A note on the fundamental theorem of algebra for the octonions

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Abstract

This paper is a revision of a portion of the author’s doctoral dissertation submitted to the University of Oregon. Using elementary concepts of $K$-theory, the Brouwer degree of the power map in the octonions is computed. Later, a proof of a weaker version of the fundamental theorem of algebra for polynomials with coefficients in the octonions is given. As a partial complement, a lower bound to the number of solutions of a homogeneous monomial equation over the octonions is obtained.

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1. Introduction

Over 60 years ago, S. Eilenberg and I. Niven [2] proved a weaker version of the fundamental theorem of algebra for the quaternions stating that for a polynomial of the form
\[ P(x) = a_0x + a_1x^2 + \cdots + a_{n-1}x^n + \text{(lower degree terms)}, \]
where the coefficients \( a_0, \ldots, a_n \) are quaternions, there exists a quaternion \( z \) such that \( P(z) = 0 \), provided that the term of degree \( n \) is unique.

Let \( \mathbb{K} = \mathbb{A}_3 \) be the algebra of octonions or Cayley numbers (for definitions and results see for example [1] or [6]). Eilenberg and Niven also point out in [2] that a polynomial map with coefficients in \( \mathbb{K} \) whose monomials are of the form \( b_0xb_1xb_2\cdots xb_k \)
is still not general enough due to the non-associativity in \( \mathbb{K} \), as the factors ought to be parenthesised in a consistent manner and hence this term might eventually involve more than \( n + 1 \) coefficients. As an example of how complicated these maps may become, a polynomial of degree 2 can have the form
\[ [(a_0(a_1x))a_2]a_3x + [b_0((xb_1x)b_2)]b_3 + c_0[cc_1c_2] + xc_3 + d. \]

In this work, only polynomials having a single term of the highest degree will be considered. The main objective of this paper is to establish the following weaker version of the fundamental theorem of algebra:

**Theorem 1.** Let \( P \) be a polynomial with coefficients in \( \mathbb{K} \) and only one term of highest degree \( n \). Then there exists an element \( z \in \mathbb{K} \) such that \( P(z) = 0 \).

Consider \( S^8 \) as the one-point compactification of \( \mathbb{K} \), that is, \( S^8 \approx \mathbb{K} \cup \{\infty\} \). Define a map \( \rho_n: S^8 \to S^8 \), given by \( \rho_n(x) = x^n \) for \( x \in \mathbb{K} \) and \( \rho_n(\infty) = \infty \). Likewise, extend \( P \) to a map \( P^+: S^8 \to S^8 \) by defining \( P^+(\infty) = \infty \). The proof of Theorem 1 will then depend upon the three following partial results. The proofs of the first two are entirely analogous to the ones provided by Eilenberg and Niven and are omitted; the proof of Lemma 4 offered in Section 3 utilises \( K \)-theoretical tools.

**Lemma 2.** The function \( P^+: S^8 \to S^8 \) defined above is a continuous map.

**Lemma 3.** The mappings \( P^+, \rho_n: S^8 \to S^8 \) are homotopic.

**Lemma 4.** The map \( \rho_n \) has degree \( n \) in the sense of Brouwer.

An interesting point worth noting is that only the proof of Lemma 2 requires that \( P \) have only one term of highest degree. It ought to be remarked that this last lemma does not necessarily mean that the equation \( x^n = a \) has exactly \( n \) solutions for all \( a \in \mathbb{K} \), but at least this number, and the actual number of solutions, if finite, differs from \( n \) by an even number; which is the case of pairs of elements in the preimage having opposite signs in their local
indices. The following result, proved in Section 4, establishes bounds for the number of solutions of some monomial equations.

**Theorem 5.** Let \( a \in \mathbb{A}_k \) be a non-real element of any Cayley–Dickson algebra. Then the equation \( x^n = a \) has exactly \( n \) solutions.

In the case where \( a \) is real, examples are given of polynomial equations for which an infinite number of solutions can be found.

### 2. Some homotopies of maps into rotation groups

Let \( T \) be a topological space and consider two maps \( A, B: T \to SO(2m) \). From these, construct a map \( T \to SO(4m) \) given by

\[
\begin{bmatrix}
AB & 0 \\
0 & I_{2m}
\end{bmatrix},
\]

where \( I_{2m} \) is the \( 2m \times 2m \) identity matrix and the juxtaposition \( AB \) signifies the map \( T \to SO(2m) \) taking the element \( x \in T \) to the product matrix \( A(x)B(x) \). Now perform the following elementary operations, where the symbol \( (C_i)(X) \mapsto C_j \) denotes the elementary operation of right multiplying the \( i \)th column by the matrix \( X \) and adding the product to the \( j \)th column, etc:

\[
\begin{bmatrix}
AB & 0 \\
0 & I_{2m}
\end{bmatrix}
\xrightarrow{(C_2)(B)+C_1 \mapsto C_1}
\begin{bmatrix}
AB & 0 \\
B & I_{2m}
\end{bmatrix}
\xrightarrow{(-A)(R_2)+R_1 \mapsto R_1}
\begin{bmatrix}
0 & -A \\
B & I_{2m}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -A \\
B & I_{2m}
\end{bmatrix}
\xrightarrow{1-A \mapsto I_{2m}}
\begin{bmatrix}
A & 0 \\
I_{2m} & B
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -A \\
B & I_{2m}
\end{bmatrix}
\xrightarrow{(-B)^{T}+C_1 \mapsto C_1}
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}.
\]

In other words, the matrix in the original map is similar to the last given block matrix. Moreover, this process actually provides a guideline to construct the following homotopies \( T \times [0,1] \to SO(4m) \):

\[
H_1(\cdot,t) = \begin{bmatrix}
AB & 0 \\
tB & I_{2m}
\end{bmatrix},
\]

\[
H_2(\cdot,t) = \begin{bmatrix}
(1-t)AB & -tA \\
B & I_{2m}
\end{bmatrix},
\]

\[
H_3(\cdot,t) = \begin{bmatrix}
0 & -A \\
B & I_{2m}
\end{bmatrix}
\begin{bmatrix}
\cos\left(\frac{\pi}{2}t\right)I_{2m} & \sin\left(\frac{\pi}{2}t\right)I_{2m} \\
-\sin\left(\frac{\pi}{2}t\right)I_{2m} & \cos\left(\frac{\pi}{2}t\right)I_{2m}
\end{bmatrix},
\]

\[
H_4(\cdot,t) = \begin{bmatrix}
A \\
R_m(1-t)B
\end{bmatrix},
\]

\[
H_5(\cdot,t) = \begin{bmatrix}
A \\
(1-t)I_{2m}B
\end{bmatrix},
\]
where \( R_m(t) \in SO(2m) \) is the block–diagonal matrix comprised by the blocks

\[
\begin{bmatrix}
\cos(\pi t) & \sin(\pi t) \\
-\sin(\pi t) & \cos(\pi t)
\end{bmatrix}.
\]

The homotopies \( H_1, \ldots, H_5 \) are well defined since \( \det(-X) = 1 \) for all \( X \in SO(2m) \) and since the determinant of the matrix of size \( 2m \times 2m \) whose blocks are the matrices \( W, X, Y, Z \) of size \( m \times m \), is computed as

\[
\det\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{cases} \\
\det(WZ - XY) & \text{if } YZ = ZY, \\
\det(WZ - YX) & \text{if } WY = YW, \\
\det(ZW - YX) & \text{if } WX = XW, \\
\det(ZW - XY) & \text{if } XZ = ZX. \\
\end{cases}
\]

For instance, since \( I_{2m} \) commutes with any matrix, then

\[
\det\begin{bmatrix} 0 & -A \\ B & I_{2m} \end{bmatrix} = \det(0I_{2m} + AB) = 1
\]

and

\[
\det\begin{bmatrix} \cos(\frac{\pi}{2} t)I_{2m} & \sin(\frac{\pi}{2} t)I_{2m} \\ -\sin(\frac{\pi}{2} t)I_{2m} & \cos(\frac{\pi}{2} t)I_{2m} \end{bmatrix} = \det(\cos^2(\frac{\pi}{2} t)I_{2m} + \sin^2(\frac{\pi}{2} t)I_{2m}) = 1,
\]

whence \( H_3(\cdot, t) = 1 \). Applying successively the homotopies \( H_1, \ldots, H_5 \), it is obtained that the maps \( T \to SO(4m) \) given, respectively, by

\[
\begin{bmatrix} AB & 0 \\ 0 & I_{2m} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}
\]

are homotopic. As a particular case, the maps \( T \to SO(4m) \) given by

\[
\begin{bmatrix} A^k & 0 \\ 0 & I_{2m} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & A^{k-1} \end{bmatrix}
\]

are homotopic for any positive integer \( k \). Hence, by further applying this homotopy in every \( 4m \times 4m \) block along the diagonal, one concludes that the maps \( T \to SO(n(2m)) \) given by

\[
\begin{bmatrix} A^n & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & I_{2m} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & A \end{bmatrix}
\]

are homotopic, where the undesignated blocks are equal to zero.

### 3. Proofs of Lemma 4 and Theorem 1

Recall the Hopf fibration \( \sigma: S^{15} \to S^8 \), given by the Hopf construction of the multiplication of octonions:

\[
\sigma(x, y) = (|x|^2 - |y|^2, 2xy),
\]
where $x, y \in \mathbb{K}$ and $|x|^2 + |y|^2 = 1$. This fibration is known to have fibres diffeomorphic to $S^7$ [5,7], that is, given by the intersection of $S^{15}$ with an eight-dimensional vector subspace of $\mathbb{K} \times \mathbb{K} \cong \mathbb{R}^{16}$. As such, these fibres constitute the sphere bundle of an 8-plane vector bundle $\zeta$ over $S^8$. Now consider the power map $\rho_n: S^8 \to S^8$ defined in Section 1 and construct the pullback diagram

$$
\begin{array}{ccc}
(\rho_n)_{\zeta} & \rightarrow & \zeta \\
\downarrow & & \downarrow \\
S^8 & \rightarrow & S^8 \\
\end{array}
$$

Notice that the clutching function for $\zeta$ is precisely the left adjoint map of the octonion multiplication

$$S^7 \to SO(8),$$

$$a \mapsto A,$$

with $A(x) = ax$, and also that this vector bundle is not zero in $\widetilde{KO}(S^8)$ (see for example [3, p. 129]). Since the map $\rho_n$ is of course the power map $x \mapsto x^n$ when restricted to the set of unit octonions $S^7$, by the general theory of vector bundles and the alternativity property $a(ab) = a^2b$ of the octonions [1], it follows that the clutching function for $(\rho_n)_{\zeta}$ is the map

$$\beta: S^7 \to SO(8),$$

$$a \mapsto A^n.$$

Now, in $\widetilde{KO}(S^8) \cong \widetilde{KO}^{-8}(pt) \cong \widetilde{KO}(pt) \cong \mathbb{Z}$, the vector bundles $(\rho_n)_{\zeta}$ and $(\rho_n)_{\zeta} \oplus (n - 1)e^8$ are of course stably equivalent where $e^8$ is a trivial 8-plane bundle. The clutching function of this last bundle is the map $S^7 \to SO(8n)$ given by

$$a \mapsto \begin{bmatrix} A^n & I_8 & \cdots & I_8 \\ \cdots & \cdots & \cdots & \cdots \\ I_8 & \cdots & \cdots & I_8 \end{bmatrix},$$

which, by the discussion from Section 2, is homotopic to the map

$$a \mapsto \begin{bmatrix} A & A & \cdots & A \\ \cdots & \cdots & \cdots & \cdots \\ A & \cdots & \cdots & A \end{bmatrix},$$

but this last is clearly the clutching function of the bundle $n\zeta$. This proves that, in $\widetilde{KO}(S^8)$, $(\rho_n)_{\zeta} = n\zeta$ and concludes the proof of Lemma 4.

Now suppose that $P$ does not have any roots. Hence $P: \mathbb{K} \to \mathbb{K}\setminus\{0\}$. Consider then the diagram

$$
\begin{array}{ccc}
S^7 & \stackrel{\rho_n}{\rightarrow} & \mathbb{K}\setminus\{0\} \\
\downarrow & & \downarrow \rho \\
\mathbb{K} & \to & \mathbb{K}\setminus\{0\} \\
\end{array}
$$
which by Lemma 3 is commutative up to homotopy and thus exhibits $\rho_n$ as homotopically trivial since $K$ is contractible. This contradicts Lemma 4 and shows that the assumption that $P$ has no roots was erroneous, thus establishing Theorem 1.

4. Roots of elements in Cayley–Dickson algebras

The proof of Theorem 5 now follows. For $n$ a positive integer, consider the equation

$$x^n = a,$$  \hspace{1cm} (1)

where $a \in A_k$ is fixed. This section contains a discussion on the number of possible solutions of (1).

Assume that $a$ is not real. Notice first that no real element $x \in A_k$ could possibly be a solution of (1). Next, by means of the decomposition $A_k = 1 \oplus 1^\perp$, express $x = x_1 + x_2 u$ where $x_1, x_2 \in \mathbb{R}$ and $u \in \{x \in A_k | \langle x, 1 \rangle = 0 \text{ and } |x|^2 = 1 \} \approx S^{2k-2}$, that is, a unit length purely imaginary element of $A_k$. Notice that $u^2 = u(-u) = -|u|^2 = -1$ for any $u \in S^{2k-2}$ and hence, in an entirely analogous manner as in $\mathbb{C}$, it follows that $x^n = R + Su$ where the real coefficients $R$ and $S$ can be obtained via binomial coefficients. Expressing the non–real octonion $a = a_1 + a_2 v$ similarly as above and with $v \neq 0$ yields the equations:

$$R = a_1, \quad S = a_2 \quad \text{and} \quad u = v.$$

The third equation allows one to work in the subalgebra of $A_k$ generated by 1 and $u$, which is isomorphic to the complex numbers [4] and thus has exactly $n$ solutions due to de Moivre’s formula. Hence, Eq. (1) has exactly $n$ solutions if $a$ is not real.

On the other hand, if $a$ is real then one can find a distinct solution set to Eq. (1) containing $n$ elements for every choice of an element $u \in S^{2k-2}$ simply by working on the subalgebra generated by 1 and $u$. All such solution sets will contain the real solutions of (1), but the non-real ones will be distinct, except in the case of antipodal elements, because the subalgebra of $A_k$ generated by 1 and $u$ and that generated by 1 and $-u$ coincide. This means that Eq. (1) has at least as many non-real solutions as there are points in $S^{2k-2} / \mathbb{Z}_2 \approx \mathbb{R}P^{2k-2}$ if $a$ is real.

The case of nonhomogeneous equations goes beyond the scope of this work. However, following the ideas in the previous paragraph, it can be remarked that the cyclotomic equation $x^2 + x + 1 = 0$ has the solutions $-\frac{\sqrt{3}}{2} + \frac{1}{2} u$ for any $u \in S^{2k-2}$.

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