PARTITIONS OF POINTS INTO INTERSECTING TETRAHEDRA

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Reay has conjectured that any set of \((m - 1)(d + 1) + k + 1\) points in general position in \(\mathbb{R}^d\) can be partitioned into \(m\) disjoint subsets \(S_1, S_2, \ldots, S_m\) in such a way that \(\bigcap_{i=1}^m \text{Conv}(S_i)\) is \(k\)-dimensional. We prove this conjecture in the particular case \(d = 3\).

1. Introduction

Radon's theorem asserts that any set \(S\) of \(d + 2\) points in \(\mathbb{R}^d\) has a partition into two subsets \(S_1\) and \(S_2\) such that \(\text{Conv}(S_1) \cap \text{Conv}(S_2) \neq \emptyset\), where \(\text{Conv}(S_i)\) denotes the convex hull of \(S_i\). This central theorem in the theory of convexity has been extended in many directions. One of the most interesting generalizations is the following theorem, proved by Tverberg in 1966, and now considered as a classical result in combinatorial geometry.

**Theorem 1 (Tverberg's theorem [5, 6]).** Any set \(S\) of \((m - 1)(d + 1) + 1\) points in \(\mathbb{R}^d\) has a partition into \(m\) disjoint subsets \(S_1, S_2, \ldots, S_m\) such that \(\bigcap_{i=1}^m \text{Conv}(S_i) \neq \emptyset\).

Any point \(x\) belonging to \(\bigcap_{i=1}^m \text{Conv}(S_i)\), for some partition \(S_1, S_2, \ldots, S_m\) of \(S\), with \(|S| = (m - 1)(d + 1) + 1\) will be called a **Tverberg point** of \(S\).

Tverberg's theorem is best possible in the sense that, for all \(d \geq 1\) and \(m \geq 1\), \((m - 1)(d + 1) + 1\) cannot be replaced by a smaller value, and that the Tverberg point is generally unique for a given partition \(S_1, S_2, \ldots, S_m\).

In [2], Reay has asked for stronger conditions on \(S\), which ensure \(\bigcap_{i=1}^m \text{Conv}(S_i)\) to be not only non-empty, but \(k\)-dimensional. It is intuitively clear that, in general, \((m - 1)(d + 1) + k + 1\) points are needed. Also, the points have to be "sufficiently independent" since, e.g. any set of points on a line will never produce a 2-dimensional intersection. A standard way to avoid degenerate configurations is to restrict our attention to points in **general position**, i.e. no \(d + 1\) of which lie in a common hyperplane. The following conjecture was proposed by Reay in 1968.

Conjecture 1 (Reay [2]). Any set $S$ of $(m - 1)(d + 1) + k + 1$ points in general position in $\mathbb{R}^d$, with $0 \leq k \leq d$, has a partition into $m$ disjoint subsets $S_1, S_2, \ldots, S_m$ such that $\bigcap_{i=1}^{m} \text{Conv}(S_i)$ is $k$-dimensional.

(See also Problem 56 in [1]). Surprisingly enough, the notion of general position is sufficiently weak to make the conjecture difficult. Indeed, general position does not prevent flats generated by points of $S$ from having degenerate intersections. Conjecture 1 was proved by Reay [2] under the assumption that $S$ is strongly independent, i.e. for any pairwise disjoint subsets $S_1, S_2, \ldots, S_m$ of $S$, we have $\dim(\bigcap_{i=1}^{m} \text{aff}(S_i)) = \max\{-1, (\sum_{i=1}^{k} \dim \text{aff}(S_i)) - (k - 1)d\}$. (As usual, $\text{aff}(S_i)$ denotes the flat spanned by $S_i$, and $\dim \text{aff}(S_i)$ denotes its dimension). Strong independence is in fact nothing but the rigorous translation of the fact that the flats $\text{aff}(S_i)$ have non-degenerate intersections. In particular, it implies general position. We point out that the reduction to strongly independent points is an easy but important step in the proof of several Tverberg-type theorems, see e.g. [6].

In its full generality, Conjecture 1 has been solved only for $k = 0$ (this is Tverberg’s theorem), $m = 2$ (Reay [2]), $d = 2$ (Reay [2], see also Roudneff [4] for a topological extension) and $(d, m) = (3, 3)$ (Reay [3]).

The purpose of the present paper is to give a complete solution for the case $d = 3$:

Theorem 2. Any set $S$ of $4m + k - 3$ points in general position in $\mathbb{R}^3$, with $0 \leq k \leq 3$, has a partition into $m$ disjoint subsets $S_1, S_2, \ldots, S_m$ such that $\bigcap_{i=1}^{m} \text{Conv}(S_i)$ is $k$-dimensional.

2. Proof of the theorem

We only consider the cases $k = 1$, 2 or 3, since for $k = 0$, Theorem 2 is equivalent to Tverberg’s theorem. We shall prove the following stronger statement:

Theorem 3. Let $S$ be a set of $4m + k - 3$ points in general position in $\mathbb{R}^3$, with $1 \leq k \leq 3$, and let $x$ be a Tverberg point of some $A \subset S$, with $|A| = 4m - 3$. Then, there exists a partition of $S$ into $m$ disjoint subsets $S_1, S_2, \ldots, S_m$ such that $\bigcap_{i=1}^{m} \text{Conv}(S_i)$ is $k$-dimensional and contains $x$.

Proof. Let $x$ be a Tverberg point of $A$ (note that the existence of such a point is given by Theorem 1). By definition, there exist pairwise disjoint subsets $A_1, A_2, \ldots, A_m$ of $A$ such that $x \in \bigcap_{i=1}^{m} \text{Conv}(A_i)$. Choosing each $A_i$ inclusion-minimal with the property that $x \in \text{Conv}(A_i)$, we get $|A_i| \leq 4$ for all $i$, $1 \leq i \leq m$. Up to renumbering, we may assume $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. Since the points of
Partitions of points into intersecting tetrahedra

S are in general position, we notice that \( |A_1| = |A_2| = 2 \) is impossible when \( m \geq 2 \), and that \( |A_1| = 1 \) implies \( |A_m| = 4 \). Assume \( |A_1| > 1 \) and let \( V = S \setminus \bigcap_{i=1}^{m-1} A_i \). If there exists \( \{u, v\} \subseteq V \) such that \( x \in [u, v] \), then general position and the fact that \( x \notin S \) implies that \( \{u, v\} \) is the only pair of \( S \) with this property. In this case, we make the convention to replace \( A_1 \) by \( \{u, v\} \). Thus, it can always be assumed that \( x \notin [u, v] \) for all \( \{u, v\} \subseteq V \).

To prove Theorem 3, we proceed by induction on \( m \). The result is trivial when \( m = 1 \). For convenience, we remark that Theorem 3 also holds for \( m = 2 \). In this case, for any partition of \( S \) into two subsets \( S_1 \) and \( S_2 \) with \( |S_1| = k + 1 \), \( |S_2| = 4 \) and \( A_1 \subseteq S_1 \), \( A_2 \subseteq S_2 \), we have \( \dim(\text{Conv}(S_1) \cap \text{Conv}(S_2)) = k \). Let us assume \( m \geq 3 \) in the sequel.

Case 1. If \( |A_m| = 4 \), then \( x \) belongs to the interior \( \text{Int \ Conv}(A_m) \) of \( \text{Conv}(A_m) \).

By the induction hypothesis, there exists a partition of \( S \setminus A_m \) into \( m-1 \) pairwise disjoint subsets \( S_1, S_2, \ldots, S_{m-1} \) such that \( \bigcap_{i=1}^{m-1} \text{Conv}(S_i) \) is \( k \)-dimensional and contains \( x \). Clearly, \( \bigcap_{i=1}^{m-1} \text{Conv}(S_i) \cap \text{Conv}(A_m) \) is also \( k \)-dimensional and contains \( x \). Thus, Theorem 3 holds in this case.

Before going into further cases, we have to make a closer study of the situation. We assume in what follows that \( |A_i| \geq 2 \) and \( |A_i| = 3 \) for all \( i \), \( 2 \leq i \leq m \), and we denote by \( H_i \) the plane spanned by \( A_i \). For every \( \{u, v\} \subseteq V \) and \( i \), with \( 2 \leq i \leq m \), we shall say that \( \{u, v\} \) is adapted to \( A_i \) if for each \( a \in A_i \) we have \( x \notin \text{Conv}\{a, u, v\} \). Notice that if \( \{u, v\} \) is not adapted to \( A_i \), then \( H_i \) strictly separates \( u \) and \( v \). In particular, for any \( \{u, v, w\} \subseteq V \), if both \( \{u, v\} \) and \( \{v, w\} \) are non-adapted to \( A_i \), then \( \{u, w\} \) is adapted to \( A_i \). Next, remark that if \( \{u, v\} \) is non-adapted to \( A_j \), then \( \{u, v\} \) is adapted to each \( A_j \), \( j > 2 \), \( j \neq i \). For otherwise, there would exist \( a_i, a_j \in A_i \), \( a_j \in A_j \) such that \( x \in \text{Conv}\{a_i, u, v\} \cap \text{Conv}\{a_j, u, v\} \). Since \( x \notin [u, v] \), this would imply that both \( a_i \) and \( a_j \) belong to the plane spanned by \( u \), \( v \) and \( x \): a contradiction to the general position.

Now, consider the edge-colored graph \( G \) on \( V \) defined as follows: \( \{u, v\} \) is an edge of \( G \), colored with the color \( i \), if and only if \( \{u, v\} \) is non-adapted to \( A_i \), \( i > 2 \). The preceding observations show that each edge of \( G \) is colored once and that \( G \) has no monochromatic triangle. Notice that we have \( |V| = m + k - |A_1| \geq 1 \). We shall need a numbering of the vertices of \( G \) which fits with the edge-coloring of \( G \). Thus, the case \( |V| - 1 \) will be irrelevant in what follows.

If \( G \) is not the complete graph on \( V \), let us choose a vertex \( u_1 \in V \) such that \( u_1 \) is not incident to all vertices of \( V \setminus \{u_1\} \).

If \( |V| = 2 \), and \( G \) is the complete graph on \( V \), then, up to exchanging the roles of \( A_2 \) and \( A_3 \), we may assume that the only edge \( \{u_1, u_2\} \) of \( G \) is not colored with the color 2.

If \( G \) is the complete graph on \( V \), with \( |V| \geq 3 \), we choose \( u_1 \) such that the edges \( \{u_1, v\}, v \in V \setminus \{u_1\} \), are not colored with the same color. This is always possible since \( G \) has no monochromatic triangle. Actually, all the points of \( V \), except perhaps one of them, can be taken to be \( u_1 \). Also, note that at this point of the
proof, any renumbering of $A_2$, $A_3$, \ldots, $A_m$ is possible. This will be used without justification later.

The following lemma is easily proved by induction:

**Lemma 1.** Let $p = |V|$. Then, there exists a numbering $u_2, u_3, \ldots, u_p$ of the vertices of $V \setminus \{u_1\}$ such that, for every $i \geq 2$, $\{u_1, u_i\}$ is not an edge of $G$ which is colored with the color $i$. Equivalently, for all $i \geq 2$, $\{u_1, u_i\}$ is adapted to $A_i$.

**Lemma 2.** Let $i \geq 2$ and suppose that $H_i$ separates $u_1$ and $u_i$. Then, there exists $S \subseteq A_i \cup \{u_1, u_i\}$ with $|S| = 4$ and such that $x \in \text{int } \text{Conv}(S)$.

**Proof.** Let $z = H_i \cap [u_1, u_i]$. Clearly, there exists $\{a, b\} \subseteq A_i$ such that $x \in \text{Conv}(z, a, b)$. By minimality of $A_i$, we have $x \notin [a, b]$, and since $\{u_1, u_i\}$ is adapted to $A_i$, we also get $x \notin [z, a]$ and $x \notin [z, b]$. Thus, $x$ belongs to the relative interior $\text{relint } \text{Conv}(z, a, b)$ of $\text{Conv}(z, a, b)$. As $H_i$ strictly separates $u_1$ and $u_i$, this implies that $x \in \text{int } \text{Conv}(S)$, where $S = \{a, b, u_1, u_i\}$. This proves Lemma 2. □

**Case 2.** Some $i$, with $2 \leq i \leq m$, satisfies the hypotheses of Lemma 2. Then by taking $A_i = S$ and $A_j = A_j$ for all $j \neq i$, we are reduced, up to renumbering, to Case 1 and the conclusion follows.

We are left with the case where, for every $i \geq 2$, $u_1$ and $u_i$ are on the same side of $H_i$. For the clarity of the discussion, we distinguish between several cases.

**Case 3.** $k = 3$, $|A_i| = 3$.

For all $i \geq 1$, let $S_i = A_i \cup \{u_i\}$. As $x$ belongs to $\text{relint } \text{Conv}(A_i)$, and as $u_1$ and $u_i$ are on the same side of $H_i$, there exists $x_i \in [x, u_i]$ such that $[x, x_i] \subseteq \text{int } \text{Conv}(S_i)$, for all $i \geq 1$. Choosing $j$ such that the distance between $x$ and $x_j$ is minimum, we get $[x, x_j] \subseteq \cap_{i=1}^{n-1} \text{int } \text{Conv}(S_i)$, which completes the proof of Case 3.

**Case 4.** $k = 3$, $|A_i| = 2$.

Call $u_0$ the element of $V \setminus \{u_1, u_2, \ldots, u_m\}$ and set $S_i = A_i \cup \{u_0, u_i\}$ and $S_i = A_i \cup \{u_i\}$ for all $i \geq 2$. Choose $u_i' \in [u_0, u_i]$ sufficiently near $u_i$ so that $u_i'$ and $u_i$ remain on the same side of $H_i$ for all $i \geq 2$. The same arguments as in Case 3 show that $\cap_{j=1}^{n-2} \text{int } \text{Conv}(S_i)$ contains an interval $[x, x']$ with $x' \in [x, u_i']$.

**Case 5.** $k = 2$, $|A_i| = 2$.

Using again the arguments of Case 3, we know the existence of a point $x' \in [x, u_i]$ such that $[x, x'] \subseteq \cap_{j=1}^{n-2} \text{int } \text{Conv}(A_i \cup \{u_i\})$. As $[x, x'] \subseteq$
relint Conv($A_1 \cup \{u_1\}$), and as \( \dim \ Conv(A_1 \cup \{u_1\}) = 2 \), we conclude that \( \cap_{i=1}^m \ Conv(S_i) \) is 2-dimensional and contains \( x \).

**Case 6.** \( k = 2, |A_1| = 3 \).

Consider the pencil of planes \( \mathcal{H} = \{H_i, i \geq 1\} \), where \( H_i \) is defined here as the plane spanned by \( A_1 \). Identifying \( \mathcal{H} \) with its associated cell complex in \( \mathbb{R}^3 \), let \( C \) denote the (3-dimensional) cell of \( \mathcal{H} \) that contains \( u_1 \). The cell \( C \) possesses at least two faces (of dimension 2). One of them is supported by a plane \( H_j \) with \( j \geq 2 \). Up to renumbering, we may assume that \( j = m \). Let \( S' = S \setminus A_m \). By Case 3, \( P = \bigcap_{i=1}^{m-1} \ Conv(A_i \cup \{u_i\}) \) is a 3-dimensional polytope that contains \( x \). Moreover, every face of \( P \) which contains \( x \) is supported by a certain plane \( H_i, 1 \leq i \leq m - 1 \), and \( u_1 \) and \( P \) are on the same side of that \( H_i \). We leave to the reader to check that, if \( P \cap H_m \) was \( k' \)-dimensional, with \( k' < 2 \), then \( C \cap H_m \) would be \( k'' \)-dimensional, with \( k'' \leq k' \): a contradiction. As \( x \) belongs to \( \text{relint} \ Conv(A_m) \), we conclude as before that \( P \cap Conv(A_m) \) is 2-dimensional and contains \( x \).

**Case 7.** \( k = 1, |A_1| = 3 \).

We use the notation of Case 6.

If \( C \) has only two faces, then all of the \( H_i \)'s have a line \( L \) in common, with \( x \in L \). As \( x \in \text{relint} Conv(A_i) \) for all \( i, 1 \leq i \leq m \), it follows that \( \cap_{i=1}^m Conv(A_i) \) is 1-dimensional and contains \( x \).

Thus, assume that \( C \) possesses at least three faces. Then, there is an edge \( e \) of \( C \) such that \( e \cap H_1 = \{x\} \). Let \( H_p \) and \( H_q, p \geq 2, q \geq 2 \), denote two distinct planes of \( \mathcal{H} \) that contain \( e \). Up to renumbering, we may suppose that \( p = m - 1 \) and \( q = m \). Consider the set \( S' = S \setminus (A_{m-1} \cup A_m) \). By Case 3, \( P = \bigcap_{i=1}^{m-2} Conv(A_i \cup \{u_i\}) \) is a 3-dimensional polytope that contains \( x \). Using the arguments of Case 6, we successively obtain that \( P \cap H_{m-1} \cap H_m \) is 1-dimensional (otherwise \( e \) would not be an edge of \( C \)), and that \( P \cap Conv(A_{m-1}) \cap Conv(A_m) \) is 1-dimensional and contains \( x \) (since \( x \in \text{relint} Conv(A_{m-1}) \cap \text{relint} Conv(A_m) \)).

**Case 8.** \( k = 1, |A_1| = 2 \).

In the last case, \( H_1 \) denotes the plane spanned by \( A_1 \cup \{u_1\} \), and \( C \) denotes one of the two cells of \( \mathcal{H} \) which contain \( u_1 \). Let \( e \) be an edge of \( C \) such that \( x \in e \subseteq H_1 \), and let \( H_j, j \geq 2 \) be a plane of \( \mathcal{H} \) which also contains \( e \). Up to renumbering, we may suppose that \( j = m \). Consider \( S' = S \setminus A_m \) and let \( F = \bigcap_{i=1}^{m-1} Conv(A_i \cup \{u_i\}) \). By Case 5, we know that \( \dim(F) = 2 \) and that \( [x, x'] \subseteq F \subseteq H_1 \) for some \( x' \in [x, u_1] \). If \( F \cap e \) was not 1-dimensional, then there would exist a plane \( H_i \) with \( i \neq 1, j \), such that \( H_i \cap e = \{x\} \) and such that \( H_i \) strictly separates \( e \) and \( F \); a contradiction to the definition of \( e \). Thus, \( \dim(F \cap e) = 1 \). Since \( e \cap Conv(A_m) \), we conclude that \( F \cap Conv(A_m) \) is also 1-dimensional and contains \( x \), which completes the proof. \( \square \)
References