equation of parallel transport is a first order system that gives a prolongation of the conformal Killing equation on differential forms. Parallel sections of this connection are related bijectively to solutions of the conformal Killing equation. We construct other conformally invariant connections, also giving prolongations of the conformal Killing equation, that bijectively relate solutions of the conformal Killing equation on k-forms to a twisting of the conformal Killing equation on $(k-\ell)$-forms for various integers ℓ. These tools are used to develop a helicity raising and lowering construction in the general setting and on conformally Einstein manifolds.

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1. Introduction

On a pseudo-Riemannian n-manifold a tangent vector field v is an infinitesimal conformal automorphisms if the Lie derivative of the metric $\mathcal{L}_v g$ is proportional to the metric g. This is the conformal Killing equation. Here *conformal Killing equation* will also be used to mean a generalisation of this to k-forms, $1 \leqslant k \leqslant n-1$: a differential k-form σ is a conformal Killing form if, with respect to the $O(g)$-decomposition of $T^*M \otimes \Lambda^k T^*M$, the Cartan part of the Levi-Civita covariant derivative $\nabla\sigma$ is zero. Equivalently for any tangent field u we have

$$\nabla_u \sigma = \varepsilon(u)\tau + \iota(u)\rho \tag{1}$$

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where, on the right-hand side τ is a $(k-1)$-form, ρ is a $(k+1)$-form, and $\varepsilon(u)$ and $\iota(u)$ indicate, respectively, the exterior multiplication and the interior multiplication of $g(u,\,)$. An important property of the conformal Killing equation (1) is that it is conformally invariant.

The main aims of this paper are: 1. to derive a conformally invariant connection which is "equivalent" to the conformal Killing equation in that its parallel sections are naturally in one-one correspondence with solutions of the conformal Killing equation (1), 2. to derive a conformally invariant connection $\tilde{\nabla}$ that in a similar way relates solutions of the conformal Killing equation on k-forms to solutions of the conformal Killing equation on $(k-\ell)$-forms (for suitable positive and negative integers ℓ) twisted by the connection $\tilde{\nabla}$, and 3. to apply these ideas to a programme of helicity raising and lowering; here solutions of (1) are combined non-linearly to yield new solutions. For the third part our aim is to illustrate the ideas; no attempt has been made to be complete.

Conformal Killing 2-forms were introduced by Tachibana in [28] and the generalisation to higher valence followed shortly after [18]. Coclosed conformal Killing forms are Killing forms (or sometime called Killing–Yano forms). The latter satisfy the equation which generalises the Killing equation on vector fields, that is (1) with τ identically 0. This equation has been studied intensively in the physics literature in connection with its role generating quadratic first integrals of the geodesic equation. Aside from this connection, and a role in the higher symmetries of other equations [2], the broader geometric meaning of higher rank conformal Killing forms is still somewhat mysterious. The issue of global existence of conformal Killing forms in the compact Riemannian setting has been pursued recently by Semmelmann and others, see [23,24] for an indication of results and further references.

A semilinear partial differential equation is said to be of *finite type* [26] if there is a suitably equivalent finite dimensional prolonged system that is "closed" in the sense that all first partial derivatives of the dependent variables are determined by algebraic formulae in terms of these same variables. For certain linear equations it can turn out that the prolonged system is the equation of parallel transport for a vector bundle connection, so one immediately gets e.g. bounds on the size of the solution space. In [23] Semmelmann explicitly constructs a prolongation and connection along these lines for (1). This was vastly generalised in [3] which presents a uniform algorithm for explicitly computing such prolongations for a wide class of geometric semilinear overdetermined partial differential equations. This class includes the conformal Killing equation as one of the simplest cases. However neither of these treatments addresses the conformal invariance of (1). For the case of $k=1$ an equivalent conformally invariant connection was given in [15]. (See also [7] which generalises this to an analogue for all parabolic geometries.) Ab initio, given a conformally invariant equation one does not know whether there is a conformally invariant prolonged system along these lines. We show that for the conformal Killing equation there is, see Theorem 3.9. The connection obtained is described explicitly as a contorsion type modification to an exterior power (as treated in [5]) of the normal standard tractor connection of [1,9]. The power of this is that the latter is a simple well-understood connection on a bundle of relatively low rank (viz. $n+2$) and which respects a bundle metric. Describing things in terms of *form-tractors* in this way captures succinctly what conformal invariance means for the components of the prolongation. The curvature of the connection in Theorem 3.9 is a conformal obstruction to solutions of (1). By a simple variant of the treatment of obstructions to conformally Einstein metrics in Section 3.3 of [17], one may proliferate other conformally invariant local invariants which obstruct solutions.

Theorem 3.9 relates solutions of the conformal Killing equation to a twisting of the exterior derivative on functions. A generalisation of this idea is this: given two suitable distinct conformally invariant equations $A=0$ and $B=0$, consider obtaining a conformally connection $\tilde{\nabla}$ so that solutions of the equation A are bijectively related (by a prolongation) to solutions of the twisting by $\tilde{\nabla}$ of the equation B i.e. $B^{\tilde{\nabla}}=0$. (Of course there are variants of this where we replace conformal invariance by any other notion of invariance.) In Section 4 we obtain results exactly of this type, with the conformal Killing equation on forms of different ranks playing the roles of both A and B, see Theorem 4.4, and also Proposition 4.3. The remarkable feature of these results is the very simple form of the twisting connection—see (36) and (38). These results and their simplicity are exploited in Section 5 where we describe explicit conformally invariant helicity raising and lowering formulae. (Cf. [6] where a related construction is outlined in general terms.) See in particular: Theorem 5.1 which uses (almost) Einstein metrics and conformal Killing fields to generate conformal Killing fields; Theorem 5.4 where conformal Killing forms are used to generate other conformal Killing forms; and Theorem 5.4 where they are used to generate conformal Killing *tensors*, i.e. symmetric trace-free tensors S such that the symmetric trace-free part of ∇S vanishes. This idea of combining solutions to yield solutions of other equations is along the lines of helicity raising and lowering by Penrose's twistors in dimension 4.

To construct the required prolongations we develop a calculus that enables us to efficiently deal with differential forms, form-tractors, and some related bundles of arbitrary rank. The ideas originate in [5] but significant extensions have been developed in [25]. The idea is that as a first step in constructing the new connections we may take the normal tractor connection to be a "first approximation". By elementary representation theory it must agree with the required connections in the conformally flat setting. Then, employing the form calculus mentioned, we compute explicitly the tractor "contorsion" needed to adjust the normal connection. Eastwood's curved translation principle (see e.g. [11]) generates conformally invariant equations from other such equations via differential splitting operators between tractor bundles and (weighted) tensor-spinor bundles. The constructions and ideas in Sections 4 and 5 involve the refinement where one seeks to "translate solutions" of equations rather just the equations themselves. This necessarily draws on solutions of other equations and their equivalence to parallel (or suitably almost parallel) sections of tractor bundles.

2. Conformal geometry, tractor calculus and conformal Killing equation

2.1. Conformal geometry and tractor calculus

We summarise here some notation and background. Further details may be found in [10,16]. Let M be a smooth manifold of dimension $n \geqslant 3$. Recall that a *conformal structure* of signature (p,q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related signature-(p,q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. So we may equivalently view the conformal structure as the equivalence class $[g]$ of these conformally related metrics. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M,[g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. We write $\mathcal{E}^a$ for the space of sections of the tangent bundle TM and $\mathcal{E}_a$ for the space of sections of T^*M. The indices here are abstract in the sense of [21] and we follow the usual conventions from that source. So for example $\mathcal{E}_{ab}$ is the space of sections of $\otimes^2T^*M$. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This is used to identify TM with $T^*M[2]$. For many calculations we employ abstract indices in an obvious way. Given a choice of metric g from $[g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Here we are raising indices and contracting using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note $E[w]$ is trivialised by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialisation. The coupled ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection (the Riemannian curvature) is given by $[\nabla_a,\nabla_b]v^c = R_{ab}{}^c{}_d v^d$ ($[\cdot,\cdot]$ indicates the commutator bracket). This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}, \tag{2}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$ and vice versa: $\mathrm{Ric}_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}$, where we write J for the trace $P_a{}^a$ of P. The *Cotton tensor* is defined by $A_{abc} := 2\nabla_{[b}P_{c]a}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor as follows:

$$(n-3)A_{abc} = \nabla^d C_{dabc}. \tag{3}$$

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\Upsilon}g$, where Υ is a smooth function. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$. With $\Upsilon_a := \nabla_a\Upsilon$, the Schouten tensor transforms according to

$$\widehat{P}_{ab} = P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{4}$$

Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g. [1,16]. From these, one can easily compute the transformation for a general valence (i.e. rank) s section

$f_{bc\cdots d} \in \mathcal{E}_{bc\cdots d}[w]$ using the Leibniz rule:

$$\hat{\nabla}_{\bar{a}} f_{bc\cdots d} = \nabla_{\bar{a}} f_{bc\cdots d} + (w-s)\Upsilon_{\bar{a}} f_{bc\cdots d} - \Upsilon_b f_{\bar{a}c\cdots d} \cdots - \Upsilon_d f_{bc\cdots \bar{a}} + \Upsilon^p f_{pc\cdots d}\boldsymbol{g}_{b\bar{a}} \cdots + \Upsilon^p f_{bc\cdots p}\boldsymbol{g}_{d\bar{a}}. \tag{5}$$

We next define the standard tractor bundle over $(M,[g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}. \tag{6}$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [8].) On a conformal structure of signature (p,q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. Up to isomorphism this is the unique *normal conformal tractor connection* [9] and it induces a normal connection on $\bigotimes \mathcal{E}^A$ that we will also denote by ∇_a and term the (normal) tractor connection. In a conformal scale g, the metric h_{AB} and ∇_a on $\mathcal{E}^A$ are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a \tau - P_{ab}\mu^b \end{pmatrix}. \tag{7}$$

It is readily verified that both of these are conformally well defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we see that $Y_A X^A = 1$, $Z_{Ab}Z^A{}_c = \boldsymbol{g}_{bc}$, and all other quadratic combinations that contract the tractor index vanish. In (6) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant.

Given a choice of $g \in [g]$, the *tractor-D operator* $D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$ is defined by

$$D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A \Box V, \tag{8}$$

where $\Box V := \Delta V + wJV$. This is conformally invariant, as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [12]).

The curvature Ω of the tractor connection is defined on $\mathcal{E}^C$ by $[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$. Using (7) and the formulae for the Riemannian curvature yields

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C}Z_{E]}{}^e A_{eab}. \tag{9}$$

2.2. Forms and tensors

The basic tractor tools for dealing with weighted differential forms are developed in [5] and following that source we write $\mathcal{E}^k[w]$ for the space of sections of $(\Lambda^k T^*M) \otimes E[w]$ (and $\mathcal{E}^k = \mathcal{E}^k[0]$). However in order to be explicit and efficient in calculations involving bundles of possibly high rank it is necessary to introduce some further abstract index notation. In the usual abstract index conventions one would write $\mathcal{E}_{[ab\cdots c]}$ (where there are implicitly k-indices skewed over) for the space $\mathcal{E}^k$. To simplify subsequent expressions we use the following conventions. Firstly indices labelled with sequential superscripts which are at the same level (i.e. all contravariant or all covariant) will indicate a completely skew set of indices. Formally we set $a^1 \cdots a^k = [a^1 \cdots a^k]$ and so, for example, $\mathcal{E}_{a^1\cdots a^k}$ is an alternative notation for $\mathcal{E}^k$ while $\mathcal{E}_{a^1\cdots a^{k-1}}$ and $\mathcal{E}_{a^2\cdots a^k}$ both denote $\mathcal{E}^{k-1}$. Next we abbreviate this notation via multi-indices: We will use the forms indices

$$\begin{aligned}
\mathbf{a}^k &:= a^1\cdots a^k = \big[a^1\cdots a^k\big], \quad k\geqslant 0,\\
\dot{\mathbf{a}}^k &:= a^2\cdots a^k = \big[a^2\cdots a^k\big], \quad k\geqslant 1,\\
\ddot{\mathbf{a}}^k &:= a^3\cdots a^k = \big[a^3\cdots a^k\big], \quad k\geqslant 2,\\
\dddot{\mathbf{a}}^k &:= a^4\cdots a^k = \big[a^4\cdots a^k\big], \quad k\geqslant 3.
\end{aligned}$$

If, for example, $k=1$ then $\dot{\mathbf{a}}^k$ simply means the index is absent, whereas if $k=1$ then $\ddot{\mathbf{a}}$ means the term containing the index $\ddot{\mathbf{a}}$ is absent. For example, a 3-form φ can have the following possible equivalent structures of indices:

$$\varphi_{a^1a^2a^3} = \varphi_{[a^1a^2a^3]} = \varphi_{\mathbf{a}^3} = \varphi_{a^1\dot{\mathbf{a}}^3} = \varphi_{[a^1\dot{\mathbf{a}}^3]} = \varphi_{a^1a^2\ddot{\mathbf{a}}^3} \in \mathcal{E}_{\mathbf{a}^3} = \mathcal{E}^3.$$

We will also use $\boldsymbol{g}_{\mathbf{a}^k\mathbf{b}^k}$ (and similarly $\boldsymbol{g}_{\dot{\mathbf{a}}^k\dot{\mathbf{b}}^k}$) for $\boldsymbol{g}_{a^1b^1}\cdots\boldsymbol{g}_{a^kb^k}$ (where all a-indices and all b-indices are skewed over) and $\boldsymbol{g}$ denotes the conformal metric.

The corresponding notations will be used for tractor indices so e.g. the bundle of tractor k-forms $\mathcal{E}_{[A^1\ldots A^k]}$ will be denoted by $\mathcal{E}_{A^1\ldots A^k}$ or $\mathcal{E}_{\mathbf{A}^k}$.

We shall demonstrate the notation by giving the conformal transformation formulae of the Levi-Civita connection acting on conformally weighted forms. Under a rescaling $g\mapsto \widehat{g}=e^{2\Upsilon}g$ of the metric, and writing $\Upsilon_a := \nabla_a\Upsilon$, from (5) we have the following on $f_{\mathbf{a}^k}\in\mathcal{E}_{\mathbf{a}^k}[w]$:

$$\begin{aligned}
\hat{\nabla}_{a^0} f_{\mathbf{a}^k} &= \nabla_{a^0} f_{\mathbf{a}^k} + w\Upsilon_{a^0} f_{\mathbf{a}^k},\\
\hat{\nabla}^{a^1} f_{\mathbf{a}^k} &= \nabla^{a^1} f_{\mathbf{a}^k} + (n+w-2k)\Upsilon^{a^1} f_{\mathbf{a}^k}.
\end{aligned}\tag{10}$$

We need similar results for spaces with more complicated symmetries. We shall define $\mathcal{E}(1,k)$ for $k\geqslant 1$ and $\mathcal{E}(2,k)$ for $k\geqslant 2$ as follows:

$$\begin{aligned}
\mathcal{E}(1,k) &:= \{f_{c\mathbf{a}^k}\in\mathcal{E}_{c\mathbf{a}^k} \mid f_{[c\mathbf{a}^k]}=0\}\subseteq\mathcal{E}_{c\mathbf{a}^k},\\
\mathcal{E}(2,k) &:= \{\tilde{f}_{\mathbf{c}^2\mathbf{a}^k}\in\mathcal{E}_{\mathbf{c}^2\mathbf{a}^k} \mid \tilde{f}_{[\mathbf{c}^2\mathbf{a}^k]} = \tilde{f}_{c^1[c^2\mathbf{a}^k]} = \tilde{f}_{[\mathbf{c}^2\mathbf{a}^{k-1}]a^k}=0\}\subseteq\mathcal{E}_{\mathbf{c}^2\mathbf{a}^k}.
\end{aligned}$$

In other words, the subspaces $\mathcal{E}(1,k)$ and $\mathcal{E}(2,k)$ are defined by the condition that any skew symmetrisation of more than k indices vanishes. The subspaces of completely trace-free tensors in $\mathcal{E}(1,k)$ and $\mathcal{E}(2,k)$ will be denoted respectively by $\mathcal{E}(1,k)_0$ and $\mathcal{E}(2,k)_0$. Tensor products with density bundles will be denoted in an obvious way. For example $\mathcal{E}(1,k)_0[w]$ is a shorthand for $\mathcal{E}(1,k)_0\otimes\mathcal{E}[w]$.

We will later need the following identities

$$f_{a^1p\dot{\mathbf{a}}^k} = \frac{1}{k} f_{p\mathbf{a}^k} \quad\text{and}\quad \tilde{f}_{a^1qp\dot{\mathbf{a}}^k} = \frac{1}{k}\tilde{f}_{pq\mathbf{a}^k}\tag{11}$$

for $f_{c\mathbf{a}^k}\in\mathcal{E}(1,k)[w]$ and $\tilde{f}_{\mathbf{c}^2\mathbf{a}^k}\in\mathcal{E}(2,k)[w]$. This follows from the skewing $[p\mathbf{a}^k]$ which vanishes in both cases. Using the second of these we recover, for example, the well-known identities $R_{[a}{}^b{}_{c]}{}^d = \frac{1}{2}R_{ac}{}^{bd}$ and $C_{[a}{}^b{}_{c]}{}^d = \frac{1}{2}C_{ac}{}^{bd}$. Via (11), (5) and a short computation we obtain the transformations

$$\begin{aligned}
\hat{\nabla}_{a^0} f_{c\mathbf{a}^k} &= \nabla_{a^0} f_{c\mathbf{a}^k} + (w-1)\Upsilon_{a^0} f_{c\mathbf{a}^k} + \boldsymbol{g}_{ca^0}\Upsilon^p f_{p\mathbf{a}^k},\\
\hat{\nabla}^c f_{c\mathbf{a}^k} &= \nabla^c f_{c\mathbf{a}^k} + (n+w-k-1)\Upsilon^c f_{c\mathbf{a}^k},\\
\hat{\nabla}^{c^1}\tilde{f}_{\mathbf{c}^2\mathbf{a}^k} &= \nabla^{c^1}\tilde{f}_{\mathbf{c}^2\mathbf{a}^k} + (n+w-k-3)\Upsilon^{c^1}\tilde{f}_{\mathbf{c}^2\mathbf{a}^k}
\end{aligned}\tag{12}$$

for $f_{c\mathbf{a}^k}\in\mathcal{E}(1,k)_0[w]$ and $\tilde{f}_{\mathbf{c}^2\mathbf{a}^k}\in\mathcal{E}(2,k)_0[w]$.

2.3. Tractor forms

It follows from the semidirect composition series of $\mathcal{E}_A$ that the corresponding decomposition of $\mathcal{E}_{\mathbf{A}^k}$ is

$$\mathcal{E}_{[A^1\ldots A^k]} = \mathcal{E}_{\mathbf{A}^k} \simeq \mathcal{E}^{k-1}[k] \mathbin{⨭} \big(\mathcal{E}^k[k]\oplus\mathcal{E}^{k-2}[k-2]\big) \mathbin{⨭} \mathcal{E}^{k-1}[k-2].\tag{13}$$

Given a choice of metric g from the conformal class this determines a splitting of this space into four components (a replacement of the ⨭ s with $\oplus$s is effected) and the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_{\mathbf{A}^{k+1}}$, $k \geqslant 1$ as follows:

$$\begin{aligned}
\mathbb{Y}^k &= \mathbb{Y}_{A^0}{}^{a^1\cdots a^k}_{A^1\cdots A^k} = \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} = Y_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} \in \mathcal{E}^{\mathbf{a}^k}_{\mathbf{A}^{k+1}}[-k-1],\\
\mathbb{Z}^k &= \mathbb{Z}^{a^1\cdots a^k}_{A^1\cdots A^k} = \mathbb{Z}^{\mathbf{a}^k}_{\mathbf{A}^k} = Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} \in \mathcal{E}^{\mathbf{a}^k}_{\mathbf{A}^k}[-k],\\
\mathbb{W}^k &= \mathbb{W}_{A' A^0}{}^{a^1\cdots a^k}_{A^1\cdots A^k} = \mathbb{W}_{A' A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} = X_{[A'} Y_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k]}^{a^k} \in \mathcal{E}^{\mathbf{a}^k}_{\mathbf{A}^{k+2}}[-k],\\
\mathbb{X}^k &= \mathbb{X}_{A^0}{}^{a^1\cdots a^k}_{A^1\cdots A^k} = \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} = X_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} \in \mathcal{E}^{\mathbf{a}^k}_{\mathbf{A}^{k+1}}[-k+1],
\end{aligned}$$

where $k \geqslant 0$. The superscript k in $\mathbb{Y}^k$, $\mathbb{Z}^k$, $\mathbb{W}^k$ and $\mathbb{X}^k$ shows always the corresponding tensor valence. (This is slightly different than in [5], where k concerns the tractor valence.) Note that $Y = \mathbb{Y}^0$, $Z = \mathbb{Z}^1$ and $X = \mathbb{X}^0$ and $\mathbb{W}^0 = X_{[A'} Y_{A^0]}$. Using these projectors a section $f_{\mathbf{A}^{k+1}} \in \mathcal{E}_{\mathbf{A}^{k+1}}$ can be written as a 4-tuple

$$f_{\mathbf{A}^{k+1}} = \begin{pmatrix} \sigma_{\mathbf{a}^k} \\ \mu_{a^0\mathbf{a}^k} \ \varphi_{\dot{\mathbf{a}}^k} \\ \rho_{\mathbf{a}^k} \end{pmatrix} = \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} \sigma_{\mathbf{a}^k} + \mathbb{Z}^{a_0\,\mathbf{a}^k}_{A^0\mathbf{A}^k} \mu_{a_0\mathbf{a}^k} + \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k} \varphi_{\dot{\mathbf{a}}^k} + \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} \rho_{\mathbf{a}^k}$$

for forms $\sigma, \mu, \varphi, \rho$ of weight and valence according to the relationship given in (13).

The conformal transformation (6) yields the transformation formulae for the projectors:

$$\begin{aligned}
\widehat{\mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k}} &= \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} - \Upsilon_{a^0} \mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k} - k\Upsilon^{a^1} \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k} - \frac{1}{2}\Upsilon^k \Upsilon_k \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} + k\Upsilon_p \Upsilon^{a^1} \mathbb{X}_{A^0}{}^{p\dot{\mathbf{a}}^k}_{\mathbf{A}^k},\\
\widehat{\mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k}} &= \mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k} + (k+1)\Upsilon^{a^0} \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k},\\
\widehat{\mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k}} &= \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k} - \Upsilon_{a^1} \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k},\\
\widehat{\mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k}} &= \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k}
\end{aligned} \tag{14}$$

for metrics $\hat{g}$ and g from the conformal class. The normal tractor connection on $(k+1)$-form-tractors is

$$\nabla_p \begin{pmatrix} \sigma_{\mathbf{a}^k} \\ \mu_{a^0\mathbf{a}^k} \ \varphi_{\dot{\mathbf{a}}^k} \\ \rho_{\mathbf{a}^k} \end{pmatrix} = \begin{pmatrix} \nabla_p \sigma_{\mathbf{a}^k} - (k+1)\mu_{p\mathbf{a}^k} - \boldsymbol{g}_{pa^1}\varphi_{\dot{\mathbf{a}}^k} \\ \left\{ {\nabla_p \mu_{a^0\mathbf{a}^k} \atop + P_{pa^0}\sigma_{\mathbf{a}^k} + g_{pa^0}\rho_{\mathbf{a}^k}} \right\} \quad \left\{ {\nabla_p \varphi_{\dot{\mathbf{a}}^k} \atop + k P_p^{a^1} \sigma_{\mathbf{a}^k} - k\delta_p^{a^1} \rho_{\mathbf{a}^k}} \right\} \\ \nabla_p \rho_{\mathbf{a}^k} - (k+1) P_p{}^{a^0} \mu_{a^0\mathbf{a}^k} + P_{pa^1}\varphi_{\dot{\mathbf{a}}^k} \end{pmatrix} \tag{15}$$

or equivalently

$$\begin{aligned}
\nabla_p \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} &= P_{pa_0} \mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k} + kP_p{}^{a^1} \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k},\\
\nabla_p \mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k} &= -(k+1)\delta_p^{a^0} \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} - (k+1)P_p{}^{a^0} \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k},\\
\nabla_p \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k} &= -\boldsymbol{g}_{pa^1} \mathbb{Y}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} + P_{pa^1} \mathbb{X}_{A^0}{}^{a^1\dot{\mathbf{a}}^k}_{\mathbf{A}^k},\\
\nabla_p \mathbb{X}_{A^0}{}^{\mathbf{a}^k}_{\mathbf{A}^k} &= \boldsymbol{g}_{pa^0} \mathbb{Z}^{a^0\mathbf{a}^k}_{A^0\mathbf{A}^k} - k\delta_p^{a^1} \mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}^k}_{\mathbf{A}^k}.
\end{aligned}$$

2.4. The conformal Killing equation on forms

The space $\mathcal{E}_{c\mathbf{a}^k} = \mathcal{E}_c \otimes \mathcal{E}_{a^1\cdots a^k}$ is completely reducible for $1 \leqslant k \leqslant n$ and we have the $O(g)$-decomposition $\mathcal{E}_{c\mathbf{a}^k}[w] \cong \mathcal{E}_{[c\mathbf{a}^k]}[w] \oplus \mathcal{E}_{\{c\mathbf{a}^k\}_0}[w] \oplus \mathcal{E}_{\mathbf{a}^{k-1}}[w-2]$ where the bundle $\mathcal{E}_{\{c\mathbf{a}^k\}_0}[w]$ consists of rank $k+1$ trace-free tensors $T_{c\mathbf{a}^k}$ (of conformal weight w) that are skew on the indices $a^1 \cdots a^k$ and have the property that $T_{[ca^1\cdots a^k]} = 0$. (Note that the three spaces on the right-hand side are SO(g)-irreducible if $k \notin \{n/2, n/2 \pm 1\}$.) On the space $\mathcal{E}_{c\mathbf{a}^k}[w]$ there is a projection $\mathcal{P}_{\{c\mathbf{a}^k\}_0}$ to the component $\mathcal{E}_{\{c\mathbf{a}^k\}_0}[w]$ and we will use the notation

$$T_{c\mathbf{a}^k} \overset{\{c\mathbf{a}^k\}_0}{=} S_{c\mathbf{a}^k} \quad \text{or} \quad T_{c\mathbf{a}^k} =_{\{c\mathbf{a}^k\}_0} S_{c\mathbf{a}^k}$$

to mean that $\mathcal{P}_{\{c\mathbf{a}^k\}_0}(T) = \mathcal{P}_{\{c\mathbf{a}^k\}_0}(S)$. We will also use the projection $\mathcal{P}_{\{c\mathbf{a}^k\}}$ to $\mathcal{E}(1,k)[w] =: \mathcal{E}_{\{c\mathbf{a}^k\}}[w]$.

Each metric from the conformal class determines a corresponding Levi-Civita connection ∇ and for $1 \leqslant k \leqslant n-1$ and $\sigma_{\mathbf{a}^k} \in \mathcal{E}^k[k+1]$, we may form $\nabla_c \sigma_{\mathbf{a}^k}$. This is not conformally invariant. However it is straightforward to verify that its projection $\mathcal{P}_{\{c\mathbf{a}^k\}_0}(\nabla\sigma)$ is conformally invariant. That is, this is independent of the choice of metric (and corresponding Levi-Civita connection) from the conformal class. Thus the equation

$$\nabla_{\{c}\sigma_{\mathbf{a}^k\}_0} = 0, \quad 1 \leqslant k \leqslant n-1 \tag{CKE}$$

called the (form) *conformal Killing equation*, is conformally invariant. This is exactly Eq. (1) from the introduction.

Suppose $\tilde{\nabla}$ is a connection on another vector bundle (or space of sections thereof) $\mathcal{E}_\bullet$. For this connection coupled with the Levi-Civita connection let us also write $\tilde{\nabla}$. Since it is a first order equation (CKE) is *strongly invariant* (cf. [11,13]) in the sense that if now $\sigma_{\mathbf{a}^k} \in \mathcal{E}_{\mathbf{a}^k\bullet}[k+1] = \mathcal{E}_{\mathbf{a}^k}[k+1] \otimes \mathcal{E}_\bullet$ then $\tilde{\nabla}_{\{c}\sigma_{\mathbf{a}^k\}_0} = 0$ is also conformally invariant. We will also call any such equation a conformal Killing equation (or sometimes for emphasis a *coupled conformal Killing equation*).

On oriented conformal manifolds the conformal Hodge-$\star$ operator (see e.g. [5]) gives a mapping $\star : \mathcal{E}^k[k+1] \to \mathcal{E}^{n-k}[n-k+1]$, and from elementary classical $\mathrm{SO}(n)$-representation theory it follows easily that $\sigma \in \mathcal{E}^k[k+1]$ solves (CKE) for k-forms if and only if $\star\sigma$ solves the version of (CKE) for $(n-k)$-forms. Since the redundancy on oriented manifolds does us no harm, we shall ignore this and in the following simply treat the equation on k-forms for $1 \leqslant k \leqslant n-1$.

3. Invariant prolongation for conformal Killing forms

Throughout this section, and in much of the subsequent work, we will write $f_{\mathbf{a}}$ (rather than $f_{\mathbf{a}^k}$) to denote a section in $\mathcal{E}_{\mathbf{a}^k}[k+1]$. That is, the superscript of the form index $\mathbf{a}$ will be omitted but can be taken to be k (or otherwise if clear from the context).

Before we start with the construction of the prolongation, we will introduce some notation for certain algebraic actions of the curvature on tensors. Let us write $\sharp$ (which we will term *hash*) for the natural action of sections A of $\mathrm{End}(TM)$ on tensors. For example, on a covariant 2-tensor T_{ab}, we have $A\sharp T_{ab} = -A^c{}_a T_{cb} - A^c{}_b T_{ac}$. If A is skew for a metric g, then at each point, A is $\mathfrak{so}(g)$-valued. The hash action thus commutes with the raising and lowering of indices and preserves the $\mathrm{SO}(g)$-decomposition of tensors. For example the Riemann tensor may be viewed as an $\mathrm{End}(TM)$-valued 2-form R_{ab} and in this notation, for an arbitrary tensor T, we have $[\nabla_a, \nabla_b]T = R_{ab}\sharp T$. Similarly we have $C_{ab}\sharp T$ for the Weyl curvature. As a section of the tensor square of the g-skew bundle endomorphisms of TM, the Weyl curvature also has a double hash action that we denote $C\sharp\sharp T$.

We need some more involved actions of the Weyl tensor on $\mathcal{E}_{\mathbf{a}^k}[w]$ for $k \geqslant 2$. These are given by

$$\begin{aligned}
(C\blacklozenge f)_{c\dot{\mathbf{a}}} &:= \frac{k-2}{k}\left(C_{ca^2}{}^{pq} f_{pq\ddot{\mathbf{a}}} + C_{a^3a^2}{}^{pq} f_{pqc\dddot{\mathbf{a}}}\right) \in \mathcal{E}_{c\dot{\mathbf{a}}^k}[w-2], \\
(C\lozenge f)_{\mathbf{ca}} &:= C_{c^1c^2a^1}{}^p f_{p\dot{\mathbf{a}}} + C_{a^1a^2c^1}{}^p f_{pc^2\ddot{\mathbf{a}}} + \frac{k}{n-k} \boldsymbol{g}_{c^1a^1}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}} \in \mathcal{E}_{\mathbf{c}^2\mathbf{a}^k}[w],
\end{aligned} \tag{16}$$

where $\mathbf{c} = \mathbf{c}^2$ and $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$. Note that $C\lozenge f$ vanishes for $k = n-1$ since $\mathcal{E}(2, n-1)_0$ is trivial. For the sake of complete clarity we have given these explicit formulae but note that, up to a multiple, the first of these is simply $C\sharp f \in \mathcal{E}_{\mathbf{c}^2\mathbf{a}^k}$ followed by projection to $\mathcal{E}(1, k-1)[w-2]$ (the projection involves a trace), while the second is $C\sharp f$ followed by projection to $\mathcal{E}(2,k)_0[w]$. This is clear except for the final projection in each case, which we now verify.

3.1. Lemma. *Let us suppose $k \geqslant 2$. Then*

(i) $(C\blacklozenge f)_{c\dot{\mathbf{a}}} = C_{\{ca^2}{}^{pq} f_{|pq|\ddot{\mathbf{a}}\}_0} \in \mathcal{E}(1, k-1)_0[w-2]$,
(ii) $(C\lozenge f)_{\mathbf{ca}} \in \mathcal{E}(2,k)_0[w]$.

Proof. (i) It follows from (16) and the Bianchi identity that $(C\blacklozenge f)_{c\dot{\mathbf{a}}}$ is trace-free. Moreover

$$C_{\{ca^2}{}^{pq} f_{|pq|\ddot{\mathbf{a}}\}} = C_{ca^2}{}^{pq} f_{pq\ddot{\mathbf{a}}} - C_{[ca^2}{}^{pq} f_{|pq|\ddot{\mathbf{a}}]} = (C\blacklozenge f)_{c\dot{\mathbf{a}}}, \tag{17}$$

where the first equality is just the definition of the projection $\{..\}$ and the second follows from re-expressing of the skew symmetrisation $[c\dot{\mathbf{a}}]$ in the last display.

(ii) According to the definition of $\mathcal{E}(2,k)_0$, we are required to show that $(C\Diamond f)_{c^1[c^2\mathbf{a}]} = (C\Diamond f)_{[\mathbf{c}\dot{\mathbf{a}}]a^{k+1}} = 0$ (note $(C\Diamond f)_{[\mathbf{ca}]} = 0$ is obvious from (16)) and also that $C\Diamond f$ is trace-free. Both skew symmetrisations $[c^2\mathbf{a}]$ and $[\mathbf{c}\dot{\mathbf{a}}]$ kill the last term of $C\Diamond f$ in (16), because $(C\blacklozenge f)_{[\mathbf{c}\dot{\mathbf{a}}]} = 0$ according to the lemma (i). Applying the symmetrisation $[c^2\mathbf{a}]$ to the first two terms in (16) and using the Bianchi identity yields $C_{c^1[c^2a^1}{}^p f_{|p|\dot{\mathbf{a}}]} + \frac{1}{2}C_{[a^1a^2|c^1}{}^p f_{p|c^2\ddot{\mathbf{a}}]}$, where the indices c^1c^2 are *not* skewed over. This is zero because $C_{c^1[c^2a^1]}{}^p = -\frac{1}{2}C_{c^2a^1c^1}{}^p$. The second skew symmetrisation $[\mathbf{c}\dot{\mathbf{a}}]$ is similar.

It remains to prove $\boldsymbol{g}^{c^1a^1}(C\Diamond f)_{\mathbf{ca}} = 0$. Tracing the last term in (16) yields $\frac{k}{n-k}\boldsymbol{g}^{c^1a^1}\boldsymbol{g}_{c^1a^1}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}} = \frac{1}{2}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}}$ after a short computation. Further computations reveal $\boldsymbol{g}^{c^1a^1}C_{c^1c^2a^1}{}^p f_{p\dot{\mathbf{a}}} = -\frac{k-1}{2k}C_{c^2a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}}$ and $\boldsymbol{g}^{c^1a^1}C_{a^1a^2c^1}{}^p f_{pc^2\ddot{\mathbf{a}}} = -\frac{k-2}{2k}C_{a^3a^2}{}^{pq} f_{pqc^2\dddot{\mathbf{a}}} + \frac{1}{2k}C_{c^2a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}}$. Summing the last three equations, the lemma part (ii) follows from (16) for $C\blacklozenge f$. □

Introducing new variables, the equation (CKE) may be re-expressed in the form $\nabla_c\sigma_{\mathbf{a}} = \mu_{\mathbf{ca}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$, where $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$ and $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$. These capture some of the 1-jet information: we have $\mu_{a^0\mathbf{a}} = \nabla_{a^0}\sigma_{\mathbf{a}}$, and $\nu_{\dot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla^p\sigma_{p\dot{\mathbf{a}}}$. We need a further set of variables to complete (CKE) to a first order closed system. There is some choice here, but, for the purposes of studying conformal invariance, $\rho_{\mathbf{a}} := -\frac{1}{k}\nabla_{a^1}\nu_{\dot{\mathbf{a}}} + \frac{1}{nk}\nabla^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} - P_{a^1}{}^p\sigma_{p\dot{\mathbf{a}}}$ is a judicious choice. We then have the following result.

3.2. Proposition. *Solutions of the conformal Killing equation (CKE), for $1 \leqslant k \leqslant n-1$, are in 1–1 correspondence with solutions of the following system on $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$, $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$, $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$ and $\rho_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k-1]$:*

$$\begin{aligned}
&\nabla_c\sigma_{\mathbf{a}} = \mu_{\mathbf{ca}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}};\\
&\nabla_c\mu_{a^0\mathbf{a}} = (k+1)\left[\boldsymbol{g}_{ca^0}\rho_{\mathbf{a}} - P_{ca^0}\sigma_{\mathbf{a}} - \frac{1}{2}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}}\right];\\
&\nabla_c\nu_{\dot{\mathbf{a}}} = -k[\rho_{c\dot{\mathbf{a}}} + P_c{}^p\sigma_{p\dot{\mathbf{a}}}] + \frac{k(k-1)}{2(n-k)}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}};\\
&\nabla_c\rho_{\mathbf{a}} = P_{ca^1}\nu_{\dot{\mathbf{a}}} - P_c{}^p\mu_{p\mathbf{a}} + \frac{1}{2}A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} - \frac{k}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \quad \textit{for } k \geqslant 2;\\
&\nabla_c\rho_{a^1} = P_{ca^1}\nu - P_c{}^p\mu_{pa^1} + A_{a^1pc}\sigma^p \quad \textit{for } k = 1.
\end{aligned} \tag{18}$$

The mapping from solutions $\sigma_{\mathbf{a}}$ of (CKE) to solutions $(\sigma_{\mathbf{a}}, \mu_{a^0\mathbf{a}}, \nu_{\dot{\mathbf{a}}}, \rho_{\mathbf{a}})$ of the system above is

$$\sigma_{\mathbf{a}} \mapsto \left(\sigma_{\mathbf{a}}, \nabla_{a^0}\sigma_{\mathbf{a}}, \frac{k}{n-k+1}\nabla^p\sigma_{p\dot{\mathbf{a}}}, \frac{1}{nk}\nabla^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} - \frac{1}{n-k+1}\nabla_{a^1}\nabla^p\sigma_{p\dot{\mathbf{a}}} - P_{a^1}{}^p\sigma_{p\dot{\mathbf{a}}}\right). \tag{19}$$

Proof. The first equation $\nabla_c\sigma_{\mathbf{a}} = \mu_{\mathbf{ca}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$ is simply a restatement of the conformal Killing equation (CKE). This gives $\mu_{a^0\mathbf{a}}$ and $\nu_{\dot{\mathbf{a}}}$ in terms of derivatives of $\sigma_{\mathbf{a}}$. Thus the proposition is clear except that we should verify that if $\sigma_{\mathbf{a}}$ solves (CKE) then we have the second, third and fourth equations of (18).

For the second equation, let us observe $(k+2)\nabla_{[c}\nabla_{a^0}\sigma_{\mathbf{a}]} = \nabla_c\nabla_{a^0}\sigma_{\mathbf{a}} - (k+1)\nabla_{a^1}\nabla_{[a^0}\sigma_{c\dot{\mathbf{a}}]}$, and that the left-hand side vanishes due to the Bianchi identity. The first term on the right-hand side is $\nabla_c\mu_{a^0\mathbf{a}}$ thus

$$\begin{aligned}
\nabla_c\mu_{a^0\mathbf{a}} &= (k+1)\nabla_{a^1}\mu_{a^0c\dot{\mathbf{a}}} = (k+1)\nabla_{a^1}(\nabla_{a^0}\sigma_{c\dot{\mathbf{a}}} - \boldsymbol{g}_{a^0[c}\nu_{\dot{\mathbf{a}}]})\\
&= (k+1)\left(\frac{1}{2}R_{a^1a^0c}{}^p\sigma_{p\dot{\mathbf{a}}} - \frac{1}{k}\boldsymbol{g}_{ca^0}\nabla_{a^1}\nu_{\dot{\mathbf{a}}}\right),
\end{aligned}$$

where the second equality follows from the first equation in (18) and the third equality from the Bianchi identity. Now the equation for $\nabla_c\mu_{a^0\mathbf{a}}$ in (18) follows from the last display using (2) and the relation $\rho_{\mathbf{a}} = -\frac{1}{k}\nabla_{a^1}\nu_{\dot{\mathbf{a}}} - P_{a^1}{}^p\sigma_{p\dot{\mathbf{a}}}$, which we have for solutions.

The third equation in (18) concerns $\nabla_c\nu_{\dot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla_c\nabla^p\sigma_{p\dot{\mathbf{a}}}$. Commuting the covariant derivatives we get $\nabla_c\nabla^p = R_c{}^p\sharp + \nabla^p\nabla_c$ where, recall, $\sharp$ captures the action of the Riemann curvature tensor R. Therefore

$$
\begin{aligned}
(n-k+1)\nabla_c \nu_{\dot{\mathbf{a}}} &= k\big[R_c{}^p{}_p{}^q \sigma_{q\dot{\mathbf{a}}} + (k-1)R_c{}^p{}_{a^2}{}^q \sigma_{pq\ddot{\mathbf{a}}} + \nabla^p(\mu_{cp\dot{\mathbf{a}}} + \boldsymbol{g}_{c[p}\nu_{\dot{\mathbf{a}}]})\big] \\
&= k\left[-\operatorname{Ric}_c{}^p \sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}(k-1)R_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - \nabla^p \mu_{pc\dot{\mathbf{a}}} + \frac{1}{k}\nabla_c \nu_{\dot{\mathbf{a}}}\right],
\end{aligned}
$$

where we have used $\nabla^p \nu_{p\ddot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla^p\nabla^q \sigma_{qp\ddot{\mathbf{a}}} = 0$. Note that the last term here is a multiple of the left-hand side. We consider the other terms on the right-hand side. Recall that (2) gives $\operatorname{Ric}_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}$. Using (2) also for the second term on the right-hand side, and the equation for $\nabla_c \mu_{a^0\mathbf{a}}$ in (18) for the third, a computation yields

$$
\begin{aligned}
-\operatorname{Ric}_c{}^p \sigma_{p\dot{\mathbf{a}}} &= -(n-2)P_c{}^p \sigma_{p\dot{\mathbf{a}}} - J\sigma_{c\dot{\mathbf{a}}}, \\
\frac{1}{2}(k-1)R_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} &= \frac{1}{2}(k-1)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} + 2(k-1)\delta^p{}_{[c}P_{a^2]}{}^q \sigma_{pq\ddot{\mathbf{a}}} \\
&= \frac{1}{2}(k-1)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - (k-1)(P_{a^2}{}^p \sigma_{pc\ddot{\mathbf{a}}} - P_c{}^p \sigma_{p\dot{\mathbf{a}}}), \\
-\nabla^p \mu_{pc\dot{\mathbf{a}}} &= -(n-k)\rho_{c\dot{\mathbf{a}}} + J\sigma_{c\dot{\mathbf{a}}} - kP_{[c}{}^p \sigma_{|p|\dot{\mathbf{a}}]} - \frac{1}{2}(k-1)C_{[a^2 c}{}^{qp}\sigma_{|pq|\ddot{\mathbf{a}}]}.
\end{aligned}
$$

Hence the last but one display says that $\frac{n-k}{k}\nabla_c \nu_{\dot{\mathbf{a}}}$ is equal to the sum of the right-hand sides of the last display. Now using the relation $-kP_{[c}{}^p \sigma_{|p|\dot{\mathbf{a}}]} = -P_c{}^p \sigma_{p\dot{\mathbf{a}}} + (k-1)P_{a^2}{}^p \sigma_{pc\ddot{\mathbf{a}}}$ and (17) we obtain immediately the third equation in (18).

For the last equation we first make an observation about its skew-symmetric part $\nabla_{[c}\rho_{\mathbf{a}]}$. Using the definition of ρ and the Bianchi identity, we have $\nabla_{[c}\rho_{\mathbf{a}]} = -\nabla_{[c}P_{a^1}{}^p \sigma_{|p|\dot{\mathbf{a}}]}$. Using the Leibniz rule and the first equation in (18) for the right-hand side, we obtain

$$
\nabla_{[c}\rho_{\mathbf{a}]} = -\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} - P_{[c}{}^p \mu_{|p|\mathbf{a}]}, \tag{20}
$$

since the term $P_{a^1}{}^p \boldsymbol{g}_{c[p}\nu_{\dot{\mathbf{a}}]}$ vanishes after the skew symmetrisation $[c\mathbf{a}]$. Now to compute the full section $\nabla_c \rho_{\mathbf{a}}$, we shall start with the equation for $\nabla_c \nu_{\dot{\mathbf{a}}}$ from (18). We apply ∇_{a^1} to both sides of this equation and skew over all a-indices. Commuting the covariant derivatives on the left-hand side, we obtain $\nabla_{a^1}\nabla_c = \nabla_c \nabla_{a^1} + R_{a^1 c}\sharp$. The first term on the right-hand side is $-k\nabla_{a^1}\rho_{c\dot{\mathbf{a}}} = (k+1)\nabla_{[c}\rho_{\mathbf{a}]} - \nabla_c \rho_{\mathbf{a}}$. Through these observations, and using (20), we obtain

$$
\begin{aligned}
\nabla_c \nabla_{a^1}\nu_{\dot{\mathbf{a}}} + (k-1)R_{a^1 ca^2}{}^p \nu_{p\ddot{\mathbf{a}}} &= -(k+1)\left(\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} + P_{[c}{}^p \mu_{|p|\mathbf{a}]}\right) \\
&\quad - \nabla_c \rho_{\mathbf{a}} - k\nabla_{a^1}P_c{}^p \sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}.
\end{aligned}
$$

Many terms can be simplified and we shall start with the first term on the left-hand side. We have $\nabla_c \nabla_{a^1}\nu_{\dot{\mathbf{a}}} = -k(\nabla_c \rho_{\mathbf{a}} + \nabla_c P_{a^1}{}^p \sigma_{|p|\dot{\mathbf{a}}})$ which follows from the equation for $\nabla_c \nu_{\dot{\mathbf{a}}}$ in (18). Combining the last two displays we obtain

$$
\begin{aligned}
-(k-1)\nabla_c \rho_{\mathbf{a}} &= 2k\nabla_{[c}P_{a^1]}{}^p \sigma_{p\dot{\mathbf{a}}} - (k+1)\left(\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} + P_{[c}{}^p \mu_{|p|\mathbf{a}]}\right) \\
&\quad - \frac{1}{2}(k-1)R_{a^1 a^2 c}{}^p \nu_{p\ddot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}},
\end{aligned}
$$

where we have also used $R_{a^1 ca^2}{}^p = \frac{1}{2}R_{a^1 a^2 c}{}^p$. Note that for the case of (the rank of σ being) $k=1$ both sides of the equality above vanish and we get no information. Now we simplify terms on the right-hand side: the first term using the Leibniz rule and the equation for $\nabla_c \sigma_{\mathbf{a}}$, the next two terms re-expressing the skew symmetrisation $[c\mathbf{a}]$ and the first curvature term using the decomposition (2). This yields

$$
\begin{aligned}
2k\nabla_{[c}P_{a^1]}{}^p \sigma_{p\dot{\mathbf{a}}} &= kA^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + 2kP_{[a^1}{}^p \mu_{c]p\dot{\mathbf{a}}} + 2kP_{[a^1}{}^p \boldsymbol{g}_{c][p}\nu_{\dot{\mathbf{a}}]} \\
&= kA^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + kP_{a^1}{}^p \mu_{cp\dot{\mathbf{a}}} - kP_c{}^p \mu_{a^1 p\dot{\mathbf{a}}} + (k-1)\boldsymbol{g}_{ca^1}P_{a^2}{}^p \nu_{p\ddot{\mathbf{a}}}, \\
-\frac{1}{2}(k+1)A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} &= -A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}(k-1)A^p{}_{a^2 a^1}\sigma_{pc\ddot{\mathbf{a}}},
\end{aligned}
$$

$$-(k+1)P_{[c}{}^{p}\mu_{|p|\mathbf{a}]} = -P_{c}{}^{p}\mu_{p\mathbf{a}} + kP_{a^1}{}^{p}\mu_{pc\dot{\mathbf{a}}},$$
$$-\frac{1}{2}(k-1)R_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}} = -\frac{1}{2}(k-1)[C_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}} + 2\boldsymbol{g}_{ca^1}P_{a^2}{}^{p}\nu_{p\ddot{\mathbf{a}}} + 2P_{ca^1}\nu_{\dot{\mathbf{a}}}].$$

Substituting these in the previous display, the proposition for $k \geqslant 2$ follows. The case $k=1$ can be checked directly by tracing $\frac{1}{2}R_{c^0c^1}\sharp\mu_{a^0a^1} = \nabla_{c^0}\nabla_{c^1}\mu_{a^0a^1} = \nabla_{c^0}[2\boldsymbol{g}_{c^1a^0}\rho_{a^1} - 2P_{c^1a^0}\sigma_{a^1} - C_{a^0a^1c^1}{}^{p}\sigma_{p}]$. □

Remark. There is a variant of the derivation for the $k \geqslant 2$ cases, as in the proof above, which generalises the treatment of $k=1$ that we give there. However this breaks down for $k=n-1$. Dually the proof we give for $k \geqslant 2$ breaks down at $k=1$. Our proof of the $k=1$ agrees with a treatment of that case distributed privately by Mike Eastwood during the preparation of [3] and his notation and conventions influenced our treatment. Earlier alternative treatments of that case have been known to the first author for some time (see [15]).

3.3. Lemma. *Let us fix $k \geqslant 2$. If $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ is a solution of* (CKE) *then $(C\diamondsuit\sigma)_{\mathbf{ca}} = 0$.*

Proof. We shall prove the lemma using the prolongation (18). Applying ∇_{c^1} to both sides of the equation for $\nabla_{c^2}\sigma_{\mathbf{a}}$, we obtain $\nabla_{c^1}\nabla_{c^2}\sigma_{\mathbf{a}} = \nabla_{c^1}\mu_{c^2\mathbf{a}} + \boldsymbol{g}_{c^2a^1}\nabla_{c^1}\nu_{\dot{\mathbf{a}}}$. The left-hand side is equal to $\frac{k}{2}R_{c^1c^2a^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} = \frac{k}{2}C_{c^1c^2a^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} + k\boldsymbol{g}_{c^1a^1}P_{c^2}{}^{p}\sigma_{p\dot{\mathbf{a}}} + kP_{c^1a^1}\sigma_{c^2\dot{\mathbf{a}}}$, according to (2). On the other hand, from (18) the right-hand side is equal to

$$\left(-k\boldsymbol{g}_{c^1a^1}\rho_{c^2\dot{\mathbf{a}}} + kP_{c^1a^1}\sigma_{c^1\dot{\mathbf{a}}} - \frac{1}{2}\big(2C_{c^2a^1c^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} - (k-1)C_{a^2a^1c^1}{}^{p}\sigma_{pc^2\ddot{\mathbf{a}}}\big)\right)$$
$$+\boldsymbol{g}_{c^2a^1}\left(-k\rho_{c^1\dot{\mathbf{a}}} - kP_{c^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}(C\blacklozenge\sigma)_{c^1\dot{\mathbf{a}}}\right).$$

Now equating these two displays and using $C_{c^2a^1c^1}{}^{p} = -\frac{1}{2}C_{c^1c^2a^1}{}^{p}$ we obtain an identity which holds for solutions. Comparing the expression with the definition of $(C\diamondsuit\sigma)$ in (16), we see the identity is $(k-1)(C\diamondsuit\sigma)=0$. □

Note that a curvature condition, equivalent to that in Lemma 3.3, is in [18]. There the identity for solutions is stated in terms of the Riemann tensor R, rather than in terms of the Weyl tensor C. In this form it has also been derived in [22] (although we could not find the necessary restriction $k \geqslant 2$ in that source). Expressing the identity via the Weyl curvature, as we do, emphasises that this is a conformally invariant condition.

Next we observe that (19) defines a conformally invariant differential splitting operator. We define a differential operator $\mathbb{D}$ on $\mathcal{E}_{\mathbf{a}^k}[k+1]$ by

$$\sigma_{\mathbf{a}} \mapsto \sigma_{A^0\mathbf{A}} := \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} - \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}}, \tag{21}$$

where $\sigma_{\mathbf{a}}$, $\mu_{a^0\mathbf{a}}$, $\nu_{\dot{\mathbf{a}}}$ and $\rho_{\mathbf{a}}$ are given by (19). Then we have the following.

3.4. Lemma. *For $1 \leqslant k \leqslant n-1$, $\mathbb{D}$ is a conformally invariant operator $\mathbb{D}: \mathcal{E}_{\mathbf{a}^k}[k+1] \to \mathcal{E}_{A^0\mathbf{A}^k}$.*

Proof. Consider $\mathbb{D}$ for $\sigma \in \mathcal{E}_{\mathbf{a}^k}[k+1]$. Let μ, ν and ρ be given in terms of σ as in (19). In these formulae ∇ is the Levi-Civita connection for some choice of metric g from the conformal class. So μ, ν and ρ depend on the metric. If we conformally rescale the metric $g \mapsto \widehat{g} = e^{2\Upsilon}g$ then it is easy to calculate (using e.g. the transformation formulae given in [16]) that the sections $\widehat{\mu}$ and $\widehat{\nu}$ for the metric $\widehat{g}$ are given by $\widehat{\mu}_{a^0\mathbf{a}} = \mu_{a^0\mathbf{a}} + (k+1)\Upsilon_{a^0}\sigma_{\mathbf{a}}$ and $\widehat{\nu}_{\dot{\mathbf{a}}} = \nu_{\dot{\mathbf{a}}} + k\Upsilon^{p}\sigma_{p\dot{\mathbf{a}}}$, where $\Upsilon_a = \nabla_a\Upsilon$. To compute $\widehat{\rho}_{\mathbf{a}} = -\frac{1}{k}\widehat{\nabla}_{a^1}\widehat{\nu}_{\dot{\mathbf{a}}} - \widehat{P}_{a^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} + \frac{1}{nk}\widehat{\nabla}^{p}\widehat{\nabla}_{\{p}\sigma_{\mathbf{a}\}_0}$ we use the transformations

$$\widehat{\nabla}_{a^1}\widehat{\nu}_{\dot{\mathbf{a}}} = \widehat{\nabla}_{a^1}(\nu_{\dot{\mathbf{a}}} + k\Upsilon^{p}\sigma_{p\dot{\mathbf{a}}}) = \big(\nabla_{a^1} + (k-1)\Upsilon_{a^1}\big)(\nu_{\dot{\mathbf{a}}} + k\Upsilon^{p}\sigma_{p\dot{\mathbf{a}}})$$
$$= \nabla_{a^1}\nu_{\dot{\mathbf{a}}} + (k-1)\Upsilon_{a^1}\nu_{\dot{\mathbf{a}}} + k(\nabla_{a^1}\Upsilon^{p})\sigma_{p\dot{\mathbf{a}}} + k\Upsilon^{p}\nabla_{a^1}\sigma_{p\dot{\mathbf{a}}} + k(k-1)\Upsilon_{a^1}\Upsilon^{p}\sigma_{p\dot{\mathbf{a}}},$$
$$\widehat{P}_{a^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} = P_{a^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} - (\nabla_{a^1}\Upsilon^{p})\sigma_{p\dot{\mathbf{a}}} + \Upsilon_{a^1}\Upsilon^{p}\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}\Upsilon^{p}\Upsilon_{p}\sigma_{\mathbf{a}},$$
$$\widehat{\nabla}^{p}\widehat{\nabla}_{\{p}\sigma_{\mathbf{a}\}_0} = \widehat{\nabla}^{p}\nabla_{\{p}\sigma_{\mathbf{a}\}_0} = \nabla^{p}\nabla_{\{p}\sigma_{\mathbf{a}\}_0} + n\Upsilon^{p}\nabla_{\{p}\sigma_{\mathbf{a}\}_0}.$$

See (10) for the first of these, (4) for the second and (12) for the last. Summing the right-hand sides with the coefficients from (19) we get,

$$\widehat{\rho}_{\mathbf{a}} = \rho_{\mathbf{a}} - \frac{k-1}{k}\Upsilon_{a^1}\nu_{\dot{\mathbf{a}}} - \Upsilon^p\nabla_{a^1}\sigma_{p\dot{\mathbf{a}}} - k\Upsilon_{a^1}\Upsilon^p\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}\Upsilon^p\Upsilon_p\sigma_{\mathbf{a}} + \frac{1}{k}\Upsilon^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0}.$$

Recall $\frac{1}{k}\Upsilon^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} = \Upsilon^p\nabla_{\{a^1}\sigma_{p\dot{\mathbf{a}}\}_0}$ using (11) therefore $-\Upsilon^p\nabla_{a^1}\sigma_{p\dot{\mathbf{a}}} + \frac{1}{k}\Upsilon^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} = -\Upsilon^p(\mu_{a^1p\dot{\mathbf{a}}} + \boldsymbol{g}_{a^1[p}\nu_{\dot{\mathbf{a}}]})$. From this and the previous display we obtain $\widehat{\rho}_{\mathbf{a}} = \rho_{\mathbf{a}} + \Upsilon^p\mu_{p\mathbf{a}} - \Upsilon_{a^1}\nu_{\dot{\mathbf{a}}} + \frac{1}{2}\Upsilon^p\Upsilon_p\sigma_{\mathbf{a}} - k\Upsilon_{a^1}\Upsilon^p\sigma_{p\dot{\mathbf{a}}}$. Using this and the transformation properties from (14), a short computation shows that $\mathbb{D}(\sigma)$ is a section of $\mathcal{E}_{A^0\mathbf{A}^k}$ that does not depend on the choice of the metric from the conformal class. □

Remarks. 1. For $k=1$, $\mathbb{D}$ is just the $w=1$ and special case of the operator $\mathbb{D}^{\beta a}$ from Section 5.1 of [4].

2. Note that the operator $\mathbb{D}$ is not unique as an invariant differential operator "putting" $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ into the top slot of $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$ (i.e. a differential splitting operator with left inverse $F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$). $\mathbb{D}$ can be obviously modified by any multiple of $\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}C_{a^1a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}$.

Assume $k \geqslant 2$. We define a 1st order differential operator $\Phi_c : \mathcal{E}_{A^0\mathbf{A}^k} \longrightarrow \mathcal{E}_{cA^0\mathbf{A}^k}$ for our later calculations. Given a section $F_{A^0\mathbf{A}} \in \mathcal{E}_{[A^0\mathbf{A}^k]}$ which, for $g \in [g]$, is convenient to take to be in the form

$$F_{A^0\mathbf{A}} = \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}^{a^0}_{A^0\mathbf{A}}{}^{\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} - \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}}, \tag{22}$$

we set

$$\begin{aligned}\Phi_c\big(F_{A^0\mathbf{A}}\big) :={}& -\frac{1}{2}\mathbb{Z}^{a^0}_{A^0\mathbf{A}}{}^{\mathbf{a}}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \\ &+ \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\bigg[A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - \frac{1}{2}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} + \frac{k}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}\bigg].\end{aligned} \tag{23}$$

Our aim is to construct a connection ${}^k\nabla$ on $\mathcal{E}_{A^0\mathbf{A}^k}$ such that solutions $\sigma_{\mathbf{a}}$ of (CKE) correspond to sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ that are parallel according to ${}^k\nabla$. Let us start with the normal tractor connection ∇. Using the previous proposition, it is a short and straightforward calculation to show that if $\sigma_{\mathbf{a}}$ is a solution of (CKE), $k \geqslant 2$ then $\nabla_c\mathbb{D}(\sigma)_{A^0\mathbf{A}} = \Phi_c(\mathbb{D}(\sigma)_{A^0\mathbf{A}})$. Also, it is easy to verify (or see [15]) that for $k=1$, if σ_{a^1} is a solution of (CKE) then $\nabla_c\mathbb{D}(\sigma)_{A^0A^1} = \Omega_{pcA^0A^1}\sigma^p$. This leads us to the following.

3.5. Lemma. (i) *Given a metric $g \in [g]$, the mapping*

$$\sigma_{\mathbf{a}} \mapsto \mathbb{D}(\sigma)_{A^0\mathbf{A}}, \quad \textit{with inverse} \quad F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}},$$

gives a bijective mapping between sections of $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ satisfying (CKE) *and sections $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$ satisfying*

$$\begin{aligned}&\nabla_cF_{A^0\mathbf{A}} = \Phi_c(F_{A^0\mathbf{A}}), && k \geqslant 2,\\ &\nabla_cF_{A^0A^1} = \frac{1}{2}\Omega_{pcA^0A^1}\sigma^p, && k=1.\end{aligned}$$

(ii) *Upon a conformal change $g \mapsto \widehat{g} = e^{2\Upsilon}g$, Φ_c transforms to*

$$\widehat{\Phi}_c\big(F_{A^0\mathbf{A}}\big) = \Phi_c(F_{A^0\mathbf{A}}) - \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Upsilon^p(C\lozenge\sigma)_{pc\mathbf{a}}$$

where $\Upsilon_a = \nabla_a\Upsilon$ and $\sigma_{\mathbf{a}} = (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$.

Proof. We have already observed that $\nabla_c\mathbb{D}(\sigma)_{A^0\mathbf{A}} = \Phi_c(\mathbb{D}(\sigma)_{A^0\mathbf{A}})$ for solutions σ of (CKE) for $k \geqslant 2$, and also the corresponding statement for $k=1$. On the other hand, looking at the coefficients of $\mathbb{Y}$ on both sides of $\nabla_cF_{A^0\mathbf{A}} = \Phi_c(F_{A^0\mathbf{A}})$ we see this relation implies that the "top slot" $\sigma_{\mathbf{a}} := (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$ of F is a solution of (CKE). Thus the claimed bijective correspondence follows.

It remains to prove (ii). Let us consider $F_{A^0\mathbf{A}}$ of the form (22) and a rescaling $g \mapsto \widehat{g}$ as above. Collecting together the conformal transformation formulae we have:

$$\begin{aligned}
&\widehat{\mu}_{a\mathbf{a}} = \mu_{a\mathbf{a}} + (k+1)\Upsilon_{a^0}\sigma_{\mathbf{a}}, \qquad \widehat{\nu}_{\dot{\mathbf{a}}} = \nu_{\dot{\mathbf{a}}} + k\Upsilon^p\sigma_{p\dot{\mathbf{a}}},\\
&\widehat{\mathbb{Z}}^{a^0}_{A^0\mathbf{A}}{}^{\mathbf{a}} = \mathbb{Z}^{a^0}_{A^0\mathbf{A}}{}^{\mathbf{a}} + (k+1)\Upsilon^{a^0}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}},\\
&\widehat{\mathbb{W}}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}} = \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}} - \Upsilon_{a^1}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}},\\
&\widehat{A}_{ab^1b^2} = A_{ab^1b^2} + \Upsilon^p C_{pab^1b^2},\\
&\widehat{\nabla}_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} = \nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + (k-2)\Upsilon_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + \boldsymbol{g}_{ca^1}\Upsilon^r(C\blacklozenge\sigma)_{r\dot{\mathbf{a}}}.
\end{aligned}\tag{24}$$

The first two transformations are immediate from (14) since $F_{A^0\mathbf{A}}$ is (assumed to be) conformally invariant. The next two formulae are directly the properties of $\mathbb{Z}$- and $\mathbb{X}$-tractors from (14). The last but one is a simple calculation using the conformal transformation formulae from for example [16], and the last follows from Lemma 3.1(i) and (12). Applying (24) to the formula (22) for Φ_c, we obtain

$$\begin{aligned}
\widehat{\Phi}_c(F_{A^0\mathbf{A}}) - \Phi_c(F_{A^0\mathbf{A}}) = \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\bigg[&-\frac{k+1}{2}\Upsilon^{a^0}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}}\\
&-\frac{k(k-1)}{2(n-k)}\Upsilon_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + \Upsilon^q C_q{}^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}\Upsilon^q C_q{}^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}}\\
&-\frac{k}{2}C_{a^1a^2c}{}^p\Upsilon^q\sigma_{qp\ddot{\mathbf{a}}} + \frac{k(k-2)}{2(n-k)}\Upsilon_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + \frac{k}{2(n-k)}\boldsymbol{g}_{ca^1}\Upsilon^r(C\blacklozenge\sigma)_{r\dot{\mathbf{a}}}\bigg].
\end{aligned}$$

It is straightforward to verify that sum of the three terms involving $C\blacklozenge\sigma$ is equal to

$$-\frac{k}{n-k}\Upsilon^r\boldsymbol{g}_{a^1[r}(C\blacklozenge\sigma)_{c]\dot{\mathbf{a}}}.\tag{25}$$

Summing the remaining terms on the right-hand side yields

$$\begin{aligned}
&\left(-\Upsilon^q C_{qa^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{2}\Upsilon^q C_{a^2a^1c}{}^p\sigma_{pq\ddot{\mathbf{a}}}\right) + \Upsilon^q C_{ca^1q}{}^p\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}\Upsilon^q C_{a^1a^2q}{}^p\sigma_{pc\ddot{\mathbf{a}}} + \frac{k}{2}\Upsilon^q C_{a^1a^2c}{}^p\sigma_{pq\ddot{\mathbf{a}}}\\
&\quad = -\Upsilon^r[C_{rca^1}{}^p\sigma_{p\dot{\mathbf{a}}} + C_{a^1a^2[r}{}^p\sigma_{|p|c]\ddot{\mathbf{a}}}].
\end{aligned}\tag{26}$$

Now summing the last two displays and comparing the result with the definition of $C\lozenge\sigma$ in (16), the lemma (ii) follows. □

We have shown that, in contrast to $\Omega_{pcA^0A^1}\sigma^p$, Φ_c for $k \geqslant 2$ is not conformally invariant. Also note that it is not algebraic but is rather a first order differential operator. We would like to replace Φ_c with an operator which, in a suitable sense, has the same essential properties (including linearity) and yet is conformally invariant and algebraic. We deal with invariance first. For $k \geqslant 2$, we define the 1st order differential operator $\Psi_c : \mathcal{E}_{[A^0\mathbf{A}^k]} \longrightarrow \mathcal{E}_{c[A^0\mathbf{A}^k]}$, for a given choice $g \in [g]$ of the metric and a section $F_{A^0\mathbf{A}} \in \mathcal{E}_{[A^0\mathbf{A}^k]}$ (taken to be of the form (22)), by

$$\Psi_c\big(F_{A^0\mathbf{A}}\big) := \Phi_c\big(F_{A^0\mathbf{A}}\big) + \frac{1}{n-2}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\nabla^p(C\lozenge\sigma)_{pc\mathbf{a}}.\tag{27}$$

Recall that $(C\lozenge\sigma)_{[pq]\mathbf{a}} \in \mathcal{E}(2,k)_0[k+1]$ and is by construction conformally invariant. Hence we have the conformal transformation $\widehat{\nabla}^p(C\lozenge\sigma)_{pc\mathbf{a}} = \nabla^p(C\lozenge\sigma)_{pc\mathbf{a}} + (n-2)\Upsilon^p(C\lozenge\sigma)_{pc\mathbf{a}}$, according to (12). From this and the previous lemma (ii) it follows that Ψ_c is conformally invariant.

Now recall we have proved in Lemma 3.3 that $C\lozenge\sigma = 0$ for σ satisfying (CKE). Therefore $\Phi_c = \Psi_c$ in this case and we have

3.6. Lemma. *Lemma* 3.5 *part* (i) *holds if we replace the operator Φ_c by Ψ_c therein.*

Now we replace the operator Ψ_c with an algebraic alternative in the following way. From (27) and the formulae (23) for Φ_c, it is clear that in the operator Ψ_c, applied to $F_{A^0\mathbf{A}}$ in the form (22), only the coefficient of $\mathbb{X}$ contains terms of the first order. Recall that we have the decomposition $\mathcal{E}_{c\mathbf{a}^k}[k+1] \cong \mathcal{E}_{[c\mathbf{a}^k]}[k+1] \oplus \mathcal{E}_{\{c\mathbf{a}^k\}_0}[k+1] \oplus \mathcal{E}_{\mathbf{a}^{k-1}}[k-1]$. If $\sigma_{\mathbf{a}} = (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}}$ is a solution of (CKE), the parts of $\nabla_c\sigma_{\mathbf{a}}$ that lie in $\mathcal{E}_{[c\mathbf{a}^k]}[k+1]$ and $\mathcal{E}_{\mathbf{a}^{k-1}}[k-1]$ may be replaced by, respectively, $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$ and $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$, according to Proposition 3.2. Moreover, it is clear that in fact this replacement is conformally invariant for any $F_{A^0\mathbf{A}}$. Thus if we remove, from the $\mathbb{X}$-slot of the formulae for Ψ_c, all the terms depending on $\nabla_{\{c}\sigma_{\mathbf{a}\}_0}$, then the resulting operator $\widetilde{\Psi}_c$ will be algebraic, conformally invariant and will satisfy Lemma 3.6 (with $\widetilde{\Psi}_c$ replacing Ψ_c therein). We now describe $\widetilde{\Psi}_c$ explicitly.

3.7. Proposition. *The mapping*

$$\sigma_{\mathbf{a}} \mapsto \mathbb{D}(\sigma)_{A^0\mathbf{A}}, \quad \textit{with inverse} \quad F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}},$$

gives a conformally invariant bijective mapping between sections of $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{A}^k}[k+1]$ satisfying (CKE) and sections $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$ satisfying,

$$\nabla_c F_{A^0\mathbf{A}} = \widetilde{\Psi}_c(F_{A^0\mathbf{A}}), \quad 1 \leqslant k \leqslant n-1.$$

For choice $g \in [g]$ of a metric from the conformal class and a section $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$, expressed in the form (22), the conformally invariant algebraic operator $\widetilde{\Psi}_c : \mathcal{E}_{A^0\mathbf{A}^k} \to \mathcal{E}_{cA^0\mathbf{A}^k}$ is given by the formula

$$\begin{aligned}\widetilde{\Psi}_c(F_{A^0\mathbf{A}}) = {} & -\frac{1}{2}\mathbb{Z}^{a^0\mathbf{a}}_{A^0\mathbf{A}} C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\mathbb{W}_{A^0A^1}{}^{\dot{\mathbf{a}}}_{\dot{\mathbf{A}}}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \\ & + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\left[A_{a^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{2(n-k)}T(\sigma)_{c\mathbf{a}}\right],\end{aligned} \tag{28}$$

where

$$\begin{aligned}T(\sigma)_{c\mathbf{a}} = {} & \frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} + 2A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - \boldsymbol{g}_{ca^1}A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} \\ & - (C_{ca^1}{}^{pq}\mu_{pq\ddot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc}\dot{}\dot{\mathbf{a}}\dot{}) - \frac{n-k-1}{k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} \in \mathcal{E}(1,k)[k-1].\end{aligned}$$

Proof. The case $k=1$ is just reformulation of Lemma 3.5. Given Lemma 3.6, for the cases $k \geqslant 2$ this boils down to simply checking the formula for $\widetilde{\Psi}$. This is a direct computation of the formula (27) for Ψ_c and then in this formula, formally replacing each instance of $\nabla_c\sigma_{\mathbf{a}}$ by $\mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$. We need to compute only the non-algebraic terms $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$ from (23) and $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$ from (27). The latter is the subject of Lemma 3.8 below, while the former is dealt with during the proof of that same lemma, see (30). Combining these results with (23) and collecting terms yields the formula (28). □

It remains then to calculate $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$ as required in the proof of the proposition above. For this we will need the following identities. They follow from the (second) Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ after a short computation.

$$\begin{aligned}\nabla_{a^1}C_{ca^2b^1b^2} &= \frac{1}{2}\nabla_c C_{a^1a^2b^1b^2} - \boldsymbol{g}_{cb^1}A_{b^2a^1a^2} + 2\boldsymbol{g}_{a^1b^1}A_{b^2ca^2}, \\ \nabla_{a^1}C_{a^2a^3b^1b^2} &= 2\boldsymbol{g}_{a^1b^1}A_{b^2a^2a^3}.\end{aligned} \tag{29}$$

3.8. Lemma. *Assume $2 \leqslant k \leqslant n-1$. If the $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ then, up to the addition of (conformally invariant) terms involving the Weyl curvature contracted into $\nabla_{\{c}\sigma_{\mathbf{a}\}_0}$, $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}} \in \mathcal{E}(1,k)_0[k-1]$ is given by the formula*

$$\begin{aligned}& \frac{n-2}{2(n-k)}\Bigg[\frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - (C_{ca^1}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\sigma_{pqc}\dot{}\dot{\mathbf{a}}\dot{}) \\ & \quad + (n-k-1)(A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + 2A^p{}_{a^1c}\sigma_{p\dot{\mathbf{a}}}) + \frac{(n-k+1)}{k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} \\ & \quad + \frac{(k-2)}{k}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq}\dot{}\dot{\mathbf{a}}\dot{} - (k-1)\boldsymbol{g}_{ca^1}A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}\Bigg] + (n-2)A_{a^1c}{}^p\sigma_{p\dot{\mathbf{a}}}.\end{aligned}$$

Proof. Here we simply expand $\nabla^q(C\Diamond\sigma)_{qc\mathbf{a}}$ via the Leibniz rule and in the process we will formally replace each $\nabla_c\sigma_{\mathbf{a}}$ by $\mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$. We shall start with $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$. Recall $(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$ was given in (16) as a sum of two terms. Applying ∇_{a^1} to these, we obtain

$$\nabla_{a^1}C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} = \frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - A^q{}_{a^1a^2}\sigma_{cq\ddot{\mathbf{a}}} + 2A^q{}_{ca^2}\sigma_{a^1q\ddot{\mathbf{a}}} + C_{ca^2}{}^{pq}(\mu_{a^1pq\ddot{\mathbf{a}}} + \boldsymbol{g}_{a^1[p}\nu_{q\ddot{\mathbf{a}}]}),$$
$$\nabla_{a^1}C_{a^3a^2}{}^{pq}\sigma_{pqc\dddot{\mathbf{a}}} = 2A^q{}_{a^3a^2}\sigma_{a^1qc\dddot{\mathbf{a}}} + C_{a^3a^2}{}^{pq}(\mu_{a^1pqc\dddot{\mathbf{a}}} + \boldsymbol{g}_{a^1[p}\nu_{qc\dddot{\mathbf{a}}]}),$$

where we have also used (29). Now summing of the right-hand sides of the last displays yields

$$\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} = \frac{k-2}{k}\bigg[\frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + 2A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}}$$
$$- (C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc\ddot{\mathbf{a}}}) + \frac{1}{k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} - \frac{1}{k}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq\dddot{\mathbf{a}}}\bigg] \tag{30}$$

where we used $\frac{2}{k}C_{ca^2a^1}{}^q = \frac{1}{k}C_{a^1a^2c}{}^q$. Note $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \in \mathcal{E}(1,k)[k-1]$.

Now we shall compute the formula for $\nabla^q(C\Diamond\sigma)_{qc\mathbf{a}}$. According to (16), $(C\Diamond\sigma)$ is defined as sum of three terms. Applying ∇^q to the first of these, and using (3), we obtain $\nabla^q C_{qca^1}{}^p\sigma_{p\dot{\mathbf{a}}} = (n-3)A_{ca^1}{}^p\sigma_{p\dot{\mathbf{a}}} + C^q{}_{ca^1}{}^p(\mu_{qp\dot{\mathbf{a}}} + \boldsymbol{g}_{q[p}\nu_{\dot{\mathbf{a}}]})$. Similarly for the second term, we obtain

$$\nabla^q C_{a^1a^2[q}{}^p\sigma_{|p|c]\ddot{\mathbf{a}}} = \frac{1}{2}(n-3)A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + \frac{1}{2}C_{a^1a^2}{}^{qp}(\mu_{qpc\ddot{\mathbf{a}}} + \boldsymbol{g}_{q[p}\nu_{c\ddot{\mathbf{a}}]})$$
$$- \frac{1}{2}(\nabla^q C_c{}^p{}_{a^1a^2})\sigma_{pq\ddot{\mathbf{a}}} + \frac{n-k+1}{2k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}},$$

where we have used $\nabla^q\sigma_{q\dot{\mathbf{a}}} = \frac{n-k+1}{k}\nu_{\dot{\mathbf{a}}}$. Summing the right-hand sides of the last two displays with the third term $\frac{k}{n-k}\nabla^q\boldsymbol{g}_{a^1[q}(C\blacklozenge\sigma)_{c]\dot{\mathbf{a}}}$ yields

$$\nabla^q(C\Diamond\sigma)_{qc\mathbf{a}} = \frac{1}{2}(\nabla^p C_c{}^q{}_{a^1a^2})\sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{2}(C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc\ddot{\mathbf{a}}})$$
$$+ (n-3)\bigg[A_{ca^1}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}}\bigg] + \frac{n-1}{2k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}}$$
$$+ \frac{k}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} - \frac{k}{2(n-k)}\boldsymbol{g}_{ca^1}\nabla^q(C\blacklozenge\sigma)_{q\dot{\mathbf{a}}} \tag{31}$$

where we have used $C^{[q}{}_{ca^1}{}^{p]} = -\frac{1}{2}C_{ca^1}{}^{qp}$. In the last display, we need the term $\nabla^p(C\blacklozenge\sigma)_{p\dot{\mathbf{a}}}$. Using the definition (16) and applying the Leibniz rule for ∇^p, we obtain

$$\nabla^p(C\blacklozenge\sigma)_{p\dot{\mathbf{a}}} = \frac{k-2}{k}\bigg[(n-3)A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} + C^r{}_{a^2}{}^{pq}\boldsymbol{g}_{r[p}\nu_{q\ddot{\mathbf{a}}]}$$
$$+ (\nabla^r C^{pq}{}_{a^3a^2})\sigma_{pqr\dddot{\mathbf{a}}} - \frac{n-k+1}{k}C_{a^2a^3}{}^{pq}\nu_{pq\dddot{\mathbf{a}}}\bigg]$$
$$= \frac{(k-2)(n-1)}{k}\bigg[A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{k}C_{a^2a^3}{}^{pq}\nu_{pq\dddot{\mathbf{a}}}\bigg] \tag{32}$$

using (29). We will also need the identity

$$\frac{1}{2}\big(\nabla^p C_c{}^q{}_{a^1a^2}\big)\sigma_{pq\ddot{\mathbf{a}}} = +\frac{1}{4}(\nabla_c C^{pq}{}_{a^1a^2})\sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{2}\boldsymbol{g}_{ca^1}A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} + A_{a^1c}{}^p\sigma_{p\dot{\mathbf{a}}}$$

which uses (29). Now we are ready to simplify (31) using (30), (32) and the last display. Collecting terms the result is

$$\nabla^q(C\Diamond\sigma)_{qc\mathbf{a}} = \frac{n-2}{4(n-k)}\bigg[(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - 2(C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc\ddot{\mathbf{a}}})$$
$$+ 2(n-k-1)A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + \frac{2(n-k+1)}{k}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}}$$

$$+ \frac{2(k-2)}{k} \boldsymbol{g}_{ca^1} C_{a^2a^3}{}^{pq} \nu_{pq\ddot{\mathbf{a}}} - 2(k-1)\boldsymbol{g}_{ca^1} A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}\Big] + \frac{1}{(n-k)}\big[(n-k)A_{a^1c}{}^p + (k-2)A^p{}_{ca^1} + (n-3)(n-k)A_{ca^1}{}^p\big]\sigma_{p\dot{\mathbf{a}}}.$$

Now the final step is to simplify the last line using the relation $A_{ca^1}{}^p = A^p{}_{a^1c} + A_{a^1c}{}^p$ which follows directly from the definition $A_{pa^1c} := 2\nabla_{[a^1}P_{c]p}$. A short computation reveals that the last line is equal to $(n-2)A_{a^1c}{}^p + (n-2)\frac{n-k-1}{n-k}A^p{}_{a^1c}$. The lemma now follows from the last two displays. □

Summarising our results we have the following.

3.9. Theorem. *For $1 \leqslant k \leqslant n-1$, the mapping $\mathcal{E}_{\mathbf{a}^k}[k+1] \to \mathcal{E}_{A^0\mathbf{A}^k}$ given by $\sigma \mapsto \mathbb{D}(\sigma)$ defined by (21) is a conformally invariant differential operator. Upon restriction it gives a bijective mapping from solutions of the conformal Killing equation (CKE) onto sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ that are parallel with respect to the connection ${}^k\nabla_c := \nabla_c - \widetilde{\Psi}_c$ where ∇_c is the normal tractor connection and $\widetilde{\Psi}_c$ is given by (28). The connection ${}^k\nabla_c$ is a conformally invariant connection on the form-tractor bundle $\mathcal{E}_{A^0\mathbf{A}^k}$. The inverting map from sections of $\mathcal{E}_{A^0\mathbf{A}^k}$, parallel for ${}^k\nabla_c$, to solutions of (CKE) is $F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}_{\mathbf{a}}F_{A^0\mathbf{A}}$.*

Sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ which are parallel for the normal tractor connection ∇_c are mapped injectively to solutions of (CKE) by

$$F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}_{\mathbf{a}}F_{A^0\mathbf{A}},$$

and $\tilde{\Psi}_c \circ \mathbb{D}$ annihilates the range of this map.

Proof. Everything has been established in the previous lemmas except for the last claim. That parallel sections are mapped injectively to conformal Killing forms is an immediate consequence of the formula (15) for the normal tractor connection on form-tractors. (Note that the equation from the first slot of $\nabla_c F_{A^0\mathbf{A}} = 0$ is $\nabla_c\sigma_{\mathbf{a}^k} - (k+1)\mu_{c\mathbf{a}^k} + \boldsymbol{g}_{ca^1}\varphi_{\dot{\mathbf{a}}^k} = 0$. This is the same equation as from the first slot for a $(k+1)$-form-tractor parallel for ${}^k\nabla_c$, as $\widetilde{\Psi}_c$ does not affect this top slot—the coefficient of $\mathbb{Y}$.) Next it is an elementary exercise using the formula (15) to verify that if $F_{A^0\mathbf{A}}$ is parallel for the normal tractor connection, then necessarily $F_{A^0\mathbf{A}} = \mathbb{D}(\sigma)$ where $\sigma_{\mathbf{a}} = (k+1)\mathbb{X}^{A^0\mathbf{A}}_{\mathbf{a}}F_{A^0\mathbf{A}}$. On the other hand from the first part of the theorem it follows that $\mathbb{D}(\sigma)$ is parallel for ${}^k\nabla$. So $\tilde{\Psi}_c(\sigma)$ vanishes everywhere. □

Remark. Let us say (as suggested in [20]) that a conformal Killing form σ is *normal* if it has the property that $\mathbb{D}(\sigma)$ is parallel for the normal tractor connection. It follows immediately from the theorem that the operator $\tilde{\Psi}_c$ detects exactly the failure of conformal Killing forms to be normal; a conformal Killing form is normal if and only if $\tilde{\Psi}_c(\sigma)$ is zero.

If $\sigma \in \mathcal{E}^k[k+1]$ vanishes on an open set then note that $\mathbb{D}(\sigma)$ vanishes on the same open set since $\mathbb{D}$ factors through the universal jet operator j^2. On the other hand if σ is a conformal Killing form then, from the theorem $\mathbb{D}(\sigma)$ is parallel for the connection ${}^k\nabla$. So the following holds.

3.10. Corollary. *On connected manifolds M a non-trivial conformal Killing form is non-vanishing on an open dense subspace.*

4. Coupled conformal Killing equations

In this section we show that solutions $\sigma \in \mathcal{E}^k[k+1]$ of the original equation (CKE) are in bijective correspondence with solutions of the coupled conformal Killing equation $\tilde{\nabla}_{(a}\overline{\sigma}_{b)_0\mathbf{B}^{k-1}} = 0$ on $\mathcal{E}_{a\mathbf{B}^{k-1}}[2]$ for a certain conformally invariant connection $\tilde{\nabla}$. Along the way we obtain some related preliminary results that should be of independent interest.

First let us observe that for any form $\sigma \in \mathcal{E}^k[k+1]$, $1 \leqslant k \leqslant n-1$, we may form the tractor-valued forms

$$\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} = \overline{M}^{\mathbf{a}^{k-l,l}}_{\mathbf{B}^l}\sigma_{\mathbf{a}^k} \quad \text{and} \quad \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} = \underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l}\sigma_{\mathbf{a}^k}, \tag{33}$$

where the invariant differential splitting operators $\overline{M}$ and $\underline{M}$ are defined by the formulae, for $1 \leqslant l \leqslant k$,

$$\overline{M}^{\mathbf{a}^{k-l,l}}_{\mathbf{B}^l} : \mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{\mathbf{a}^{k-l}\mathbf{B}^l}[k-l+1],$$
$$\overline{M}^{\mathbf{a}^{k-l,l}}_{\mathbf{B}^l}\sigma_{\mathbf{a}^k} = (n-k+1)\mathbb{Z}^{\mathbf{b}^l}_{\mathbf{B}^l}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l} - l\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\nabla^{b^1}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l},$$

and, for $1 \leqslant l \leqslant n-k$,

$$\underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l} : \mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{\mathbf{a}^{k+l}\mathbf{B}^l}[k+l+1],$$
$$\underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l}\sigma_{\mathbf{a}^k} = (k+1)\mathbb{Z}^{\mathbf{b}^l}_{\mathbf{B}^l}\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}}\sigma_{\mathbf{a}^k} - l\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{b}}^l}\boldsymbol{g}_{\dot{\mathbf{b}}^l\dot{\mathbf{a}}^{k,l}}\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k}.$$

Here we use multi-indices

$$\mathbf{a}^{k,l} = \left[a^{k+1}\cdots a^{k+l}\right] \quad \text{and} \quad \dot{\mathbf{a}}^{k,l} = \left[a^{k+2}\cdots a^{k+l}\right].$$

The conformal invariance of $\overline{M}$ and $\underline{M}$ may be verified directly via the formulae (14).

Although $\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and $\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$, as defined in (33), are invariant for the stated ranges of l, in the sequel we shall only need the tensor valence of $\overline{\sigma}$ and $\underline{\sigma}$ to be in the interval $[1, n-1]$. Therefore we shall henceforth assume that for $\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ we have $1 \leqslant l \leqslant k-1$ and for $\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ we have $1 \leqslant l \leqslant n-k-1$, respectively.

Let us next describe $\nabla_{\{c}\overline{\sigma}_{\mathbf{a}^{k-l}\}_0\mathbf{B}^l}$ and $\nabla_{\{c}\underline{\sigma}_{\mathbf{a}^{k+l}\}_0\mathbf{B}^l}$ when σ is a solution of (CKE). (Recall that ∇ denotes the coupled Levi-Civita-normal tractor connection.) This is explicitly formulated in the proposition below. First we need the following lemma.

4.1. Lemma. *Let us suppose that σ is a solution of (CKE). Then*

$$\text{(a)}\quad \nabla_c\nabla^p\sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} (n-k+1)\left[-\frac{k-1}{n-k}C_c{}^p{}_{[a^1}{}^q\sigma_{|p|\dot{\mathbf{a}}^{k-l}|q|\dot{\mathbf{b}}^l]} - P_c{}^p\sigma_{\dot{\mathbf{a}}^{k-l}p\dot{\mathbf{b}}^l}\right].$$

$$\text{(b)}\quad \nabla_c\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k} \overset{\{c\mathbf{a}^{k+1}\}_0}{=} (k+1)[C_{ca^{k+1}a^1}{}^p\sigma_{p\dot{\mathbf{a}}^k} - P_{ca^{k+1}}\sigma_{\mathbf{a}^k}].$$

In reading (b) here recall the convention that sequentially labelled indices (at a given level) are assumed to be skewed over.

Proof. First let us note that the trace part in the first case, and skew-symmetrisation $[c\mathbf{a}^{k+1}]$ in the second case, is zero on both sides. In the subsequent discussion we use Proposition 3.2 and the notation therein.

The left-hand side of (a) is equal to $\frac{n-k+1}{k}\nabla_c\nu_{\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ up to the sign $(-1)^{k-l}$. Now (a) follows using $C_c{}^{[p}{}_{a^1}{}^{q]} = \frac{1}{2}C_{ca^1}{}^{pq}$ and the equation for $\nabla_c\nu_{\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ in (18) where $(C\blacklozenge\sigma)_{c\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ is given by Lemma 3.1(i). Note that the projection $\{..\}$ over indices in the latter lemma exactly removes the completely skew-symmetric part of $C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}$ (see (17)). Since the projection $\{c\mathbf{a}^{k-l}\}_0$ annihilates the completely skew-symmetric part $C_{[ca^2}{}^{pq}\sigma_{|pq|\ddot{\mathbf{a}}]}$ we have $(C\blacklozenge\sigma)_{c\mathbf{a}^{k-l}\dot{\mathbf{b}}^l} =_{\{c\mathbf{a}^{k-l}\}_0} C_{ca^1}{}^{pq}\sigma_{pq\dot{\mathbf{a}}^{k-l}\dot{\mathbf{b}}}$. The part (b) follows similarly from the expression for $\nabla_c\mu_{a^{k+1}\mathbf{a}^k}$ in (18). $\square$

4.2. Proposition. *The form $\sigma \in \mathcal{E}^k[k+1]$, $1 \leqslant k \leqslant n-1$ is a solution of (CKE) if and only if either of the following conditions is satisfied*:

$$\nabla_c\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} \frac{l(k-1)(n-k+1)}{n-k}\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}C_c{}^p{}_{[a^1}{}^q\sigma_{|p|\dot{\mathbf{a}}^{k-l}|q|\dot{\mathbf{b}}^l]},$$
$$\nabla_c\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k+l}\}_0}{=} -l(k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}C_{c[a^{k+1}a^1}{}^p\sigma_{|p|\dot{\mathbf{a}}^k}\boldsymbol{g}_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l}.$$

Proof. The expressions on the left-hand side can be computed by directly differentiating the expressions (33) defining $\underline{\sigma}$ and $\overline{\sigma}$ and expanding in terms of the $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{W}$, $\mathbb{Z}$ splitting operators introduced in Section 2.3. The resulting "$\mathbb{Y}$-slot" (i.e. the coefficient of $\mathbb{Y}$) on the left-hand side is zero order, as an operator on σ, and is killed by the symmetrisation $\{c\mathbf{a}^{k-l}\}$ in the case of $\nabla_c\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and by taking the trace-free part in the case of $\nabla_c\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$. Essentially the same argument shows (in both cases) that also the operator in the $\mathbb{W}$-slot vanishes. The $\mathbb{Z}$-slot is of the first order as an operator on σ. To show this vanishes requires some computation. We will need the relation

$$k\boldsymbol{g}_{c[a^1}\nabla^p\sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l]} = (k-l)\boldsymbol{g}_{ca^1}\nabla^p\sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l} + l\boldsymbol{g}_{cb^1}\nabla^p\sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}. \tag{34}$$

(Recall our convention that all sequentially labelled indices are implicitly skewed over. So the b-indices are skewed and also the a-indices are skewed.) To prove this first observe the projection to the completely skew part of the right-hand side obviously yields exactly the left-hand side. On the other hand the right-hand side is manifestly skew over the b-indices and also over the a-indices. It is easily verified that it is also skew-symmetric in the index pair a^1b^1 and so the result follows.

Using (18) for $\nabla_c\sigma_{\mathbf{a}}$, one computes the $\mathbb{Z}$-slot of $\nabla_c\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ is

$$(n-k+1)\nabla_{[c}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l]} + k\boldsymbol{g}_{c[a^1}\nabla^p\sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l]} - l\boldsymbol{g}_{cb^1}\nabla^p\sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}.$$

The first term is killed by the projection $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}}$ and the remaining part is in the trace part over $\{c\mathbf{a}^{k-l}\}$ (i.e. in particular is annihilated by $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$) due to (34). The $\mathbb{Z}$-slot of $\nabla_c\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ is

$$\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}}\nabla_{[c}\sigma_{\mathbf{a}^k]} - l\boldsymbol{g}_{cb^1}\boldsymbol{g}_{\dot{\mathbf{b}}^l\dot{\mathbf{a}}^{k,l}}\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k} + \frac{k(k+1)}{n-k+1}\boldsymbol{g}_{ca^1}\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}}\nabla^p\sigma_{p\dot{\mathbf{a}}^k}$$

(also using (18)). The last term is killed by taking the trace-free part and it is easy to show the sum of the first two terms is $\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}}\nabla_c\sigma_{\mathbf{a}^k}$ (up to a scalar multiple) which vanishes after the symmetrisation $\{c\mathbf{a}^{k+l}\}$.

At this point it is worthwhile noting that if the projection $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$ kills $\nabla_c\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ or the projection $\mathcal{P}_{\{c\mathbf{a}^{k+l}\}_0}$ kills $\nabla_c\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ then σ is a solution of (CKE); the vanishing of the $\mathbb{Z}$-slots implies $\nabla_c\sigma_{\mathbf{a}} = \mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$ in (18) since $\mathcal{P}_{\{c\mathbf{a}^k\}_0}\circ\mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$ is a non-zero multiple of $\mathcal{P}_{\{c\mathbf{a}^k\}_0}$.

It remains to evaluate the $\mathbb{X}$-slots. This can be done easily using the rules for $\nabla_c\mathbb{W}$ and $\nabla_c\mathbb{X}$ from Section 2.3. We get

$$-l\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\big[(n-k+1)P_c{}^p\sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l} + \nabla_c\nabla^p\sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}\big]$$
$$-l\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\big[(k+1)P_{c[a^{k+1}}\sigma_{\mathbf{a}^k}g_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l} + \nabla_c\nabla_{[a^{k+1}}\sigma_{\mathbf{a}^k}g_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l}\big]$$

for $\nabla_c\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and $\nabla_c\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$, respectively. Now the proposition follows using Lemma 4.1. □

For our next construction we will especially need the first case of the proposition above for $l = k-1$, that is for $\bar{\sigma}_{a^1\dot{\mathbf{B}}^k}$. We will construct a connection $\tilde{\nabla}$ on $\mathcal{E}_{a^1\dot{\mathbf{B}}^k}$ such that the equation $\tilde{\nabla}_{(c}\bar{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = 0$ is equivalent to Eq. (CKE). Reformulating the proposition for $\bar{\sigma}_{a^1\dot{\mathbf{B}}^k}$, we get that σ is a solution of (CKE) if and only if

$$\nabla_{(c}\bar{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = \frac{(k-1)(k-2)(n-k+1)}{n-k}\mathbb{X}_{B^2}{}^{\ddot{\mathbf{b}}^k}_{\ddot{\mathbf{B}}^k}C_{b^3}{}^p{}_{(c}{}^q\sigma_{a^1)_0pq\dddot{b}^k}. \tag{35}$$

This shows that $\nabla_{(c}\bar{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = 0$ is equivalent to (CKE) in the flat case. In the curved case we modify the connection ∇ in the following way. Let us consider the tensor-tractor field

$$\begin{aligned}\kappa_{cE^0E^1F^0F^1} &:= \mathbb{X}_{E^0}{}^{e^1}_{E^1}\Omega_{ce^1F^0F^1}\\ &= \mathbb{X}_{E^0}{}^{e^1}_{E^1}\mathbb{Z}^{f^0f^1}_{F^0F^1}C_{ce^1f^0f^1} - 2\mathbb{X}_{E^0}{}^{e^1}_{E^1}\mathbb{X}_{F^0}{}^{f^1}_{F^1}A_{f^1ce^1},\end{aligned}$$

where $\Omega_{ce^1F^0F^1}$ is the curvature of the normal tractor connection. By construction this is conformally invariant. We will show that the required connection $\tilde{\nabla}$ can be written in the form $\tilde{\nabla}_c = \nabla_c + x\kappa_c\sharp\sharp$, $x\in\mathbb{R}$ where (via the tractor metric) we view $\kappa_{cE^0E^1F^0F^1}$ as a 1-form taking values in $\mathrm{End}(\mathcal{E}^A)\otimes\mathrm{End}(\mathcal{E}^A)$ and $\sharp$ indicates the usual action of tractor-bundle endomorphisms (i.e. it is the tractor-bundle analogue of the $\mathrm{End}(TM)$ action defined in Section 3 and we use the same notation as for that case). To determine the parameter $x\in\mathbb{R}$, let us compute the double action:

$$\begin{aligned}\kappa_c\sharp\sharp(\bar{\sigma}_{a^1\dot{\mathbf{B}}^k}) &= \mathbb{X}^{e^1}\mathbb{Z}^{f^0f^1}C_{ce^1f^0f^1}\sharp\sharp\big[(n-k+1)\mathbb{Z}^{\dot{\mathbf{b}}^k}_{\dot{\mathbf{B}}^k}\sigma^k_{\mathbf{a}^1\dot{\mathbf{b}}^k}\big]\\ &= (k-1)(n-k+1)\mathbb{X}^{e^1}\sharp\mathbb{Z}^{\dot{\mathbf{b}}^k}_{\dot{\mathbf{B}}^k}C_{ce^1b^2}{}^q\sigma_{a^1q\ddot{\mathbf{b}}^k}\\ &= -\frac{1}{2}(k-1)(k-2)(n-k+1)\mathbb{X}_{B^2}{}^{\ddot{\mathbf{b}}^k}_{\ddot{\mathbf{B}}^k}C_c{}^p{}_{b^3}{}^q\sigma_{a^1qp\dddot{\mathbf{b}}^k}.\end{aligned}$$

The form of the right-hand side shows that $\tilde{\nabla}$ is the required connection for a suitable parameter $x \in \mathbb{R}$, and comparing with (35) yields the explicit value for x. The resulting connection is

$$\tilde{\nabla}_c = \nabla_c + \frac{2}{n-k}\kappa_c \sharp\sharp, \tag{36}$$

where on the right-hand side ∇ is the normal tractor connection. Note that this connection is obviously conformally invariant (since both κ and the tractor connection are conformally invariant). This might seem inevitable, since from its derivation (or otherwise) it is clear that Eq. (35) is conformally invariant. However (36) is an invariant connection which may turn out to have applications in other circumstances. We summarise the last result.

4.3. Proposition. *A weighted k-form $\sigma \in \mathcal{E}^k[k+1]$ is a conformal Killing k-form (i.e. solution of (CKE)) if and only if*

$$\tilde{\nabla}_{(a}\bar{\sigma}_{b)_0} = 0 \tag{37}$$

where $\tilde{\nabla}$ is the Levi-Civita connection coupled with (36) and $\bar{\sigma}$ is the conformally invariant tractor extension of σ given by (33) with $l = k-1$.

The last result generalises. First observe that as well as the action $\kappa_c\sharp\sharp$ used in (36), we can consider also the action $\omega_c\sharp\sharp$ where we view the tensor-tractor field $\omega_{cE^0E^1f^0f^1} := \mathbb{X}_{E^0}{}^{e^1}{}_{E^1} C_{ce^1f^0f^1}$ as a one form taking values in $\mathrm{End}(\mathcal{E}^A) \otimes \mathrm{End}(\mathcal{E}^a)$ and $\sharp$ indicates the usual action of tensor/tractor-bundle endomorphisms. Now for any real parameter x we obtain a connection on tensor tractor fields via the formula,

$$\nabla^x_c = \nabla_c + x(\omega_c\sharp\sharp + \kappa_c\sharp\sharp) \tag{38}$$

where ∇ indicates the usual coupled tractor-Levi-Civita connection.

4.4. Theorem. *A weighted k-form $\sigma \in \mathcal{E}^k[k+1]$ is a conformal Killing k-form if and only if either of the following conditions hold*:

$$\nabla^x_{\{c}\bar{\sigma}_{\mathbf{a}^{k-l}\}_0\mathbf{B}^l} = 0 \quad or \quad \nabla^y_{\{c}\underline{\sigma}_{\mathbf{a}^{k+l}\}_0\mathbf{B}^l} = 0$$

where $x = \frac{2}{n-k}$ and $y = \frac{2}{k}$, and $\bar{\sigma}$, $\underline{\sigma}$ are the conformally invariant tractor extensions of σ given by (33).

Proof. First let us compute the actions $\omega_c\sharp\sharp$ and $\kappa_c\sharp\sharp$ on $\bar{\sigma}$ and $\underline{\sigma}$. The result is

$$\begin{aligned}
\omega_c\sharp\sharp\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} &= -\frac{1}{2}l(k-l)(n-k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l} C_c{}^p{}_{a^1}{}^q \sigma_{p\dot{\mathbf{a}}^{k-l}q\dot{\mathbf{b}}^l},\\
\kappa_c\sharp\sharp\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} &= -\frac{1}{2}l(l-1)(n-k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l} C_c{}^p{}_{b^2}{}^q \sigma_{\mathbf{a}^{k-l}qp\ddot{\mathbf{b}}^l},\\
\omega_c\sharp\sharp\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} &= \frac{1}{2}l(k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\big[(l-1)C_{ca^{k+2}b^2a^{k+1}}\mathbf{g}_{\ddot{\mathbf{a}}^{k,l}\ddot{\mathbf{b}}^l}\sigma_{\mathbf{a}^k} + kC_{ca^{k+1}a^1}{}^p\mathbf{g}_{\dot{\mathbf{a}}^{k,l}\dot{\mathbf{b}}^l}\sigma_{p\dot{\mathbf{a}}^k}\big],\\
\kappa_c\sharp\sharp\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} &= -\frac{1}{2}l(l-1)(k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l} C_{ca^{k+2}b^2a^{k+1}}\mathbf{g}_{\ddot{\mathbf{a}}^{k,l}\ddot{\mathbf{b}}^l}\sigma_{\mathbf{a}^k}.
\end{aligned}$$

Now the value $y = \frac{2}{k}$ follows immediately from Proposition 4.2. In the case of $\bar{\sigma}$, we can reformulate Proposition 4.2 in the following way: σ is a solution of (CKE) if and only if

$$\nabla_c\bar{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} \frac{l(n-k+1)}{n-k}\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\big[(k-l)C_c{}^p{}_{a^1}{}^q\sigma_{p\dot{\mathbf{a}}^{k-l}q\dot{\mathbf{b}}^l} + (l-1)C_c{}^p{}_{b^2}{}^q\sigma_{\mathbf{a}^{k-l}qp\dot{\mathbf{b}}^l}\big],$$

cf. (34). Thus the value $x = \frac{2}{n-k}$ follows. □

Remark. Note that the connections (38) preserve the $\mathrm{SO}(p,q)$ symmetry type (over tensor indices) and $\mathrm{SO}(p+1, q+1)$ symmetry type of the any tensor-tractor field they act on. The coupled tractor-Levi-Civita connection ∇ has this property. Then the $\omega_c\sharp\sharp$ action preserves these symmetries since ω_c is a 1-form taking values in the tensor product of orthogonal tractor endomorphisms tensor with orthogonal tensor endomorphisms. Similarly κ_c is a 1-form taking values in the tensor square of orthogonal tractor endomorphisms.

Note also that the action $C_{ab}\sharp$ of the Weyl tensor on tensors may in a natural way be viewed as a conformal action of the tractor curvature $\Omega_{ab}\sharp$ on *tensors*. (For example contract each tensor index "c" into a $Z_C{}^c$ and then apply the usual action of $\Omega_{ab}\sharp$ on these tractor indices. Finally remove each the new tractor index by contracting with $Z^C{}_e$. The result is conformally invariant since $\Omega_{ab}{}^C{}_D X^D = 0$.) If we extend the action $\Omega_{ab}\sharp$ to tensors in this way, then the connections ∇^x and ∇^y become $\nabla^x_c = \nabla_c + x\kappa_c\sharp\sharp$ and $\nabla^y_c = \nabla_c + y\kappa_c\sharp\sharp$ with x and y as above.

5. Applications: Helicity raising and lowering and almost Einstein manifolds

In the first part here we will assume the structure is almost Einstein in the sense of [14]. This is a manifold with a conformal structure and a section $\alpha \in \mathcal{E}[1]$ satisfying $[\nabla_{(a}\nabla_{b)_0} + P_{(ab)_0}]\alpha = 0$. Equivalently there is a standard tractor I_A that is parallel with respect to the normal tractor connection ∇. It follows that $I_A := \frac{1}{n}D_A\alpha = Y_A\alpha + Z_A^a\nabla_a\alpha - \frac{1}{n}X_A(\Delta + P)\alpha$, for some section $\alpha \in \mathcal{E}[1]$, and so $X^A I_A = \alpha$ is non-vanishing on an open dense subset of M and on this subset $g = \alpha^{-2}\boldsymbol{g}$ is an Einstein metric (where, recall $\boldsymbol{g}$ is the conformal metric). In particular any conformally Einstein manifold is almost Einstein.

In this setting we immediately have the theorem which follows. Recall that in a particular choice of metric a k-form σ is a Killing form if it is a solution of (1) with τ identically 0. Let us term a k-form σ a *dual-Killing form* if it is a solution of (1) where instead ρ is identically 0 (since on oriented manifolds the Hodge dual of a Killing form is a dual-Killing form and vice versa).

5.1. Theorem. *Let us consider a k-form $\sigma_{\mathbf{a}^k} \in \mathcal{E}^k[k+1]$. Then, for $k \in \{1,\dots,n\}$,*

$$\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}} := \alpha\nabla^p\sigma_{\mathbf{a}^{k-1}p} - (n-k+1)(\nabla^p\alpha)\sigma_{\mathbf{a}^{k-1}p} \in \mathcal{E}^{k-1}[k]$$

is conformally invariant. For $k \in \{0,\dots,n-1\}$,

$$\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}} := \alpha\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k} - (k+1)(\nabla_{a^{k+1}}\alpha)\sigma_{\mathbf{a}^k} \in \mathcal{E}^{k+1}[k+2]$$

is conformally invariant. If σ is a solution of (CKE) then we have the following equivalences:

$$\begin{aligned}
&\nabla_{\{c}\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}\}_0} = 0 \Longleftrightarrow C_{ca^1}{}^{pq}\sigma_{\dot{\mathbf{a}}^{k-1}pq} \overset{\{c\mathbf{a}^{k-1}\}_0}{=} 0,\\
&\nabla_{\{c}\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}\}_0} = 0 \Longleftrightarrow C_{ca^{k+1}a^1}{}^{p}\sigma_{\dot{\mathbf{a}}^{k}p} \overset{\{c\mathbf{a}^{k+1}\}_0}{=} 0
\end{aligned}\tag{39}$$

for $2 \leqslant k \leqslant n-1$ and $1 \leqslant k \leqslant n-2$, respectively. In the case that the first curvature condition is satisfied then the corresponding conformal Killing form $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}}$ is a Killing form away from the zero set of α, and in the Einstein scale $g = \alpha^{-2}\boldsymbol{g}$. In the case that the second curvature condition is satisfied then the corresponding conformal Killing form $\underline{\underline{\sigma}}_{\mathbf{a}^{k-1}}$ is a dual-Killing form away from the zero set of α, and in the Einstein scale $g = \alpha^{-2}\boldsymbol{g}$.

Proof. The first part of the proposition follows from relations $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}} = I^B\overline{\sigma}_{\mathbf{a}^{k-1}B}$ and $\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}} = I^B\underline{\sigma}_{\mathbf{a}^{k+1}B}$ where the forms $\overline{\sigma}_{\mathbf{a}^{k-1}B}$ and $\underline{\sigma}_{\mathbf{a}^{k+1}B}$ are defined by (33) in Section 4. The result (39) follows from Proposition 4.2 and continuity, since the tractor I^B is parallel and $I^B X_B$ is non-vanishing on an open dense set in the manifold. For the final points note that, from the formulae for $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}}$ and $\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}}$ given in the first part of the theorem, it is clear that these are, respectively, coclosed and closed in the Einstein scale $g = \alpha^{-1}\boldsymbol{g}$ given off the zero set of α. $\square$

Remarks. 1. Note that the first curvature condition on the right-hand side of (39) is that $(C\blacklozenge\sigma) = 0$. That is that the projection of $C\sharp\sigma$ to $\mathcal{E}(1,k-1)[k-1]$ should vanish everywhere. Similarly the second is simply that the (unique up to a multiple) projection of $C\sharp\sigma$ to $\mathcal{E}(1,k+1)_0[k+1]$ should vanish everywhere. Note that in the case that the manifold is oriented then the second curvature condition is exactly that the Hodge dual of σ satisfies the first condition (as applied to $(n-k)$-form solutions of (CKE)).

2. Note that on an almost Einstein manifold with a conformal Killing k-form such that $(C\blacklozenge\sigma) = 0$ then, according to the theorem, on the open dense set where α is non-vanishing there is a scale so that $\overline{\overline{\sigma}}$ is a Killing form. But the

section α does not necessarily give a global metric whereas the form $\bar{\bar{\sigma}}$ is a globally defined conformal Killing form. A similar comment applies to $\underline{\underline{\sigma}}$.

5.2. Corollary. *If σ_{ab} is a conformal Killing 2-form then*

$$\bar{\bar{\sigma}}_a = \alpha\nabla^p\sigma_{ap} - (n-1)(\nabla^p\alpha)\sigma_{ap}$$

is a conformal Killing vector field. If $\sigma'_{\mathbf{a}^{n-2}}$ is a conformal Killing $(n-2)$-form then

$$\underline{\underline{\sigma}}'_{\mathbf{a}^{n-1}} := \alpha\nabla_{a^{n-1}}\sigma'_{\mathbf{a}^{n-2}} - (n-1)(\nabla_{a^{n-1}}\alpha)\sigma'_{\mathbf{a}^{n-2}} \ \in \mathcal{E}^{n-1}[n]$$

is a conformal Killing $(n-1)$-form. Away from the zero set of α, $\bar{\bar{\sigma}}_a$ is a Killing vector for the Einstein metric $g = \alpha^{-2}\mathbf{g}$, while in this scale $\underline{\underline{\sigma}}'_{\mathbf{a}^{n-1}}$ is a dual-Killing form.

Proof. This is just the theorem above for $k=2$. The condition $C_{(ab)_0}{}^{pq}\sigma_{pq}$ is trivially satisfied, and, hence, so too is the dual condition (cf. point 1. of the remark above). □

Note that a weaker form of the first part of the corollary has been proved (by a direct computation) in [22, 7.2].

Remark. Note that according to the Corollary, on Einstein 4-manifolds a non-parallel conformal Killing 2-form implies the existence of either a non-trivial Killing vector field or a non-trivial dual-Killing 3-form. Thus if the 4-manifold is also oriented then, in any case, a non-parallel conformal Killing 2-form determines a non-trivial Killing vector field.

The first part of the theorem is valid also for $k=1$ in the sense, that if σ_a satisfies (CKE) then $\bar{\bar{\sigma}} := \alpha\nabla^p\sigma_p - n(\nabla^p\alpha)\sigma_p$ is (conformally invariant and) another almost Einstein scale. This is easily seen as follows. Let us write $\sigma_{CD} := \mathbb{D}^a_{CD}\sigma_a$, where $\mathbb{D}$ was defined for Lemma 3.4. Then by Lemma 3.5 we have

$$\nabla_a\sigma_{CD} = \frac{1}{2}\Omega^p{}_{aCD}\sigma_p. \tag{40}$$

Note that $I^D\sigma_{CD}$ is parallel with respect to the normal tractor connection ∇ since $\nabla_a I^D\mathbb{D}^a_{CD}\sigma_a = (\nabla_a\sigma_{CD})I^D = \frac{1}{2}\sigma^p\Omega_{paCD}I^D = 0$. Then the result follows from Theorem 3.1 of [17] since $\bar{\bar{\sigma}} = X^C I^D\sigma_{CD}$.

Some related results follow. Following [17] we term a metric (or conformal structure) *weakly generic* if the Weyl curvature is injective as bundle map $TM \to \otimes^3 TM$.

5.3. Proposition. (i) *If σ_a is a non-homothetic conformal Killing vector field (i.e. a solution of (CKE) with non-constant $\nabla_a\sigma^a$) on an Einstein manifold then there exists a non-trivial solution $\tilde{\sigma}_a$ of (CKE) which is exact for the Einstein scale (i.e. a conformal gradient field).*

(ii) *If a weakly generic conformally Einstein manifold M admits a conformal Killing vector field σ^a, then σ^a is a homothety for any Einstein metric in the conformal class.*

Proof. Let us write $I^1_D := I_D$ and $I^2_C := \sigma_{CP}I^P$, where $\sigma_{CP} = \mathbb{D}^a_{CP}\sigma_a$. These parallel tractors determine a parallel tractor 2-form tractor $I^1_{[C}I^2_{D]}$. Let us write $\tilde{\sigma}_a := \frac{1}{2}\mathbb{X}^{CD}{}_a I^1_{[C}I^2_{D]}$. (Note that from the last part of Theorem 3.9 it follows immediately that $\tilde{\sigma}_a$ is a conformal Killing field hence $\Omega^p_{aCD}\tilde{\sigma}_p = 0$ by (40). Thus $C_{abc}{}^p\tilde{\sigma}_p = 0$.)

Since I^1_D and I^2_C are parallel and the top slot of I^2_C is $\bar{\bar{\sigma}} = X^C I^D\sigma_{CD}$ it follows (Theorem 3.1 of [17]) that $I^2_C = \frac{1}{n}D_C\bar{\bar{\sigma}}$. To compute $\tilde{\sigma}_a$ let us write explicitly

$$I^1_D = Y_D\alpha + Z^d_D\nabla_d\alpha - \frac{1}{n}X_D(\Delta + J)\alpha,$$

$$I^2_C = Y_C\bar{\bar{\sigma}} + Z^c_C\nabla_c\bar{\bar{\sigma}} - \frac{1}{n}X_C(\Delta + J)\bar{\bar{\sigma}}.$$

Here we have used the formula (8) for the tractor D operator. Now it follows easily that $\tilde{\sigma}_a$ is $(\nabla_a\alpha)\bar{\bar{\sigma}} - \alpha(\nabla_a\bar{\bar{\sigma}})$ up to a (non-zero) scalar multiple. (From this formula, it is also easy to verify by a direct computation that $\tilde{\sigma}_a$ satisfies (CKE).) In the Einstein scale α we have $\nabla\alpha = 0$, whence $\tilde{\sigma}_a = -\nabla_a(\alpha\bar{\bar{\sigma}}) = -\nabla_a(\alpha^2\nabla^p\sigma_p)$.

(ii) This is an immediate consequence of part (i) since it is well known (and an easy exercise to verify) that any conformal gradient field $\tilde{\sigma}^a$ necessarily satisfies $C_{ab}{}^c{}_p\tilde{\sigma}^p = 0$. □

One can easily access further results along these lines, but manifolds admitting a conformal gradient field are rather well understood and there are many classification results due to, for example, H.W. Brinkman, J.P. Bourguignon, D.V. Alekseevskii and others. For some recent progress and indication of the state of art see [19].

Theorem 5.1 exploited the standard tractor I_A which (corresponds to an almost Einstein scale α and) is parallel with respect to the normal tractor connection ∇. Here we drop the assumption that the manifold is almost Einstein and assume instead that the manifold is equipped with a conformal Killing field σ^a. Then we use the tractor $\sigma_{AB} := \mathbb{D}^p_{AB}\sigma_p$ (given by (21)) provided by the conformal Killing form σ_a. This is not, in general, parallel with respect to the normal tractor connection ∇. Rather, we obtained (40) in Lemma 3.5.

5.4. Theorem. *For each pair $\sigma \in \mathcal{E}^1[2]$ and $\tau \in \mathcal{E}^k[k+1]$*

$$\check{\bar{\tau}}_{\mathbf{a}^{k-2}} := 2\sigma^p\nabla^q\tau_{\mathbf{a}^{k-2}pq} + (n-k+1)(\nabla^p\sigma^q)\tau_{\mathbf{a}^{k-2}pq}, \quad k \in \{2,\dots,n\}$$

is a conformally invariant section of $\mathcal{E}^{k-2}[k-1]$, and

$$\check{\underline{\tau}}_{\mathbf{a}^{k+2}} := 2\sigma_{a^{k+1}}\nabla_{a^{k+2}}\tau_{\mathbf{a}^k} + (k+1)(\nabla_{a^{k+1}}\sigma_{a^{k+2}})\tau_{\mathbf{a}^k}, \quad k \in \{0,\dots,n-2\}$$

is a conformally invariant section of $\mathcal{E}^{k+2}[k+3]$. If σ and τ are solutions of (CKE) then the following is satisfied: for $3 \leqslant k \leqslant n-1$ $\check{\bar{\tau}}_{\mathbf{a}^{k-2}}$, is a solution of (CKE) if and only if

$$(n-k+1)C^r{}_c{}^{pq}\tau_{\mathbf{a}^{k-2}pq}\sigma_r + (k-2)C_{ca^1}{}^{pq}\tau_{p\dot{\mathbf{a}}^{k-2}qr}\sigma^r \overset{\{c\mathbf{a}^{k-2}\}_0}{=} 0$$

and, for $1 \leqslant k \leqslant n-3$, $\check{\underline{\tau}}_{\mathbf{a}^{k+2}}$, is a solution of (CKE) if and only if

$$2C_{ca^{k+1}a^1}{}^p\tau_{p\dot{\mathbf{a}}^k}\sigma_{a^{k+2}} - C^p_{ca^{k+1}a^{k+2}}\tau_{\mathbf{a}^k}\sigma_p \overset{\{c\mathbf{a}^{k+2}\}_0}{=} 0.$$

Proof. The first part of the proposition follows from relations $\check{\bar{\tau}}_{\mathbf{a}^{k-2}} = 2\bar{\tau}_{\mathbf{a}^{k-2}RS}\sigma^{RS}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+2}} = 2\underline{\tau}_{\mathbf{a}^{k+2}RS}\sigma^{RS}$. The second part is a result of a direct computation. Using Proposition 4.2 and (40) we obtain the following:

$$\begin{aligned}
2\nabla_c\bar{\tau}_{\mathbf{a}^{k-2}RS}\sigma^{RS} &\overset{\{c\mathbf{a}^{k-2}\}_0}{=} 2(\nabla_c\bar{\tau}_{\mathbf{a}^{k-2}RS})\sigma^{RS} + 2\bar{\tau}_{\mathbf{a}^{k-2}RS}\nabla_c\sigma^{RS}\\
&\overset{\{c\mathbf{a}^{k-2}\}_0}{=} \frac{4(n-k+1)}{n-k}\mathbb{X}_{RS}{}^s\big[(k-2)C_c{}^p{}_{a^1}{}^q\tau_{p\dot{\mathbf{a}}^{k-2}qs} - C_c{}^p{}_s{}^q\tau_{p\dot{\mathbf{a}}^{k-2}qa^1}\big]\sigma^{RS} + \bar{\tau}_{\mathbf{a}^{k-2}RS}\Omega^p{}_c{}^{RS}\sigma_p\\
&\overset{\{c\mathbf{a}^{k-2}\}_0}{=} \frac{n-k+1}{n-k}\big[(n-k+1)C^s{}_c{}^{pq}\tau_{\mathbf{a}^{k-2}pq}\sigma_s + (k-2)C_{ca^1}{}^{pq}\tau_{p\dot{\mathbf{a}}^{k-2}qs}\sigma^s\big],\\
2\nabla_c\underline{\tau}_{\mathbf{a}^{k+2}RS}\sigma^{RS} &\overset{\{c\mathbf{a}^{k+2}\}_0}{=} 2(\nabla_c\underline{\tau}_{\mathbf{a}^{k+2}RS})\sigma^{RS} + 2\underline{\tau}_{\mathbf{a}^{k+2}RS}\nabla_c\sigma^{RS}\\
&\overset{\{c\mathbf{a}^{k+2}\}_0}{=} -4(k+1)\mathbb{X}_{RS}^{\,s}C^p_{ca^{k+1}a^1}\tau_{p\dot{\mathbf{a}}^k}\boldsymbol{g}_{a^{k+2}s}\sigma^{RS} + \underline{\tau}_{\mathbf{a}^{k+2}RS}\Omega^p{}_c{}^{RS}\sigma_p\\
&\overset{\{c\mathbf{a}^{k+2}\}_0}{=} -(k+1)[2C^p_{ca^{k+1}a^1}\tau_{p\dot{\mathbf{a}}^k}\sigma_{a^{k+2}} - Cca^{k+1}a^{k+2p}\tau_{\mathbf{a}^k}\sigma_p]. \qquad \square
\end{aligned}$$

Note for the cases of a conformal Killing 3-form τ the first curvature condition of the theorem is satisfied by any conformal gradient vector field σ.

Now it is obvious how to obtain more general results for couples of conformal Killing forms $\sigma \in \mathcal{E}^l[l+1]$ and $\tau \in \mathcal{E}^k[k+1]$ where $1 \leqslant k, l \leqslant n-1$. We set $\sigma_{\mathbf{A}^{l+1}} := \mathbb{D}\sigma$ and define $\check{\bar{\tau}}_{\mathbf{a}^{k-l-1}} := \bar{\tau}_{\mathbf{a}^{k-l-1}\mathbf{A}^{l+1}}\sigma^{\mathbf{A}^{l+1}}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+l+1}} := \underline{\tau}_{\mathbf{a}^{k+l+1}\mathbf{A}^{l+1}}\sigma^{\mathbf{A}^{l+1}}$ for $0 \leqslant k-l-1 \leqslant n$ and $0 \leqslant k+l+1 \leqslant n$, respectively. The case $l = 1$ is described in the previous Theorem and in general, the obstructions for $\check{\bar{\tau}}_{\mathbf{a}^{k-l-1}}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+l+1}}$ to be solutions of (CKE) are very similar to the cases

$l = 1$. (In the proof of these new cases, we replace $\nabla_c \sigma^{RS}$ by $\nabla_c \sigma^{\mathbf{A}^{l+1}}$. The latter is, in general quite complicated but we actually need only '$\mathbb{Z}$-slot' and '$\mathbb{Y}$-slot' which are similar to the case $l = 1$.)

5.5. Corollary. *Let $\sigma_a \in \mathcal{E}_a[2]$ be a solution of (CKE) and write $\mu_{bc} := \nabla_{[b}\sigma_{c]}$ (in a choice of scale). Then the section*

$$\sigma_{a^0}\mu_{a^1a^2}\cdots\mu_{a^{2p-1}a^{2p}} \in \mathcal{E}^{2p+1}[2p+2], \quad p \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor$$

is conformally invariant. If $\sigma_{a^0}C_{a^1a^2c}{}^d\sigma_d = 0$ then it is a solution of (CKE) for any $1 \leqslant p \leqslant \lfloor \frac{n-2}{2} \rfloor$.

Proof. For $p = 1$, this is Theorem 5.4 applied to $\tau := \sigma \in \mathcal{E}^1[2]$. If the curvature condition is satisfied then it is easily checked that applying the same theorem to σ_a and $\tau := \sigma_{a^0}\mu_{a^1a^2}$, we obtain the case $p = 2$. Repeating this procedure, the general case follows. □

Let us note there are several results in [23] related to those in this section, see Propositions 3.4 and 3.5 in [23]. These concern a special case satisfying that $\nabla_c\mu_{a^0a^1}$ is pure trace (which implies that σ_a is an eigensection for the Schouten tensor viewed as a section of $\mathrm{End}(TM)$). This immediately yields $\sigma_{a^0}C_{a^1a^2c}{}^p\sigma_p = 0$ using (18).

Our last application concerns conformal Killing m-tensors. These are valence m symmetric trace-free tensors $t_{b\cdots c} \in \mathcal{E}_{(b\cdots c)_0}[2m]$ which are solutions of the conformally invariant equation $\nabla_{(a}t_{b\cdots c)_0} = 0$. Obviously, any conformal Killing form $\sigma_a \in \mathcal{E}_a[2]$ yields a conformal Killing tensor $\sigma_{(a}\cdots\sigma_{b)_0}$. Note that generalising the $m = 2$ version of this observation we have the following. If $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ is conformal Killing form then $\sigma_{(a}{}^{\dot{\mathbf{c}}}\sigma_{b)_0\dot{\mathbf{c}}} \in \mathcal{E}_{(ab)_0}[4]$, is a conformal Killing 2-tensor. (The special case of this where σ is a conformal Killing 2-form appeared in [27, 4.1(4)].) This follows from (18) by a direct computation or from the relation $\sigma_{(a}{}^{\dot{\mathbf{e}}}\sigma_{b)_0\dot{\mathbf{e}}} = \frac{1}{(n-k+1)^2}\bar{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\bar{\sigma}_{b)_0\dot{\mathbf{E}}}$ (which holds since X_A and Z_a^A are orthogonal), and Propositions 4.2 and 4.3. The point here is that one applies the normal tractor ∇_c connection to $\bar{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\bar{\sigma}_{b)_0\dot{\mathbf{E}}}$ to obtain $2\bar{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\nabla_b\bar{\sigma}_{c)_0\dot{\mathbf{E}}}$ after the projection to $\mathcal{E}_{(abc)_0}[4]$. Then from Proposition 4.2 and again the orthogonality of X_A and Z_a^A we may replace ∇ by $\tilde{\nabla}$ to obtain $2\bar{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\tilde{\nabla}_b\bar{\sigma}_{c)_0\dot{\mathbf{E}}}$. But then by Proposition 4.3 the last expression vanishes. It is clear this example generalises and so we have the following theorem.

5.6. Theorem. *Suppose $\sigma^1, \ldots, \sigma^m$ is a collection of conformal Killing forms of respective ranks $r_1, \ldots, r_m$ where $(\sum^m r_i) - m$ is an even number. Then $\sigma^1_{(a}\cdot\sigma^2_b\cdots\sigma^m_{c)_0}$ is a conformal Killing m-tensor. Here $\sigma^1_a\cdot\sigma^2_b\cdots\sigma^m_c$ indicates any contraction of the collection $\sigma^1, \ldots, \sigma^m$ over the suppressed indices.*

Of course it will often be the case that a given contraction $\sigma^1_a\cdot\sigma^2_b\cdots\sigma^m_c$ vanishes upon projection to the trace-free part. However it is easy to proliferate non-trivial examples.

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References

[1] T.N. Bailey, M.G. Eastwood, A.R. Gover, Thomas' structure bundle for conformal, projective and related structures, Rocky Mountain J. Math. 24 (1994) 1191–1217.

[2] I.M. Benn, P. Charlton, Dirac symmetry operators from conformal Killing–Yano tensors, Classical Quantum Gravity 14 (1997) 1037–1042.

[3] T. Branson, A. Čap, M.G. Eastwood, A.R. Gover, Prolongations of geometric overdetermined systems, Internat. J. Math. 17 (2006) 641–664.

[4] T. Branson, A.R. Gover, Electromagnetism, metric deformations, ellipticity and gauge operators on conformal 4-manifolds, Diff. Geom. Appl. 17 (2002) 229–249.

[5] T. Branson, A.R. Gover, Conformally invariant operators, differential forms, cohomology and a generalisation of Q curvature, Comm. Partial Differential Equations 30 (2005) 1611–1669.

[6] D.M.J. Calderbank, T. Diemer, Differential invariants and curved Bernstein–Gelfand–Gelfand sequences, J. Reine Angew. Math. 537 (2001) 67–103.

[7] A. Čap, Infinitesimal automorphisms and deformations of parabolic geometries, Preprint math.DG/0508535, http://www.arxiv.org.

[8] A. Čap, A.R. Gover, Tractor bundles for irreducible parabolic geometries, in: Global Analysis and Harmonic Analysis, Marseille-Luminy, 1999, in: Smin. Congr., vol. 4, Soc. Math. France, Paris, 2000, pp. 129–154.
[9] A. Čap, A.R. Gover, Tractor calculi for parabolic geometries, Trans. Amer. Math. Soc. 354 (2002) 1511–1548.
[10] A. Čap, A.R. Gover, Standard tractors and the conformal ambient metric construction, Ann. Global Anal. Geom. 24 (2003) 231–259.
[11] M.G. Eastwood, Notes on conformal differential geometry, Supp. Rend. Circ. Matem. Palermo, Ser. II Suppl. 43 (1996) 57–76.
[12] A.R. Gover, Aspects of parabolic invariant theory, Rend. Circ. Mat. Palermo (2) Suppl. 59 (1999) 25–47.
[13] A.R. Gover, Conformal de Rham Hodge theory and operators generalising the Q-curvature, Rend. Circ. Mat. Palermo (2) Suppl. 75 (2005) 109–137.
[14] A.R. Gover, Almost conformally Einstein manifolds and obstructions, in: Differential Geometry and its Applications, Proceedings of the 9th International Conference on Differential Geometry and its Applications, Prague 2004, Charles University, Prague, 2005, pp. 243–255, math.DG/0412393, http://www.arxiv.org.
[15] A.R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, Math. Ann. 336 (2006) 311–334, math.DG/0506037, http://www.arxiv.org.
[16] A.R. Gover, L.J. Peterson, Conformally invariant powers of the Laplacian, Q-curvature and tractor calculus, Comm. Math. Phys. 235 (2003) 339–378.
[17] A.R. Gover, P. Nurowski, Obstructions to conformally Einstein metrics in n dimensions, J. Geom. Phys. 56 (2006) 450–484.
[18] T. Kashiwada, On conformal Killing tensor, Natur. Sci. Rep. Ochanomizu Univ. 19 (1986) 67–74.
[19] W. Kühnel, H.-B. Rademacher, Essential conformal fields in pseudo-Riemannian geometry. II, J. Math. Sci. Univ. Tokyo 4 (1997) 649–662.
[20] F. Leitner, Conformal Killing forms with normalisation condition, Rend. Circ. Mat. Palermo (2) Suppl. 75 (2005) 279–292.
[21] R. Penrose, W. Rindler, Spinors and Space–Time. Vol. 1. Two-Spinor Calculus and Relativistic Fields, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984, x+458 pp.
[22] U. Semmelmann, Conformal Killing forms on Riemannian manifolds, habilitation, Ludwig Maximilians University in Munchen, 2001.
[23] U. Semmelmann, Conformal Killing forms on Riemannian manifolds, Math. Z. 245 (2003) 503–527.
[24] U. Semmelmann, Killing forms on G2 and Spin 7 manifolds, J. Geom. Phys. 56 (2006) 1752–1766, math.DG/0410065, http://www.arxiv.org.
[25] J. Šilhan, Invariant operators in conformal geometry, PhD thesis, University of Auckland, 2006.
[26] D.C. Spencer, Overdetermined systems of linear partial differential equations, Bull. Amer. Math. Soc. 75 (1969) 179–239.
[27] S.E. Stepanov, On conformal Killing 2-form of electromagnetic field, J. Geom. Phys. 33 (2000) 191–209.
[28] S. Tachibana, On conformal Killing tensor in a Riemannian space, Tohoku Math. J. 21 (1969) 56–64.