Rings with involution and chain conditions*

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Abstract


The structure of involution rings with d.c.c. and a.c.c. on *-biideals is investigated. If an involution ring \( A \) has d.c.c. on *-biideals, then its Jacobson radical is nilpotent, and \( A \) is an artinian ring with artinian radical. If an involution ring has a.c.c. on *-biideals, then its Baer radical is finitely generated as an abelian group. For a polynomial ring \( A[x] \) over a nonassociative involution ring \( A \) a criterion is given to satisfy a.c.c. on *-biideals. In particular, a polynomial ring \( A[x] \) over an associative involution ring \( A \) has a.c.c. on *-biideals if and only if \( A \) is finite and semiprime; this characterization can be considered as an involutive counterpart of the Hilbert Basis Theorem. These results are valid also for rings without involution, and in this way (commutative) rings with a.c.c. on biideals are characterized. Also examples are given for disproving some expectations.

1. Introduction

In the theory of rings with involution instead of one-sided ideals (which make no sense) *-biideals (that is, biideals closed under involution) can be used successfully in describing their structure (cf. for instance [1], [8] and [9]). In this paper we investigate the effect of imposing descending chain condition (d.c.c.) and ascending chain condition (a.c.c.) on *-biideals for a ring with involution.

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For rings (without involution) the notion of biideals has been introduced in [6], but only few results have been achieved concerning the structure of rings satisfying chain conditions on biideals (for instance, [7] and [11]). Biideals are generalizations of one-sided ideals, so d.c.c. and a.c.c. on biideals are stronger requirements than those on one-sided ideals.

A ring will always mean an associative ring, unless otherwise stated (see Sections 3 and 4). We say that a ring $A$ is a ring with involution, if on $A$ there is defined an involution $^*$ subjected to the usual identities,

$$(a + b)^* = a^* + b^* , \quad (ab)^* = b^*a^* , \quad a^{**} = a ,$$

for all $a, b \in A$. For a subset $M$ of a ring $A$ with involution, $M^*$ will stand for the subset of all involutive images of elements of $M$. An ideal $I$ of a ring $A$ with involution is called a $^*$-ideal (and is denoted by $I \triangleleft^* A$), if $I^* \subseteq I$. Let us recall that a biideal $B$ of $A$ is a subring $B$ such that $BAB \subseteq B$. If, in addition, also $B^* \subseteq B$ holds, then we say that $B$ is a $^*$-biideal of $A$. $^*$-subgroup, $^*$-subring and $^*$-subalgebra are defined analogously.

A (not necessarily associative) ring $A$ is said to be semiprime, if $I \triangleleft A$ and $I^2 = 0$ imply $I = 0$ (see [13, Section 8.2]). For a ring $A$ with involution semiprimeness is equivalent to the condition

$I \triangleleft^* A$ and $I^2 = 0$ imply $I = 0$.

For associative rings the upper radical of all semiprime rings is referred to as the Baer radical, and is denoted by $\beta$. As is well-known the $\beta$-radical of a ring with involution is a $^*$-ideal. For a nonassociative ring $A$ the Baer ideal $\beta(A)$ is defined as the smallest ideal of $A$ such that $A/\beta(A)$ is semiprime. It can be easily checked that $\beta(A)$ is a $^*$-ideal in the ring $A$ with involution.

For associative rings with involution the Andrunakievich Lemma is valid: if $K \triangleleft^* I \triangleleft^* A$ and $\tilde{K}$ denotes the $^*$-ideal of $A$ generated by $K$, then $\tilde{K}^3 \subseteq K$.

If the ring $A$ with involution has d.c.c. on $^*$-biideals, then $\beta(A)$ coincides with the Jacobson radical of $A$ and it is contained in the maximal torsion ideal of $A$. This statement has been proved in [9, Theorem 1] under more general terms.

If a semiprime ring $A$ with involution has d.c.c. on $^*$-biideals, then $A$ is an artinian ring, as it has been proved in [1, Theorem 1].

In our investigations we shall deal also with the additive group $A^+$ of a ring $A$. We say that a ring $A$ is reduced, if its additive group $A^+$ is reduced, that is, its maximal divisible subgroup is 0. An abelian group $A$ is said to be an essential extension of its subgroup $G$, if $G \cap H \neq 0$ for every subgroup $H \neq 0$ of $A$. An abelian group $A$ is an essential extension of a finite group if and only if $A$ has d.c.c. on subgroups, as it has been proved in [12 Theorem 4.4] (cf. also [3, Theorem 25.1]).

As usual, $\mathbb{Z}$ will denote the ring of integers.
2. Descending chain condition on \(*\)-biideals

In this section we shall impose d.c.c. on \(*\)-biideals for the considered ring with involution, and among others, we shall prove that in this case the Jacobson radical is nilpotent. As an easy consequence we shall describe the structure of rings with involution satisfying d.c.c on \(*\)-biideals.

**Lemma 1.** Let $A$ be a ring with involution. If the Baer radical $\beta(A)$ of $A$ is not finite, then there exists a \(*\)-ideal $I$ of $A$ such that $I$ is not finite, $I^3 = 0$ and the annihilator

$$J = \{ x \in \beta(A) \mid lx + xI = 0 \}$$

of $I$ in $\beta(A)$ is contained in $I$.

**Remark.** For commutative rings the corresponding assertion was proved by Arnautov [2].

**Proof.** Let us consider the set

$$\{ K \triangleleft^* A \mid K^3 = 0 \},$$

and by the Zorn Lemma let us choose a maximal element $I$ of this set. We shall prove that $I$ is a desired \(*\)-ideal of $A$.

First, let us observe that $J \triangleleft^* A$. Assume that $Z \subseteq Z$, and consider the sets

$$S_k = \{ M \triangleleft^* A \mid M \subseteq J, M \not\subseteq I, M^k = 0 \}$$

for $k = 1, 2, \ldots$. If each $S_k$ is empty, then all nilpotent \(*\)-ideals of $A$ contained in $J$, are contained also in $I$. Since $J \not\subseteq I$ by the hypothesis, and since $J \subseteq \beta(A)$, it follows that

$$0 \neq J/(J \cap J) \subseteq \beta(A)/(J \cap J) \triangleleft^* A/(J \cap J).$$

Hence there exists a nilpotent \(*\)-ideal $M/(I \cap J)$ of $A/(I \cap J)$ of degree $k \geq 2$ such that $M \subseteq J$, $M \not\subseteq I$ and $M^k \subseteq I \cap J$. Since

$$M^{2k} \subseteq (I \cap J)^2 \subseteq I \cdot J = 0,$$

$M$ is a nilpotent \(*\)-ideal of $A$ such that $M \subseteq J$, but $M \not\subseteq I$, a contradiction. Thus there exists an integer $k \geq 2$ with $S_k \neq \emptyset$, and without loss of generality we may assume that $k$ is the least such number. So for any $M \in S_k$ we have $M \subseteq J$, $M \not\subseteq I$, $M^k = 0$. Since $k \geq 2$, we have
and by the minimality of \( k \) we conclude \( M^2 \subseteq I \). Furthermore, we have

\[
(M + I)^3 \subseteq M^3 + MI + IM + I^3 \subseteq II + I + I^3 = 0.
\]

Since \( M \not\subseteq I \), it follows \( M + I \not\subseteq I \) contradicting the choice of \( I \). Thus \( J \subseteq I \) is valid.

Secondly, we shall prove that \( |I| = \infty \). Suppose that \( |I| < \infty \), and let us consider the right and left annihilators

\[
R = \{ x \in \beta(A) \mid lx = 0 \} \quad \text{and} \quad L = \{ x \in \beta(A) \mid xI = 0 \}
\]

of \( I \) in \( \beta(A) \). Obviously \( J = R \cap L \). We claim that the index \( |\beta(A) : J| \) of \( J^+ \) in the additive group of \( \beta(A) \) is finite. For this purpose let us define for each element \( a \in I \) the additive group homomorphism

\[
\tilde{a} : \beta(A) \to I
\]

by \( \tilde{a}(x) = ax \) for every \( x \in \beta(A) \). Obviously \( \ker \tilde{a} \) is the right annihilator of the element \( a \) in \( \beta(A) \) and

\[
|\text{im } \tilde{a}| \leq |I| < \infty.
\]

On the other hand, we have

\[
|\text{im } \tilde{a}| = |\beta(A) : \ker \tilde{a}|,
\]

and therefore the latter is finite. Since

\[
R = \bigcap (\ker \tilde{a} \mid a \in I),
\]

the finiteness of \( I \) yields that also \( |\beta(A) : R| \) is finite. Similarly we obtain \( |\beta(A) : L| < \infty \), and so also the index \( |\beta(A) : J| \) of \( J = R \cap L \) in \( \beta(A) \) has to be finite.

The finiteness of \( |\beta(A) : J| \) yields that \( J \) is infinite because \( |\beta(A)| = \infty \). Since \( |I| < \infty \), it follows \( J \not\subseteq I \), a contradiction. Thus also \( |I| = \infty \) and the lemma is proved. \( \square \)

**Lemma 2.** Let \( A \) be a ring with involution satisfying d.c.c. on \( ^* \)-bideals. If \( J < I^* \) and \( J^2 = 0 \), then \( J^+ \) satisfies d.c.c. on subgroups. If, in addition, \( A \) is reduced, then \( J \) is finite.

**Proof.** In view of [9, Theorem 1], \( J^+ \) is a torsion group. Let us consider the \( ^* \)-ideals
of \( A \) for each prime \( p \). \( J \neq 0 \) implies that \( M_p \neq 0 \) for at least one prime. Because \( A \) has d.c.c. on \(*\)-biideals, there are only finitely many primes \( p \) with \( M_p \neq 0 \). Assume that for some \( p \) the \(*\)-ideal \( M_p \) is infinite, and define \( G_0 = M_p \). Let \( W \) be any subgroup of \( G_0 \) of finite index \( >1 \). Such a subgroup \( W \) exists, because \( M_p \) is an infinite elementary \( p \)-group. Then also the index \( |G_0 : (W \cap W^*)| \) is finite, and therefore \( |W \cap W^*| = \infty \). Defining \( G_1 = W \cap W^* \) and continuing this procedure we get an infinite strictly descending chain \( G_0 \supset G_1 \supset \cdots \) of \(*\)-biideals of \( A \), contradicting the assumption. Thus each \( M_p \) is finite.

Let \( T_p \) denote the \( p \)-component of \( J \). Since \( T_p \) is an essential extension of \( M_p \), in view of [12, Theorem 4.4] the finiteness of \( M_p \) implies that \( T_p \) satisfies d.c.c. on subgroups. Since there are only finitely many primes with \( T_p \neq 0 \) and since \( J^+ \) is a torsion group, it follows \( J = \bigoplus T_p \) and so also \( J^+ \) satisfies d.c.c. on subgroups.

If \( A \) is reduced, then by the d.c.c. on subgroups \( J \) has to be finite. 

**Theorem 3.** If a ring \( A \) with involution satisfies d.c.c. on \(*\)-biiideals, then its Jacobson radical \( \mathcal{J}(A) \) satisfies d.c.c. on subgroups, and hence \( \mathcal{J}(A) \) is nilpotent. If, in addition, \( \mathcal{J}(A) \) is reduced, then \( \mathcal{J}(A) \) is finite.

**Proof.** In view of [9, Theorem 1] we have \( \mathcal{J}(A) = \beta(A) \). Hence it suffices to prove that \( \beta(A) \) satisfies d.c.c. on subgroups and that \( \beta(A) \) is finite whenever \( \mathcal{J}(A) \) is reduced.

Let \( D \) denote the maximal divisible \(*\)-ideal of \( A \). First we prove that for \( \tilde{A} = A/D \) the Baer radical \( \beta(\tilde{A}) \) is finite. Assume that \( |\beta(\tilde{A})| = \infty \). Applying Lemma 1, there exists a \(*\)-ideal \( I = I/D \) of \( A \) such that \( |I| = \infty \), \( I^+ = 0 \). Hence \( \tilde{I}^4 = 0 \) and an application of Lemma 2 yields that \( \tilde{I}^2 \) is finite. Applying again Lemma 2 for

\[
J = \tilde{I}/\tilde{I}^2 <_{\text{d.c.c.}} \tilde{A}/\tilde{I}^2,
\]

we conclude that \( \tilde{I}/\tilde{I}^2 \) has d.c.c. on subgroups. Hence also \( \tilde{I} \) has d.c.c. on subgroups. Since \( \tilde{I} \) is reduced, it follows that \( \tilde{I} \) is finite, contradicting \( |\tilde{I}| = \infty \). Thus \( \beta(\tilde{A}) \) is finite.

Next, let us consider the \(*\)-ideal \( K = \beta(A) \cap D \) of \( A \). Since \( K^2 = 0 \), Lemma 2 is applicable yielding that \( K \) satisfies d.c.c. on subgroups. As we have already seen,

\[
\beta(A)/K \cong (\beta(A) + D)/D \subseteq \beta(A/D) = \beta(\tilde{A})
\]

is finite, so \( \beta(A) \) satisfies d.c.c. on subgroups. Hence, as is well-known, \( \beta(A) = \mathcal{J}(A) \) is nilpotent (cf. for instance [4, Theorem 30.6]).

**Corollary 4.** A ring \( A \) with involution has d.c.c. on \(*\)-biiideals if and only if \( A \) is an artinian ring with artinian radical.
Proof. By [9, Theorem 1] we have $A/\mathcal{J}(A) = A/\beta(A)$, and hence by [1, Theorem 1] it follows that $A/\mathcal{J}(A)$ is an artinian ring. Since by Theorem 3 $\mathcal{J}(A)$ is artinian, the necessity follows.

For the sufficiency we recall that by [5] (cf. also [4, Theorem 67.4]) $A$ is the direct sum of finitely many matrix rings over infinite division rings and of a ring having d.c.c. on subgroups. Hence the sufficiency follows immediately.

Remark. The structure of artinian rings with artinian radical has been fully described by Kertész and Widiger [5] (see also [4, Section 67]).

3. Ascending chain condition on *-biideals

We start this section with a lemma which is well-known for rings (without involution).

Lemma 5. If a ring $A$ with involution satisfies a.c.c. on *-ideals, then its Baer radical $\beta(A)$ is nilpotent.

Proof. $\beta(A) \neq 0$ contains clearly a nilpotent *-ideal $I \neq 0$. By the Andrunakievich Lemma we may assume that $I$ is a *-ideal of $A$ as well. Because of the a.c.c. on *-ideals in $A$, $I$ can be chosen as a maximal nilpotent *-ideal of $A$. If $\beta(A)/I \neq 0$, then repeating the preceding argument we get a nilpotent *-ideal $K/I \neq 0$ of $A/I$ such that $K \subseteq \beta(A)$. Consequently $K$ is a nilpotent *-ideal of $A$ contained in $\beta(A)$, and containing $I$. Hence by the maximality of $I$ it follows $K = I$, contradicting $K/I \neq 0$. Thus $I = \beta(A)$ is nilpotent.

Theorem 6. Let $A$ be a ring with involution. If $A$ satisfies a.c.c. on *-biideals, then the additive group of the Baer radical $\beta(A)$ of $A$ is finitely generated.

Proof. By Lemma 5, $I = \beta(A)$ is nilpotent; let $n \geq 2$ be the degree of nilpotence of $I$. First we prove that $J^+ = (I^{n-1})^+$ is finitely generated. Let us assume that this is not the case and consider a finitely generated subgroup $M_1 \neq 0$ of $J$. Now $G_1 = M_1 + M_1^*$ is again a finitely generated subgroup of $J$ and by $J^2 = 0$ it follows that $G_1$ is a *-biideal of $A$. Since $J$ is not finitely generated, there exists a finitely generated subgroup $M_2$ of $J$ properly containing $G_1$. Hence $G_2 = M_2 + M_2^*$ is a finitely generated *-subgroup of $J$ as well as a *-biideal of $A$. Continuing this procedure, we get an infinite strictly ascending chain of *-biideals, a contradiction. Thus $J = I^{n-1}$ is finitely generated as an abelian group.

We proceed the proof by induction on $n$. If $n = 2$, then the assertion has been proved. Suppose that the assertion is true for $n-1$. The factor ring $A/I^{n-1}$ satisfies a.c.c. on *-biideals, and its Baer radical $\beta(A)/I^{n-1}$ has nilpotence degree $n-1$. Hence by the hypothesis the additive group of $\beta(A)/I^{n-1}$ is finitely
generated, and so together with $I^n-1$ also the additive group of $\beta(A)$ is finitely generated. □

**Example 7.** There exists a ring $A$ with involution such that $A$ as a ring is left and right noetherian, but $A$ does not satisfy a.c.c. on $*$-biideals. The example will be a commutative ring. Let us consider the ring $\mathbb{C}[x]$ of complex polynomials and let the involution $*$ be the complex conjugation on the coefficients and $x^* = x$ on the indeterminate. The factor ring $\mathbb{C}[\bar{x}] = \mathbb{C}[x]/(x^2) = \mathbb{C}[\bar{x}]$ is clearly noetherian, and its Baer radical is $\mathbb{C}[\bar{x}]$ whose additive group is not finitely generated. Hence by Theorem 6 the ring $\mathbb{C}[\bar{x}]$ does not satisfy a.c.c. on $*$-biideals.

At this place we pose the following problem:

**Problem 1.** Let $A$ be a semiprime ring with involution and satisfying a.c.c. on $*$-biideals. Does $A$ satisfy a.c.c. on biideals?

In the sequel we shall consider not necessarily associative rings. For the sake of convenience we introduce some notations. For any subset $S$ of a ring $A$

$$\langle S \rangle, \quad [S], \quad (S)$$

will denote the subring, the right ideal and the left ideal of $A$ generated by $S$, respectively. Clearly we have

$$[S] = \mathbb{Z} \cdot S + [SA] \quad \text{and} \quad (S) = \mathbb{Z} \cdot S + (AS).$$

Further, we define

$$[S] = [S](S).$$

Obviously $S^2 \subseteq [S]$ holds.

For nonassociative rings we generalize the notion of a biideal in the following manner. A subring $G$ of $A$ is said to be a biideal of $A$, if $[G] \subseteq G$. Obviously


holds. Further we put

$$\langle G \rangle = G(AG) + [GA]G + [GA](AG),$$

and so $[G] = G^2 + \langle G \rangle$. Clearly

$$[G] \cap G^\perp \subseteq G \quad \text{and} \quad [G] \subsetneq G + [G]$$
are valid. Moreover, as one can easily check, \( \{G\} \subseteq (AG) \) and \( \{G\} \subseteq (GA) \) are valid, and hence we have

\[
G\{G\} \subseteq G(AG) \subseteq \{G\} \quad \text{and} \quad \{G\} \subseteq (GA)G \subseteq \{G\}.
\]

Thus also

\[
\{G\} \cap G < G \quad \text{and} \quad \{G\} < G + \{G\}
\]

hold.

If \( G \) is a subring closed under involution, then so are \( [G] \) and \( \{G\} \) and we have

\[
\{G\} \cap G < * G \quad \text{and} \quad \{G\} \cap G < * G.
\]

**Proposition 8.** If \( A \) is a ring with involution and \( A \) has a.c.c. on \( * \)-biideals, then for every \( * \)-subring \( G \) of \( A \) the additive group of the factor ring \( G/([G] \cap G) \cong (G + [G])/[G] \) is finitely generated.

**Proof.** Since \( G^2 \subseteq [G] \), every \( * \)-subgroup of \( G + [G] \) containing \( [G] \) is a \( * \)-biideal. If \( (G + [G])/[G] \) is not finitely generated, then we can construct an infinite strictly ascending chain

\[
[G] \subset G_1 \subset \cdots \subset G_n \subset \cdots
\]

of \( * \)-biideals of \( G + [G] \) as in the proof of Theorem 6. This contradicts the assumption. \( \Box \)

**Proposition 9.** Let \( A \) be a ring with involution such that \( A \) as a ring is left and right noetherian. If for every \( * \)-subring \( G \) of \( A \) the additive group of \( G/([G] \cap G) \) is finitely generated, then \( A \) has a.c.c. on \( * \)-biideals.

**Proof.** Let \( G_1 \subseteq \cdots \subseteq G_n \subseteq \cdots \) be an ascending chain of \( * \)-biideals of \( A \). Since \( A \) is right and left noetherian, there exists a natural number \( n \) such that

\[
(G_k) = (G_n) \quad \text{and} \quad [G_k) = [G_n).
\]

for all \( k > n \). Hence it follows

\[
[G_k) = [G_k)(G_k) = [G_n)G_n] = [G_n] .
\]

Considering \( G = \bigcup_{i=1}^\infty G_i \), we have

\[
[G] = [G_n] = [G_k]
\]
for all $k > n$. Since $G$ is a $*$-biideal of $A$, it follows that $[G] \subseteq G$, and we have got an ascending chain

$$G_n/[G] \subseteq \cdots \subseteq G_k/[G] \subseteq \cdots$$

of $*$-subgroups in $G/[G]$. By the assumption $G/[G]$ is finitely generated, and therefore the ascending chain must terminate. Thus there exists an $m \geq n$ such that $G_m = G_k$ for all $k > m$, and the assertion has been proved. \( \square \)

In proving the next theorem, we make use of the following auxiliary statements.

**Lemma 10.** If $A$ is a ring with a.c.c. on biideals, then every ideal $I$ of $A$ with $I^2 = 0$ is a finitely generated abelian group.

**Proof.** Every additive subgroup $G$ of $I$ is a biideal of $A$ because $[G] \subseteq I^2 = 0$. Hence by the assumption $I'$ has a.c.c. on subgroups, and therefore $I'$ is finitely generated, as claimed. \( \square \)

**Lemma 11.** Let $G$ be a subring of a ring $A$. The additive group $G/([G] \cap G)$ is finitely generated if and only if $G/\{G\} \cap G)$ is finitely generated.

**Proof.** By $\{G\} \subseteq [G]$ the factor ring $G/([G] \cap G)$ is a homomorphic image of $G/\{G\} \cap G)$, and hence the sufficiency follows.

For the necessity let us put $\tilde{G} = G/\{G\} \cap G)$. By the modularity we have

$$[G] \cap G = (G^2 + \{G\}) \cap G = G^2 + (\{G\} \cap G).$$

Now it follows that $\tilde{G}/\tilde{G}^2 \cong G/([G] \cap G)$, and hence $\tilde{G}/\tilde{G}^2$ is finitely generated. Hence there exist elements $a_1, \ldots, a_n \in \tilde{G}$ such that

$$\tilde{G} = \sum_{i=1}^{n} a_i \mathbb{Z} + \tilde{G}^2. \quad (\ast)$$

By definition we have $G^2 \subseteq \{G\}$ which implies $G^2 = 0$. Since by (\ast) we have

$$\tilde{G}^2 = \sum_{i,j=1}^{n} a_i a_j \mathbb{Z},$$

it follows that

$$\tilde{G} = \sum_{i=1}^{n} a_i \mathbb{Z} + \sum_{i,j=1}^{n} a_i a_j \mathbb{Z}$$

proving that $\tilde{G}$ is finitely generated.
We say that an element $a$ of a ring $A$ with involution is *almost regular*, if $a \in \{(a, a^*)\}$. If $A$ consists of almost regular elements, then we say that $A$ is an *almost regular ring* with involution.

For any prime $p$ let $\mathbb{F}_p$ denote the prime field, $\mathbb{F}_p[x]$ the ring of polynomials and $\mathbb{F}_p(x)$ the field of rational functions over $\mathbb{F}_p$. If $A$ is a ring with involution $^*$, then the involution can be extended to the polynomial ring $A[x]$ by defining either $x^* = x$ or $x^* = -x$. Speaking of a polynomial ring $A[x]$ over a ring $A$ with involution, we assume that the involution is extended to $A[x]$ in one of these two possibilities.

**Theorem 12.** Let $A$ be a not necessarily associative ring with involution, and let $A_p$ denote the $p$-torsion ideal of $A$. The following conditions are equivalent:

(i) $A[x]$ has a.c.c. on $^*$-biideals,

(ii) $A$ is semiprime, finite and

$$A_p(x) = A_p[x] \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p(x)$$

is almost regular for every prime $p$.

**Proof.** (i) $\Rightarrow$ (ii) Let $I$ be a $^*$-ideal of $A$ such that $I^2 = 0$. Then $I[x]$ is a $^*$-ideal of $A[x]$ and $(I[x])^2 = 0$. Hence by Lemma 10, $I[x]$ is finitely generated as an abelian group. Since

$$I[x] = I \oplus Ix \oplus Ix^2 \oplus \cdots$$

group direct decomposition holds, $I[x]$ is finitely generated if and only if $I = 0$. Thus $A$ is semiprime.

In the factor ring $A[x]/(x^2)$, which has also a.c.c. on $^*$-biideals, $A\bar{x}$ (where $\bar{x} = x + (x^2)$) is an ideal such that $(A\bar{x})^2 = 0$. Hence by Lemma 10 it follows that $(A\bar{x})^+ \approx A^+$ is finitely generated. Consequently the torsion ideal $T$ of $A$ is finite.

Putting $C = A/T$, by the isomorphism $C[x] \cong A[x]/T[x]$ it follows that $C[x]$ has a.c.c. on $^*$-biideals. Hence $C^+$ is a finitely generated free abelian group, and therefore there exists an element $a \in C \setminus 2C$. Let us consider the factor ring $D = C[x]/(4Cx^2)$, and set $y = x + (4Cx^2)$. We claim that the $^*$-subgroup

$$G = \sum_{i=1}^n (\mathbb{Z}2ay^i + \mathbb{Z}2a^*y^i)$$

of $D$ is a $^*$-biideal of $D$. Let $w(x_1, \ldots, x_n)$ be any nonassociative monomial of indeterminates $x_1, \ldots, x_n$, $n \geq 2$. Introducing $e = \text{id}$ or $^*$ and substituting

$$x_1 = 2a^i y^j, \quad x_2 = 2a^i y^j, \quad i, j \geq 1$$
and $x_3 = d_3, \ldots, x_n = d_n$ with $d_3, \ldots, d_n \in D$, and having in mind that $y$ is a central element, we get

$$d = w(2a^iy^j, 2a^iy^j, d_3, \ldots, d_n) = w(4a^iy^2, a^iy^j, d_3, \ldots, d_n).$$

Since $4a^ix^2 \in (4Cx^2)$, we conclude $4a^iy^2 = 0$ and so also $d = 0$. Hence $G$ is a $*$-biideal of $D$. Moreover, every $*$-subgroup of $G$ is a $*$-biideal of $D$ too. Hence $G$ must be a finitely generated abelian group, for $D$ has a.c.c. on $*$-biideals. Thus by the definition of $G$ there exists a natural number $n$ such that

$$G = \sum_{i=1}^{n} (\mathbb{Z}2ay^i + \mathbb{Z}2a^iy^i).$$

Consequently

$$2ay^{n+1} = \sum_{i=1}^{n} (m_i2ay^i + k_ia^iy^i)$$

holds with suitable integers $m_1, \ldots, m_n$ and $k_1, \ldots, k_n$. Thus

$$2ax^{n+1} - \sum_{i=1}^{n+1} (m_i2ax^i + k_i2a^ix^i) \in (4Cx^2).$$

Since the ideal $(4Cx^2)$ is homogeneous, it follows that $2ax^{n+1} \in (4Cx^2)$. Hence

$$2ax^{n+1} \in 4Cx^{n+1}$$

is valid implying $a \in 2C$, which contradicts the choice of $a$. Thus $C = 0$ follows, and so $A = T$ holds which has been already proved to be finite.

Next we prove the almost regularity of the ring $A_p(x)$. Let $z \in A_p(x)$ be an arbitrary element. Then $z$ can be expressed in the form

$$z = \sum_{i=1}^{n} u_i \otimes h_ig_i^{-1}, \quad u_i \in A_p[\chi], \quad h_ig_i \in \mathbb{F}_p[\chi], \quad i = 1, \ldots, n.$$

Denoting the least common multiple of $g_1, \ldots, g_n$ by $q$ and setting $f = qf^*$ we have

$$h_ig_i^{-1} = f_i f^{-1}$$

with appropriately chosen $f_1, \ldots, f_n \in \mathbb{F}_p[\chi]$. Now

$$z = \left( \sum_{i=1}^{n} a_i f_i \right) \otimes f^{-1}$$
is valid. Let us consider the *-subalgebra $B$ of $A_p[x]$ over $\mathbb{F}_p[x]$ generated by the element $a = \sum_{i=1}^n a_i$. It follows from Proposition 8 and Lemma 11 that the factor ring $B/(\langle B \rangle \cap B)$ is finitely generated as an abelian group. Since it is $p$-torsion, it must be finite. Hence it is a finite $\mathbb{F}_p[x]$-module, and therefore it has a nonzero annihilator $N$. Let $g(x) \in N$ be a nonzero polynomial. Then $ag(x) \in \{B\}$ holds. By the definition of $\{B\}$ it follows that $\{B\} = \{(a, a^*)\}$. Hence $ag(x) \in \{(a, a^*)\}$ and also

$$a \otimes g(x) = ag(x) \otimes 1 \in \{(a, a^*)\} \otimes 1$$

$$\subseteq \{(a \otimes 1, a^* \otimes 1)\}_{A_p(x)} = \{(a \otimes f^{-1}, a^* \otimes f^{-1})\}_{A_p(x)}$$

hold. Since $\{(a \otimes f^{-1}, a^* \otimes f^{-1})\}_{A_p(x)}$ is an $F_p(x)$-submodule of $A_p(x)$, we may multiply by $(gf^{-1})$ obtaining

$$a \otimes f^{-1}(x) \in \{(a \otimes f^{-1}, a^* \otimes f^{-1})\}_{A_p(x)},$$

which proves that $A_p(x)$ is indeed almost regular.

(ii) $\Rightarrow$ (i) Since $A$ is finite and semiprime, we have an additive group direct decomposition

$$A[x] = \bigoplus_{i=1}^n A_{p_i}[x].$$

Hence every additive subgroup $S$ of $A[x]$ has the form

$$S = \bigoplus_{i=1}^n (S \cap A_{p_i}[x]),$$

and therefore without loss of generality we may confine ourselves to the case $A = A_{p_i}$.

Since $A$ is a finite-dimensional $\mathbb{F}_p$-algebra, $A[x]$ is a finitely generated $\mathbb{F}_p[x]$-module. Thus it is left and right noetherian as a ring without involution.

We claim that $B/(\langle B \rangle \cap B)$ is finite for every *-subring $B$ of $A[x]$. Let $C$ denote the *-subalgebra of $A[x]$ over $\mathbb{F}_p[x]$ generated by $B$. Clearly $\{C\} = \{B\}$ holds, and hence we have

$$B/(\langle B \rangle \cap B) \cong (B + \{B\})/\{B\} \subseteq (C + \{C\})/\{C\}.$$

It is enough to prove that $(C + \{C\})/\{C\}$ is finite. First we show that $(C + \{C\})/\{C\}$ is a torsion $\mathbb{F}_p[x]$-module. Let $c \in C$ be an arbitrary element. Since $A(x)$ is almost regular, we have
c \otimes 1 \in \{ (c \otimes 1, c^* \otimes 1) \}_{A(x)},

and so

\[ c \otimes 1 = \sum_{i=1}^{n} d_i \otimes h_i g_i^{-1}, \]

\[ d_i \in \{(c, c^*)\}, \quad h_i, g_i \in F_p[x], \quad i = 1, \ldots, n. \]

As we have earlier seen, we may write

\[ cf \otimes 1 = a \otimes f^{-1} \]

with suitable elements \( a \in \{(c, c^*)\} \) and \( f \in F_p[x]. \) Now we have

\[ cf \otimes 1 = c \otimes f = a \otimes 1, \]

which yields

\[ cf = a \in \{(c, c^*)\} \subseteq \{C\}. \]

Thus \( c + \{C\} \) is a torsion element in \((C + \{C\})/\{C\},\) proving that \((C + \{C\})/\{C\}\) is a torsion \(F_p[x]\)-module.

Since \( A[x] \) is a finitely generated \(F_p[x]\)-module, so are \( C \) and \((C + \{C\})/\{C\}\) as well. This latter is also a torsion \(F_p[x]\)-module, and therefore it has a nonzero annihilator \(N.\) Hence \((C + \{C\})/\{C\}\) is a finitely generated \(F_p[x]/N\)-module. \(F_p[x]/N\) is a finite ring, and so also \((C + \{C\})/\{C\}\) must be finite for every *-subring \(B\) of \(A[x].\)

Finally an application of Proposition 9 and Lemma 11 yields the implication (ii) \(\Rightarrow\) (i). \(\Box\)

In the case of an associative ring \(A\) the rings \(A_p(x)\) of Theorem 12(ii) are always almost regular, as we shall see. Hence we get the following characterization of finite semiprime associative rings with involution which can be considered as an analogon of the Hilbert Basis Theorem for involution rings and *-biideals.

**Corollary 13.** For an associative ring \(A\) with involution the following conditions are equivalent:

(i) \(A[x]\) has a.c.c. on *-biideals,

(ii) \(A\) is semiprime and finite.

**Proof.** \(A_p(x) = A_p[x] \otimes F_p[x] \otimes F_p(x)\) is a finite-dimensional semiprime algebra over \(F_p(x),\) and hence it is always almost regular. \(\Box\)
4. Rings (without involution)

All the earlier results remain valid if we consider rings without involution (and hence biideals for *-biideals, subgroups for *-subgroups, etc.). The proofs become somewhat simpler.

The corresponding assertion to Corollary 4 is known, and it was proved by Widiger [10, Satz 10.5] and generalized to alternative rings in [11, Theorem 13]. For rings (without involution), however, we can prove more results.

**Theorem 14.** For a not necessarily associative ring $A$ the following conditions are equivalent:

(i) $A$ has a.c.c. on biideals,

(ii) $A$ is left and right noetherian and for every subring $G$ of $A$ the additive group of the factor ring $G/([G] \cap G)$ is finitely generated.

**Proof.** (i) $\Rightarrow$ (ii) Obvious in view of the corresponding version of Proposition 8.

(ii) $\Rightarrow$ (i) Clear by Proposition 9 (without involution).

**Corollary 15.** For a commutative associative ring $A$ the following conditions are equivalent:

(i) $A$ has a.c.c. on biideals,

(ii) $A$ is noetherian and for every ideal $I$ of $A$ the additive group of the Baer radical $\beta(A/I)$ is finitely generated,

(iii) $A$ is noetherian and for every ideal $I$ of $A$ the abelian group $I/I^2$ is finitely generated.

**Proof.** (i) $\Rightarrow$ (ii) Obvious by Theorem 14.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) In view of Theorem 14 it suffices to prove that for every subring $G$ of $A$ the additive subgroup of $G/(G \cap G)$ is finitely generated. By the commutativity we have $[G] = G^2 + G^2A$ and hence $G$ is an ideal of $A$. For $I = GA + G$ we have $I^2 \subseteq A$ and $I^2 = G^2 + G^2A = [G]$. Hence

$$G/([G] \cap G) \cong (G + [G])/[G] \subseteq I/I^2$$

is valid. Since by (iii) the additive group of $I/I^2$ is finitely generated, also $(G/([G] \cap G))^+$ is finitely generated. An application of Theorem 14 yields (i).

**Example 16.** Let $F$ be a finite field and $F[x]$ the polynomial ring over $F$. Every ideal $I$ of $F[x]$ has a.c.c. on biideals. First we prove that $I$ is noetherian. Let

$$0 \neq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$$
be an ascending chain of ideals of $I$, and let $J(I_1)$ denote the sum of all ideals of $A$ contained in $I_1$. By the Andrunakievich Lemma it follows that
\[ I_1^3 \subseteq (I_1 \cdot F[x])^3 \subseteq I_1, \]
and so we have
\[ 0 \neq (I_1 \cdot F[x])^3 \subset J(I_1). \]
Consequently the factor ring $F[x]/J(I_1)$ is finite. Hence the chain
\[ I_1/J(I_1) \subseteq \cdots \subseteq I_n/J(I_n) \subseteq \cdots \]
must terminate, that is, $I$ is noetherian. Next, we prove that for every ideal $K$ of $I$ the Baer radical $\beta(I/K)$ is finite. Without loss of generality we may confine ourselves to the case $K \neq 0$ by Theorem 6 (without involution). Using again the Andrunakievich Lemma, it follows that $(K \cdot F[x])^3 \subseteq K$, and hence the factor ring $I/K$ is a homomorphic image of $I/(K \cdot F[x])^3$ which is finite by
\[ I/(K \cdot F[x])^3 \subseteq F[x]/(K \cdot F[x])^3. \]
Hence also $I/K$ is finite implying that $\beta(I/K)$ is finite. An application of Corollary 15 yields the assertion.

**Problem 2.** Does there exist an associative simple prime non-artinian ring with a.c.c. on bi-ideals?

**References**


