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MINIMIZING THE JUMP NUMBER FOR PARTIALLY-ORDERED SETS: A GRAPH-THEORETIC APPROACH, II

Maciej M. SYSŁO

Institute of Computer Science, University of Wrocław, 51151 Wrocław, Poland

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This paper is a continuation of another author's work (Order 1 (1985) 7–19), where arc diagrams of posets have been successfully applied to solve the jump number problem for N-free posets. Here, we consider arbitrary posets and, again making use of arc diagrams of posets, we define two special types of greedy chains: strongly and semi-strongly greedy. Every strongly greedy chain may begin an optimal linear extension (Theorem 1 and Corollary 1). If a poset has no strongly greedy chains, then it has an optimal linear extension which starts with a semi-strongly greedy chain (Theorem 2). Therefore, every poset has an optimal linear extension which consists entirely of strongly and semi-strongly greedy chains. This fact leads to a polynomial-time algorithm for the jump number problem in the class of posets whose arc diagrams contain a bounded number of dummy arcs.

1. Preliminaries

A partially ordered set $(P, \leqslant)$, simply written as P and called a *poset*, consists of a set P and a binary relation $\leqslant$ which is reflexive, antisymmetric, and transitive. Throughout this paper, all sets are finite. We say that q *covers* p, $p, q \in P$, if $p < q$ and $p \leqslant r < q$ implies $p = r$. Let us denote $N_c^-(q) = \{p \in P: p$ is covered by q in $P\}$, $N_c^+(q) = \{p \in P: p$ covers q in $P\}$, $N^-(q) = \{p \in P: p < q$ in $P\}$, and $N^+(q) = \{p \in P; q < p$ in $P\}$, where $q \in P$. A total ordering $L = p_1p_2 \ldots p_m$ of P is a *linear extension* of P if $p_i < p_j$ in P implies $i < j$. Let $\mathscr{L}(P)$ denote the set of all linear extensions of a poset P. A pair (p_i, p_{i+1}) forms a *jump* in L if p_i is not covered by p_{i+1} in P. The jumps split L into chains C_i of P, we denote such a decomposition by $L = C_0 \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_k$. The number of jumps in L, $s(P, L)$ equals to k and the *jump number* $s(P)$ of P equals to the minimum of $s(P, L)$ taken over all linear extensions L of P. The *jump number problem* consists in evaluating $s(P)$ and constructing a linear extension of P with $s(P)$ jumps. Let $\mathcal{O}(P)$ denote the set of all *optimal* linear extensions of P, i.e., $s(P, L) = s(P)$ for every $L \in \mathcal{O}(P)$. Although the problem is NP-complete (see Pulleyblank [4]), some of its special instances can be efficiently solved by polynomial-time algorithms.

A jump (p_i, p_{i+1}) in a linear extension $L = p_1p_2 \ldots p_m$ of P is *greedy* if p_i is not covered in P by any element $q \in P - L_i$ such that $N^-(q) \subseteq \{p_1, p_2, \ldots, p_i\}$, where $L_i = p_1p_2 \ldots p_i$. A linear extension L is *greedy* if all jumps in L are greedy. For the purpose of this paper, it will be convenient to use the notion of a greedy chain. A chain C in P is a *greedy chain* if $N^-(p) \cup \{p\} = C$, where $p = \sup C$, and for no element $q \in N_c^+(p)$, the chain $C \cup \{q\}$ has this property. Note that if $L = C_0 \oplus C_1 \oplus \cdots \oplus C_k$ is a greedy linear extension of P, then C_0 is a greedy chain in P and C_i is a greedy chain in the poset $P - \bigcup_{j<i} C_j$ for $i = 1, 2, \ldots, k$. Let $\mathscr{G}(P)$ denote the set of all greedy linear extensions of P. A poset P is a *greedy poset* if every greedy linear extension of P is optimal, that is, if $\mathscr{G}(P) \subseteq \mathscr{O}(P)$.

Every linear extension of a poset can be transformed to a greedy one. More precisely, if a linear extension $L = C_0 \oplus C_1 \oplus \cdots \oplus C_k$ of P is not greedy, let C_i be the first chain in L which is not greedy in P_i. Clearly, we can extend C_i to a greedy chain C_i' in P_i and then modify $C_j' = C_j - C_i'$ for $j > i$. Thus we obtain a linear extension $L' = C_0 \oplus C_1 \oplus \cdots \oplus C_i' \oplus \cdots \oplus C_k'$ for which $s(P, L') \leqslant s(P, L)$. We can repeat this procedure and transform L to a greedy linear extension, denoted by $G(L)$ for which $s(P, G(L)) \leqslant s(P, L)$. We now formulate this folklore observation in a formal form and then we shall prove the existence of special greedy linear extensions for posets, see Theorem 3, which can be used to reduce significantly the solution space for the jump number problem.

Lemma 1. *Every linear extension L of a poset P can be transformed to a greedy linear extension $G(L)$ such that $s(P, G(L)) \leqslant s(P, L)$.*

Hence, every poset has an optimal greedy linear extension. A greedy linear extension of a poset P can be obtained by the following algorithm:

Greedy algorithm

Choose a greedy chain C_0 in P. Then, for $i = 1, 2, \ldots,$ while $Q = P - \bigcup_{j<i} C_j \neq \emptyset$, choose a greedy chain C_i in Q.

It is still an open problem for which posets each greedy linear extension is optimal. Rival in [5] proved that the greedy algorithm solves the jump number problem for N-free posets P and moreover, that $\mathscr{O}(P) = \mathscr{G}(P)$ in this case. A poset P is N-free if its Hasse diagram G (called also a vertex diagram) is N-free, i.e., if G contains no induced subgraph isomorphic to N, see Fig. 1. Another proof of Rival's result, completely based on graph-theoretic considerations, appeared in Sysło [6], where we make a significant use of digraph representations of posets in which the poset elements are assigned to arcs.

The purpose of this paper is to introduce new types of greedy chains, called strongly and semi-strongly greedy which then are used to define a subclass $\mathscr{G}_s(P)$ of $\mathscr{G}(P)$ such that $\mathscr{G}_s(P) \cap O(P) \neq \emptyset$ for every P. Both types of chains are derived from arc representations of posets. We prove that every strongly greedy chain of a poset P may begin an optimal linear extension of P (Theorem 1) and that every poset P has an optimal linear extension which begins with a semi-strongly greedy

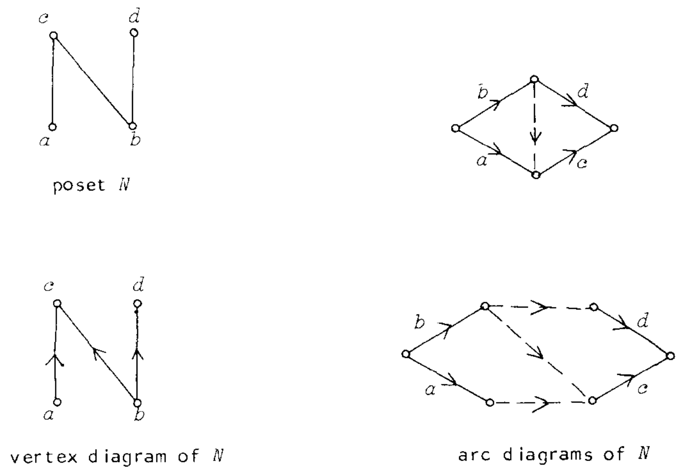


Fig. 1. Poset N and its diagrams.

chain (Theorem 2). Then we show that every poset has an optimal linear extension consisting of strongly and semi-strongly greedy chains what leads to a polynomial-time algorithm for solving the jump number problem in the class of posets which need a bounded number of dummy arcs in their arc representations.

We shall now define some basic notions related to digraphs which then will be used to represent posets. A digraph $D=(V, A)$ consists of a set of *vertices* V and a set of *arcs* A, where $A \subseteq V \times V$. An arc is denoted by (u, v), $u, v \in V$. A digraph may contain *loops*, that is arcs of the form (u, u), $u \in V$. If a digraph D is allowed to contain also multiple (parallel) arcs than it is defined as $D=(V, A, t, h)$, where V is the vertex set, A is the arc set, and t, h are two incidence mappings t, $h: A \rightarrow V$, where t stands for *tail* and h stands for *head*. An arc a in A is of the form $a=(t(a), h(a))$. In what follows, we shall always omit the mappings t and h if a digraph has no parallel arcs. A sequence of arcs $\pi=(a_1, a_2, \ldots, a_l)$ is a path of length l if $h(a_i)=t(a_{i+1})$ for $i=1, 2, \ldots, l-1$. The path π *begins* with the arc a_1 and also π begins with the vertex $t(a_1)$. Similarly, π *terminates* with the arc a_l and with the vertex $h(a_l)$. We shall denote also $t(\pi)=t(a_1)$ and $h(\pi)=h(a_l)$. A path is a *cycle* if $h(a_l)=t(a_1)$. A digraph is *acyclic* if it contains no cycle of length greater than 1. We denote by $\operatorname{tc} D=(V, \operatorname{tc} A, t^*, h^*)$ the digraph which is the *transitive closure* of D, that is, $(a_1, a_2, \ldots, a_l)$, $l \geqslant 1$, is a path in D if and only if, $\operatorname{tc} A$ contains an arc b such that $t^*(a_1)=t^*(b)$ and $h^*(a_l)=h^*(b)$. Let us denote $A^*=\operatorname{tc}(A) \cup \{(v, v): v \in V\}$.

In the next section we define vertex and arc diagrams of posets and present some basic properties of the latter. In Section 3, new types of greedy chains (paths) are introduced and their role in solving the jump number problem is discussed.

2. Vertex- and arc-diagrams of posets

Let $(P, \leqslant)$ be a poset. In digraph representations of P, poset elements can be represented by vertices and by arcs of digraphs. In a vertex representation, the vertex set of a digraph corresponds to the ground set P and in the other case P corresponds to a subset of the arc set. Comparability digraphs and Hasse diagrams are the most popular vertex representations of posets. In the *comparability digraph* $D(P) = (P, A)$ of a poset P we have $(p, q) \in A$ if and only if $p \leqslant q$ in P. The *vertex diagram* (known also as a *Hasse diagram*), $D^{\mathrm{H}}(P) = (P, A^{\mathrm{H}})$ of P is the transitive reduction of $D(P)$ without loops, that is $(p, q) \in A^{\mathrm{H}}$ if and only if q covers p in P. Both vertex representations are *unique*, acyclic and contain no parallel arcs.

An arc representation of a poset is called in the sequel an arc diagram. An acyclic digraph $D^{\mathrm{A}}(P) = (X, R, t, h)$ without loops (in general however, with parallel arcs) together with a mapping $\phi : P \to R$ is an *arc diagram* of a poset P if for every $p, q \in P$, $p \neq q$, we have

$$p < q \text{ if and only if } (h^*(\phi(p)), t^*(\phi(q))) \in R^*, \tag{1}$$

where t^*, h^* are the incidence mappings of $\operatorname{tc} D^{\mathrm{A}}(P)$ and $R^* = \operatorname{tc}(R) \cup \{(x, x) : x \in X\}$. The mapping ϕ may be thought of as an assignment of poset elements to certain arcs of $D^{\mathrm{A}}(P)$ such that the poset P is isomorphic to the set $\phi(P)$ with the relation on $\phi(P)$ defined by the paths of $D^{\mathrm{A}}(P)$. Formally, an arc diagram is a pair $(D^{\mathrm{A}}(P); \phi)$. For the sake of simplicity, however, we shall quite often denote an arc diagram by a digraph $D^{\mathrm{A}}(P)$ in which certain arcs are labelled by the poset elements in the way that preserves the relation between them along the paths of $D^{\mathrm{A}}(P)$.

Let us denote $S = R - \phi(P)$. An arc $a \in R$ is a *poset arc* if $a \in \phi(P)$, and otherwise a is called a *dummy arc*, since its only role in $D^{\mathrm{A}}(P)$ is to preserve the relation between the elements of P. If $\leqslant_R$ is the relation on R such that $q \leqslant_R r$ if and only if either $q = r$ or $(h^*(q), t^*(r)) \in R^*$, then clearly $(R, \leqslant_R)$ is an *enlargement* of $(P, \leqslant)$ since its restriction to P is exactly $(P, \leqslant)$. Note that $(D^{\mathrm{A}}(P); \psi)$ is an arc diagram for $(R, \leqslant_R)$, where $\psi(r) = r$ for every $r \in R$.

A path π in $D^{\mathrm{A}}(P)$ consisting of poset arcs is called a *poset path*. The relation $\leqslant_R$ can be naturally extended to the relation $\leqslant_{XR}$ on $X \cup R$. We have $x \leqslant_{XR} y$ for $x, y \in (X \cup R)$ if and only if there exists in $D^{\mathrm{A}}(P)$ a path starting with x which terminates with y. In this case, if $x \neq y$ we say that x is *below* y and y is *above* x. For the sake of simplicity, we denote both extensions $\leqslant_R$ and $\leqslant_{XR}$ of $\leqslant$ also by $\leqslant$.

Unlike the two vertex representations described above, every poset has an infinite number of arc diagrams. Figure 1 shows two arc diagrams of the poset N, which is the smallest poset whose arc diagram requires a dummy arc.

Posets which have arc diagrams with no dummy arcs can be characterized as those which are N-free. Equivalently, a poset P is N-free if and only if $D^{\mathrm{H}}(P)$ is a line digraph (see [7, Section 2]). Note that for every poset P the enlargement $(R, \leqslant_R)$ of $(P, \leqslant)$ generated by an arc diagram $D^A(P)$ of P is N-free. In what follows, we shall always assume that every N-free poset P is represented by an arc diagram with no dummy arcs. In the other case, for the sake of simplicity, we assume that an arc diagram $(D^A(P) = (X, R, t, h); \phi)$ of a poset P is in a *compact* form, that is, it satisfies the following conditions:

(i) $D^A(P)$ has exactly one source and exactly one sink;

(ii) Every dummy arc is *essential* for $D^A(P)$ to satisfy the condition (1), that is, for every dummy arc a of $D^A(P)$, $((X, R - a, t, h); \phi)$ does not satisfy (1), and

(iii) No dummy arc of $D^A(P)$ can be contracted so that the resulting digraph is an arc diagram of P. (A *contraction* of an arc a in a digraph D depends on removing a from D and identifying its tail and head.)

It is easy to see that every arc diagram of a poset P can be transformed to one which has properties (i), (ii) and (iii). To this end, we first remove or contract dummy arcs provided the resulting digraph is still an arc diagram of P. If $((X, R - \{a\}, t, h); \phi)$ for $a \in R - \phi(P)$ is also an arc diagram of P, then the dummy arc a is *redundant* and it can be removed from R. A dummy arc a can be contracted if either $t(a)$ is not a tail of another arc or $h(a)$ is not a head of another arc. Finally, after performing all possible removals and contractions we merge all sources and all sinks of the resulting diagram into one source and one sink, respectively. Note that a compact arc diagram has no dummy arcs incident with the source or with the sink.

We refer the reader to Section 4 of [7] which contains a survey of methods for constructing arc diagrams of posets.

3. Greedy chains

3.1. *Greedy chains*

We now define in an arc diagram $D^A(P) = (X, R, t, h)$ of a poset P a counterpart of a greedy chain of P and then restrict our attention to special types of greedy chains.

Let $\pi = (a_1, a_2, \ldots, a_k)$, $k \geqslant 1$ be a path in $D^A(P)$ which satisfies the following conditions:

(i) No vertex of π is a head of any arc $b \in R$, $b \neq a_j$ for every j $(0 \leqslant j < k)$;

(ii) a_k is a poset arc, and

(iii) π cannot be extended to a path which satisfies (i) and (ii), and has more poset arcs than π has.

A path π in $D^A(P)$ which satisfies (i)–(iii) is called a *greedy path* of $D^A(P)$. We have a very useful property of greedy paths.

Lemma 2. *If $D^{A}(P)$ is a compact arc diagram of P then a greedy path in $D^{A}(P)$ contains no dummy arcs.*

Proof. By condition (ii), if a greedy path π contains a dummy arc a then a is not the last arc of π. Hence, none of the vertices $t(a)$ and $h(a)$ is a head of an arc of $D^{A}(P)$, therefore a can be contracted, what contradicts the assumption that $D^{A}(P)$ is a compact arc diagram. □

Let C_π denote the sequence of arcs of a path π. It is easy to see that C_π is a greedy chain in P if π is a greedy path in $D^{A}(P)$. On the other hand, every greedy chain C of P generates a greedy path in $D^{A}(P)$—let $\pi(C)$ denote the corresponding path. Hence we have

Lemma 3. *There exists a one-to-one correspondence between greedy chains of P and greedy paths in a compact arc diagram $D^{A}(P)$ of P.*

3.2. *Strongly greedy chains*

We now define in $D^{A}(P)$ a special class of greedy paths which have no known counterparts in the terms of (greedy) chains of posets. A greedy path π is said to be *strongly greedy* if additionally to (i)–(iii) it satisfies:

(iv) either (1) $h(\pi)$ is the sink of $D^{A}(P)$, or

(2) $h(\pi)$ is the head of a poset arc b ($b \neq a_k$) such that no poset path terminating with b has a vertex which is incident with a dummy arc.

In particular, π is strongly greedy if π is greedy and $h(\pi)$ terminates a greedy path π' ($\pi' \neq \pi$) no vertex of which is a tail of a dummy arc.

In an arc diagram shown in Fig. 2, each of the greedy paths (p_4, p_5), (p_1), (p_2) and (p_9, p_{10}) is strongly greedy, however (p_6) is not strongly greedy.

A strong greedy path, if it exists in $D^{A}(P)$, can be used to reduce the size of P in the jump number problem. Namely, we have the following theorem.

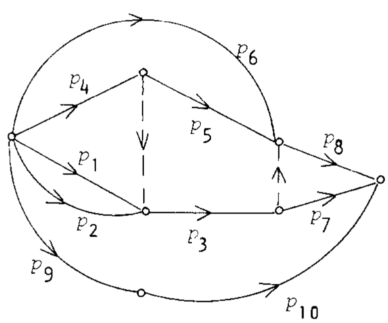


Fig. 2.

Theorem 1. *Let π be a strongly greedy path in a compact arc diagram $D^{A}(P)$ of a poset P. Then every greedy linear extension L of P can be transformed to a greedy linear extension L^* of P which begins with the chain C_π and $s(P, L^*) \leq s(P, L)$.*

Proof. Since π is a greedy path, C_π is a greedy chain in P and therefore there exists a greedy linear extension of P which begins with C_π. Let $L = C_0 \oplus C_1 \oplus \cdots \oplus C_s$ be an arbitrary greedy linear extension of P, where C_i is a greedy chain in $P_i = P - \bigcup_{j<i} C_j$. If $C_l = C_\pi$ for certain l, then evidently $L' = C_l \oplus C_0 \oplus C_1 \oplus \cdots \oplus C_{l-1} \oplus C_{l+1} \oplus \cdots \oplus C_s$ is also a linear extension of P and we set in this case $L^* = G(L')$. Otherwise, let l be the maximal index such that $C_l \cap C_\pi \neq \emptyset$. Note that if $C_i \cap C_\pi \neq \emptyset$ (for $i < l$), then $C_i \cap C_\pi$ is an initial segment of C_i and $C_i \cap C_\pi \neq C_i$ since C_π is a greedy chain in P and L is a greedy linear extension of P. We can have two cases: either $C_l \cap C_\pi = C_l$, or $C_l \cap C_\pi \subsetneqq C_l$. Note that if $h(C_\pi)$ is the sink of $D^{A}(P)$, then $C_l \cap C_\pi = C_l$.

In the former case we construct $L' = C'_0 \oplus C'_1 \oplus \cdots \oplus C'_s$ from L in the following way:

(1) assign C_π to C'_0,
(2) assign $C_i - C_\pi$ to C'_{i+1} for $i = 0, 1, \ldots, l-1$,
(3) assign C_i to C'_i for $i = l+1, \ldots, s$.

It is easy to verify that also in this case L' is a linear extension of P, therefore we may set $L^* = G(L')$.

Let us assume now that $C_l \cap C_\pi \neq \emptyset$ and $C_l \cap C_\pi \neq C_l$. That is, $\pi(C_l)$ goes beyond $h(\pi)$ and therefore $\pi(C_l)$ contains the last arc in L which terminates at $h(\pi)$. A simple transformation described above which brings C_π to the beginning of L may result in a worse linear extension, see an example which follows the proof. Let k be the maximal index such that $k < l$, $C_k \cup (C_l - C_\pi)$ forms a chain in P and $\pi(C_k)$ is a part of a poset path terminating at $h(\pi)$ no vertex of which is a tail of a dummy arc. The existence of such k is guaranteed by the strong greediness of π.

To construct L' in this case, we apply to L the above steps (1)–(3), and then partition C'_{k+1} among C'_i $(k+2 \leq i \leq l)$ and $C_l - C_\pi$ since in general, $C'_{k+1} \cup (C_l - C_\pi)$ may not be a greedy chain in any subposet $P - \bigcup_{j<i} C'_j$ for $i \geq k+1$. To this end, we first assign temporarily C'_{k+1} to D, then C'_i to C'_{i-1} for $i = k+2, \ldots, l$ and $C_l - C_\pi$ to C'_l.

Let $D = (a_1, a_2, \ldots, a_{n'})$. If $D \cup C'_l$ is not a greedy chain in $P - \bigcup_{j<l} C'_j$, then this is due to the existence of an element in D which is smaller than an element in C'_i for certain i, $k+1 \leq i \leq l-1$. Let $f(a_j)$ denote the minimal index i such that C'_i contains an element c and $a_j < c$, where $j = 1, 2, \ldots, n'$ and $i = k+1, \ldots, l$. We have $f(a_{j_1}) \leq f(a_{j_2})$ for $j_1 < j_2$ because $a_{j_1} < a_{j_2}$. Now, the chain D can be partitioned into subchains $D = (D_1, D_2, \ldots, D_n)$, where $D_j = \{b : f(b) = i_j, b \in D\}$, $k+1 \leq i_j \leq l$ for $j = 1, 2, \ldots, n$. We claim that $D_j \cup C'_{i_j}$ is an initial segment of a greedy chain in $P - (\bigcup_{i<i_j} C'_i \cup \bigcup_{i<j} D_i)$ for every $j = 1, 2, \ldots, n$.

We show first that $\sup D_j$ is covered by $\inf C'_{i_j}$. Since D is a part of a chain such

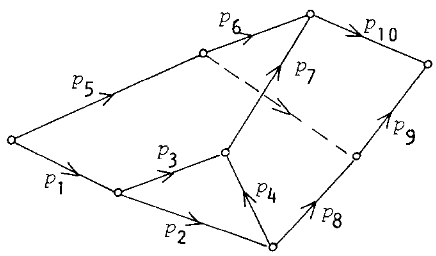


Fig. 3.

that no vertex of $\pi(D)$ is a tail of a dummy arc, if $b<c$ for $b \in D$ and $c \in P - \bigcup_{i \leq k} C_i'$, then there exists c', $c' \leq c$ such that $b < c'$, c' covers b in P and $h(\phi(b)) = t(\phi(c'))$ in $D^{A}(P)$. Hence, if $b \in D_j$ then $c' \in C_{i_j}'$ and $c' = \inf C_{i_j}'$, since otherwise $D = C_{k+1}'$ would not be a greedy chain in $P - \bigcup_{i \leq k} C_i'$. Therefore, $\sup D_j$ is covered by $\inf C_{i_j}'$ for every $j = 1, 2, \ldots, n$.

Secondly, by the definition of C_{i_j}' and D_j, there is no element $y \in P - (\bigcup_{i<i_j} C_i' \cup \bigcup_{i<j} D_i)$ such that $y < x$ for any element x in $D_j \cup C_{i_j}'$. Hence, $D_j \cup C_{i_j}'$ is an initial segment of a greedy chain in $P - (\bigcup_{i<i_j} C_i' \cup \bigcup_{i<j} D_i)$.

We can now again modify L' by assigning $D_j \cup C_{i_j}'$ to C_{i_j}' for $j = 1, 2, \ldots, n$. Finally, we take $G(L')$ as L^*. □

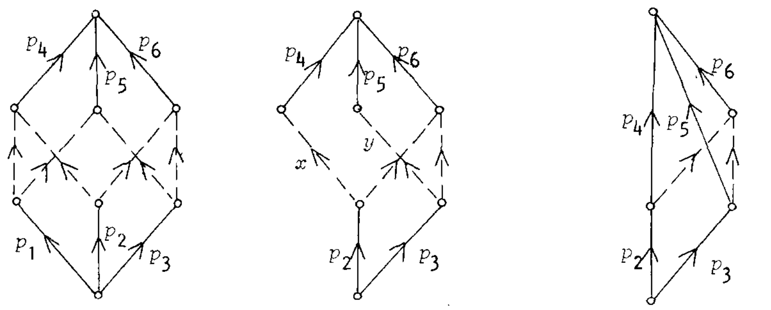
To illustrate the transformation described in the above proof, let us consider the poset P whose arc diagram is shown in Fig. 3 and its greedy linear extension $L = p_1p_3 \oplus p_2p_4p_7 \oplus p_8 \oplus p_5p_6p_{10} \oplus p_9$. Let us take a strongly greedy path $\pi = (p_5, p_6)$. Hence, $k = 1$, $l = 3$ and $D = p_2p_4p_7$. The chain D can be partitioned into (p_2, p_4p_7) such that $p_2 \cup C_2' = p_2p_8$ and $p_4p_7 \cup C_3' = p_4p_7p_{10}$ are segments of greedy chains in $P - \{p_5, p_6, p_1, p_3\}$ and $P - \{p_5, p_6, p_1, p_3, p_2, p_8\}$, respectively. Note that p_2p_8 is indeed only an initial segment of a greedy chain in $P - \{p_5, p_6, p_1, p_3\}$. Hence, we obtain $L' = p_5p_6 \oplus p_1p_3 \oplus p_2p_8 \oplus p_4p_7p_{10} \oplus p_9$ and $L^* = G(L') = p_5p_6 \oplus p_1p_3 \oplus p_2p_8p_9 \oplus p_4p_7p_{10}$.

As an immediate consequence of Theorem 1 we obtain

Corollary 1. *If a compact arc diagram $D^{A}(P)$ of a poset P admits a strongly greedy path π then P has an optimal greedy linear extension which begins with the chain C_π and $s(P) = s(P - C_\pi) + 1$.*

This corollary can be utilized in every method for solving the jump number problem to reduce a poset as long as its arc diagram contains a strongly greedy path.

If π is a greedy path in $D^{A}(P)$ then a non-terminal vertex of π can be only a tail of another (poset or dummy) arc of $D^{A}(P)$. Hence, an arc diagram $D^{A}(P - C_\pi)$ of the poset $P - C_\pi$ for a greedy path π can be constructed from



$D_1 = D^A(C_3)$ $\quad$ $D_2 = D^A(C_3-\{p_1\})$ $\quad$ D_3 - a compact arc diagram of $C_3-\{p_1\}$

(a) (b) (c)

Fig. 4.

$D^A(P)$ by:

(1) removing the arcs of the path π;
(2) removing all dummy arcs with tails at non-terminal vertices of π;
(3) removing all dummy arcs with tails at $h(\pi)$ if there is no arc coming to $h(\pi)$;
(4) removing all isolated vertices which may result in steps (1)–(3);
(5) merging all vertices of indegree 0 into one source.

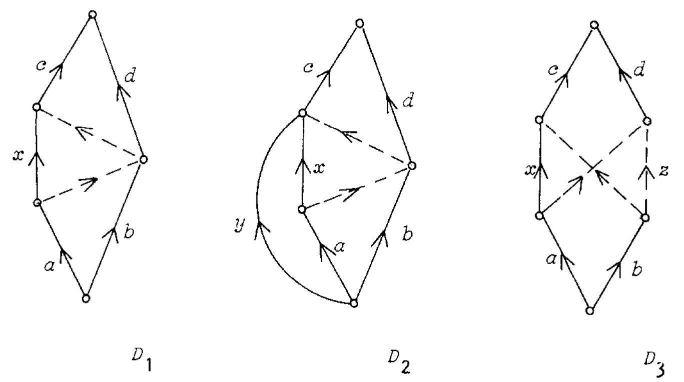
Figure 4(b) illustrates the result of applying steps (1)–(5) to the arc diagram of the 3-crown C_3 (shown in Fig. 4(a)) and its greedy path $\pi = (p_1)$. Although, D_2 is an arc diagram of $C_3 - \{p_1\}$, it is not a compact one, since the dummy arcs x and y may be contracted. After performing these two contractions, we obtain a compact arc diagram D_3 of $C_3 - \{p_1\}$ shown in Fig. 4(c). Therefore, to obtain a compact arc diagram of a poset $P - C_\pi$ for a greedy path π from a compact arc diagram of P, one has first to apply the above steps (1)–(5) to $D^A(P)$ and then to remove or contract redundant dummy arcs from the resulting diagram.

Let us note that every greedy path in $D^A(P)$ for an N-free poset P is strongly greedy (we remind that in this case P has an arc diagram with no dummy arcs). Note also, that if π is a strongly greedy path in $D^A(P)$ of an N-free poset P, then $P - C_\pi$ is also N-free. Moreover, the removal of π from $D^A(P)$ reduces by 1 exactly one vertex-indegree (that of $h(\pi)$) which is at least 2. Thus, we obtain the result which was originally proved by solving a certain Eulerian completion problem.

Corollary 2 ([6]). *If a poset P is N-free then $s(P) = |P| - n + 1$, where n denotes the number of vertices in the compact arc diagram of P.*

3.3. *Semi-strongly greedy chains*

The application of Theorem 1 and Corollary 1 is restricted by the fact that not every poset has a strongly greedy path in its arc diagram, see Fig. 5(a). Compare also that no k-crown for $k \geqslant 3$ admits a strongly greedy path in its arc diagrams. We will now show that every arc diagram of a non N-free poset contains a certain path whose extension results in another special type of greedy paths which together with strongly greedy paths can be very useful in designing an efficient method for solving the jump number problem.



(a) (b) (c)

Fig. 5.

Lemma 4. *Every arc diagram $D^A(P)$ of a poset P contains a path from the source of length at least* 1 *whose no vertex is a head of a dummy arc. Moreover, if $D^A(P)$ contains a dummy arc, it has such a path whose end-vertex is a tail of at least one dummy arc.*

Proof. The lemma is trivially true if $D^A(P)$ contains no dummy arcs. Otherwise, note first that every vertex of $D^A(P)$ lies on a path from the source. To find the required path, let us remove from $D^A(P)$ the source and subsequently all vertices which become of in-degree 0 and which are not incident with dummy arcs. When this procedure stops, every vertex of in-degree 0 is a tail of a dummy arc. Since the reduced digraph is acyclic, it contains at least one such vertex, say u. Hence every path from the source of $D^A(P)$ to u satisfies the lemma conditions. □

We shall now show that if an arc diagram $D^A(P)$ of P admits no strongly greedy paths then it contains a greedy extension of a path as defined in Lemma 4. Formally, we define a greedy path π to be *semi-strongly greedy* if π is not strongly greedy and instead of (iv) it satisfies the following condition:

(iv′) π has a vertex which is a tail of a dummy arc but is not a head of a dummy arc.

If π is a semi-strongly greedy path in $D^{A}(P)$, then C_π is called a semi-strongly greedy chain in P. As an easy consequence of Lemma 4 we have

Corollary 3. *If an arc diagram $D^{A}(P)$ of a poset P admits no strongly greedy path then $D^{A}(P)$ contains a semi-strongly greedy path.*

Proof. Let us assume that $D^{A}(P)$ contains no strongly greedy path. Hence, $D^{A}(P)$ has at least one dummy arc. By Lemma 4, there exists in $D^{A}(P)$ a path ρ whose no vertex is the head of a dummy arc and $h(p)$ is the tail of a dummy arc. Moreover, the construction of ρ given in the proof of Lemma 4 ensures that no vertex of any path in $D^{A}(P)$ which terminates with $h(p)$ is incident with any dummy arc of $D^{A}(P)$. Hence, no vertex of ρ is a head of a poset arc, since otherwise $D^{A}(P)$ would contain a strongly greedy path. Therefore, ρ is an initial segment of a greedy path which can be extended to one which satisfies condition (iv′). □

To illustrate both new kinds of greedy paths let us consider the arc diagrams in Figs. 5(a), (b) and (c). The path $\pi_1 = (a, x)$ is greedy in all diagrams D_1, D_2 and D_3. However, π_1 is strongly greedy only in D_2 (due to the existence of poset path (y)) and semi-strongly greedy in D_1 and D_3. The path $\pi_2 = (b)$ is greedy but neither strongly and nor semi-strongly greedy in both diagrams D_1 and D_2 and it is semi-strongly greedy in D_3. Note that D_1 and D_3 are arc diagrams of the same poset, however D_3 is not a compact arc diagram since the dummy arc z can be contracted. The diagram obtained by contracting z in D_3 is isomorphic to D_1.

One may draw a conclusion from Corollaries 1 and 2 that a strongly greedy chain of a poset contributes one to its jump number and this jump can be attributed to the poset arc which is an end-arc of the corresponding path in an arc diagram. In the case of other greedy paths, a jump may or may not be caused by a poset arc. While we cannot avoid the former jumps we should minimize the number of those which can be attributed to dummy arcs. Semi-strongly greedy paths seem to be good candidates for removing from arc diagrams dummy arcs which will cause no jumps. This heuristic conclusion is justified by Theorem 2.

Let π be a semi-strongly greedy path in an arc diagram $D^{A}(P)$ of a poset P. We denote by $p(\pi)$ the first arc of π whose head is a tail of a dummy arc. Note that $p(\pi)$ is unique for a given π but two semi-strongly greedy paths π and π', $\pi \neq \pi'$ can have $p(\pi) = p(\pi')$. If C is a semi-strongly greedy chain of P then for the sake of simplicity we denote $p(\pi(C))$ by $p(C)$.

Theorem 2. *If an arc diagram $D^{A}(P)$ of a poset P contains no strongly greedy path then for every greedy linear extension L of P there exists a greedy linear extension L^* of P which starts with a semi-strongly greedy chain and $s(P, L^*) \leq s(P, L)$.*

Proof. Let $(D^{\mathrm{A}}(P);\phi)$ be an arc diagram of a poset P and let $L=C_0\oplus C_1\oplus\cdots\oplus C_s$ be a greedy linear extension of P. Since $D^{\mathrm{A}}(P)$ contains a semi-strongly greedy path, we can determine k to be the smallest index such that C_k contains an arc $p(C)$ for a certain semi-strongly greedy chain C in P. Note that, since C is a greedy chain in P, we have $\{q\in C_k: q\leqslant p(C)\}\subseteq C$, i.e., the segment of C_k below $p(C)$ belongs also to C. We can have two main cases.

1. There exists a semi-strongly greedy chain C of P such that $p(C)\in C_k$ and $C_k\cap C=C_k$.

Since both chains are greedy (C in P and C_k in $P-\bigcup_{j<k}C_j$), C_k is a terminal segment of C. Moreover, C_j for $j<k$ can have only an initial segment common with C. Hence

$$L'=C\oplus(C_0-C)\oplus(C_1-C)\oplus\cdots\oplus(C_{k-1}-C)\oplus C_{k+1}\oplus\cdots\oplus C_s \quad (2)$$

is a linear extension of P and $s(P,L')=s(P,L)$. Therefore, we can take $L^*=G(L')$.

2. $C_k\cap C\neq C_k$ for every semi-strongly greedy chain C of P such that $p(C)\in C_k$. We can have two subcases.

2(a) There exists a semi-strongly greedy chain C of P with $p(C)\in C_k$ such that $C_k\cap C$ is a terminal segment of C.

To transform L into L^* in this case, we first construct L' with C as the first chain (similarly as in Case 1) and then partition the elements of C_k-C among the chains C_j, $0\leqslant j\leqslant k-1$. Let us denote $C_k=(b_1,b_2,\ldots,b_{n'})=(C_k\cap C, b_l, b_{l+1},\ldots,b_{n'})=(C_k', C_k'')$, where $l<n'$. Since $C\cup b_l$ is not a greedy chain in P and C_k is a greedy chain in $P-\bigcup_{i<k}C_i$, for each b_j, $l\leqslant j\leqslant n'$ there exists $c_j\in\bigcup_{i<k}C_i$ such that $c_j<b_j$. Let $g(b_j)$ denote the maximal index i such that C_i contains an element c for which $c<b_j$. We have $g(b_{j_1})\leqslant g(b_{j_2})$ for $j_1<j_2$ because $b_{j_1}<b_{j_2}$. Hence, the chain C_k'' can be partitioned into subchains $C_k''=(E_1,E_2,\ldots,E_n)$ such that $E_j=\{b: g(b)=i_j,\ b\in C_k''\}$, where $1\leqslant i_j\leqslant k-1$ for $j=1,2,\ldots,n$. We now claim that $(C_{i_j}-C)\cup E_j$ is an initial segment of a greedy chain in $P-(C\cup\bigcup_{i<i_j}C_i\cup\bigcup_{i<j}E_i)$ for every $j=1,2,\ldots,n$. To this end, we first show that $\sup(C_{i_j}-C)$ is covered by $\inf E_j$. Suppose that $\sup(C_{i_j}-C)\not\prec\inf E_j$ and let us denote $x_j=\sup\{q\in C_{i_j}: q<p \text{ for } p\in E_j\}$. Since E_j contains at least one element b which is greater than an element of C_{i_j}, x_j is well-defined for each j. Evidently $x_j<\sup C_{i_j}$ because otherwise we would have $\sup C_{i_j}<\inf E_j$. If there exists $z_j\in P$ such that $x_j<z_j<\inf E_j$, then $z_j\in C_i$ for certain i, $i>i_j$. Hence, for $b\in E_j$ we would have $g(b)\geqslant i>i_j$—a contradiction with the definition of i_j and E_j. Therefore, $\inf E_j$ would have to cover x_j. Moreover, the poset arcs in $D^{\mathrm{A}}(P)$ corresponding to x_j and $\inf E_j$ cannot share a vertex because otherwise C_{i_j} would not be a greedy chain in $P-\bigcup_{i<i_j}C_i$ since $p(C)<\inf E_j$ and $i_j<k$ therefore the element of C_{i_j} which covers x_j would be greater than $p(C)$, where we have $p(C)\in C_k$ and $j<k$. Thus, the head of $\phi(x_j)$ is a tail of a dummy arc which starts a dummy path from $h(\phi(x_j))$ to $t(\inf E_j)$. This means that C_{i_j} is a semi-strongly greedy chain in $P-\bigcup_{i<i_j}C_i$—a contradiction since C_k is the first chain in L which

contains $p(C)$ for a certain semi-strongly greedy chain C. Therefore, we have $\sup(C_{i_j} - C) < \inf E_j$. By the same argument as above there exists no z such that $\sup(C_{i_j} - C) < z < \inf E_j$. Hence, $\sup(C_{i_j} - C)$ is covered by $\inf E_j$. Now, by the definition of E_j, there is no element $y \in P - (C \cup \bigcup_{i<i_j} C_i \cup \bigcup_{i<j} E_i)$ such that $y < b$ for any element b of E_j. Hence, $(C_{i_j} - C) \cup E_j$ is an initial segment of a greedy chain in $P - (C \cup \bigcup_{i<i_j} C_i \cup \bigcup_{i<j} E_i)$.

Therefore,

$$L' = C \oplus C'_0 \oplus C'_1 \oplus \cdots \oplus C'_{k-1} \oplus C'_{k+1} \oplus \cdots \oplus C'_s$$

is a linear extension of P, where first for every i, $0 \leqslant i \leqslant s$ we assign $C_i - C$ to C'_i and then for $j = 1, 2, \ldots, n$ we assign $C'_{i_j} \cup E_j$ to C'_{i_j}. We have $s(P, L') = s(P, L)$ and if we take $L^* = G(L')$ then $s(P, L^*) \leqslant s(P, L)$.

2(b) $C_k \cap C$ is an interior segment of C for every semi-strongly greedy chain C of P such that $p(C) \in C_k$.

Let us choose as C a semi-strongly greedy chain of P such that $p(C) \in C_k$ and $C_k \cap C$ is of maximal length among all such semi-strongly greedy chains. We can also write $C_k = (b_1, b_2, \ldots, b_{n'}) = (C_k \cap C, b_l, b_{l+1}, \ldots, b_{n'}) = (C'_k, C''_k)$, where $l < n'$. Since $C_k \cap C$ is an interior segment of C, $\sup(C_k \cap C) < \sup C$ and $\sup(C_k \cap C) < \sup C_k$. Let $q \in C_k$ and q covers $\sup(C_k \cap C)$. It is clear that $\phi(\sup(C_k \cap C))$ and $\phi(q)$ do not share a vertex in $D^{\mathrm{A}}(P)$ since otherwise $(C_k \cap C) \cup q$ would be a segment of a semi-strongly greedy chain C' for which $|C_k \cap C'| > |C_k \cap C|$. Therefore, $\phi(\sup(C_k \cap C))$ and $\phi(q)$ are separated in $D^{\mathrm{A}}(P)$ by a path consisting entirely of dummy arcs. Moreover, for every element $b_j \in C''_k$ there exists $c_j \in \bigcup_{i<k} C_i$ such that $c_j < b_j$. We now proceed similarly as in the Case 2(a). First, we put C in the beginning of L', subtract C from each C_i, $i = 0, 1, \ldots, s$ and then partition $C_k - C$ among the chains C_j, $0 \leqslant j \leqslant k-1$. Finally we take $G(L')$ as G^*. □

Arc diagrams in Fig. 6 illustrate the Cases 2(a) and (b) of the above proof. The poset P_1 has exactly one semi-strongly greedy chain $C = p_3p_4$. Let us consider

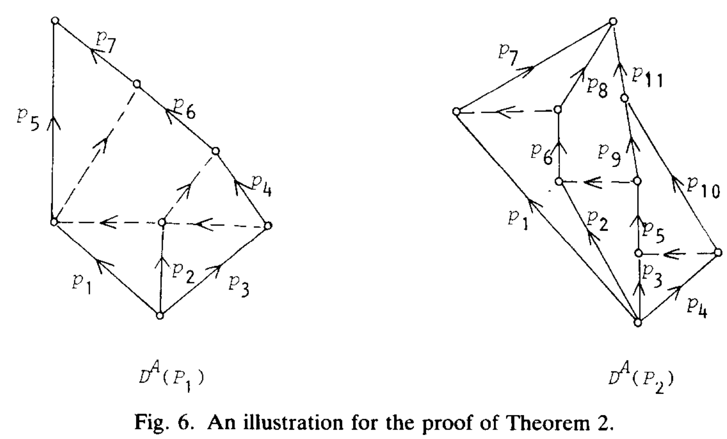


Fig. 6. An illustration for the proof of Theorem 2.

$L_1 \in \mathscr{G}(P_1)$ such that $L_1 = p_2 \oplus p_1 \oplus p_3p_4p_6p_7 \oplus p_5 = C_0 \oplus C_1 \oplus C_2 \oplus C_3$. We have $p(C) = p_3$ and $k = 2$. Hence, we obtain $E_1 = p_6$ and $E_2 = p_7$, where $i_1 = 0$ and $i_2 = 1$. Therefore, $L_1' = p_3p_4 \oplus p_2p_6 \oplus p_1p_7 \oplus p_5$ and $L_1^* = L_1'$. If we take $L_2 \in \mathscr{G}(P_1)$ such that $L_2 = p_1 \oplus p_2 \oplus p_3p_4p_6p_7 \oplus p_5$, then also $p(C) = p_3$, $k = 2$ but $E_1 = p_6p_7$, where $i_1 = 1$. Hence $L_2' = p_3p_4 \oplus p_1 \oplus p_2p_6p_7 \oplus p_5$. Since p_1 is not a semi-strongly greedy chain in $P_1 - \{p_3, p_4\}$ a similar procedure can be applied to $P_1 - \{p_3, p_4\}$. Finally, we obtain $L_2'' = p_3p_4 \oplus p_2p_6 \oplus p_1p_7 \oplus p_5$ which is L_1^*. The poset P_2 has also exactly one semi-strongly greedy chain $C = p_4p_{10}$. Let us consider $L \in \mathscr{G}(P_2)$ such that $L = p_2 \oplus p_3 \oplus p_1 \oplus p_4p_5p_6p_8 \oplus p_7 \oplus p_{10} \oplus p_9p_{11}$. We have $p(C) = p_4$, $k = 3$, and $E_1 = p_5p_6p_8$ and $i_1 = 1$. Hence

$$L' = p_4p_{10} \oplus p_2 \oplus p_3p_5p_6p_8 \oplus p_1 \oplus p_7 \oplus p_9p_{11} \quad \text{and}$$

$$L^* = G(L') = p_4p_{10} \oplus p_2 \oplus p_3p_5p_6p_8 \oplus p_1p_7 \oplus p_9p_{11}.$$

Since p_2 is not a semi-strongly greedy chain in $P_2 - \{p_4, p_{10}\}$ we can again apply a similar procedure to $P_2 - \{p_4, p_{10}\}$. Continuing this process we obtain a linear extension $p_4p_{10} \oplus p_3p_5p_9p_{11} \oplus p_2p_6p_8 \oplus p_1p_7$.

In contrast to Corollary 1 which allows to begin an optimal linear extension of a poset P with an arbitrary greedy chain provided P admits such a chain, Theorem 2 does not determine which semi-strongly greedy chain is a proper candidate for the first chain in an optimal linear extension, if P has no strongly greedy chains.

Figure 7 shows an arc diagram of a poset P, which has two semi-strongly

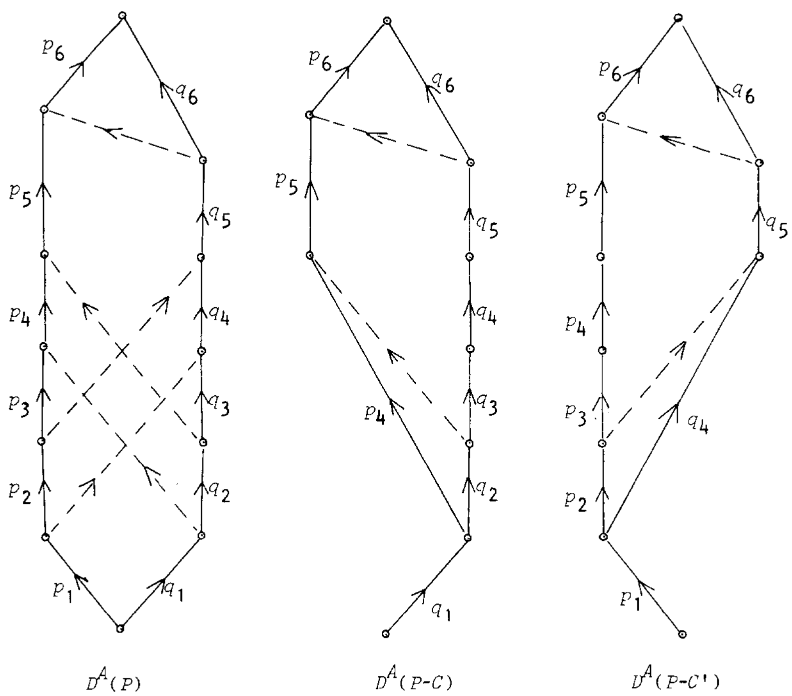


Fig. 7.

greedy chains $C = p_1p_2p_3$ and $C' = q_1q_2q_3$, and the arc diagrams of the posets $P - C$ and $P - C'$. One can easily check that $s(P) = 2$ and for every greedy linear extension L which starts with the chain C' we have $s(P, L) \geq 3$, therefore the semi-strongly greedy chain C' begins no optimal linear extension of P.

In spite of the above drawback of Theorem 2, we can define a *semi-strongly greedy linear extension* of a poset P to be a linear extension $L = C_0 \oplus C_1 \oplus \cdots \oplus C_s$ such that each chain C_i, $i = 0, 1, 2, \ldots, s$ is strongly greedy in $P_i = P - \bigcup_{j<i} C_i$ or semi-strongly greedy in P_i if P_i admits no strongly greedy chains. Corollary 1 and Theorem 2 lead immediately to the following conclusion

Theorem 3. *Every poset P has an optimal semi-strongly greedy linear extension and*

$$s(P, L) \leq s(P) + \tfrac{1}{2}(a(D^{\wedge}(P)) - 1)$$

for every semi-strongly greedy linear extension L of P, where $a(D)$ is the number of dummy arcs in an arc diagram D of P.

Proof. The existence for a poset P of an optimal linear extension which is semi-strongly greedy follows directly from Corollary 1 and Theorem 2. To prove the inequality, note first that if an arc diagram $D^{\wedge}(P)$ of P contains exactly one dummy arc, then $s(P, L) = s(P)$ for every semi-strongly greedy linear extension L of P. If $D^{\wedge}(P)$ contains more than one dummy arc then with each semi-strongly greedy path (chain) at least two dummy arcs are involved of which one has no effect on the number of jumps. Since every strongly greedy path contributes exactly one to the jump number and the number of such paths that a poset can have is at most $s(P)$, we obtain the inequality. □

The bound in Theorem 3 is tight. Figure 8 shows a diagram of a poset which has a semi-strongly greedy linear extension $L = p_3p_4p_5 \oplus p_1p_2 \oplus p_6 \oplus p_7p_8p_9$ such that $s(P, L) = s(P) + \frac{1}{2}(a(D^{\wedge}(P)) - 1) = 2 + 1 = 3$.

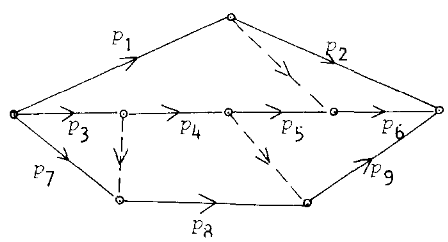


Fig. 8.

4. Conclusions

The discussion in the preceding sections leads to a polynomial-time algorithm for computing the jump number of a poset with a bounded number of dummy arcs in its arc diagram. Note first that if a poset P has an arc diagram with at most k dummy arcs than P has at most $k!$ semi-strongly greedy linear extensions which differ in the order of traversing the tails of dummy arc (or equivalently, of taking semi-strongly greedy chains). Strongly greedy chains, by Corollary 1, can be taken in arbitrary order. Hence, one can easily design an algorithm (for instance, brand and bound) which for posets with at most k dummy arcs in their arc diagrams solves the jump number problem in time bounded by a polynomial function of $|P|$.

A detailed description of an algorithm which finds the solution to the jump number problem among semi-strongly greedy linear extensions, analysis of its performance and comparison with other polynomial-time algorithms for this problem presented in [2] and [3] are contained in [9].

We conclude this paper with yet another comment and a problem. The dimension of a poset P is defined as the minimum cardinality of a *realizer* for P which is a subset of $\mathscr{L}(P)$ whose intersection is P. Bouchitté et al. [1] proved that the greedy dimension is also a well-defined notion in the sense that every poset P has a realizer consisting of greedy linear extensions. We note here that a poset may not be however expressible as the intersection of its semi-strongly greedy linear extensions. The poset N, which has exactly one semi-strongly greedy linear extension may serve as a counter-example, see the diagram of N. It remains an open question which posets admit realizers in the family of their semi-strongly greedy linear extensions. Every N-free poset P has such a realizer since greedy and semi-strongly greedy linear extensions coincide for P in this case.

Strongly and semi-strongly greedy chains in posets can be defined without referring to arc representations of posets, see [8] for an alternate approach.

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