On cyclic branched coverings of hyperbolic links

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Abstract

Two of the main methods for the construction of closed orientable 3-manifolds, and in particular of hyperbolic 3-manifolds, are surgery on links and branched coverings of links. If the link is hyperbolic, i.e., has hyperbolic complement, by results of Thurston most of the resulting 3-manifolds are hyperbolic. In the present paper, for a fixed integer \( n > 2 \), we consider hyperbolic 3-manifolds \( M_{n,k} \) which are cyclic \( n \)-fold branched coverings of a hyperbolic link with two components. Our main theorem relates the classification up to isometry or homeomorphism of these manifolds to the symmetry group of the link and allows a complete classification of these manifolds in various cases; as an example, we consider cyclic branched coverings of the Whitehead link. The classification resembles the classification of lens spaces which are the cyclic branched coverings of the Hopf link (which is not hyperbolic, however); it generalizes to links with more than two components. For the proof, which consists in a mixture of geometric and algebraic arguments, we extend methods used in [5] in a special case.

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1. Introduction

Let \( L = K_1 \cup K_2 \) be an oriented link in the 3-sphere \( S^3 \) with two components. Denote by \( O_n(L) \) the 3-orbifold whose underlying topological space is the 3-sphere and whose singular set, of branching index \( n \), is the link \( L \) (see [6] for the theory of orbifolds). Let

\[
\pi_1 O_n(L) = \pi_1(S^3 - L) / \langle m_1^n, m_2^n \rangle
\]

denote the orbifold fundamental group of \( O_n(L) \) where \( m_1 \) and \( m_2 \) are oriented meridians of \( K_1 \) and \( K_2 \), respectively. The abelianized group \( (\pi_1 O_n(L))_{ab} \) is isomorphic to \( \mathbb{Z}_n \times \mathbb{Z}_n \).
generated by the images of the two meridians $m_1$ and $m_2$. For an integer $k \neq 0$ with $(k, n) = 1$, consider the map

$$\psi_{n,k} : \pi_1 O_n(L) \longrightarrow (\pi_1 O_n(L))_{ab} \longrightarrow \mathbb{Z}_n$$

defined by $\psi_{n,k}(m_1) = \bar{1}$, $\psi_{n,k}(m_2) = \bar{k}$. Denote by $M_{n,k}$ the closed orientable 3-manifold which is the cyclic branched covering (or covering in the sense of orbifolds) of $O_n(L)$ corresponding to the kernel of $\psi_{n,k}$, with $\pi_1 M_{n,k} \cong \text{kernel } \psi_{n,k}$. The preimage of $K_i$ is a knot $\tilde{K}_i$ in $M_{n,k}$, $i = 1, 2$, and the complement of $L = \tilde{K}_1 \cup \tilde{K}_2$ in $M_{n,k}$ is the regular (unbranched) covering of the complement $S^3 - L$ of $L$ corresponding to the kernel of the composition (which we denote by the same letter)

$$\psi_{n,k} : \pi_1 (S^3 - L) \longrightarrow \pi_1 O_n(L) \longrightarrow \mathbb{Z}_n.$$ 

Also, $\tilde{L}$ is the fixed point set of the cyclic group of covering transformations acting on $M_{n,k}$, with quotient $O_n(L)$.

In a geometric way the manifolds $M_{n,k}$ can be constructed as follows. Choose a Seifert surface for each component of the link (the Seifert surface for one component ignores the other component – the two surfaces will intersect transversely). The manifold $M_{n,k}$ can then be constructed explicitly by taking $n$ copies of the complement of the union of the Seifert surfaces, numbering them $0, 1, \ldots, n - 1$, and gluing them together according to the following rule. When you pass through the first Seifert surface in the “positive” direction, you pass from copy $i$ to copy $i + 1$ (mod $n$); when you pass through the second Seifert surface in the “positive” direction, you pass from copy $i$ to copy $i + k$ (mod $n$).

Now suppose $L$ is a hyperbolic link, that is a link with hyperbolic complement $S^3 - L$. By Thurston’s hyperbolic surgery theorem (see [6,1] for the manifold version which generalizes to orbifolds), $O_n(L)$ is a hyperbolic 3-orbifold for large $n$. In fact, by Thurston’s orbifold geometrization theorem [7] and Dunbar’s list of geometric nonhyperbolic 3-orbifolds (with underlying topological space the 3-sphere, see [2]), $n \geq 4$ will suffice (noting that for $n \geq 3$, $O_n(L)$ contains no incompressible euclidean 2-suborbifolds). Also, for $n$ sufficiently large, the curves $K_1$ and $K_2$ are the two unique geodesics of shortest length in the hyperbolic 3-orbifold $O_n(L)$. In general, for an explicitly given link $L$ the hyperbolicity of $L$ and also of the orbifolds $O_n(L)$ can be checked by direct and more elementary methods, for example by computer using Weeks’ SnapPea program (see, e.g., [4]).

We shall assume in the following that $O_n(L)$ is a hyperbolic 3-orbifold. Then the manifolds $M_{n,k}$ are closed hyperbolic 3-manifolds, of the same volume for fixed $n$, and the covering transformations are isometries. The isometry groups of $O_n(L)$ and $M_{n,k}$ are finite groups (see [6]). The isometry group of $O_n(L)$ injects into the symmetry group $\pi_0(\text{Diff}(S^3, L))$ of the link $L$; in fact, as a consequence of Mostow’s rigidity theorem, this injection is an isomorphism.

Our main result is the following
Theorem 1. Let \( L = K_1 \cup K_2 \) be a hyperbolic link with two components. Let \( n \geq 2 \) be such that the 3-orbifold \( \mathcal{O}_n(L) \) is also hyperbolic, and let \( k, k' \neq 0 \) be integers with \( (k, n) = (k', n) = 1 \).

(a) Suppose there exists a prime \( p \) which divides \( n \) but does not divide the order of the symmetry group of the link \( L \); alternatively, suppose that \( n \) is sufficiently large such that \( K_1 \) and \( K_2 \) are the two unique shortest geodesics of \( \mathcal{O}_n(L) \). Then, if the hyperbolic 3-manifolds \( M_{n,k} \) and \( M_{n,k'} \) are isometric (or equivalently, homeomorphic), one of the following conditions holds.

(i) \( k \equiv k' \pmod{n} \);

(ii) \( k \equiv -k' \pmod{n} \) and there exists an isometry of \( \mathcal{O}_n(L) \) fixing \( K_1 \) and \( K_2 \) and reversing the orientation of exactly one of \( K_1 \) and \( K_2 \);

(iii) \( kk' \equiv 1 \pmod{n} \) and there exists an isometry of \( \mathcal{O}_n(L) \) exchanging \( K_1 \) and \( K_2 \) whose square preserves orientations of both \( K_1 \) and \( K_2 \);

(iv) \( kk' \equiv -1 \pmod{n} \) and there exists an isometry of \( \mathcal{O}_n(L) \) exchanging \( K_1 \) and \( K_2 \) whose square reverses orientations of both \( K_1 \) and \( K_2 \).

(b) Conversely, if one of the conditions (i)–(iv) holds then the hyperbolic 3-manifolds \( M_{n,k} \) and \( M_{n,k'} \) are isometric.

The proof of Theorem 1 uses and extends methods from [5] where a class of hyperbolic 3-manifolds with totally geodesic boundary is considered which are cyclic branched coverings of a link in the 3-ball.

The number theoretical conditions \( k \equiv \pm k' \pmod{n} \) and \( kk' \equiv \pm 1 \pmod{n} \) in the theorem correspond exactly to the conditions in the classification of lens spaces, which are the cyclic branched coverings of the Hopf link (which is not hyperbolic). It is a natural question if also in our situation the stronger assertion holds that the manifolds \( M_{n,k} \) and \( M_{n,k'} \) are homeomorphic if and only if one of the conditions (i)–(iv) holds, that is if the extra condition on the prime number \( p \) or the two shortest geodesics is really necessary. At moment we do not know the answer, but we suspect that for some link and small value of \( n \), there exist cases of homeomorphisms not given by the above number theoretical conditions. To construct such an example, one has to find a hyperbolic 3-manifold \( M \) with two cyclic groups of isometries of order \( n \) which are not conjugate by an isometry of \( M \), but whose quotient orbifolds are the 3-sphere with the same link \( L \) as singular or branch set (this is a consequence of the proof of Theorem 1).

In the last section, as an example we shall classify the cyclic branched coverings of the Whitehead link, thus solving a problem which remained open in a paper by Helling, Kim, and Mennicke [3], where an explicit construction of these hyperbolic manifolds is given using Poincaré's theorem on fundamental polyhedra. The symmetry group of the Whitehead link is the dihedral group of order 8, so the only values of \( n \) where Theorem 1 does not apply directly are the powers of two; by a refinement of the proof of Theorem 1 one gets a solution also for these values. The symmetry groups of the hyperbolic 2-component links up to 9 crossings (with the exceptions of the links 9_24 and 9_40), together with the behavior of each symmetry on the components of the link, are given in [4];
these listed groups have only elements of order 2 and 4 (the two excluded links have also symmetries of order 3).

The proof of Theorem 1 gives also the following

**Theorem 2.** Let \( L \) and \( L' \) be hyperbolic links. Given \( n \geq 2 \), we denote by \( M \), respectively \( M' \), hyperbolic 3-manifolds which are \( n \)-fold cyclic branched coverings as above of the hyperbolic 3-orbifolds \( O_n(L) \) respectively, \( O_n(L') \). Suppose there exists a prime \( p \) which divides \( n \) but does not divide the order of the symmetry group of either \( L \) or \( L' \); alternatively, suppose that \( n \) is sufficiently large such that \( L \) respectively \( L' \) are systems of shortest geodesics of \( O_n(L) \), respectively \( O_n(L') \). Then if \( M \) and \( M' \) are isometric (or homeomorphic) the links \( L \) and \( L' \) are equivalent.

2. Proof of Theorem 1

To simplify notations, for fixed \( n \) and \( k \) set \( O := O_n(L) \), \( M := M_{n,k} \) and \( \psi := \psi_{n,k} \); by \( H = H_{n,k} \cong \mathbb{Z}_n \) we denote the finite cyclic group of covering transformations of the cyclic branched covering \( M \) of \( O \). The fixed point set of each nontrivial element of \( H \) is the preimage \( \tilde{L} = \tilde{K}_1 \cup \tilde{K}_2 \) of \( L \) in \( M \). If we denote by \( h = h_{n,k} \) a generator of \( H \) acting as a rotation of minimal angle \( 2\pi/n \) around \( \tilde{K}_1 \) (say in clockwise direction) then, by the definition of \( \psi \), the transformation \( h^k \) is the rotation of minimal angle around \( \tilde{K}_2 \) (where, under the isomorphism \( H \cong \mathbb{Z}_n = \text{image} \psi \), the element \( h \in H \) corresponds to the element \( 1 \in \mathbb{Z}_n \); replacing \( \mathbb{Z}_n \) by \( H \), we have \( \psi(m_1) = h, \ \psi(m_2) = h^k \)).

Now, in order to prove the first direction of the theorem, suppose that \( M = M_{n,k} \) and \( M' := M_{n,k'} \) are isometric. For \( M' \), we have the corresponding surjection

\[
\psi' : \pi_1 O \to \mathbb{Z}_n \cong H' = \langle h' \rangle,
\]

with \( \psi'(m_1) = h', \ \psi'(m_2) = (h')^{k'} \) and kernel \( \psi' \cong \pi_1 M' \).

Let \( \delta : M \to M' \) be an isometry. Crucial for the proof is the following lemma whose proof we postpone to the end of this section.

**Lemma 1.** The isometry \( \delta : M \to M' \) can be chosen such that \( \delta H \delta^{-1} = H \).

The lemma implies that \( \delta \) projects to an isometry \( \gamma \) of the orbifold \( O \) (in particular mapping its singular set, the link \( L \), to itself). We have a commutative diagram (up to conjugation)

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1 M & \rightarrow & \pi_1 O & \rightarrow & H = \langle h \rangle & \rightarrow & 1 \\
\downarrow \delta_* & & \downarrow \gamma_* & & \downarrow \beta & & \\
1 & \rightarrow & \pi_1 M' & \rightarrow & \pi_1 O & \rightarrow & H' = \langle h' \rangle & \rightarrow & 1
\end{array}
\]

where \( \delta_* \) and \( \gamma_* \) are the isomorphisms induced on fundamental groups and \( \beta \) is induced by conjugation with \( \delta \) or by \( \gamma_* \) from the diagram (equivalently, one may use the analogous diagram for the corresponding unbranched coverings of \( S^3 - L \) thus avoiding
the formalism of orbifold fundamental groups and their induced maps). Now, in analogy to the four cases of the theorem, up to conjugation in \( \pi_1 \mathcal{O} \) there are the following possibilities, with \( \varepsilon = \pm 1 \).

(i) \( \gamma_\ast(m_1) = m_1^\varepsilon \), \( \gamma_\ast(m_2) = m_2^\varepsilon \).

(ii) \( \gamma_\ast(m_1) = m_1^\ast \), \( \gamma_\ast(m_2) = m_2^{-\ast} \).

(iii) \( \gamma_\ast(m_1) = m_2^\varepsilon \), \( \gamma_\ast(m_2) = m_1^\varepsilon \).

(iv) \( \gamma_\ast(m_1) = m_2^\varepsilon \), \( \gamma_\ast(m_2) = m_1^{-\varepsilon} \).

Suppose we are in case (iv). Then
\[
\beta(h) = \beta(\psi(m_1)) = \psi'(\gamma_\ast(m_1)) = \psi'(m_2^\varepsilon) = (h')^{\varepsilon k'},
\]
\[
\beta(h^k) = \beta(\psi(m_2)) = \psi'(\gamma_\ast(m_2)) = \psi'(m_1^{-\varepsilon}) = (h')^{-\varepsilon}.
\]
It follows \((h')^{\varepsilon k'} = (h')^{-\varepsilon}\), therefore \(kk' \equiv -1 \text{ (mod } n)\) and we are in case (iv) of the theorem. The other cases are proved in a similar way. This proves part (a) of the theorem.

In order to prove part (b), suppose we are in one of the cases (ii), (iii) or (iv) of the theorem; in particular, we have an isometry \( \gamma \) of \( \mathcal{O} \) with the specified properties. Define \( \beta : H \to H' \) by \( \beta(h) := \psi'(\gamma_\ast(m_1)) \). Then, by the numerical conditions in the theorem, \( \beta \circ \psi = \psi' \circ \gamma_\ast \) and we have a commutative diagram as above. It follows that \( \gamma \) can be lifted to an isometry \( \delta : M \to M' \). This finishes the proof of Theorem 1. It remains the

**Proof of Lemma 1.** Suppose first that \( K_1 \) and \( K_2 \) are the two unique shortest geodesics of the hyperbolic orbifold \( \mathcal{O} \). Then their preimages \( \tilde{K}_1 \) and \( \tilde{K}_2 \), respectively \( \tilde{K}_1' \) and \( \tilde{K}_2' \), are the two unique shortest geodesics in \( M \), respectively \( M' \); it follows \( \delta(\tilde{L}) = \tilde{L}' \) which implies \( \delta H \delta^{-1} = H' \) because \( H \) and \( H' \) are rotation groups of the same order around their respective axes \( \tilde{L} \) and \( \tilde{L}' \).

Now suppose that there exists a prime \( p \) which divides \( n \) but does not divide the order of the symmetry group of \( L \) (or equivalently, of the isometry group of \( \mathcal{O} \)). Suppose, in contradiction to the lemma, that \( H \) and \( \tilde{H} := \delta H \delta^{-1} \) are not conjugate in the isometry group \( \text{Iso}(M) \) of \( M \); then no nontrivial subgroup of \( H \) is conjugate to a subgroup of \( \tilde{H} \) (otherwise a conjugating isometry would map the fixed point set of \( H \) to that of \( \tilde{H} \) and therefore give a conjugation also of these groups). In particular, the \( p \)-Sylow subgroups \( H_p \), respectively \( \tilde{H}_p \), of \( H \), respectively \( \tilde{H} \), are not conjugate. This implies that \( H_p \) (and also \( \tilde{H}_p \)) is not a \( p \)-Sylow subgroup of \( \text{Iso}(M) \) (by a Sylow theorem any two \( p \)-Sylow subgroups are conjugate). Again by the Sylow theorem, \( H_p \) is a subgroup of some \( p \)-Sylow subgroup \( \Sigma_p \) of \( \text{Iso}(M) \), in fact a proper subgroup by the above. We need the following

**Lemma 2.** Let \( \Sigma_p \) be a finite \( p \)-group and \( H_p \) be a proper subgroup of \( \Sigma_p \). Then the normalizer of \( H_p \) in \( \Sigma_p \) contains \( H_p \) as a proper subgroup.

We omit the proof of Lemma 2 which is an easy consequence of the fact that the center of a finite \( p \)-group is nontrivial (if the center of \( \Sigma_p \) is not contained in \( H_p \) the lemma is clear, otherwise one divides out the center and applies an induction hypothesis to the factor groups of lower orders).
Applied to the above situation Lemma 2 says that there exists an element in the normalizer of $H_p$ in $\Sigma_p$ which is not in $H_p$. This element normalizes also $H$ (because it maps the fixed point set of $H_p$ to itself which coincides with that of $H$) and therefore projects to an isometry of $O$ whose order is some nontrivial power of $p$. This contradicts the choice of $p$ and finishes the proof of Lemma 1. □

3. Cyclic branched coverings of the Whitehead link

Now let $L$ be the Whitehead link, see Fig. 1. By [6], $L$ is hyperbolic, by [3] the 3-orbifolds $O_n(L)$ are hyperbolic for $n \geq 3$. As above, we denote by $M_{n,k}$ the $n$-fold branched covering of $O_n(L)$ corresponding to the kernel of the surjection $\psi_{n,k} : \pi_1 O_n(L) \rightarrow \mathbb{Z}_n$ defined as in the introduction, $(n,k) = 1$. For $n \geq 3$, the $M_{n,k}$'s are closed hyperbolic 3-manifolds which have been explicitly constructed in [3] using Poincaré's Theorem on fundamental polyhedra.

The symmetry group of the Whitehead link $L$ is isomorphic to the dihedral group $D_4$ of order 8, see [4], so Theorem 1 applies for each $n \geq 3$ which is not a power of 2. By a slightly more careful analysis of the situation we get the following

**Theorem 3.** Let $n \geq 2$ and $k,k' \neq 0$, with $(n,k) = (n,k') = 1$. The cyclic branched coverings $M := M_{n,k}$ and $M' := M_{n,k'}$ of the Whitehead link are isometric (or homeomorphic) if and only if $k \equiv \pm k' \pmod{n}$ or $kk' \equiv \pm 1 \pmod{n}$.

**Proof.** Suppose first that one of the numerical conditions is satisfied. This direction of the theorem has also been proved in [3]. It follows from Theorem 1 noting that all types (ii)–(iv) of isometries of $O_n(L)$ in Theorem 1 really occur, see [4] or [3].

Now, for the other direction, suppose that $\delta : M \rightarrow M'$ is an isometry. By Theorem 1, we can assume that $n = 2^m$ is a power of two. If $\delta$ projects to an isometry of $O_n(L)$, the proof of Theorem 1 applies. Therefore we can assume that $H$ and $\tilde{H} = \delta H \delta^{-1}$ are not conjugate in the isometry group $\text{Iso}(M)$ of $M$ (using the same notations as in the proof of Theorem 1); as in the proof of Lemma 1 this implies $H \cap \tilde{H} = 1$. We can also assume that $H$ and a conjugate of $\tilde{H}$ (which we denote again by $\tilde{H}$) are proper subgroups of the

![Fig. 1. Whitehead link $5^2_1$.](image-url)
same 2-Sylow subgroup $\Sigma_2$ of $\text{Iso}(M)$. Let $\alpha$ be an element in the normalizer $N(H)$ of $H$ in $\Sigma_2$. Then $\alpha$ projects to an isometry $\gamma$ of $\mathcal{O}_n(L)$ and we have a commutative diagram

$$
\begin{array}{ccc}
1 & \rightarrow & \pi_1 M \\ & \downarrow & \downarrow \alpha' \\
& \rightarrow & \pi_1 \mathcal{O}_n(L) & \rightarrow & H & \rightarrow & 1
\end{array}
$$

Now the proof of Theorem 1, with $k' = k$, implies $k \equiv \pm k \pmod{n}$ or $kk = k^2 \equiv \pm 1 \pmod{n}$. The cases $k \equiv -k \pmod{n}$ and $k^2 \equiv -1 \pmod{n}$ cannot occur for algebraic reasons therefore we are left with the possibilities $\gamma_*(m_1) = m_1^\pm$, $\gamma_*(m_2) = m_2^\pm$ (case (i) in the proof of Theorem 1) and $\gamma_*(m_1) = m_1^\pm$, $\gamma_*(m_2) = m_2^\pm$ (if $k^2 \equiv 1 \pmod{n}$: case (iii)). By [4] or [3] the subgroup of isometries of $\mathcal{O}_n(L)$ satisfying one of these two properties is the dihedral group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of order 4; in particular $H = \langle h \rangle$ has index 2 or 4 in its normalizer $N(H)$ in $\Sigma_2$, with factor group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $y$ be an element in the normalizer of $N(H)$ in $\Sigma_2$. Then $yhy^{-1} = hz$, for some $z \in N(H)$, and $yh^2y^{-1} = (hz)^2 = h(wh^{-1})z^2 \in H$. Therefore $y$ normalizes a nontrivial subgroup of $H$ and then also the whole group $H$, that is $y \in N(H)$. Now Lemma 2, applied to the subgroup $N(H)$ of $\Sigma_2$, implies $N(H) = \Sigma_2$, therefore $H$ (and also $\tilde{H}$) has index 2 or 4 in $\Sigma_2$. But then $H \cap \tilde{H} \neq 1$ which is a contradiction (note that we can assume $n = 2^m \geq 8$ because for $n = 2$ and $n = 4$ there is nothing to prove). This finishes the proof of Theorem 3. \(\square\)

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Note added in proof

(a) In the proof of Theorem 1, using instead of Lemma 2 the stronger assertion 1.5 on p. 88 of the book of Suzuki (Group Theory 1, Springer 1982), it follows that in Theorems 1 and 2 in the condition on the divisibility one can replace the prime number $p$ by the highest power of $p$ dividing $n$. Therefore Theorems 1 and 2 remain valid under the hypothesis that $n$ does not divide the order of the (finite) isometry or symmetry group of the hyperbolic link $L$ respectively of $L_1$ or $L_2$. Note also that Theorem 3 is an immediate consequence of this stronger version of Theorem 1 (the classification in the only left case $n = 8$ follows from 1.5 in Suzuki’s book because the isometry group of the Whitehead link has no element of order 8).

(b) Theorem 3 on the classification of the cyclic branched coverings of the Whitehead link has been proved independently in a recent preprint of D.A. Derevnin, On distinguishing of cyclic coverings of the Whitehead link (Preprint Sonderforschungsbereich 343, Universität Bielefeld 1995), by different computational methods.
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