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Reflection Principles for Harmonic and Polyharmonic Functions

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1. INTRODUCTION

An arbitrary point of the Euclidean space $\mathbf{R}^{n+1}$ $(n \geqslant 1)$ is represented by $M = (X, y)$, where $X \in \mathbf{R}^n$ and $y \in \mathbf{R}$. The mirror image of M in the hyperplane $\mathbf{R}^n \times \{0\}$ and the projection of M onto this hyperplane are denoted by $M^* = (X, -y)$ and $M_0 = (X, 0)$, respectively. Throughout this paper Ω is an open subset of $\mathbf{R}^{n+1}$ which is symmetric in the sense that $M^* \in \Omega$ whenever $M \in \Omega$. We write

$$\Omega^+ = \{M \in \Omega: y > 0\}, \qquad \Omega^0 = \{M \in \Omega: y = 0\}, \qquad \Omega^- = \{M \in \Omega: y < 0\}$$

and assume throughout that Ω^0 is nonempty.

We are concerned with extending to the whole of Ω a polyharmonic function in Ω^+ which has certain partial derivatives "vanishing" on Ω^0. Here we explain the sense in which these derivatives will be required to "vanish." Let $|\cdot|$ be Euclidean norm on $\mathbf{R}^{n+1}$ and let σ denote n-dimensional surface-area measure. If $P \in \mathbf{R}^n \times \{0\}$ and a and r are positive numbers, we put

$$\tau(P, a, r) = \{M \in \mathbf{R}^{n+1}: y = a, |M_0 - P| < r\}.$$

A function $f: \Omega^+ \to \mathbf{R}$ will be said to be locally convergent in mean (l.c.i.m.) to 0 on Ω^0 if for each point P of Ω^0 there is a positive number r such that

$$\int_{\tau(P,a,r)} |f(M)|\, d\sigma(M) \to 0 \qquad (a \to 0+).$$

In view of the results that follow, it is worthwhile to observe that if f is continuous in Ω^+ and has the properties

(i) each point $(X, 0)$ of Ω^0 has a neighborhood $\mathcal{N}_{(X,0)}$ such that f is bounded in $\Omega^+ \cap \mathcal{N}_{(X,0)}$,

(ii) for each point $(X, 0)$ of Ω^0

$$\lim_{y \to 0+} f(X, y) = 0,$$

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then f is l.c.i.m. to 0 on Ω^0. This follows from Lebesgue's bounded convergence theorem.

In [2] we proved

THEOREM A. *If h is harmonic in Ω^+ and is l.c.i.m. to 0 on Ω^0, then h has a harmonic continuation h^* to Ω such that $h^*(M) = -h^*(M^*)$ for each M in Ω.*

This improves the classical reflection principle for harmonic functions. In this paper we prove several generalizations of Theorem A.

Let p be a positive integer. As usual, we say that a function w is p-harmonic in Ω if w is real-analytic and $\Delta^p w \equiv 0$ in Ω, where Δ^p is the pth iterated Laplacian operator on $n + 1$ variables. By strengthening the conditions on Ω, we can prove a general existence theorem for polyharmonic continuations. We say that Ω is cylindrical if $(X, y_2) \in \Omega$ whenever $y_1 \leqslant y_2 \leqslant y_3$ and (X, y_1), $(X, y_3) \in \Omega$. For brevity, we write $\partial^\alpha = \partial^\alpha/\partial y^\alpha$ $(\alpha = 0, 1, 2,...)$.

THEOREM 1. *Let Ω be cylindrical, and let w be p-harmonic in Ω^+. If there exist p distinct nonnegative integers $\alpha_1, ..., \alpha_p$ such that $\partial^{\alpha_j} w$ is l.c.i.m to 0 on Ω^0 for each $j = 1, ..., p$, then w has a unique p-harmonic continuation to Ω.*

For certain values of $\alpha_1, ..., \alpha_p$ we can give explicit formulas for the continuation. In particular, when $p = 1$ (the harmonic case) we can give such a formula for any value of α_1.

THEOREM 2. *Let Ω be cylindrical, and let h be harmonic in Ω^+. If there is a nonnegative integer α such that $\partial^\alpha h$ is l.c.i.m. to 0 on Ω^0, then, for each nonnegative integer β, $\partial^\beta h$ has a continuous extension $(\partial^\beta h)_*$ to $\Omega^+ \cup \Omega^0$. Further, if h' is defined in Ω by*

$$\begin{aligned} h'(M) &= 0 && (\alpha = 0, 1) \\ &= \sum_{k=0}^{\frac{1}{2}\alpha-1} \{(2k)!\}^{-1} y^{2k} (\partial^{2k} h)_* (M_0) && (\alpha = 2, 4, 6,...) \\ &= \sum_{k=0}^{\frac{1}{2}(\alpha-3)} \{(2k+1)!\}^{-1} y^{2k+1} (\partial^{2k+1} h)_* (M_0) && (\alpha = 3, 5, 7,...), \end{aligned}$$

then the function h^*, defined in Ω by

$$\begin{aligned} h^*(M) &= h(M) && (M \in \Omega^+) \\ &= (\partial^0 h)_* (M) && (M \in \Omega^0) \qquad (1) \\ &= (-1)^{\alpha+1} h(M^*) + 2h'(M) && (M \in \Omega^-), \end{aligned}$$

is harmonic in Ω.

In the case $\alpha = 0, 1$, the condition that Ω is cylindrical may be dropped. With this weaker hypothesis, the case $\alpha = 0$ reduces to Theorem A.

THEOREM 3. *Let w be p-harmonic in Ω^+. If $\partial^{2j}w$ is l.c.i.m. to 0 on Ω^0 for each $j = 0,\ldots, p-1$, then the p-harmonic continuation w^* of w to Ω satisfies $w^*(M) = -w^*(M^*)$ for each M in Ω.*

THEOREM 4. *Let w be p-harmonic in Ω^+. If $\partial^{2j+1}w$ is l.c.i.m. to 0 on Ω^0 for each $j = 0,\ldots, p-1$, then the p-harmonic continuation w^* of w to Ω satisfies $w^*(M) = w^*(M^*)$ for each M in Ω.*

THEOREM 5. *Let w be p-harmonic in Ω^+. If $\partial^{j}w$ is l.c.i.m. to 0 on Ω^0 for each $j = 0,\ldots, p-1$, then the p-harmonic continuation w^* of w to Ω is defined by*

$$\begin{aligned} w^*(M) &= w(M) && (M \in \Omega^+) \\ &= 0 && (M \in \Omega^0) \\ &= \sum_{k=0}^{p-1} y^{p+k}(k!)^{-2} \Delta^k\{(-y)^{k-p}\, w(M^*)\} && (M \in \Omega^-). \end{aligned} \tag{2}$$

Theorem 5 will follow from Theorem 1 and the known

THEOREM B. *If w is p-harmonic in Ω^+ and*

$$\lim_{M\to P} y^{1-p}w(M) = 0 \qquad (P \in \Omega^0),$$

then the function w^*, defined in Ω by (2), *is p-harmonic in Ω.*

Theorem B is due in its full generality to Huber [6]. The case $n = p = 2$ is due to Duffin [4].

2. PROOF OF THEOREM 2

We require some preliminary results. The closure of a subset ω of R^{n+1} is denoted by $\bar{\omega}$.

LEMMA 1. *Let ω be an open, symmetric, cylindrical subset of $\mathbf{R}^{n+1}$ such that $\bar{\omega} \subset \Omega$ and ω^0 is bounded. If h is harmonic in Ω^+ (resp. Ω), then there exists a harmonic function H in ω^+ (resp. ω) such that $\partial^1 H = h$ in ω^+ (resp. ω).*

A weak form of this lemma has been proved in the case $n = 2$ by Duffin [4]. Our proof is modeled on his.

Let a be a positive number such that the closure of the set

$$\Gamma = \{(X, a): (X, 0) \in \omega\}$$

is contained in Ω. Define K in $\Gamma \times \Gamma$ by

$$\begin{aligned} K(M, P) &= -\tfrac{1}{2} \mid M - P \mid && (n = 1) \\ &= -(2\pi)^{-1} \log \mid M - P \mid && (n = 2) \\ &= \{(n - 2)\, s_n\}^{-1} \mid M - P \mid^{2-n} && (n \geqslant 3), \end{aligned}$$

where s_n is the surface area of the unit sphere in $\mathbf{R}^n$, and given a bounded, continuous function $f: \Gamma \to \mathbf{R}$, define $U_f: \Gamma \to \mathbf{R}$ by

$$U_f(M) = \int_\Gamma K(M, P) f(P)\, d\sigma(P).$$

Now, identifying Γ with a bounded open subset of $\mathbf{R}^n$ in the obvious way, writing $\Delta' = \Delta - \partial^2$, and using a familiar technique of differentiation (see, e.g., Helms [5, pp. 122–124] for the cases $n \geqslant 2$ or Wermer [9, Sect. 3] for the case $n = 3$), we have in Γ

$$\Delta' U_f(M) = -f(M).$$

Define H in ω^+ by

$$H(M) = \int_a^y h(X, t)\, dt + U_{\partial^1 h}(X, a). \tag{4}$$

Then $\partial^1 H = h$ in ω^+ and

$$\begin{aligned} \Delta H(M) &= \partial^1 h(M) + \int_a^y \Delta' h(X, t)\, dt - \Delta' U_{\partial^1 h}(X, a) \\ &= \partial^1 h(M) - \int_a^y \partial^2 h(X, t)\, dt - \partial^1 h(X, a) = 0. \end{aligned}$$

The passage of Δ' under the first integral sign in (4) is justified by the fact that h and all its derivatives are locally bounded in ω^+.

The above proof remains valid when Ω^+ and ω^+ are replaced by Ω and ω.

LEMMA 2. *Let Ω be cylindrical. If h harmonic in Ω^+ and for some nonnegative integer α, $\partial^\alpha h$ has a harmonic continuation to Ω, then so also has h.*

The proof is by induction on α. The case $\alpha = 0$ is trivial. Suppose that $\alpha \geqslant 1$ and that the result holds for $\alpha - 1$. Let $(\partial^\alpha h)^*$ be the harmonic continuation of

$\partial^\alpha h$ to Ω, and let ω satisfy the hypotheses of Lemma 1. By Lemma 1, there is a harmonic function H in ω such that $\partial^1 H = (\partial^\alpha h)^*$ in ω. Hence $\partial^1(H - \partial^{\alpha-1}h) = 0$ in ω^+, so that $H = \partial^{\alpha-1}h + H'$ in ω^+, where H' is a harmonic function in ω and is independent of y. It follows that $H - H'$ is a harmonic continuation of $\partial^{\alpha-1}h$ to ω. Letting ω vary and using the fact that harmonic continuations are unique, we see that $\partial^{\alpha-1}h$ has a harmonic continuation to Ω. By the induction hypothesis, h has the required continuation.

LEMMA 3. *Let Ω be cylindrical. If h is harmonic in Ω and there exists an integer $\alpha > 1$ such that $\partial^\alpha h = 0$ on Ω^0, then the function h', defined in Ω by*

$$h'(M) = \sum_{k=0}^{\frac{1}{2}\alpha-1} \{(2k)!\}^{-1} y^{2k}\partial^{2k}h(M_0) \qquad (\alpha = 2, 4, 6,\ldots)$$

$$= \sum_{k=0}^{\frac{1}{2}(\alpha-3)} \{(2k+1)!\}^{-1} y^{2k+1}\partial^{2k+1}h(M_0) \qquad (\alpha = 3, 5, 7,\ldots),$$

is harmonic in Ω. Further, if β is a nonnegative integer, then $\partial^\beta h' = \partial^\beta h$ on Ω^0 when $\alpha - \beta = 2, 4, 6,\ldots$, and $\partial^\beta h' = 0$ on Ω^0 otherwise.

We give a proof only for $\alpha = 2, 4, 6,\ldots$; the proof for $\alpha = 3, 5, 7,\ldots$ is similar. Suppose that $\alpha = 2m$ and that $\partial^{2m}h = 0$ on Ω^0. Then h' is given by

$$h'(M) = \sum_{k=0}^{m-1} \{(2k)!\}^{-1} y^{2k}\partial^{2k}h(M_0),$$

so that

$$\Delta h'(M) = \sum_{k=1}^{m-1} \{(2k-2)!\}^{-1} y^{2k-2}\partial^{2k}h(M_0) + \sum_{k=0}^{m-1} \{(2k)!\}^{-1} y^{2k}\Delta'\partial^{2k}h(M_0).$$

Since $\Delta'\partial^{2k}h = -\partial^{2k+2}h$ for each k, it now follows that

$$\Delta h'(M) = -\{(2m-2)!\}^{-1} y^{2m-2}\partial^{2m}h(M_0) = 0.$$

Clearly, if $\beta > 2m-2$, then $\partial^\beta h' \equiv 0$ in Ω. If $0 \leqslant \beta \leqslant 2m-2$, then

$$\partial^\beta h'(M) = \sum_{k=[\frac{1}{2}\beta+\frac{1}{2}]}^{m-1} \{(2k-\beta)!\}^{-1} y^{2k-\beta}\partial^{2k}h(M_0).$$

If β is odd, then each term in this sum vanishes on Ω^0. If β is even, then each term except the first vanishes on Ω^0, and this term agrees with $\partial^\beta h$ on Ω^0.

We can now prove Theorem 2. By Theorem A, $\partial^\alpha h$ has a harmonic continuation to Ω. Hence, by Lemma 2, h has such a continuation h_*, say. For each β, define $(\partial^\beta h)_*$ in Ω by $(\partial^\beta h)_* = \partial^\beta h_*$. We now have somewhat more than the first part of the theorem.

In proving the second part, we work first with even values of α. Since h has a harmonic continuation h_* to Ω and $(\partial^\beta h)_* = \partial^\beta h_*$ in Ω, it follows from Lemma 3 that $h - h'$ is harmonic in Ω^+ and vanishes on Ω^0. Hence, by the classical reflection principle, $h - h'$ has a harmonic continuation $(h - h')^*$ to Ω satisfying $(h - h')^*(M) = -(h - h')^*(M^*)$ for each M in Ω. It follows that $(h-h')^* + h'$ is harmonic in Ω and agrees with h in Ω^+. Since $h' = h = (\partial^0 h)_*$ on Ω^0 and $h'(M) = h'(M^*)$ for each M in Ω, the function $(h - h')^* + h'$ is equal to the function h^* defined by (1). •

Next we work with $\alpha = 1$. It is enough to show that if $P \in \Omega^0$, then there is a neighborhood of P in which the function h^*, defined by (1) (with $\alpha = 1$) is harmonic. By Theorem A, $\partial^1 h$ has a harmonic continuation $(\partial^1 h)^*$ to Ω satisfying $(\partial^1 h)^*(M) = -(\partial^1 h)^*(M^*)$ for each M in Ω. Let c be a positive number such that the closure of the cylinder

$$C = \{M \in \mathbf{R}^{n+1} : | M_0 - P | < c, | y | < c\} \tag{5}$$

is contained in Ω. Then the function h^* of (1) satisfies

$$\bar{h}(M) = h(X, c) + \int_c^y (\partial^1 h)^*(X, t)\, dt \tag{6}$$

in C. Since $(\partial^1 h)^*$ and all its derivatives are bounded in C, we may differentiate under the integral sign in (6), to obtain

$$\begin{aligned} \Delta\bar{h}(M) &= \Delta' h(X, c) + \partial^1(\partial^1 h)^*(M) + \int_c^y \Delta'(\partial^1 h)^*(X, t)\, dt \\ &= -\partial^2 h(X, c) + \partial^1(\partial^1 h)^*(M) - \int_c^y \partial^2(\partial^1 h)^*(X, t)\, dt = 0. \end{aligned}$$

Finally suppose that h satisfies the hypotheses of the theorem for some $\alpha = 3, 5, 7, \ldots$. Since h has a harmonic continuation h_* to Ω and $(\partial^\beta h)_* = \partial^\beta h_*$ in Ω, it follows from Lemma 3 that $h - h'$ is harmonic in Ω^+ and that $\partial^1(h - h')$ vanishes on Ω^0. By the result for $\alpha = 1$, $h - h'$ has a harmonic continuation $(h - h')^*$ to Ω satisfying $(h - h')^*(M) = (h - h')^*(M^*)$ for each M in Ω. It follows that $(h - h')^* + h'$ is harmonic in Ω and agrees with h in Ω^+. Since $h'(M) = -h'(M^*)$ for each M in Ω, $(h - h')^* + h'$ is equal to the function h^* defined in (1).

3. Proof of Theorem 1

Again, we require some preliminary results.

Lemma A. *Let Ω be cylindrical. If w is p-harmonic in Ω^+, then there exist harmonic functions $h_0, \ldots, h_{p-1}$ in Ω^+ such that*

$$w(M) = \sum_{i=0}^{p-1} y^i h_i(M). \tag{7}$$

A proof may be found in Kraft [7]. It is easy to show, conversely, that any sum of the form (7) is p-harmonic in Ω^+.

The main result preparatory to the proof of Theorem 1 is the

PROPOSITION. *Let Ω be cylindrical. Suppose that w is p-harmonic in Ω^+ and is given by* (7). *If w is l.c.i.m. to* 0 *on Ω^0, then so also is h_0.*

To prove the proposition, we need some more notations and results. Let $B(M, r)$ denote the open ball of center M and radius r. If $f : \Omega \to \mathbf{R}$ is continuous on Ω and $\bar{B}(M, r) \subset \Omega$, let $A(f; M, r)$ be the volume mean of f over $B(M, r)$, i.e.,

$$A(f; M, r) = (v(r))^{-1} \int_{B(M,r)} f(N)\, dN,$$

where $v(r)$ is the volume of $B(M, r)$. Then, for fixed r, $A(f; \cdot, r)$ is defined on a subset of Ω and is continuous there. Hence, we can define recursively a sequence of iterated means $(A_j(f; \cdot, r))$ by

$$A_0(f; \cdot, r) = f, \qquad A_j(f; M, r) = A(A_{j-1}(f; \cdot, r); M, r) \qquad (j \geqslant 1).$$

Of course, as j increases, the domain of definition of $A_j(f; \cdot, r)$ contracts.

LEMMA B. *Let w be p-harmonic in Ω^+. If $r > 0$, $M \in \Omega^+$ and* $\operatorname{dist}(M, \mathbf{R}^{n+1} \backslash \Omega^+) > (p-1)\, r$, *then*

$$r^{2p-2} \Delta^{p-1} w(M) = (2n+6)^{p-1} \sum_{j=0}^{p-1} (-1)^{p-1-j} \binom{p-1}{j} A_j(w; M, r).$$

A proof of the corresponding result for a polyharmonic function in $\mathbf{R}^n$ may be found in [1, Theorem 2].

LEMMA 4. *Let $f : \Omega^+ \to \mathbf{R}$ be continuous in Ω^+. Suppose that $P \in \Omega^0$, that $\epsilon > 0$ and that there exist numbers a and r with $0 < a < r$ such that*

$$\int_{\tau(P,b,r)} |f(M)|\, d\sigma(M) < \epsilon$$

whenever $0 < b < a$. *Then*

$$\int_{\tau(P,b,\rho)} |A_j(f; M, c)|\, d\sigma(M) < \epsilon \qquad (j = 0, 1, 2, \ldots)$$

whenever $0 < (j+1)\, b < a$, $0 < c$, $jc < b$, *and* $0 < \rho < r - aj/(j+1)$.

The proof is by induction on j. When $j = 0$ there is nothing to prove. Suppose that the result holds for some j. Denote by O the origin of $\mathbf{R}^{n+1}$ and by η the

$(n+1)$th coordinate of a point Q. If $0 < (j+2)\,b < a$, $0 < (j+1)\,c < b$, and $0 < \rho < r - a(j+1)/(j+2)$, then

$$v(c) \int_{\tau(P,b,\rho)} | A_{j+1}(f; M, c)| \, d\sigma(M) \leqslant \int_{\tau(P,b,\rho)} \int_{B(M,c)} | A_j(f; Q, c)| \, dQ \, d\sigma(M)$$

$$= \int_{\tau(P,b,\rho)} \int_{B(O,c)} | A_j(f; M+Q, c)| \, dQ \, d\sigma(M)$$

$$= \int_{B(O,c)} \int_{\tau(P,b,\rho)} | A_j(f; M+Q, c)| \, d\sigma(M) \, dQ$$

$$= \int_{B(O,c)} \int_{\tau(P+Q_0,b+\eta,\rho)} | A_j(f; M, c)| \, d\sigma(M) \, dQ$$

$$\leqslant \int_{B(O,c)} \int_{\tau(P,b+\eta,c+\rho)} | A_j(f; M, c)| \, d\sigma(M) \, dQ. \tag{8}$$

The change of order of integration is justified by the local boundedness of $A_j(f; \cdot, c)$. Now $c + \rho < b/(j+1) + \rho < r - a(j+2)^{-1}(\{j+1) - (j+1)^{-1}\} = r - aj/(j+1)$. Further, for each Q in $B(O, c)$, we have $\square\, b + \eta > b - c > jc$ and $b + \eta < b + c < b(j+2)/(j+1) < a/(j+1)$. Hence, by the induction hypothesis, for each such Q, the inner integral in (8) is less than ϵ. The result for $j+1$ follows, and the proof of the lemma is complete.

We proceed to the proof of the proposition. This is by induction on p. When $p = 1$ there is nothing to prove. Suppose that $p > 1$ and that the result is true for $p - 1$. Let w satisfy the hypotheses of the proposition. Suppose that $P \in \Omega^0$ and that $\epsilon > 0$. Then there exist numbers a and r with $0 < a < r$ such that

$$\int_{\tau(P,b,r)} | w(M)| \, d\sigma(M) < \epsilon$$

whenever $0 < b < a$. A particular consequence of Lemma 4 is that

$$\int_{\tau(P,b,r_0)} | A_j(w; M, b/p)| \, d\sigma(M) < \epsilon \qquad (j = 0, 1, \ldots, p-1), \tag{9}$$

where $r_0 = \frac{1}{2}r - \frac{1}{2}a$, whenever $0 < b < a/p$. A simple calculation gives $\Delta^{p-1}w = 2^{p-1}(p-1)!\,\partial^{p-1}h_{p-1}$, so that, by Lemma B and (9), whenever $0 < b < a/p$

$$b^{2p-2} \int_{\tau(P,b,r_0)} | \partial^{p-1}h_{p-1}(M)| \, d\sigma(M) \leqslant K \sum_{j=0}^{p-1} \int_{\tau(P,b,r_0)} | A_j(w; M, b/p)| \, d\sigma(M)$$

$$< Kp\epsilon,$$

where K is a number depending only on n and p. Hence

$$b^{2p-2} \int_{\tau(P,b,r_0)} |\,\partial^{p-1} h_{p-1}(M)|\, d\sigma(M) \to 0 \qquad (b \to 0+).$$

From this and the equation

$$\partial^{p-2} h_{p-1}(M) = \partial^{p-2} h_{p-1}(X, a/p) + \int_{a/p}^{y} \partial^{p-1} h_{p-1}(X, t)\, dt$$

it follows easily that

$$b^{2p-3} \int_{\tau(P,b,r_0)} |\,\partial^{p-2} h_{p-1}(M)|\, d\sigma(M) \to 0 \qquad (b \to 0+).$$

Repeating this argument a further $p - 2$ times, we arrive eventually at the relation

$$b^{p-1} \int_{\tau(P,b,r_0)} |\, h_{p-1}(M)|\, d\sigma(M) \to 0 \qquad (b \to 0+). \tag{10}$$

Let u denote the sum of the first $p - 1$ terms in (7). Then $w(M) = u(M) + y^{p-1} h_{p-1}(M)$. From (10) and the fact that w is l.c.i.m. to 0 on Ω^0 it follows that u is l.c.i.m. to 0 on Ω^0. Now u is $(p - 1)$-harmonic, so, by the induction hypothesis, h_0 is l.c.i.m. to 0 on Ω^0. The proof of the proposition is complete.

We can now prove Theorem 1. Let w be given in Ω^+ by (7). If $\gamma_j = \min(\alpha_j, p - 1)$, then, by Leibniz's theorem, in Ω^+

$$\partial^{\alpha_j} w(M) = \sum_{i=0}^{\gamma_j} \alpha_j!\{(\alpha_j - i)!\}^{-1} \partial^{\alpha_j - i} h_i(M) + \sum_{k=1}^{p-1} y^k U_k(M), \tag{11}$$

where $U_1, \ldots, U_{p-1}$ are harmonic in Ω^+. Now define harmonic functions H_j, V_j in Ω^+ by

$$H_j = \sum_{i=0}^{\gamma_j} \alpha_j!\{(\alpha_j - i)!\}^{-1} \partial^{p-1-i} h_i, \qquad V_j = \sum_{i=0}^{\gamma_j} \alpha_j!\{(\alpha_j - i)!\}^{-1} \partial^{\alpha_j - i} h_i.$$

Since $\partial^{\alpha_j} w$ is l.c.i.m. to 0 on Ω^0, by (11) and the proposition, V_j is l.c.i.m. to 0 on Ω^0. By Theorem A, V_j has a harmonic continuation to Ω. If $\alpha_j \geqslant p - 1$, then $V_j = \partial^{\alpha_j - p + 1} H_j$, and therefore, by Lemma 2, H_j has a harmonic continuation to Ω; if $\alpha_j < p - 1$, then $H_j = \partial^{p-1+\alpha_j} V_j$, and again H_j has a harmonic continuation to Ω. Now the coefficient of $\partial^{p-1-i} h_i$ in H_j is 1 if $i = 0$ and is $\alpha_j(\alpha_j - 1) \cdots (\alpha_j - i + 1)$ otherwise. Hence, to prove that $\partial^{p-1-i} h_i$ can be expressed as a linear combination of the H_j, it is enough to show that the $p \times p$ determinant D, whose jth row is

$$1 \square \alpha_j \square \alpha_j(\alpha_j - 1) .\square.\square. \alpha_j(\alpha_j - 1) \ldots (\alpha_j - p + 2)$$

is nonzero. Now it is easy to show that D is equal to the determinant whose (j, i)th entry is α_j^{i-1}, and since the α_j are distinct, this determinant is nonzero (see, e.g. [3, p. 303]). Hence $\partial^{p-1-i}h_i$ $(i = 0, 1,..., p-1)$ can be expressed as a linear combination of the H_j $(j = 1,..., p)$. Since each H_j has a harmonic continuation to Ω, so also has each $\partial^{p-1-i}h_i$. By Lemma 2, each h_i has such a continuation h_i^*. The p-harmonic continuation w^* which we seek is given in Ω by

$$w^*(M) = \sum_{i=0}^{p-1} y^i h_i^*(M).$$

The uniqueness of the continuation follows from the fact that p-harmonic functions are real-analytic.

4. Proof of Theorems 3 and 4

These theorems are proved simultaneously by induction. The case $p = 1$ of Theorem 3 is Theorem A, and the case $p = 1$ of Theorem 4 is Theorem 2 with $\alpha = 1$. Now suppose that $p > 1$ and that both theorems holds for $p - 1$.

We show first that, under these assumptions, Theorem 3 holds for p. Let w satisfy the hypotheses of Theorem 3. Then, by Lemma A and the subsequent remark, we may write

$$w(M) = h(M) + y \sum_{i=1}^{p-1} y^{i-1} h_i(M) = h(M) + yv(M), \qquad \text{say,}$$

in Ω^+, where $h, h_1, ..., h_{p-1}$ are harmonic and v is $(p-1)$-harmonic in Ω^+. As in the proof of Theorem 1, each h_i has a harmonic continuation to Ω. Hence v has a $(p-1)$-harmonic continuation to Ω, and it follows that

$$\lim_{M \to P} y\, \partial^\alpha v(M) = 0 \qquad (\alpha = 0, 1, 2,...) \tag{12}$$

for each P on Ω^0. By the proposition, h is l.c.i.m. to 0 on Ω^0, and therefore, by Theorem A, h has a harmonic continuation h^* to Ω satisfying $h^*(M) = h^*(M^*)$ for each M in Ω. Hence

$$\lim_{M \to P} \partial^\alpha h(M) = 0 \qquad (\alpha = 0, 2, 4,...) \tag{13}$$

for each P on Ω^0. Now, in Ω^+

$$\partial^\alpha w(M) = \partial^\alpha h(M) + \alpha \partial^{\alpha-1} v(M) + y \partial^\alpha v(M) \qquad (\alpha = 1, 2, 3,...). \tag{14}$$

From (12)–(14) and the hypotheses on w it follows that $\partial^{\alpha-1}v$ is l.c.i.m. to 0 on Ω^0

for $\alpha = 2, 4, \ldots, 2p - 2$. By the induction hypothesis, v has a $(p-1)$-harmonic continuation v^* to Ω satisfying $v^*(M) = v^*(M^*)$ for each M in Ω. The required continuation w^* is given by $w^*(M) = h^*(M) + yv^*(M)$.

Next we show that, under the same assumptions as above, Theorem 4 holds for p. Let w now satisfy the hypotheses of Theorem 4. Then $\partial^1 w$ satisfies the hypotheses of Theorem 3, and, by the result of the last paragraph, has a p-harmonic continuation $(\partial^1 w)^*$ to Ω satisfying $(\partial^1 w)^*(M) = -(\partial^1 w)^*(M^*)$ for each M in Ω. Suppose that $P \in \Omega^0$, and let C be the cylinder given by (5). Define w^* in C by

$$w^*(M) = w(X, c) + \int_c^y (\partial^1 w)^*(X, t)\, dt.$$

Then it is easy to verify that $w^*(M) = w^*(M^*)$ for each M in C. Also

$$\Delta^p = \sum_{k=0}^{p} \binom{p}{k} \Delta'^k \partial^{2p-2k},$$

so that, since $(\partial^1 w)$ and all its derivatives are bounded in C, we have when $M \in C$

$$\begin{aligned}
&\Delta^p w^*(M) \\
&= \Delta'^p w(X, c) + \sum_{k=0}^{p-1} \binom{p}{k} \Delta'^k \partial^{2p-2k-1} (\partial^1 w)^*(M) + \int_c^y \Delta'^p (\partial^1 w)^*(X, t)\, dt \\
&= \sum_{k=0}^{p-1} \binom{p}{k} \Delta'^k \partial^{2p-2k+1} w(X, c) + \sum_{k=0}^{p-1} \binom{p}{k} \Delta'^k \partial^{2p-2k-1} (\partial^1 w)^*(M) \\
&\quad - \int_c^y \sum_{k=0}^{p-1} \binom{p}{k} \Delta'^k \partial^{2p-2k} (\partial^1 w)^*(X, t)\, dt = 0.
\end{aligned}$$

Hence w^* is p-harmonic in C. It follows that w has a continuation of the required type into some neighborhood of each point of Ω^0, and therefore w has such a continuation into Ω. The induction is complete.

5. Proof of Theorem 5

By Theorem 1, w has a p-harmonic continuation to Ω. In particular, this implies that $\partial^\alpha w$ has a continuous extension $(\partial^\alpha w)_*$ to $\Omega^+ \cup \Omega^0$ for each non-negative α. Since, when $\alpha = 0, 1, \ldots, p-1$, $\partial^\alpha w$ is l.c.i.m. to 0 on Ω^0, we have for such α, $(\partial^\alpha w)_* = 0$ on Ω^0, i.e.,

$$\lim_{M \to P} \partial^\alpha w(M) = 0 \qquad (\alpha = 0, 1, \ldots, p-1)$$

for each $P \in \Omega^0$. From this it follows easily that (3) holds, and hence, by Theorem B, that w has the continuation w^* given by (2).

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