ALGEBRAIC VECTOR BUNDLES ON PROJECTIVE SPACES: A PROBLEM LIST

ROBIN HARTSHORNE

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The idea for this paper arose at a conference on algebraic vector bundles on projective spaces held at Oxford in May 1978.* The recently discovered connection between some partial differential equations of mathematical physics and certain holomorphic vector bundles on complex projective 3-space has provided an enormous boost to a subject which was already of interest to algebraic geometers. Conversely one may hope that the results obtained by abstract mathematical methods may be of use to the physicists.

It is too early to write a comprehensive summary of this rapidly changing field. However, I thought it might be useful to mark the present state of affairs in the subject and to stimulate further research by providing a list of problems. So, accepting the risk that it will soon be outdated, I have drawn up a list of problems which seem to me representative of current trends of research in the subject, including some suggested by other participants in the Oxford conference. I will not attempt to define all the terms used, nor to summarize all previous work—for that I refer to the annotated bibliography at the end and the further references contained in those papers.

The problems will be stated mostly in the language of abstract algebraic geometry [27]: projective space $\mathbb{P}_k^n$ is taken over an algebraically closed groundfield $k$ of characteristic $p \geq 0$; a vector bundle is a locally free coherent algebraic sheaf; in the case where the groundfield is $\mathbb{C}$, the category of algebraic vector bundles is equivalent to the category of holomorphic vector bundles. There are several different definitions of stable vector bundle in current use. For the sake of definiteness, we will use the definition of Mumford and Takemoto: a vector bundle $E$ of rank $r$ on $\mathbb{P}_k^n$ is stable if for every coherent subsheaf $F$ of $E$, of rank $s$, $c_1(F)/s < c_1(E)/r$, where $c_1$ denotes the first Chern class. Replacing $<$ by $\leq$ gives the definition of semistable.

**Problem 1. Does there exist an indecomposable vector bundle of rank 2 on $\mathbb{P}_k^n$?**

This is part of the more general problem to find vector bundles of small rank on large projective spaces. It is known that there are indecomposable bundles of rank 2 on $\mathbb{P}_k^n$ for $n = 2, 3, 4$; of rank 3 on $\mathbb{P}_k^n$ for $n = 2, 3, 4, 5$; of rank $n - 1$ on $\mathbb{P}_k^n$ for all $n \geq 2$ and over a field of characteristic 2 there is one of rank 2 on $\mathbb{P}_k^3$. Other than these there are no known indecomposable bundles of rank 1 or $< r < n - 1$ on $\mathbb{P}_k^n$. The problem is intriguing because there is no particular reason why they should not exist, yet they have proven extremely difficult to find.

A recent result of Grauert and Schneider [20] says that an indecomposable rank 2 bundle on $\mathbb{P}_k^n$ for $n \geq 5$ must be stable.† This narrows the search considerably. In particular it implies that the Chern classes must satisfy $c_1^2 < 4c_2$.

The problem can be translated into questions of linear algebra, to find matrices satisfying certain conditions. Unfortunately these seem very difficult.

In another direction, one can ask about the existence of topological $C^2$-bundles on...
Rees [48] and Smith [54] have shown for every $n \geq 5$ there exist nontrivial $C'$-bundles on $\mathbb{P}^n$ with Chern classes $c_1 = c_2 = 0$. The result of Grauert and Schneider would imply that these could not be holomorphic.

References: [7, 12, 13, 26, 32, 33, 56, 57, 59, 80, 82, 90].

**Problem 2.** Find more rank 2 vector bundles on $\mathbb{P}^n$.

The only indecomposable rank 2 bundles known on $\mathbb{P}^n$ are the bundle of Horrocks and Mumford [32], plus those obtained from it by tensoring with a line bundle or pulling back by a finite morphism from $\mathbb{P}^n$ to itself and, in characteristic 2, some other bundles constructed by Horrocks [34].

**Problem 3.** Does there exist a nonsingular subvariety $Y \subset \mathbb{P}^n$ of dimension 4 which is not a complete intersection?

According to a result of Barth, any such $Y$ would be the zero set of a section of an indecomposable rank 2 vector bundle on $\mathbb{P}^n$ and conversely any such vector bundle would give such a $Y$. So this problem is equivalent to the problem of finding an indecomposable rank 2 vector bundle on $\mathbb{P}^n$. Schneider [53] has shown that if such a subvariety $Y$ exists, its degree must be $> 514$.

This problem is a special case of the more general problem of finding subvarieties of small codimension in projective space. In particular, if $Y$ is a nonsingular subvariety of $\mathbb{P}^n$ and if $\dim Y > \frac{1}{2} n$, is $Y$ necessarily a complete intersection?

References: [6, 26].

**Problem 4.** Let $X$ be a nonsingular projective variety of dimension $n$ with a fixed very ample divisor $H$. Do the stable vector bundles $E$ on $X$ of fixed rank $r$ and with fixed Hilbert polynomial $P$ form a bounded family?

Another way of phrasing this question is to ask whether the moduli space of those bundles is of finite type over $k$. For Maruyama [41] has shown that these bundles have a coarse moduli scheme, which is locally of finite type over $k$. The answer is known to be yes if $n \leq 2$ or if $r \leq 2$ and also for $r = 3, 4$ in characteristic 0.

References: [6, 26].

**Problem 5.** Let $E$ be a stable (respectively, semistable) vector bundle of rank $r$ on $\mathbb{P}^n$ with $n \geq 3$. Is it true that for almost all hyperplanes $H \subset \mathbb{P}^n$ the restriction of $E$ to $H$ is stable (respectively, semistable)?

If so, this would be an important step for an inductive solution of the problem of boundedness (Problem 4). It has been proved by Barth for $r = 2$, with one exception, the nullcorrelation bundle (respectively, by Maruyama, whenever $r < n$).

References: [7, 29, 43].

**Problem 6.** Study the compactification of the moduli space of stable vector bundles obtained by allowing semistable torsion-free sheaves as well.

This is a rather vague problem, but the point is that to obtain a real understanding of stable vector bundles, it is important to consider also semistable bundles and stable and semistable torsion-free sheaves. In particular, all those results which have been proved so far for vector bundles should be done also for torsion-free sheaves. This will doubtless involve new technical difficulties, but should provide a better understanding of questions such as how the moduli spaces of bundles with different Chern classes fit together. It may also have some significance for mathematical physics by providing a model for "gauge fields with singularities."

References: [40–42].

**Problem 7.** Describe explicitly the moduli space $M(c_1, c_2)$ of stable rank 2 bundles on $\mathbb{P}^1$ with Chern classes $c_1$, $c_2$. 
In particular one can ask what are the irreducible components of \( M \), what are their dimensions, is there a universal family of bundles on \( M \) (in which case \( M \) would be a fine moduli space)? At least one knows under what conditions \( M(c_1, c_2) \) is nonempty, namely if and only if \( c_1^2 < 4c_2 \) and \( c_1c_2 = 0 \mod (2) \).

The corresponding problem for stable rank 2 bundles on \( P^2 \) has been treated quite satisfactorily by Barth (for \( c_1 \) even) and Hulek (for \( c_1 \) odd). Barth shows for \( c_1 = 0 \) that the moduli space \( M(0, c_2) \) for \( c_2 > 0 \) is an irreducible rational nonsingular quasiprojective variety of dimension \( 4c_2 - 3 \). He also parametrizes the bundles in terms of quadratic forms on certain vector spaces.

On \( P^3 \) the situation is much more complicated. Restricting to the case \( c_1 = 0 \) for simplicity, in which case \( c_2 > 0 \), it is known that every irreducible component of \( M(0, c_2) \) has dimension \( \geq 8c_2 - 3 \). On the other hand, the moduli space is known to be disconnected for \( c_2 \geq 3 \) and for every odd \( c_2 \geq 5 \) there exist irreducible components of dimension \( > 8c_2 - 3 \). The maximum possible dimension is not known but has been guessed by Barth. A complete description of the moduli space \( M(0, c_2) \) has been given only for \( c_2 = 1, 2 \).

Horrocks has shown that every rank 2 bundle on \( P^3 \) can be represented by a monad. This essentially reduces the question to linear algebra. For small values of \( c_2 \) one can list what types of monads might occur, but the question of existence of the bundles is difficult. Only in the special case of stable bundles with \( c_1 = 0 \) and \( H^1(E(-2)) = 0 \) (which includes all the bundles coming from instantons) does the monad take on a particularly manageable form.

References: [7, 9–13, 17, 28, 29, 35, 75, 84, 88].

Problem 8. Describe explicitly the moduli space \( M'(k) \) of \( SU(2) \)-instantons with instanton number \( k \).

According to Atiyah and Ward [2], this problem is equivalent to classifying stable rank 2 holomorphic vector bundles \( E \) on \( P^3 \) with \( c_1 = 0, c_2 = k \), having a real structure \( \sigma \) with \( \sigma^2 = -1 \) and no real jumping lines. Atiyah and Hitchin, and independently Drinfeld and Manin, showed that these bundles satisfy \( H^1(E(-2)) = 0 \) and so can be represented by very special monads of Horrocks. The moduli space \( M'(k) \) is known to be a real analytic manifold of (real) dimension \( 8k - 3 \), but except for the cases \( k = 1, 2 \) it is not yet known whether it is connected. It is a union of a subset of the connected components of the real part of the complex moduli space \( M(0, k) \) of Problem 7.

The monads lead to a very explicit problem in linear algebra over the quaternions which in principle describes the moduli space. Using this one can show \( \pi_k(M'(2)) = \mathbb{Z}_2 \) and it is conjectured that \( \pi_k(M'(k)) = \mathbb{Z}_2 \) for all \( k \geq 2 \). The precise topological structure (e.g. homotopy type) of the spaces \( M'(k) \) should be of importance to the physicists in their evaluation of Feynman integrals.

References: [2, 3, 5, 15, 17, 28, 46, 65, 68].

Problem 9. Find the least integer \( t \), as a function of \( c_1 \) and \( c_2 \), such that for each stable vector bundle \( E \) of rank 2 on \( P^3 \) with \( c_1 \) and \( c_2 \) as Chern classes, \( H^0(E(t)) \neq 0 \).

Another approach to the problem of classification of rank 2 bundles on \( P^3 \) is to associate to the bundle a curve in \( P^3 \) obtained as the zero set of a section of some twist \( E(t) \) of the bundle. To do this efficiently, we want to use the least possible \( t \) (which can be bounded in terms of \( c_1 \) and \( c_2 \) because of boundedness of the family [Problem 4]). Then in principle the classification of bundles is reduced to the classification of curves (also difficult).

In case \( c_1 = 0 \), I conjecture that the least \( t > \sqrt{3c_2 + 1} - 2 \) will do. For the bundles coming from instantons this bound works and is the best possible. For arbitrary stable bundles, this bound works for \( c_2 \leq 9 \), but the best bound I can prove for arbitrary \( c_2 \) is of the order of \( (c_2)^{2/3} \).

References: [29].
PROBLEM 10. Let \( S = k[x_0, x_1, x_2, x_3] \) be the homogeneous coordinate ring of \( \mathbb{P}^3 \). Characterize those finite-length graded \( S \)-modules \( N \) such that there exists a rank 2 (respectively, stable) vector bundle \( E \) on \( \mathbb{P}^3 \) with \( N = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^3, E(n)) \).

The module \( N \) is an important invariant of the bundle \( E \). In fact Horrocks [31] has shown that a bundle \( E \) (of unspecified rank) is determined up to direct sums with line bundles by the module \( N \) together with the module \( N' = \bigoplus_{n \in \mathbb{Z}} H^2(\mathbb{P}^3, E(n)) \) and an element \( \beta \in \text{Ext}^2_s(N', N) \). In the case of rank 2, \( N' \) is the dual of \( N \), up to shift, so that \( E \) is almost determined by \( N \).

If \( C \) is a curve in \( \mathbb{P}^3 \) obtained as the zero set of a section of a twist of \( E \), then the module \( N \) appears (up to shift) as \( \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^3, \mathcal{J}_C(n)) \) where \( \mathcal{J}_C \) is the ideal sheaf of \( C \). Rao [45] has shown that every finite length graded \( S \)-module \( N \) arises in this way from some irreducible nonsingular curve in \( \mathbb{P}^3 \) and that \( N \) characterizes the liaison equivalence class of \( C \). On the other hand, he shows that not every \( N \) arises from a rank 2 bundle as above.

On \( \mathbb{P}^3 \), a rank 2 bundle \( E \) is completely determined by the corresponding module \( N \) and indeed this is the starting point of Barth’s classification of bundles on \( \mathbb{P}^3 \) [8].

PROBLEM 11 (Barth). Let \( N \) be the graded \( S \)-module associated to a stable rank 2 bundle \( E \) on \( \mathbb{P}^3 \) with \( c_1 = 0 \), as in Problem 10. Is \( N \) generated as an \( S \)-module by its elements in degrees \( < 0 \)?

This is true for those bundles \( E \) satisfying \( H^0(E(-2)) = 0 \); in that case \( N \) is generated by elements in degree \(-1\). If true in general, this would limit the types of monads needed to construct all rank 2 stable bundles on \( \mathbb{P}^3 \).

References: [9–11].

PROBLEM 12. Given integers \( d, k > 0 \), find the maximum genus \( g \) of an irreducible nonsingular curve \( C \) of degree \( d \) in \( \mathbb{P}^3 \) which is not contained in any surface of degree \( < k \).

For \( k = 1 \), \( g = \frac{1}{2}(d - 1)(d - 2) \). For \( k = 2 \) the bound \( g \leq \frac{1}{2}d^2 - d + 1 \) was given by Castelnuovo. For general \( k \) and for \( d > k(k - 1) \) the bound

\[
g \leq \frac{d^2}{2k} + \frac{1}{2}d(k - 4) + 1
\]

was given by Halphen and has recently been proved by Harris and Gruson and Peskine.

If \( d \leq \frac{1}{3}(k^2 + 4k + 6) \) then an easy argument using the Riemann–Roch theorem and Clifford’s theorem shows that

\[
g \leq d(k - 1) - \binom{k + 2}{3} + 1,
\]

but it is not known if this is the best possible.

In the remaining range \( \frac{1}{3}(k^2 + 4k + 6) \leq d \leq k(k - 1) \) the bound is not known, but I conjecture that

\[
g \approx \frac{1}{4}d(k - 7 + \sqrt{(12d - 3k^2 - 6k + k)}) + 1,
\]

and there are many examples of curves giving equality. If true, this conjecture would imply an affirmative solution to the conjecture in problem 9.

References: [21–25, 27, Ch. IV, §6, 29].

PROBLEM 13 (Barth). Let \( C \subset \mathbb{P}^3 \) be a space curve whose canonical bundle \( \omega_C = \mathcal{O}_C(2m - 4) \) for some \( m \) and assume that \( C \) has degree \( d > m^2 \) and that \( C \) does not lie
on any surface of degree \( m \). Then is the natural map

\[
H^q(C_{\mathbb{P}^1}(1)) \otimes H^q(C_{\mathbb{C}}(m + k - 1)) \to H^q(C_{\mathbb{C}}(m + k))
\]

surjective for all \( k \geq 0 \)?

This is a translation of Problem 11 into space curves.

References: [1, 9, 11].

**Problem 14.** For which values of \( c_1, c_2, c_3 \) does there exist a stable rank 3 bundle \( E \) on \( \mathbb{P}^3 \) with the given Chern classes?

This is the first unknown case of the general question, what restrictions does stability impose on the Chern classes of a vector bundle on \( \mathbb{P}^n \)? Topology imposes certain congruences on the Chern classes of (topological) \( C^\infty \)-bundles on \( \mathbb{P}^n \). In the case of rank 3 bundles on \( \mathbb{P}^3 \) this is \( c_1 c_2 = c_3 \) (mod 2). Vogelaar has recently shown that for every \( c_1, c_2, c_3 \) satisfying this congruence, there exists an algebraic rank 3 vector bundle on \( \mathbb{P}^3 \) with those Chern classes. On the other hand, one can see easily that stability implies \( c_1^2 < 3c_2 \). It remains to determine the possible values of \( c_3 \) for stable bundles.

In addition to this question about rank 3 bundles, there are many other untouched questions about rank 3 and higher rank bundles on \( \mathbb{P}^3 \), similar to Problems 7–10. For example, one can ask about the moduli space of \( SU(3) \)-instantons, which correspond to certain stable rank 3 bundles with \( c_1 = c_3 = 0, c_2 \geq 2 \).

References: [4, 58, 62].

**Problem 15.** Let \( E \) be a stable rank \( r \) bundle on \( \mathbb{P}^n \) with first Chern class \( c_1 \) satisfying \( -r < c_1 \leq 0 \). For any line \( L \subseteq \mathbb{P}^n \), let \( E|_L = \bigoplus_{a_i} \mathcal{O}_L(a_i) \) with \( a_1 \geq a_2 \geq \cdots \geq a_r \). What are the possible values of \( a_1 \) for the general line \( L \)?

The decomposition of \( E \) restricted to a line and how that decomposition changes as the line varies has been an important tool in studying vector bundles of rank 2. For example, one defines a *jumping line* for \( E \) to be a line where the \( a_i \) are different from the \( a_i \) of a general line. A *uniform* bundle is defined as one with no jumping lines.

In the case of stable rank 2 bundles on \( \mathbb{P}^n \), the theorem of Grauert and Müllich says that for the general line, if \( c_1 = 0 \), then \( a_1 = a_2 = 0 \); if \( c_1 = -1 \), then \( a_1 = 0, a_2 = -1 \). However, over a field of characteristic \( p > 0 \) other things may happen and for rank \( r \geq 3 \) over any field the answer is not yet known.*

References: [7, 18, 19, 37, 49, 62].

**Problem 16.** Extend the results about stable bundles proved so far only over \( \mathbb{C} \) to the case of a (groundfield) \( k \) of arbitrary characteristic.

This means replacing those arguments which involve complex analysis or characteristic zero hypotheses with methods of abstract algebraic geometry and noting what new phenomena may occur in characteristic \( p > 0 \).

References: [37, 51, 73].

**Problem 17.** Let \( L \) be a line in \( \mathbb{P}^1 \) and let \( \mathfrak{X} \) be the formal completion of \( \mathbb{P}^1 \) along \( L \). Study vector bundles on \( \mathfrak{X} \) whose restriction to \( L \) is trivial.

This is the analogue, in abstract algebraic geometry, of the problem of studying holomorphic vector bundles on a tubular neighborhood of \( L \) in \( \mathbb{P}^2 \). It is suggested via the Penrose transform by localization at a point in Minkowski space, which corresponds to localization near a line in \( \mathbb{P}^2 \).

For this algebraic problem, one loses the second Chern class of a bundle and the moduli spaces become infinite-dimensional. However, a global bundle on \( \mathbb{P}^1 \) is

*This problem is now solved (over \( \mathbb{C} \)) by Spindlzl[83].
determined by its restriction to \( \mathfrak{X} \), so this local problem may provide a new perspective on the global moduli problem.

**PROBLEM 18.** Take two projective 3-spaces with coordinates \( x_i \) and \( x'_i \) respectively and let \( F \subset P^3 \times P^3 \) be the hypersurface defined by \( \Sigma x x'_i = 0 \). Study vector bundles on \( F \) whose restriction to a general \( P^3 \times P^3 \) in \( F \) is trivial and which admit an extension to the third infinitesimal neighborhood of \( F \) in \( P^3 \times P^3 \).

This problem arises from the work of Witten[63] and independently Isenberg, Yasskin, and Green[36] on non-self-dual solutions of the Yang–Mills equation. Actually their problem is only local, in the neighborhood of some \( P^3 \times P^3 \) inside \( F \), but global results of an algebro-geometric nature might be useful.

**PROBLEM 19.** By restricting bundles on \( P^3 \) to a fixed curve \( C \) in \( P^3 \), establish some relationship between the moduli spaces of stable vector bundles on \( P^3 \) with the moduli of stable bundles on \( C \).

This seems a natural thing to try since so much is known now about the moduli spaces of vector bundles on curves. The first step would be to determine under what conditions the restriction to \( C \) of a stable bundle on \( P^3 \) is stable on \( C \).

References: [44, 71, 76–78, 87]

**PROBLEM 20 (Barth).** Give a description of the versal deformation space of a vector bundle on \( P_c^3 \) with orthogonal (respectively, symplectic) structure.

This problem arises naturally in Barth’s work and would generalize the case of deformations of bundles on \( P_c^3 \) without additional structure considered by Brieskorn.

References: [7, 11].

**PROBLEM 21 (Barth).** Let \( S_n \) be the vector space of symmetric \( n \times n \) \( \mathbb{C} \)-matrices and let \( \Lambda_n \) be the vector space of alternating \( n \times n \) \( \mathbb{C} \)-matrices. Let \( A \) be a sufficiently general element of \( H^0(P_c^1, S_n \otimes_c \mathcal{O}_P(1)) \). Then there is an exact sequence of sheaves on \( P_c^1 \)

\[
0 \to \mathcal{O}_P(1) \oplus \mathcal{O}_P(2-n) \to S_n \otimes \mathcal{O}_P(1) \to \Lambda_n \otimes \mathcal{O}_P(2) \to 0,
\]

where \( \phi \) is given by \( (I, A, A^2, \ldots, A^{n-1}) \) and \( \psi(B) = [B, A] \) for any \( B \). What is the coboundary map

\[
\delta: H^0(\Lambda_n \otimes \mathcal{O}_P(2)) \to H^1(\otimes \mathcal{O}_P(k))?
\]

This problem arises out of the linear algebra translation of Problem 7 for stable rank 2 bundles \( E \) on \( P_c^3 \) satisfying \( c_1 = 0 \) and \( H^1(E(-2)) = 0 \). For \( n = 1, 2, 3 \), the fact that \( \delta \) is obviously zero shows that the subset of \( M(0, n) \) corresponding to these bundles is connected.

References: [3, 10, 17, 46].

**PROBLEM 22 (Atiyah).** For any rank 2 holomorphic vector bundle \( E \) on \( P_c^3 \) corresponding to an instanton, consider the restriction of \( E \) to a fixed plane \( P_c^2 \). This gives a mapping from the \( 8k-3 \) real-dimensional moduli space \( M'(k) \) of instantons (Problem 8) to the \( 4k-3 \) complex-dimensional moduli space \( M''(k) \) of stable vector bundles on \( P_c^2 \) with \( c_1 = 0 \) and \( c_2 = k \). It is a fibering with fiber \( SL(2, \mathbb{C})/SU(2) \). Is the image of this mapping equal to the Zariski-open subset \( V \) of \( M''(k) \) corresponding to stable bundles which are trivial on the unique real line in \( P_c^2 \)?

If so, this would give a method of studying the real moduli space of instantons in terms of the complex moduli space of bundles on \( P^2 \) discussed by Barth[8].
Problem 23 (Horrocks). Let \( A_n \) be a regular local ring of dimension \( n + 1 \), \( \mathfrak{m} \) its maximal ideal and \( X_n = \text{Spec} \ A - \{ \mathfrak{m} \} \). Are there any nontrivial bundles \( E \) on \( X_n \) which are infinitely extendable, in the sense that for each \( n' > n \), there exists a bundle \( E' \) on \( X_n \) whose restriction to \( X_n \) is \( E \), where \( X_n \) is embedded in \( X_n' \), as the subset defined by the vanishing of \( n' - n \) regular parameters of \( A_n \)?

The corresponding question on projective space has a negative answer: any infinitely extendable bundle \( E \) on \( P^n \) (meaning for each \( n' \) there is a bundle \( E' \) on \( P^n \) restricting to \( E \) on \( P^n \)) is a direct sum of line bundles. This was proved in the case of rank 2 by Barth and Van de Ven and for arbitrary rank by Sato. An independent proof was given by Tyurin. There is also an analogous statement for subvarieties: if \( Y_n \subseteq P^n \) is a nonsingular subvariety and if for each \( n' > n \) there is a nonsingular subvariety \( Y_{n'} \subseteq P^n \) such that \( Y_n \) is the tranversal intersection of \( Y_{n'} \) with \( P^n \), then \( Y_n \) is a complete intersection.

References: [5a, 6, 26, 50–52, 60].

Problem 24 (Horrocks). Let \( A \) be a regular local ring of dimension \( n + 1 \). What is the minimum rank \( r(p) \) of the \( p^n \) module of syzygies of a nonzero artinian \( A \)-module?

It is known that

\[
\begin{align*}
r(2) &= n \\
r(p) &= r(n + 1 - p) \\
r(p) &> \frac{n - 1}{n + 1 - p} \\
r(p) &\leq \binom{n}{p - 1}
\end{align*}
\]

(principal ideal theorem) 
(duality) 
(Lebelt[39]) 
(Koszul complex).

Problem 25 (Horrocks). What is the minimum rank \( s(p) \) of bundles on the punctured spectrum \( X_n \) (using notation above) with projective dimension \( n + 1 - p \) (>0)?

This is a generalization of Problem 24. Clearly \( s(p) \leq r(p) \). Lebelt’s inequality holds for \( s \) also.

Reference: [14a].

Problem 26 (Horrocks). Find the least \( b - a - c \) such that there exists a sequence

\[ aA_n \rightarrow bA_n \rightarrow cA_n, \]

with \( \beta \alpha = 0 \), \( \alpha \), \( \beta \) locally split on \( X_n \) but not globally split.

The first unsolved case is \( n = 4 \).

Note Added in Proof (2 April 1979). To bring this problem list up to date, we include new and additional Refs. [64–90] below.

Unfortunately, Grauert and Schneider have not yet been able to repair the error in their paper[20]. Therefore that result and its consequences (in particular[53]) must remain conjectural.

Barth has pointed out that in Problem 5, one must expect some exceptions, generalizing the nullcorrelation bundle in rank 2. Thus, for example, for stable bundles of rank \( r \) on \( P^2 \) with \( c_1 = 0 \) one should require \( c_2 \geq r \).

With regard to Problem 7, see the recent work of Hulek[35], Le Potier[75], Strømme[84] and Tyurin[88].

References


Some results about the generators of the ideal of a subcanonical space curve \( C \), i.e. \( \omega_C \approx \mathcal{O}_C(m) \) for some \( m \), generalizing classical results about canonical curves.


Shows how the Penrose “twistor” approach to space-time can be used to interpret self-dual solutions of the Yang–Mills equation in terms of holomorphic vector bundles on \( P^3 \).

A construction in terms of linear algebra for all vector bundles on $\mathbb{P}^r$ coming from instantons. Announcement of the results in Drinfeld and Manin[17].


A self-contained account of the ideas of R. Penrose connecting four-dimensional Riemannian geometry with three-dimensional complex analysis. Applied in particular to show existence and dimension of moduli space of instantons corresponding to any compact gauge group.


Announcement of results in [4].


Investigates topological properties of the moduli space $M_k$ of SU(2)-instantons with instanton number $k$. In particular for $k \geq q$, the homology $H_k(M_k)$ contains all the $q^n$ homology of the $k^n$ component of the space of all maps of $S^4 \to SU(2)$. Conjecture: $\pi_4(M_k) = \mathbb{Z}_k$ for $k \geq 2$.


Any infinitely extendable rank 2 bundle on $\mathbb{P}^n$ is a direct sum of line bundles.


If $Y$ is a nonsingular subvariety of degree $d$ and dimension $n$ in a projective space $\mathbb{P}^r$, and if $n \geq 5(d - 1)/2$, then $Y$ is a complete intersection.


Contains general properties of stable bundles; proof of the theorem of Grauert–Müllich; and the theorem that if $E$ is stable on $\mathbb{P}^r$, $r > 3$, then the restriction to a general hyperplane $H$ is also stable (with one exception).


Shows that the moduli space of stable rank 2 vector bundles on $\mathbb{P}_2$, with $c_1 = 0$, $c_2 \geq 2$, is a smooth, rational, connected variety of dimension $4c_2 - 3$ and gives more detailed description in the cases $c_2 = 2, 3, 4, 5$.


Contains exhaustive tables of various data about stable rank 2 bundles on $\mathbb{P}_3$ with $c_1 = 0, c_2 = 1, 2, \ldots, 8$; in particular, a list of possible monads and some examples of families of dimension $> 8c_2 - 3$.


Contains a careful treatment with full proofs of the construction, originally due to Horrocks, which associates to each stable orthogonal or symplectic bundle on $\mathbb{P}^r$ or $\mathbb{P}^r$ a special self-dual monad.


Constructs a canonical “two-sided resolution” of any coherent sheaf on $\mathbb{P}^r$ which generalizes the monad construction of Horrocks on $\mathbb{P}^r$.

13. A. A. Beilinson: The derived category of coherent sheaves on $\mathbb{P}^r$, preprint.

An expanded version of Beilinson[12] with more details and examples.


An algebraic discussion of the “syzygy problem,” which asks over a regular local ring whether every nonfree $j$th syzygy module must have rank $> j$. The corresponding problem for a vector bundle $E$ (not a sum of line bundles) on $\mathbb{P}^r$ is this: if $H^i(E(m)) = 0$ for all $i = 1, 2, \ldots, j - 2$ and for all $m \in \mathbb{Z}$, does $E$ have rank $> j$?


Discusses the matrix formulation of the problem of classifying instantons and gives a rather explicit description of the moduli of SU(2)-instantons for $k = 2, 3$.


Announcement of results in [17].

17. V. G. Drinfeld and Ju. I. Manin: Instantons and Sheaves on $\mathbb{C}P^1$, preprint.

Contains the complete proof that every bundle on $\mathbb{C}P^1$ arising from an instanton is represented by a special monad of the form $\mathbb{A} \otimes \mathbb{F}(-1) \to \mathbb{B} \otimes \mathbb{F} \to \mathbb{C} \otimes \mathbb{F}(1)$, $\mathbb{A}, \mathbb{B}, \mathbb{C}$, vector spaces. In particular, proves the critical $H^l(E(-2)) = 0$ and constructs the monad by the method of Beilinson.


Extends earlier work of Van de Ven (rank 2 on $\mathbb{P}^r$) and Sato (rank $r = n$ on $\mathbb{P}^r$) to prove that every *uniform* bundle of rank 3 on $\mathbb{P}^r$ (meaning the isomorphism class of the restriction to a line is independent of the line chosen) is homogeneous. See also[69].
Gives a construction for rank 2 vector bundles from a codimension 2 subvariety. Also shows that the restriction of a stable rank 2 bundle to a general line is $E \bigoplus C$ if $c_1 = 0$; $E \bigoplus C(1)$ if $c_1 = 1$.

The main result states that any indecomposable rank 2 vector bundle on $\mathbb{P}^n$, for $n \geq 4$, must be stable. Unfortunately there is an error on p. 85 which the authors have not yet been able to repair, so the question remains open.

Using the result of Laudal[38], they determine the maximum genus $g$ of a nonsingular curve $C$ in $\mathbb{P}^n$ of degree $d$, not contained in any surface of degree $<k$, in the range $d > k(k - 1)$.

Contains a good discussion of Halphen’s work on curves in $\mathbb{P}^n$, what he stated and to what extent proofs of those statements are known.

A treatise on the classification of curves in $\mathbb{P}^n$ including tables of all curves of degree $\leq 20$. Unfortunately, some proofs rely on general position arguments which do not live up to modern standards of rigor.

Contains a derivation of the maximum genus $g$ of an irreducible curve $C$ of degree $d$ in $\mathbb{P}^n$, not contained in any surface of degree $<k$, provided $d > k^2$.

Constructs examples of space curves of large genus very close to the bound conjectured in Problem 12.

New expanded version of [24].

A survey article discussing various questions concerning subvarieties of small codimension and vector bundles of small rank on projective spaces. The conjecture “$Y$ nonsingular $\subseteq \mathbb{P}^n$, dim $Y > n$ & $Y$ a complete intersection” remains open.


Statement of the algebro-geometric problem of rank 2 vector bundles on $\mathbb{P}^1$ corresponding to $SU(2)$-instantons and announcement of the results of Hartshorne[29], in particular, the classification of instanton bundles with $c_1 = 2$.

Complete discussion of basic properties and examples of stable rank 2 vector bundles on $\mathbb{P}^1$ and their moduli spaces. Main results include a bound on $t$ for $H^1(E(t)) \neq 0$ (Problem 9) and complete classification of stable bundles with $c_1 = 0, c_2 = 2$, showing their moduli space is connected.


This paper analyzes vector bundles on the punctured spectrum of a regular local ring $A$ in terms of certain complexes of modules on $A$, with applications to vector bundles on $\mathbb{P}^n$ and $\mathbb{P}^3$.

The only known example of an indecomposable rank 2 bundle on $\mathbb{P}^n$ (except in characteristic 2).

Uses the theory of group representations to construct a number of rank 3 bundles on $\mathbb{P}^5$, indecomposable except in characteristic 2.

Contains several methods and in particular constructs many new rank 2 bundles on $\mathbb{P}^n$ over a field of characteristic 2, some of which have negative discriminant and so cannot be stable.

35. K. W. Hulek: Stable rank-2 vector bundles on $\mathbb{P}^3$ with $c_1(F)$ odd, preprint.
Representing each such bundle by a monad and using techniques similar to Barth[8], he shows that the moduli variety $M(1, n)$ is irreducible, nonsingular, and rational. He gives also a detailed description of $M(-1, 2)$ and $M(-1, 3)$.

An interpretation of non-self-dual Yang–Mills fields as holomorphic vector bundles on a 5-complex-dimensional manifold $\Lambda \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Generализет the theorem of Grauert-Mülich to rank 2 stable vector bundles on $\mathbb{P}^n$ over a field of characteristic $p > 0$. See [72, 73].

Contains the following result, used by Gruson and Peskine in [21]: if $C$ is an irreducible curve in $\mathbb{P}^3$ of
degree \( d \), not contained in any surface of degree \( < k \) and if \( d > k(k - 1) \), then the general hyperplane section of \( C \) is not contained in any curve of degree \( < k \).


40. M. MARUYAMA: Openness of a family of torsion free sheaves, J. Math. Kyoto Univ. 16 (1976), 627-637. The properties of a coherent sheaf being stable or semistable are open conditions (this is needed to construct moduli spaces).


44. S. RAMANAN: Vector bundles on algebraic curves. ICM 1978 Helsinki.


54. L. SMITH: Complex 2 plane bundles over \( CP(n) \), Manuscr. math. 24 (1978), 221-228.

55. L. SWITZ: Complex 2 plane bundles over \( CP(n) \), Manuscr. math. 24 (1978), 221-228.


60. A. N. TYURIN: Finite dimensional vector bundles over infinite varieties, Izv. Akad. Nauk Ser. Mat. 40 (1976), 1248-1268. Math. U.S.S.R. Izvestija 10 (1976), 1187-1204. Shows that any vector bundle of finite rank over $\mathbb{P}^n$ is a direct sum of line bundles. In other words, given an infinite tower of bundles $E_n$ on $\mathbb{P}^n$ with $E_n|_{\mathbb{P}^{n-1}} = E_{n-1}$ for each $n$, each $E_n$ is a sum of line bundles.
62. J. H. RAWNSLEY: On the Atiyah-Hitchin-Drinfeld-Manin vanishing theorem for cohomology group of $\mathbb{P}^n$, preprint. Gives a construction for bundles of rank $r$ on $\mathbb{P}^n$ in terms of subvarieties of codimension 2. Application: there exists a rank 3 bundle on $\mathbb{P}^3$ with Chern classes $c_1, c_2, c_3$ if and only if $c_1^2c_2 = c_3$ (mod 7).
63. E. WITTEN: An interpretation of classical Yang-Mills theory, preprint. The second order classical Yang-Mills equation leads to a problem about holomorphic vector bundles on a certain hypersurface in $\mathbb{P}^k \times \mathbb{P}^r$.
64. C. W. BERNARD, N. H. CHRIST, A. H. GUTH and E. J. WEINBERG: Pseudoparticle parameters for arbitrary gauge groups, Phys. Rev. D 16 (1977), 2967-2977. Applies the Atiyah-Singer index theorem to determine the number of parameters needed to describe all self-dual Yang-Mills solutions on $S^4$ relative to an arbitrary gauge group $G$.
70. G. ELENCWAJG and O. FORSTER: Bounding cohomology groups of vector bundles on $\mathbb{P}^n$ (in preparation). The set of semistable vector bundles of rank $r$ on $\mathbb{P}^n$ with given $c_1$ and $c_2$ forms a bounded family. In particular the higher Chern classes are bounded by $c_1$ and $c_2$. The proof uses the result of Spindler[83].
71. D. GIESEKER: On a theorem of Bogomolov on Chern classes of stable bundles, preprint. Proves two theorems about semistable bundles in characteristic 0: (1) If $E$ is semistable on a surface, then $S^r(E)$ is also semistable; (2) If $E$ is semistable on a surface, then $C(E) = 2(h^r - 1)(c_1(E))$. The latter result is proved by reduction (mod $p$).
74. O. A. LAUDAL: Formula moduli of algebraic structures, Lecture Notes in Math, Springer, Berlin (to appear). Detailed theory of a general global deformation theory, used in particular to prove the results of [38].
75. J. LE POTIER: Fibres stables de rang 2 sur $\mathbb{P}^4(C)$, preprint. Let $M(c_1, c_2)$ denote the moduli space of stable rank 2 vector bundles on $\mathbb{P}^4(C)$ with given $c_1, c_2$. Then $M(c_1, c_2)$ admits a universal family if and only if either $c_1$ is odd, or $c_2$ is even and $c_1^2 - 4c_2$ is odd. $M(0, c_2)$ is simply connected except for $c_2 = 2$, in which case $\pi_1 = \mathbb{Z}/32$. The homotopy group $\pi(M(0, c_2)) = \mathbb{Z}/32$ for $c_2 = 2$, $\mathbb{Z}/2 \times \mathbb{Z}$ for $c_2$ even $\equiv 0$; $\mathbb{Z}$ for $c_2$ odd.
76. D. MUMFORD: P. E. NEWSTEAD: Periods of a moduli space of bundles on curves, Amer. J. Math. 90 (1968), 1200-1208. Let $C$ be a curve of genus $g \geq 2$, let $x_0 \in C$ be a fixed point, and let $S$ be the space of rank 2 stable vector bundles $E$ on $C$ with det $E = \mathcal{O}(x_0)$. Then $S$ is a nonsingular projective variety, it has a universal family, and the second intermediate Jacobian variety of $S$ is the Jacobian variety of $C$.
77. P. E. NEWSTEAD: A non-existence theorem for families of stable bundles, J. London Math. Soc. (2) 6 (1973), 259-266. Non-existence of a universal family for the variety of moduli of stable vector bundles of rank $n$ and degree 0 on a curve of genus $g \geq 2$.
79. J. H. RAWNSLEY: On the Atiyah–Hitchin–Drinfeld–Manin vanishing theorem for cohomology groups of instanton bundles, preprint. Using Dolbeault cohomology groups, he gives a direct proof that $H^*(E(-2))$ is isomorphic to the kernel of $D^*D + R/6$ and so vanishes, where $E$ is an instanton bundle on $\mathbb{P}^4(C)$.
80. E. REES: Complex bundles with two sections, Proc. Camb. Phil. Soc. 71 (1972), 457-462. Studies when the conditions $c_1 = c_2 = 0$ are sufficient to knock off a trivial bundle of rank 2 from a bundle of rank $n$ on a complex manifold $X$ of dimension $n$. Application: On $\mathbb{P}^4$, there is a $C^*$-bundle with given $c_1, c_2 = 0$ such that $c_2 = 0$ (mod 12).

If \( E \) is a stable rank 2 vector bundle on \( \mathbb{P}^1 \), and \( c_1 \equiv 2c_2 + c_3 \), then \( E \) is positive.

82. M. Schneider: Holomorphic vector bundles on \( \mathbb{P}^1 \). Séminaire Bourbaki 530 (1978–79).

A survey of recent results, with an extensive bibliography.


If \( E \) is a semistable rank \( r \) bundle on \( \mathbb{P}^r \), and if for a general line \( L \), \( E|_L \cong \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_r) \) with

\[
\sum a_i = a, \quad \text{and} \quad a_i - a_{i-1} \leq 1 \quad \text{for each} \quad i = 1, \ldots, r - 1.
\]

84. S. A. Strömme: On the moduli space for stable rank two vector bundles on \( \mathbb{P}^2 \) with odd first Chern class, preprint.

Proves that \( M(-1,n) \) is irreducible and rational, by a method independent of that of Hulek.[35]

85. G. Trautmann: Darstellung von Vektorraumbündeln über \( \mathbb{C}^n - \{0\} \). *Archiv der Math.* 24 (1973), 303–313.

This does for \( \mathbb{C}^n - \{0\} \) what [59] does for \( \mathbb{P}^n \).


A report on deformation theory of vector bundles, with applications to the moduli space described in [59].


A survey of moduli of vector bundles on curves, including the Narasimhan-Ramanan theorem.

88. A. N. Tyurin: Rationality of the regular components of the variety of moduli of vector bundles of rank 2 on \( \mathbb{P}^1 \), preprint.

Claims that the variety of moduli of rank 2 bundles \( E \) on \( \mathbb{P}^1 \) with \( c_1 = 0, c_2 > 0 \) and \( H^r(E(-2)) = 0 \) is an irreducible, rational variety. Unfortunately, the proof is incorrect.


Any uniform vector bundle of rank 2 on \( \mathbb{P}^n \) is either a direct sum of line bundles or (in case \( n = 2 \)) a twist of the tangent bundle. In particular, uniform bundles of rank 2 are homogeneous.


Constructs an indecomposable rank \( n - 1 \) bundle on \( \mathbb{P}^r \) for any \( n \geq 3 \).

*University of California,*
*Berkeley, California*