MATHEMATICS

THE LINEAR MODULUS OF AN ORDER BOUNDED LINEAR TRANSFORMATION. II

BY

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4. *Order bounded linear integral operators*

Let (X, Λ, μ) be a (totally) σ-finite measure space (i.e., μ is a non-negative and countably additive measure on the σ-algebra Λ of subsets of the non-empty set X, such that X is a finite or countable union of sets of finite measure). By $M(X, \mu)$, or briefly $M(X)$, we shall denote the set of all μ-almost everywhere finitevalued and μ-measurable real functions on X, with identification of μ-almost everywhere equal functions. The set $M(X)$ is a real vector space under the usual operations, and $M(X)$ is even a Riesz space with respect to the partial ordering defined by saying that $f \leqslant g$ in $M(X)$ means that $f(x) \leqslant g(x)$ holds for μ-almost every $x \in X$. It is well-known that the Riesz space $M(X)$ is Dedekind complete, i.e., every subset of $M(X)$ that is bounded from above has a supremum. Some caution is necessary in handling this statement. Given the set $(f_\alpha : \alpha \in \{\alpha\})$ in $M(X)$ such that the set is bounded from above, the supremum $f = \sup f_\alpha$ is not necessarily the pointwise supremum of the functions $f_\alpha (\alpha \in \{\alpha\})$. In the first place, the pointwise supremum need not be measurable. Secondly, even if the pointwise supremum is measurable, it need not be equal to the function $f = \sup f_\alpha$ in $M(X)$. As an example, take ordinary Lebesgue measure on the interval $[0, 1]$ and, for x and α running through $[0, 1]$, let $f_\alpha(x) = 0$ for $x \neq \alpha$ and $f_\alpha(x) = 1$ for $x = \alpha$. Then, since every f_α is μ-equal to the function identically zero, we have $f = \sup f_\alpha$ identically equal to zero, but the pointwise supremum of the functions f_α is identically equal to one.

Assume now that (X, Λ, μ) and (Y, Γ, ν) are both (totally) σ-finite measure spaces. The corresponding Riesz spaces of real measurable functions will be denoted by $M(X)$ and $M(Y)$ respectively. Let $T(x, y)$ be a real and $(\mu \times \nu)$-measurable function on the Cartesian product $X \times Y$. For any $f \in M(Y)$ the function $T(x, y) f(y)$ is then $(\mu \times \nu)$-measurable, which implies that for μ-almost every $x \in X$ the function $T(x, y) f(y)$, as a function of y, is ν-measurable. It follows that

$$h(x) = \int_Y |T(x, y) f(y)| \, d\nu(y) \tag{1}$$

makes sense for these values of x, and the resulting function $h(x)$ is μ-measurable on X. All this is included in a complete formulation of Fubini's theorem on repeated integration. Of course, it is not necessary that the function h assumes a finite value for μ-almost every $x \in X$, i.e., h is not necessarily a member of $M(X)$. The set of all $f \in M(Y)$ with the property that the corresponding function h satisfies $h \in M(X)$ will be called the *Y-domain* of $T(x, y)$, or simply the domain of $T(x, y)$ if there is no danger of confusion between X and Y. Obviously, the domain of $T(x, y)$, written as dom (T), is an ideal in the Riesz space $M(Y)$. It follows that dom (T) in its own right is a Dedekind complete Riesz space.

Let $f \in$ dom (T), and let h be given by (1). It follows already from what was observed above that for any $x \in X$ for which $h(x)$ is finite the function g, defined by

$$g(x) = \int_Y T(x, y) f(y)\, d\nu(y),$$

is finite. The μ-measurability of $g(x)$ follows by observing that

$$g(x) = \int_Y (T(x, y) f(y))^+\, d\nu(y) - \int_Y (T(x, y) f(y))^-\, d\nu(y)$$

holds for μ-almost every $x \in X$, and both terms on the right are μ-measurable (once again by applying Fubini's theorem to the non-negative and $(\mu \times \nu)$-measurable functions under the signs of integration).

If f and g have the same meaning as above, the mapping $T: f \to g$ is now a linear mapping from the Riesz space dom (T) into the Riesz space $M(X)$. We shall say that T is a *linear integral operator* and, as usual with integral operators, $T(x, y)$ is called the *kernel* of T.

Denote the integral operator with kernel $|T(x, y)|$ by A. It is evident that dom $(A) =$ dom (T), and A is not only a linear mapping, but even a positive linear mapping from dom $(A) =$ dom (T) into $M(X)$. Obviously, $A - T$ is also a positive linear mapping from dom (T) into $M(X)$, so T is majorized by A, i.e., T is order bounded. We summarize some of these facts in the following theorem.

THEOREM 4.1. *Given the $(\mu \times \nu)$-measurable real function $T(x, y)$, the integral operator T, defined by*

$$(Tf)(x) = \int_Y T(x, y) f(y)\, d\nu(y),$$

is an order bounded linear mapping from dom (T) *into $M(X)$. If L is any Riesz subspace of* dom (T), *then the restriction of T on L is of course an order bounded linear mapping from L into $M(X)$.*

Given the integral operator T as in the last theorem, it is a natural question to ask what the mappings T^+, T^- and $|T|$ are, and it is an obvious conjecture that these mappings are integral operators with kernels $T^+(x, y)$, $T^-(x, y)$ and $|T(x, y)|$ respectively. We shall prove that the conjecture is true. One simple remark first. The operator P with kernel $T^+(x, y)$ is certainly a majorant of T as well as of the null operator, and hence

$$P \geqslant \sup (T, 0) = T^+.$$

Similarly, the operator N with kernel $T^-(x, y)$ satisfies $N \geqslant T^-$, and so the operator A with kernel $|T(x, y)|$ satisfies $A \geqslant |T|$.

THEOREM 4.2. *Given the integral operator T with kernel $T(x, y)$ as in the preceding theorem, the linear mappings T^+, T^- and $|T|$ are also integral operators with kernels $T^+(x, y)$, $T^-(x, y)$ and $|T(x, y)|$ respectively.*

PROOF. Let $0 \leqslant f \in \operatorname{dom}(T)$ be given. Then the function

$$\int_Y |T(x, y)| f(y)\, d\nu(y)$$

is non-negative, μ-measurable and μ-almost everywhere finite on X. The function f will be kept fixed throughout the proof. We have to prove that

$$(T^+ f)(x) = \int_Y T^+(x, y)\, f(y)\, d\nu(y)$$

holds for μ-almost every $x \in X$. For this purpose, write $T_1(x, y) = T(x, y)\, f(y)$. Then

$$T_1^+(x, y) = T^+(x, y)\, f(y)$$

and

$$T^+ f = \sup (Tg : 0 \leqslant g \leqslant f) = \sup (T_1 h : 0 \leqslant h \leqslant e) = T_1^+ e,$$

where e is the function on Y satisfying $e(y) = 1$ for every $y \in Y$. Hence, we may just as well prove that

$$(T_1^+ e)(x) = \int_Y T_1^+(x, y)\, d\nu(y)$$

holds for μ-almost every $x \in X$. It follows from

$$T_1^+ e = \sup (T_1 g : 0 \leqslant g \leqslant e)$$

that for $0 \leqslant g \leqslant e$ we have

$$(T_1^+ e)(x) \geqslant \int_Y T_1(x, y)\, g(y)\, d\nu(y)$$

for almost every $x \in X$. Hence, if E is any μ-measurable subset of X such that the functions $T_1^+ e$ and $\int_Y |T_1(x, y)|\, d\nu$ are μ-summable over E, and if g and h are measurable functions on Y and X respectively satisfying

$$0 \leqslant g \leqslant e \text{ and } 0 \leqslant h \leqslant \chi_E,$$

then

$$\int_X (T_1^+ e)\, h\, d\mu \geqslant \iint T_1(x, y)\, g(y)\, h(x)\, d(\mu \times \nu). \tag{2}$$

Now, let $s(x, y)$ be a $(\mu \times \nu)$-step function of the particular form

$$s(x, y) = \sum_{n=1}^{p} a_n \chi_{A_n}(x) \chi_{B_n}(y),$$

with all coefficients a_n real, all A_n of finite μ-measure and all B_n of finite ν-measure. Assume also that

$$0 \leqslant s(x, y) \leqslant \chi_E(x) e(y),$$

where E is the same subset of X as above. Then $s(x, y)$ can be written as

$$s(x, y) = \sum_{n=1}^{q} \chi_{E_n}(x) t_n(y),$$

with $E_1, \ldots, E_q$ disjoint subsets of E, each E_n of finite μ-measure, and $t_n(y) \leqslant e(y)$ for $n = 1, \ldots, q$. Substituting now $h(x) = \chi_{E_n}(x)$ and $g(y) = t_n(y)$ in (2), we obtain

$$\int_{E_n} (T_1^+ e) \, d\mu \geqslant \iint T_1(x, y) \chi_{E_n}(x) t_n(y) \, d(\mu \times \nu)$$

for $n = 1, \ldots, q$, so by addition

$$\int_E (T_1^+ e) \, d\mu \geqslant \iint T_1(x, y) s(x, y) \, d(\mu \times \nu). \tag{3}$$

At the next step, let $\sigma(x, y)$ be a function satisfying

$$0 \leqslant \sigma(x, y) \leqslant \chi_E(x) e(y),$$

and such that there exists a sequence $(s_n(x, y) : n = 1, 2, \ldots)$ of step functions of the kind considered above satisfying $0 \leqslant s_n \uparrow \sigma$ on $E \times Y$. Any function $\sigma(x, y)$ of this kind is sometimes called a σ-function. It follows from (3) by means of the theorem on dominated convergence that

$$\int_E (T_1^+ e) \, d\mu \geqslant \iint T_1(x, y) \sigma(x, y) \, d(\mu \times \nu). \tag{4}$$

Finally, let $p(x, y)$ be any $(\mu \times \nu)$-measurable function satisfying

$$0 \leqslant p(x, y) \leqslant \chi_E(x) e(y).$$

It is well-known from the theory of product measures that $p(x, y)$ differs at most on a set of $(\mu \times \nu)$-measure zero from the limit function of an appropriate decreasing sequence of σ-functions. Hence, we may assume that there exists a sequence $(\sigma_n(x, y) : n = 1, 2, \ldots)$ of σ-functions such that $\sigma_n(x, y) \downarrow p(x, y)$ holds on $X \times Y$. For $n = 1, 2, \ldots$, we set

$$\sigma_n'(x, y) = \min \left(\sigma_n(x, y), \chi_E(x) e(y)\right).$$

The functions σ_n' are σ-functions satisfying

$$0 \leqslant \sigma_n'(x, y) \leqslant \chi_E(x) e(y)$$

and $\sigma_n' \downarrow p$. Hence, once more by the theorem on dominated convergence, it follows from (4) that

$$\int_E (T_1^+ e)\, d\mu \geqslant \iint T_1(x, y)\, p(x, y)\, d(\mu \times \nu). \tag{5}$$

Now, let $p(x, y) = 1$ at all points of $E \times Y$ where $T_1(x, y) > 0$, and $p(x, y) = 0$ everywhere else on $X \times Y$. Then $p(x, y)$ satisfies the conditions required in (5), so

$$\int_E (T_1^+ e)\, d\mu \geqslant \iint T_1^+(x, y)\, \chi_E(x)\, d(\mu \times \nu) = \int_E \{\int_Y T_1^+(x, y)\, d\nu(y)\}\, d\mu(x). \tag{6}$$

On the other hand, since the integral operator with kernel $T_1^+(x, y)$ is a majorant of T_1^+, we have also the inverse inequality, and so there is equality in (6). The same equality persists to hold if E is replaced by any μ-measurable subset of E. This implies that

$$(T_1^+ e)(x) = \int_Y T_1^+(x, y)\, d\nu(y) \tag{7}$$

holds for μ-almost every $x \in E$. Finally, we return to the conditions which E had to satisfy. It was required that $T_1^+ e$ and $\int_Y |T_1(x, y)|\, d\nu(y)$ are μ-summable over E. Both functions, call them p and q, are members of $M^+(X)$, and it is easy to see that there exists a sequence $E_n \uparrow X$ such that p and q are μ-summable over each E_n. Indeed, let $A_n = (x : p(x) \leqslant n)$ and $B_n = (x : q(x) \leqslant n)$ for $n = 1, 2, \ldots$, and let $(C_n : n = 1, 2, \ldots)$ be a sequence of subsets of X of finite measure such that $C_n \uparrow X$. Then

$$E_n = A_n \cap B_n \cap C_n \quad (n = 1, 2, \ldots)$$

satisfies the required conditions. Hence, (7) holds μ-almost everywhere on each E_n, and so (7) holds μ-almost everywhere on X. This concludes the proof that T^+ is an integral operator with kernel $T^+(x, y)$. The proof for T^- is similar, and the desired result for $|T|$ follows then by addition. Note that the transition from an arbitrary $f \geqslant 0$ to the function e is not merely for notational convenience, but is of essential importance where we introduce the functions σ_n'.

We make some additional remarks.

(i) If $M(X)$ and $M(Y)$ denote the complex Riesz spaces of all complex functions that are measurable and almost everywhere finite on X and Y respectively, and if $T(x, y)$ is complex and $(\mu \times \nu)$-measurable on $X \times Y$, then the mapping T with kernel $T(x, y)$ is an order bounded linear transformation from dom (T) into $M(X)$. It is proved similarly as above that the linear modulus $|T|$ of T is the integral transformation with kernel $|T(x, y)|$.

(ii) In a recent book "Integral Operators in Spaces of Summable Functions" by M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik and P. E. SOBOLEVSKI ([3], 1966) the following situation is considered. Let Ω be a subset of R^n of finite Lebesgue measure, and let $t(x, y)$ be a real Lebesgue measurable function on $\Omega \times \Omega$ such that the integral operator T with kernel $t(x, y)$ is norm bounded from the real space $L^p(\Omega)$ into the real space $L^q(\Omega)$, where p and q are given real numbers satisfying $1 \leqslant p, q \leqslant \infty$. In Theorem 4.2 the authors state that T is order bounded if and only if T is majorized by a positive transformation T_0 from L^p into L^q. In the proof it is asserted that the linear modulus $|T|$, defined by

$$|T|f = \sup\,(|Tg| : -f \leqslant g \leqslant f) \text{ for } f \geqslant 0,$$

is the integral operator with kernel $|t(x, y)|$. Their proof, however, contains an error. We reproduce part of the argument. For $f \geqslant 0$ in dom (T), we have

$$\int_\Omega |t(x, y)| f(y)\, dy < \infty$$

for almost every $x \in \Omega$. Take such a point x, say $x = x_0$. Hence, for x_0 and f fixed, the function $|t(x_0, y)| f(y)$ is an L^1-function of y on Ω, so $F_{x_0}(g)$, defined for all L^∞-functions $g(y)$ by

$$F_{x_0}(g) = \int_\Omega t(x_0, y) f(y) g(y)\, dy,$$

is a bounded linear functional on $L^\infty(\Omega)$ with norm

$$\|F_{x_0}\| = \int_\Omega |t(x_0, y)|\, f(y)\, dy.$$

Also, by the definition of the norm of a bounded linear functional, we have

$$\|F_{x_0}\| = \sup_{|h(y)| \leqslant 1} \left| \int t(x_0, y) f(y) h(y)\, dy \right| = \sup_{|g(y)| \leqslant f(y)} \left| \int t(x_0, y) g(y)\, dy \right|.$$

This shows that the pointwise supremum of the set $(|Tg| : |g| \leqslant f)$ is almost everywhere on Ω equal to $\int_\Omega |t(x, y)| f(y) dy$, and the authors conclude from this (formula 4.11 on p. 77) that

$$(|T|f)(x) = \int_\Omega |t(x, y)| f(y)\, dy.$$

The conclusion is false because, as observed earlier, the function sup $(|Tg| : |g| \leqslant f)$ in the Riesz space $L^q(\Omega)$ may be properly smaller than the pointwise supremum. The correct conclusion would be, therefore, that

$$(|T|f)(x) \leqslant \int_\Omega |t(x, y)|\, f(y)\, dy,$$

which was known from the beginning.

(iii) We present an example, showing that a norm bounded linear integral transformation from an L^2-space into itself is not necessarily order bounded from L^2 into L^2, although it is order bounded from L^2

into a certain larger space. To be specific, let $X = Y$ be the set of all integers, equipped with the counting measure μ (i.e., $\mu(m) = 1$ for every point of $X = Y$). Furthermore, let the kernel $T(m, n)$ be defined on $X \times Y$ by

$$T(m, n) = \frac{1}{\pi} \cdot \frac{1}{m + n + \frac{1}{2}}$$

for all integers m, n.

In order that $y = (\dots, \eta_{-1}, \eta_0, \eta_1, \eta_2, \dots)$ be a member of $\text{dom}_Y(T)$, it is necessary and sufficient that

$$\sum_{n=-\infty}^{\infty} |T(m, n)\eta_n| < \infty$$

holds for every m, and obviously this is equivalent to requiring that

$${\sum_{n=-\infty}^{\infty}}' |n^{-1}\eta_n| < \infty,$$

where $\sum'$ indicates that the term with $n = 0$ is omitted. For any number p satisfying $1 \leqslant p < \infty$ the sequence space l^p is included in $\text{dom}_Y(T)$, and so T is an order bounded integral operator from l^p into $M(X)$. For the present purposes, we restrict our attention to the case that $p = 2$. The following holds.

The mapping T is a unitary transformation from the Hilbert space l^2 into itself, and T is equal to its own inverse (i.e., T^2 is equal to the identity transformation in l^2).

The proof is straightforward (we refer to a paper by E. C. TITCHMARSH [7]).

The mapping T, although a norm bounded linear mapping from l^2 into itself, is not an order bounded linear mapping from l^2 into itself.

We briefly indicate the proof. It will be sufficient to show that the mapping $|T|$ with kernel $|T(m, n)|$ fails to map l^2 into itself. Assume, on the contrary, that $|T|$ maps l^2 into l^2. Then the inner product $(|T|y, x)$ exists as a finite number for all x and y in l^2, i.e.,

$$\sum_n |\xi_n| \left(\sum_k |T(n, k)\eta_k|\right) < \infty$$

holds for all $x = (\xi_n)$ and $y = (\eta_n)$ in l^2. We set

$$\xi_n = \begin{cases} (n^{\frac{1}{2}} \log n)^{-1} & \text{for } n = 2, 3, \dots, \\ 0 & \text{for } n = 1, 0, -1, -2, \dots \end{cases}$$

and

$$\eta_n = \begin{cases} (|n|^{\frac{1}{2}} \log |n|)^{-1} & \text{for } n = -2, -3, \dots, \\ 0 & \text{for } n = -1, 0, 1, 2, \dots . \end{cases}$$

Then x and y are members of l^2 with non-negative coordinates, and

$$\pi(|T|y, x) = \sum_{n=2}^{\infty} \xi_n \left(\sum_{k=2}^{\infty} \frac{\eta_{-k}}{|n-k+\frac{1}{2}|}\right) \geqslant \sum_{n=2}^{\infty} \xi_n \left(\sum_{k=n+1}^{\infty} \frac{\eta_{-k}}{k-n-\frac{1}{2}}\right)$$

$$= \sum_{n=2}^{\infty} \xi_n \left(\sum_{k=1}^{\infty} \frac{\eta_{-(k+n)}}{k-\frac{1}{2}}\right) = \sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \left(\sum_{n=2}^{\infty} \xi_n \eta_{-(n+k)}\right)$$

$$\geqslant \sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \left(\sum_{n=2}^{\infty} \xi_{n+k}^2\right) = \sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \left(\sum_{n=k+2}^{\infty} \xi_n^2\right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \left(\sum_{n=k+2}^{\infty} \frac{1}{n \log^2 n}\right) \geqslant \sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \int_{k+2}^{\infty} \frac{dx}{x \log^2 x}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k-\frac{1}{2}) \log (k+2)} = \infty,$$

against the assumption that $(|T|y, x)$ is finite.

It has been shown thus that T is order bounded from l^2 into $M(X, \mu)$ and norm bounded from l^2 into l^2, but not order bounded from l^2 into l^2.

There exist several variants. The results can be extended from $p=2$ to arbitrary p satisfying $1<p<\infty$, but the proofs become more difficult. All variants are known under the name of the *Hilbert transform*. We still observe it can be shown that the mapping T corresponding to the kernel $T(m, n)$ is norm bounded from l^1 into l^p for any $p>1$. Hence, by Theorem 2.3, T is order bounded from l^1 into l^p.

(iv) It was shown in the proof of Theorem 4.2 that for $0 \leqslant f \in \operatorname{dom}(T)$ we have

$$\int_Y T^+(x, y) f(y)\, d\nu(y) = \sup_{0 \leqslant h \leqslant f} \int_Y T(x, y) h(y)\, d\nu(y), \tag{8}$$

where the supremum is to be understood not as a pointwise supremum, but as a supremum in the space $M(X)$. Now, let $0 \leqslant g \in M(X)$ be such that

$$\iint_{X \times Y} T^+(x, y) g(x) f(y)\, d(\mu \times \nu) < \infty.$$

One might believe that (8) implies

$$\begin{cases} \iint_{X\times Y} T^+(x, y) g(x) f(y)\, d(\mu\times\nu) = \\ \qquad = \sup_{0\leqslant h\leqslant f,\, 0\leqslant k\leqslant g} \iint_{X\times Y} T(x, y) k(x) h(y)\, d(\mu\times\nu). \end{cases} \tag{9}$$

This is not true, however, as the following example shows. Let $X = Y = [0, 1]$ with Lebesgue measure, let

$$T(x, y) = \begin{cases} 1 & \text{for } 0 \leqslant y \leqslant 1-x, \\ -1 & \text{for } 1 \geqslant y > 1-x, \end{cases}$$

and let $f(y)=g(x)=1$ for all y and x. It is obvious then that the left hand side of (9) is

$$\int_0^1 \int_0^1 T^+(x,y)\,dxdy = \tfrac{1}{2}.$$

In order to determine the right hand side of (9) we first state the following simple lemma, the proof of which we leave to the reader.

On the interval $[0,1]$, *let* $q(x)$ *be non-increasing and let* $k(x)$ *satisfy* $0 \leqslant k(x) \leqslant 1$ *with* $\int_0^1 k(x)\,dx = \alpha$. *Then, writing* $k_1(x)=1$ *for* $0 \leqslant x \leqslant \alpha$ *and* $k_1(x)=0$ *for* $\alpha < x \leqslant 1$, *we have*

$$\int_0^1 q(x)\,k(x)\,dx \leqslant \int_0^1 q(x)\,k_1(x)\,dx = \int_0^\alpha q(x)\,dx.$$

Now, let $0 \leqslant h(y) \leqslant 1$ and $0 \leqslant k(x) \leqslant 1$ on $[0,1]$ with $\int_0^1 h(y)\,dy = \beta$ and $\int_0^1 k(x)\,dx = \alpha$. Then, since

$$q(x) = \int_0^1 T(x,y)\,h(y)\,dy$$

is a non-increasing function of x, the lemma shows that

$$\int_0^1 \int_0^1 T(x,y)\,k(x)\,h(y)\,dxdy \leqslant \int_0^1 \int_0^1 T(x,y)\,k_1(x)\,h(y)\,dxdy.$$

Applying the lemma once more, we obtain

$$\int_0^1 \int_0^1 T(x,y)\,k_1(x)\,h(y)\,dxdy \leqslant \int_0^1 \int_0^1 T(x,y)\,k_1(x)\,h_1(y)\,dxdy =$$
$$= \int_0^\alpha \int_0^\beta T(x,y)\,dxdy.$$

It is easy to see that

$$\max_{0 \leqslant \alpha \leqslant 1,\, 0 \leqslant \beta \leqslant 1} \int_0^\alpha \int_0^\beta T(x,y)\,dxdy = \tfrac{1}{3};$$

the maximum is attained for $\alpha=\beta=\frac{2}{3}$. Hence, the right hand side of (9) is $\frac{1}{3}$. As shown above, the left hand side of (9) is $\frac{1}{2}$.

(v) Given the subset D of the Riesz space L, the set D^d of all elements g satisfying $g \perp f$ for all $f \in D$ is called the *disjoint complement* of D. In an (Archimedean) Riesz space the subset D is sometimes said to be *order dense* if D^d consists only of the null element. In the case of the integral operator T the set $\mathrm{dom}_Y(T)$ is order dense in $M(Y)$, therefore, if and only if the only function disjoint to all functions in $\mathrm{dom}_Y(T)$ is the null function. This is equivalent to saying that any subset E of Y of positive ν-measure contains a subset F such that $0 < \nu(F) < \infty$ and $\chi_F \in \mathrm{dom}_Y(T)$, where χ_F denotes the characteristic function of F.

We can interchange the roles of X and Y. Thus, $\mathrm{dom}_X(T)$ is the set of all $f \in M(X)$ such that

$$h(y) = \int_X |T(x,y)\,f(x)|\,d\mu(x) \in M(Y),$$

and the mapping $f \to g$, defined by

$$g(y) = \int_X T(x,y)\,f(x)\,d\mu(x),$$

is an order bounded linear mapping from $\mathrm{dom}_X(T)$ into $M(Y)$. The following theorem holds.

THEOREM 4.3. *The set* $\mathrm{dom}_X(T)$ *is order dense in* $M(X)$ *if and only if* $\mathrm{dom}_Y(T)$ *is order dense in* $M(Y)$.

PROOF. We briefly indicate a (purely measure theoretic) proof. First of all, we need a so-called exhaustion lemma, for the proof of which we refer to ([9], Theorem 67.3).

Let (P) be some property which any ν-measurable subset of Y does or does not possess, where it is understood that (P) is a property of the equivalence classes modulo sets of measure zero rather than of the individual sets. Assume, furthermore, that

(i) *if* E_1 *and* E_2 *are subsets of Y possessing (P), then* $E_1 \cup E_2$ *possesses (P),*
(ii) *if E possesses (P), then any measurable subset of E possesses (P),*
(iii) *any set of positive measure has a subset of positive measure possessing (P).*

Then there exists a sequence $(Y_n: n=1, 2, \ldots)$ *of measurable sets of finite measure such that* $Y_n \uparrow Y$, *and every* Y_n *has the property (P).*

Assuming now that $\mathrm{dom}_Y(T)$ is order dense in $M(Y)$, it follows easily that there exists a sequence $Y_n \uparrow Y$ such that $\nu(Y_n) < \infty$ and $\chi_{Y_n} \in \mathrm{dom}_Y(T)$ for $n=1, 2, \ldots$. Indeed, let us say that the ν-measurable subset E of Y has property (P) whenever $\chi_E \in \mathrm{dom}_Y(T)$. Evidently, the property (P) satisfies the conditions in the exhaustion lemma, so the existence of the sequence $(Y_n: n=1, 2, \ldots)$ with the desired properties follows.

We need another measure theoretic lemma, as follows.

Let (X, Λ, μ) *be a σ-finite measure space, and let* $(g_n: n=1, 2, \ldots)$ *be a sequence of functions in* $M^+(X)$. *Then there exist strictly positive coefficients* $(a_n: n=1, 2, \ldots)$ *such that*

$$s = \sum_1^\infty a_n g_n \in M^+(X).$$

For the proof, assume first that $\mu(X) < \infty$. Then, for n fixed, the sets $E_{nk} = (x: g_n(x) \geqslant k)$ satisfy $\mu(E_{nk}) \downarrow 0$ as $k \to \infty$. We determine $k = k(n)$ such that

$$\mu(E_{n, k(n)}) < 2^{-n},$$

and we set $a_n = (2^n \cdot k(n))^{-1}$. Then $a_n g_n(x) \leqslant 2^{-n}$ holds for all x in the complement of $E_{n, k(n)}$, so

$$\sum_{n=n_0+1}^\infty a_n g_n(x) \leqslant 2^{-n_0}$$

holds, except on a set of measure at most 2^{-n_0}.

Given $\varepsilon > 0$, we now determine the natural number n_0 such that $2^{-n_0} < \varepsilon$. Then

$$s(x) = \sum_1^{n_0} a_n g_n(x) + \sum_{n_0+1}^\infty a_n g_n(x) = s_{n_0}(x) + r_{n_0}(x),$$

where $r_{n_0}(x) \leqslant 2^{-n_0} < \varepsilon$ holds for all x, except on a set of measure at most $2^{-n_0} < \varepsilon$, and where $s_{n_0}(x)$, being a finite sum of functions in $M^+(X)$, is itself a function in $M^+(X)$. It follows that the set $(x: s(x) = +\infty)$ is of measure less than ε. This holds for every $\varepsilon > 0$, so $(x: s(x) = +\infty)$ is of measure zero. In other words, we have $s \in M^+(X)$.

Now assume $\mu(X) = \infty$, and let $X = \bigcup_1^\infty D_n$ with all D_n mutually disjoint and of finite measure. We write

$$\varphi(x) = \sum_1^\infty c_n \chi_{D_n}(x),$$

where all coefficients c_n are strictly positive and such that $\int_X \varphi(x) d\mu < \infty$. Then, setting

$$\mu_1(E) = \int_E \varphi(x)\, d\mu$$

for every μ-measurable set E, it is evident that μ_1 is a finite measure in X (i.e., $\mu_1(X) < \infty$) such that any subset E of X is μ_1-measurable if and only if E is μ-measurable. Furthermore, sets of μ_1-measure zero are the same as sets of μ-measure zero. Hence, we have $M^+(X, \mu_1) = M^+(X, \mu)$. It follows that $g_n \in M^+(X, \mu_1)$ holds for $n = 1, 2, \ldots$, so in view of what we proved already there exist strictly positive coefficients $(a_n: n = 1, 2, \ldots)$ such that

$$s = \sum_1^\infty a_n g_n \in M^+(X, \mu_1) = M^+(X, \mu).$$

This concludes the proof of the lemma.

We return to the integral operator T. It will be sufficient to prove that order denseness of $\text{dom}_Y (T)$ implies order denseness of $\text{dom}_X (T)$. In view of the order denseness of $\text{dom}_Y (T)$ there exists a sequence $Y_n \uparrow Y$ such that $\nu(Y_n) < \infty$ and $\chi_{Y_n} \in \text{dom}_Y (T)$ for $n = 1, 2, \ldots$. Writing $D_1 = Y_1$ and $D_n = Y_n - Y_{n-1}$ for $n = 2, 3, \ldots$, we have $\chi_{D_n} \in \text{dom}_Y (T)$ for all n. It follows that

$$g_n(x) = \int_Y |T(x, y)| \chi_{D_n}(y)\, d\nu(y) \in M^+(X)$$

for $n = 1, 2, \ldots$, and so, by the last lemma, there exist strictly positive coefficients $(a_n: n = 1, 2, \ldots)$ such that

$$s(x) = \sum_1^\infty a_n g_n(x) \in M^+(X). \tag{10}$$

Writing $t(y) = \sum_1^\infty a_n \chi_{D_n}(y)$, it is evident that $t \in M^+(Y)$ and $t(y) > 0$ for every $y \in Y$. Furthermore, we have

$$s(x) = \int_Y |T(x, y)|\, t(y)\, d\nu(y). \tag{11}$$

Now, let $\varphi \in M^+(X)$ be an auxiliary function satisfying $\varphi(x) > 0$ for all $x \in X$ and $\int \varphi(x) d\mu < \infty$. We set

$$f(x) = \begin{cases} \varphi(x)/s(x) & \text{for } s(x) > 0, \\ 1 & \text{for } s(x) = 0. \end{cases}$$

Then $f \in M^+(X)$, and $f(x) > 0$ except at the points x where $s(x) = +\infty$. This shows that $0 < f(x) < \infty$ holds μ-almost everywhere on X. Furthermore, we have

$$\int f(x)\, s(x)\, d\mu \leqslant \int \varphi(x)\, d\mu < \infty,$$

so

$$\iint |T(x, y)|\, f(x)\, t(y)\, d(\mu \times \nu) < \infty.$$

By Fubini's theorem this implies that

$$\int_X |T(x, y)|\, f(x)\, t(y)\, d\mu(x) \in M^+(Y),$$

so in view of $t(y) > 0$ for every $y \in Y$ it follows that

$$\int_X |T(x, y)|\, f(x)\, d\mu(x) \in M^+(Y).$$

This shows that $f \in \mathrm{dom}_X\,(T)$. But $f(x) > 0$ holds for μ-almost every $x \in X$, and so $\mathrm{dom}_X\,(T)$ is order dense in $M(X)$. This completes the proof.

Properties of integral operators of the same kind as considered here were also investigated by A. Aronszajn and P. Szeptycki [1]. Our theorem that $\mathrm{dom}_X\,(T)$ is order dense in $M(X)$ if and only if $\mathrm{dom}_Y\,(T)$ is order dense in $M(Y)$ corresponds to their theorem (Proposition 4.2) that $T(x, y)$ is non-singular if and only if $T(y, x)$ is non-singular. Our proof is purely measure theoretic; their proof uses the fact that if $\varphi \in M^+(Y)$ is chosen such that $\varphi(y) > 0$ for all y and $\int_Y \varphi d\nu = 1$, and if

$$\varrho_Y(f) = \int_Y \frac{|f(y)|}{1 + |f(y)|}\, \varphi(y)\, d\nu(y)$$

for $f \in M(Y)$, then $\varrho_Y(f - g)$ is a distance function in $M(Y)$ with respect to which $M(Y)$ is an F-space (in the terminology of S. Banach). Similarly, $M(X)$ has a metric generated by a function ϱ_X. Furthermore, if on $\mathrm{dom}_Y\,(T)$ we define ϱ_T by

$$\varrho_T(f) = \varrho_Y(f) + \varrho_X(|T| \cdot |f|),$$

then $\mathrm{dom}_Y\,(T)$ is an F-space with respect to the metric generated by ϱ_T (so, in particular, $\mathrm{dom}_Y\,(T)$ is a complete metric space with respect to this metric). The proof of completeness corresponds with the part in our proof (near the formulas (10) and (11)) where the functions $s(x)$ and $t(y)$ are introduced, and where it is shown that $s = |T|t$.

Since in the paper by Aronszajn and Szeptycki the Riesz space aspects are not mentioned, the paper has no proof that the operator sup $(T, -T)$ has the kernel $|T(x, y)|$.

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