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Transport equations and quasi-invariant flows on the Wiener space

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Abstract

We shall investigate on vector fields of low regularity on the Wiener space, with divergence having low exponential integrability. We prove that the vector field generates a flow of quasi-invariant measurable maps with density belonging to the space $\mathbf{L}\log\mathbf{L}$. An explicit expression for the density is also given.

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1. Introduction

It is well known that for a bounded smooth vector field Z on $\mathbb{R}^d$, there exists a unique flow of diffeomorphisms $U_t : \mathbb{R}^d \to \mathbb{R}^d$ generated by Z, and for any $u_0 \in C^1(\mathbb{R}^d)$, the function $u_t(x) := u_0(U_t(x))$ solves uniquely the transport equation

$$\frac{\partial u}{\partial t} - Z \cdot \nabla u = 0, \quad t > 0, \quad u|_{t=0} = u_0. \tag{1.1}$$

Conversely, let u^i be the solution of the transport equation (1.1) with $u^i|_{t=0} = x^i$, i.e. the ith coordinate of x $(1 \leqslant i \leqslant d)$, then $U_t := (u_t^1, \dots, u_t^d)$ is the flow associated to the vector field Z.

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Eq. (1.1) is a first-order linear PDE and allows to avoid to estimate the Jacobian of Z when a smoothing procedure is applied. Based on these facts, DiPerna and Lions [10] established a theory which enables them to get a flow of measurable maps associated to the vector field with only Sobolev regularity, and it was shown that if the divergence of Z is bounded, then the flow leaves the Lebesgue measure quasi-invariant. Cipriano and Cruzeiro [7] took the standard Gaussian measure γ on $\mathbb{R}^d$ as the reference measure, and they obtained similar results under the hypotheses that the divergence $\mathrm{div}_\gamma(Z)$ of Z relative to γ is exponentially integrable. On the other hand, L. Ambrosio [2,3] took advantage the use of continuity equations

$$\frac{\partial \mu_t}{\partial t} + D_x \cdot (Z\mu_t) = 0, \quad t > 0, \quad \mu|_{t=0} = \mu_0, \tag{1.2}$$

which allowed him to construct quasi-invariant flows associated to vector fields Z with only BV regularity. In this spirit, Ambrosio and Figalli [4] proved recently the existence and uniqueness of the so-called L^r-regular flow associated to vector fields Z on abstract Wiener spaces, mainly under the low Sobolev regularity of Z and the integrability of $e^{\lambda\,\mathrm{div}_\mu(Z)}$ for large enough $\lambda > 0$.

In recent years, there are also intensive studies [5,14,16] on the existence and uniqueness of flows associated to vector fields in infinite dimensional space, i.e. the Wiener space (W, H, μ). In a pionner work, Cruzeiro [8] obtained a flow of quasi-invariant measurable maps U_t generated by smooth vector fields Z on the Wiener space $X = C_0([0,1], \mathbb{R}^d)$, provided that $\mathrm{div}_\mu(Z)$ belongs to the exponential class. Using the same method, this result was generalized to smooth tangent processes ξ on X in [6], and some conditions were weakened by Gong and Zhang [12]. In this work, we follow the method of [10] and start from a finite dimensional result: Theorem 2.2 in [7]. To prove the uniqueness of the transport equation (1.1) related to the vector field Z on W, as in [4], the key step is the estimation of the commutator:

$$B_\varepsilon(v, Z) = \langle Z, \nabla P_\varepsilon v\rangle - P_\varepsilon\big(\langle \nabla v, Z\rangle\big),$$

where v is a function and P_ε is the Ornstein–Uhlenbeck semigroup on the Wiener space W. The main difference with respect to [4] is that we assume that the divergence $\mathrm{div}_\mu(Z)$ satisfies

$$\int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu < +\infty \quad \text{for a small } \lambda_0 > 0, \tag{1.3}$$

instead of a large enough λ_0. In our case, the L^p estimate for the density K_t holds for a small time t, but fails to be true for a large time; instead we will prove that K_t belongs to $\mathbf{L}\log\mathbf{L}$ for all t.

The paper is organized as follows. In Section 2, we recall some elements in Malliavin calculus and put forward the connection between the commutator estimate in [4] and the geometric analysis on the Wiener space. In Section 3, we deal with transport equations on the Wiener space in full generality for its own interest. Our main contribution is the Section 4: under the hypothesis (1.3), we construct first a family of functions, seen as the coordinate functions under Haar basis, via transport equations; the flow U_t will be defined by Random series, similar to Lévy's construction for the Brownian motion.

2. Malliavin calculus and commutator estimate

Let (W, H, μ) be an abstract Wiener space, i.e. W is a separable Banach space, H is a separable Hilbert space and μ is a Borel probability on W, such that H is continuously and densely embedded into W and for any $\ell \in W^*$ (dual space of W), we have

$$\int_W e^{\sqrt{-1}\ell(w)}\,d\mu(w) = e^{-|\ell|_H^2/2},$$

where $|\cdot|_H$ is the norm in H. In the following, we fix an orthonormal basis $\{h_i\colon i \geqslant 1\}$ of H, with $h_i \in W^*$ for all $i \geqslant 1$.

We refer to [13] or to a short book [11] for the background in Malliavin calculus. For a $Z \in L^p(W, K)$, where $p > 1$ and K is a separable Hilbert space, we say that $Z \in \mathbb{D}_1^p(W, K)$ if there exists $\nabla Z \in L^p(W, H \otimes K)$ such that for each $h \in H$ and $p' < p$,

$$\langle \nabla Z, h\rangle = D_h Z = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Z(w + \varepsilon h) \quad \text{holds in } L^{p'}.$$

The space $\mathbb{D}_1^p(W, K)$ is complete under the norm: $\|Z\|_{1,p}^p = \|Z\|_{L^p}^p + \|\nabla Z\|_{L^p}^p$. A K-valued functional Z is called cylindrical if there are $N, M \geqslant 1$, $f_i \in C_b^\infty(\mathbb{R}^M)$ and $k_i \in K$ $(1 \leqslant i \leqslant N)$, such that

$$Z = \sum_{i=1}^N f_i\big(h_1(w), \ldots, h_M(w)\big)k_i.$$

By the Schmidt orthogonalization procedure, we may always assume that $\{k_1, \ldots, k_N\}$ is an orthonormal system. Note that $Z : W \to K$ is Fréchet differentiable of any order. We denote by $\mathrm{Cylin}(W, K)$ the space of K-valued cylindrical functionals. If $K = \mathbb{R}$, we simply write $\mathbb{D}_1^p(W)$ and $\mathrm{Cylin}(W)$. It is known that K-valued cylindrical functions are dense in $\mathbb{D}_1^p(W, K)$. A basic result in Malliavin calculus is that $\mathrm{div}_\mu(Z) \in L^p(W)$ exists for $Z \in \mathbb{D}_1^p(W, H)$: $\int_W F\,\mathrm{div}_\mu(Z)\,d\mu = \int_W \langle \nabla F, Z\rangle_H\,d\mu$ and there exists $C_p > 0$, such that

$$\big\|\mathrm{div}_\mu(Z)\big\|_{L^p} \leqslant C_p\|Z\|_{\mathbb{D}_1^p}. \tag{2.1}$$

The Ornstein–Uhlenbeck semigroup P_ε on W is defined by the Mehler formula:

$$P_\varepsilon F(x) = \int_X F\big(e^{-\varepsilon}x + \sqrt{1 - e^{-2\varepsilon}}\,y\big)\,d\mu(y). \tag{2.2}$$

Here are some basic properties of the semigroup P_ε:

Proposition 2.1.

(1) *For any $\varepsilon > 0$, $P_\varepsilon(\mathrm{Cylin}(W)) \subset \mathrm{Cylin}(W)$.*
(2) *For any $\varepsilon > 0$ and $p \in [1, +\infty)$, we have for all $u \in L^p(W)$, $\|P_\varepsilon u\|_{L^p} \leqslant \|u\|_{L^p}$ and $\lim_{\varepsilon\to 0}\|P_\varepsilon u - u\|_{L^p} = 0$.*
(3) *P_ε is self-adjoint in $L^2(W)$. Furthermore, for any $p \in\,]1, +\infty[$ and $u \in L^p(W)$, $v \in L^{p'}(W)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we have*

$$\int_W u P_\varepsilon v\,d\mu = \int_W v P_\varepsilon u\,d\mu.$$

(4) *For every* $\varepsilon > 0$ *and* $p > 1$, *we have* $P_\varepsilon u \in \mathbb{D}_1^p(W)$ *for any* $u \in L^p(W)$, *and there is* $C_{p,\varepsilon} > 0$ *such that*

$$\|\nabla P_\varepsilon u\|_{L^p(W,H)} \leqslant C_{p,\varepsilon}\|u\|_{L^p(W)}.$$

Notice that this last result was due to H. Sugita [15]. Let V_n be the subspace of H spanned by $\{h_1, \dots, h_n\}$ and π_n the orthogonal projection from H to V_n. Then π_n can be extended to W, and $\gamma_n := (\pi_n)_*\mu$ is the standard Gaussian measure on V_n. Let $\mathbb{E}^{V_n}$ be the conditional expectation with respect to the σ-field generated by cylindrical functions of the form $F = f \circ \pi_n$. For $Z \in \mathbb{D}_1^p(W, H)$ and any $n \geqslant 1$, there exists $Z_n : V_n \to V_n$ satisfying

$$\mathbb{E}^{V_n}\big(\pi_n(Z)\big) = Z_n \circ \pi_n \tag{2.3}$$

and

$$\operatorname{div}_{\gamma_n}(Z_n) \circ \pi_n = \mathbb{E}^{V_n}\big(\operatorname{div}_\mu(Z)\big). \tag{2.4}$$

It is well known that

$$\lim_{n\to+\infty} \|Z_n \circ \pi_n - Z\|_{\mathbb{D}_1^p(W,H)} = 0. \tag{2.5}$$

Now given a cylindrical vector field $Z : W \to H$:

$$Z(w) = \sum_{i=1}^N f_i\big(h_1(w), \dots, h_M(w)\big)h_i \in W^*,$$

we consider the quantity, for $x, y \in W$,

$$\begin{aligned} A_Z(x, y) &= \big\langle Z'(x) \cdot y, y\big\rangle - \sum_{i=1}^N (D_{h_i} f_i)(x) \\ &= \sum_{i=1}^N \big\langle \nabla f_i(x), y\big\rangle\langle h_i, y\rangle - \sum_{i=1}^N (D_{h_i} f_i)(x), \end{aligned} \tag{2.6}$$

where $Z'(x) : W \to W^*$ denotes the Fréchet differential of $x \to Z(x)$.

Proposition 2.2. *Define* $(\nabla Z(x))^*$ *by* $\langle(\nabla Z(x))^*, h_1 \otimes h_2\rangle_{H\otimes H} = \langle \nabla Z(x), h_2 \otimes h_1\rangle_{H\otimes H}$. *Then*

$$\int_W \big|A_Z(x, y)\big|^2\, d\mu(y) = \big|\nabla Z(x)\big|_{H\otimes H}^2 + \big\langle \nabla Z(x), \big(\nabla Z(x)\big)^*\big\rangle_{H\otimes H}. \tag{2.7}$$

Proof. Using the second expression in (2.6),

$$\begin{aligned} \big|A_Z(x, y)\big|^2 = &\sum_{i,j=1}^N \big\langle \nabla f_i(x), y\big\rangle\big\langle \nabla f_j(x), y\big\rangle\langle h_i, y\rangle\langle h_j, y\rangle \\ &- 2\left(\sum_{i=1}^N \big\langle \nabla f_i(x), y\big\rangle\langle h_i, y\rangle\right)\left(\sum_{i=1}^N (D_{h_i} f_i)(x)\right) + \left(\sum_{i=1}^N (D_{h_i} f_i)(x)\right)^2. \end{aligned}$$

We obtain by the integration by parts formula,

$$\sum_{i,j=1}^{N}\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),y\rangle\langle h_i,y\rangle\langle h_j,y\rangle\,d\mu(y)$$
$$=\sum_{i,j=1}^{N}\int_W D_{h_j}\big[\langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),y\rangle\langle h_i,y\rangle\big]\,d\mu(y),$$

which is equal to

$$\sum_{i,j=1}^{N}\bigg(\int_W \langle\nabla f_i(x),h_j\rangle\langle\nabla f_j(x),y\rangle\langle h_i,y\rangle\,d\mu(y)+\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),h_j\rangle\langle h_i,y\rangle\,d\mu(y)$$
$$+\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),y\rangle\langle h_i,h_j\rangle\,d\mu(y)\bigg). \tag{2.8}$$

Again by integrating by parts,

$$\int_W \langle\nabla f_i(x),h_j\rangle\langle\nabla f_j(x),y\rangle\langle h_i,y\rangle\,d\mu(y)=\langle\nabla f_i(x),h_j\rangle\int_W D_{h_i}\langle\nabla f_j(x),y\rangle\,d\mu(y)$$
$$=D_{h_j}f_i(x)D_{h_i}f_j(x). \tag{2.9}$$

Similarly,

$$\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),h_j\rangle\langle h_i,y\rangle\,d\mu(y)=D_{h_j}f_j(x)D_{h_i}f_i(x). \tag{2.10}$$

Combining equalities (2.8), (2.9) and (2.10) lead to

$$\sum_{i,j=1}^{N}\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),y\rangle\langle h_i,y\rangle\langle h_j,y\rangle\,d\mu(y)$$
$$=\sum_{i,j=1}^{N}D_{h_j}f_i(x)D_{h_i}f_j(x)+\bigg(\sum_{i=1}^{N}D_{h_i}f_i(x)\bigg)^2+\sum_{i=1}^{N}\big|\nabla f_i(x)\big|_H^2,$$

since

$$\int_W \langle\nabla f_i(x),y\rangle\langle\nabla f_j(x),y\rangle\langle h_i,h_j\rangle\,d\mu(y)=\delta_{ij}\int_W \langle\nabla f_i(x),y\rangle^2\,d\mu(y)=\delta_{ij}\big|\nabla f_i(x)\big|_H^2$$

and

$$\int_W \langle\nabla f_i(x),y\rangle\langle h_i,y\rangle\,d\mu(y)=\langle\nabla f_i(x),h_i\rangle=D_{h_i}f_i(x).$$

Therefore

$$\int_W \big|A_Z(x,y)\big|^2\,d\mu(y)=\sum_{i,j=1}^{N}D_{h_j}f_i(x)D_{h_i}f_j(x)+\sum_{i=1}^{N}\big|\nabla f_i(x)\big|_H^2. \tag{2.11}$$

Now $\nabla Z(x)=\sum_{i=1}^N \nabla f_i(x)\otimes h_i$. We have

$$\left|\nabla Z(x)\right|_{H\otimes H}^2=\sum_{i=1}^N\left|\nabla f_i(x)\right|_H^2$$

and

$$\left\langle \nabla Z(x),\left(\nabla Z(x)\right)^*\right\rangle_{H\otimes H}=\sum_{i,j=1}^N D_{h_j}f_i(x)D_{h_i}f_j(x).$$

So, according to equality (2.11), we get (2.7). □

Proposition 2.3. *Denote by* $I(x,y)=\langle Z'(x)\cdot y,y\rangle-\langle Z(x),x\rangle$*, then for any* $1<p\leqslant 2$,

$$\|I\|_{L^p(W\times W)}\leqslant C_p\|Z\|_{\mathbb{D}_1^p(W,H)}. \tag{2.12}$$

Proof. We deduce from Proposition 2.2 that

$$\left\|A_Z(x,\cdot)\right\|_{L^2(W)}\leqslant\sqrt{2}\left|\nabla Z(x)\right|_{H\otimes H}. \tag{2.13}$$

Note that $\mathrm{div}_\mu(Z)=\sum_i f_i(x)\langle h_i,x\rangle-\sum_i D_{h_i}f_i(x)$. Then

$$\begin{aligned} I(x,y)&=\left\langle Z'(x)\cdot y,y\right\rangle-\sum_{i=1}^N D_{h_i}f_i(x)+\sum_{i=1}^N D_{h_i}f_i(x)-\sum_{i=1}^N f_i(x)\langle h_i,x\rangle\\ &=A_Z(x,y)-\mathrm{div}_\mu(Z)(x)\end{aligned}$$

and

$$\|I\|_{L^p(W\times W)}\leqslant\|A_Z\|_{L^p(W\times W)}+\left\|\mathrm{div}_\mu(Z)\right\|_{L^p(W)}. \tag{2.14}$$

For any $1<p\leqslant 2$, we have

$$\|A_Z\|_{L^p(W\times W)}=\left(\int_W\left\|A_Z(x,\cdot)\right\|_{L^p(W)}^p\,d\mu(x)\right)^{1/p}\leqslant\left(\int_W\left\|A_Z(x,\cdot)\right\|_{L^2(W)}^p\,d\mu(x)\right)^{1/p}.$$

This plus (2.13) gives

$$\|A_Z\|_{L^p(W\times W)}\leqslant\left(\int_W\left(\sqrt{2}\left|\nabla Z(x)\right|_{H\otimes H}\right)^p d\mu(x)\right)^{1/p}\leqslant\sqrt{2}\,\|Z\|_{\mathbb{D}_1^p(W,H)}.$$

Now by (2.1) and (2.14), we know that there exists $C_p>0$ such that

$$\|I\|_{L^p(W\times W)}\leqslant C_p\|Z\|_{\mathbb{D}_1^p(W,H)},$$

which completes the proof. □

Remark 2.4. For $p=2$, the following equality holds

$$\int_{X\times X}\left|I(x,y)\right|^2 d\mu(x)\,d\mu(y)=\int_X|\nabla Z|_{H\otimes H}^2\,d\mu+2\int_X\left|\mathrm{div}_\mu(Z)\right|^2 d\mu-\int_X|Z|_H^2\,d\mu. \tag{2.15}$$

Proof. In fact it is obvious that $\int_X A_Z(x,y)\,d\mu(y)=0$. Then

$$\int_{X\times X} \big|I(x,y)\big|^2\,d\mu(x)\,d\mu(y)=\int_{X\times X}\big|A_Z(x,y)\big|^2\,d\mu(x)\,d\mu(y)+\int_X\big|\mathrm{div}_\mu(Z)\big|^2\,d\mu.$$

But according to (2.7),

$$\int_{X\times X}\big|A_Z(x,y)\big|^2\,d\mu(x)\,d\mu(y)=\int_X|\nabla Z|^2_{H\otimes H}\,d\mu+\int_X\big\langle\nabla Z,(\nabla Z)^*\big\rangle_{H\otimes H}\,d\mu.$$

A version of the Weitzenböck formula on the Wiener space (see [13,11]) reads as

$$\int_X\big|\mathrm{div}_\mu(Z)\big|^2\,d\mu=\int_X|Z|^2_H\,d\mu+\int_X\big\langle\nabla Z,(\nabla Z)^*\big\rangle_{H\otimes H}\,d\mu.$$

The result (2.15) follows. □

Proposition 2.5. *(See [4].) Set* $\tilde Z(x,y)=\langle Z(x),y\rangle$ *and*

$$O_\varepsilon(x,y)=\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y,-\sqrt{1-e^{-2\varepsilon}}\,x+e^{-\varepsilon}y\big).$$

Then

$$P_\varepsilon\big(\mathrm{div}_\mu(Z)\big)(x)=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W\tilde Z\big(O_\varepsilon(x,y)\big)\,d\mu(y). \tag{2.16}$$

Proof. We have

$$\begin{aligned}P_\varepsilon(D_{h_i}f_i)(x)&=\int_W D_{h_i}f_i\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\,d\mu(y)\\&=\frac{1}{\sqrt{1-e^{-2\varepsilon}}}\int_W f_i\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\langle h_i,y\rangle\,d\mu(y)\end{aligned}$$

and

$$P_\varepsilon\big(f_i\langle h_i,\cdot\rangle\big)(x)=\int_W f_i\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\big\langle h_i,e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big\rangle\,d\mu(y).$$

Therefore

$$\begin{aligned}&P_\varepsilon\big(f_i\langle h_i,\cdot\rangle-D_{h_i}f_i\big)(x)\\&\quad=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\int_W f_i\big(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y\big)\big\langle h_i,-\sqrt{1-e^{-2\varepsilon}}\,x+e^{-\varepsilon}y\big\rangle\,d\mu(y),\end{aligned} \tag{2.17}$$

since

$$e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}\,y-\frac{1}{\sqrt{1-e^{-2\varepsilon}}}y=-\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\big(-\sqrt{1-e^{-2\varepsilon}}\,x+e^{-\varepsilon}y\big).$$

Summing up both sides of (2.17), we get the result. □

Let $R_{-\pi/2}: W \times W \to W \times W$ be the rotation defined by $(x, y) \to (y, -x)$. Using the notation $\tilde{Z}$, the term $I(x, y)$ in Proposition 2.12 can be expressed by

$$I = \tilde{Z}'(R_{-\pi/2}),$$

where the prime denotes the Fréchet differential.

Theorem 2.6. *Let $v \in \mathrm{Cylin}(W)$ and define $B_\varepsilon(v, Z) = \langle Z, \nabla P_\varepsilon v\rangle - P_\varepsilon(\langle \nabla v, Z\rangle)$. Then for any $p, q, r \geqslant 1$ with $1 < p \leqslant 2$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have*

$$\big\| B_\varepsilon(v, Z)\big\|_{L^r} \leqslant C_p \|v\|_{L^q} \|Z\|_{\mathbb{D}_1^p}. \tag{2.18}$$

Proof. We have

$$\langle Z, \nabla P_\varepsilon v\rangle = \frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} \int_W v\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\tilde{Z}(x, y)\, d\mu(y),$$

where $\tilde{Z}$ is defined in Proposition 2.5. Replacing Z by vZ in (2.16), we obtain

$$P_\varepsilon\big(\mathrm{div}_\mu(vZ)\big) = -\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} \int_W v\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\tilde{Z}\big(O_\varepsilon(x, y)\big)\, d\mu(y).$$

Note that $\mathrm{div}_\mu(vZ) = v\,\mathrm{div}_\mu(Z) - \langle \nabla v, Z\rangle$, then

$$B_\varepsilon(v, Z) = \langle Z, \nabla P_\varepsilon v\rangle + P_\varepsilon\big(\mathrm{div}_\mu(vZ)\big) - P_\varepsilon\big(v\,\mathrm{div}_\mu(Z)\big). \tag{2.19}$$

The delicate term is

$$\begin{aligned} B_\varepsilon^1 &:= \langle Z, \nabla P_\varepsilon v\rangle + P_\varepsilon\big(\mathrm{div}_\mu(vZ)\big) \\ &= \frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} \int_W v\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\big(\tilde{Z}(x, y) - \tilde{Z}\big(O_\varepsilon(x, y)\big)\big)\, d\mu(y). \end{aligned}$$

Note that $\frac{d}{d\varepsilon}O_\varepsilon = \frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} R_{-\pi/2} \circ O_\varepsilon$. Setting $\tilde{v}(x, y) = v(x)$, we can write B_ε^1 in the form

$$B_\varepsilon^1 = -\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} \int_W \tilde{v}\big(O_\varepsilon(x, y)\big)\left(\int_0^1 \frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}} I\big(O_{\varepsilon s}(x, y)\big)\, ds\right) d\mu(y). \tag{2.20}$$

It follows that

$$\big|B_\varepsilon^1(x)\big|^r \leqslant \left(\frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}}\right)^r \int_W \big|\tilde{v}\big(O_\varepsilon(x, y)\big)\big|^r \cdot \left|\int_0^1 \frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}} I\big(O_{\varepsilon s}(x, y)\big)\, ds\right|^r d\mu(y),$$

therefore

$$\begin{aligned} \big\|B_\varepsilon^1\big\|_{L^r} &\leqslant \frac{e^{-\varepsilon}}{\sqrt{1-e^{-2\varepsilon}}} \big\|\tilde{v}\big(O_\varepsilon(\cdot,\cdot)\big)\big\|_{L^q(W\times W)} \left\|\int_0^1 \frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}} I\big(O_{\varepsilon s}(\cdot,\cdot)\big)\, ds\right\|_{L^p(W\times W)} \\ &\leqslant \|v\|_{L^q(W)} \|I\|_{L^p(W\times W)}, \end{aligned}$$

since, by the invariance of $\mu \otimes \mu$ under $O_{\varepsilon s}$,

$$\left\| I\left(O_{\varepsilon s}(\cdot,\cdot)\right)\right\|_{L^p(W\times W)} = \|I\|_{L^p(W\times W)}$$

and

$$\int_0^1 \frac{\varepsilon e^{-\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}\,ds \leqslant e^{\varepsilon}\int_0^1 \frac{\varepsilon e^{-2\varepsilon s}}{\sqrt{1-e^{-2\varepsilon s}}}\,ds = e^{\varepsilon}\sqrt{1-e^{-2\varepsilon}}.$$

Combining with (2.19) and by Proposition 2.1(2), we get

$$\left\|B_\varepsilon(v,Z)\right\|_{L^r} \leqslant \|v\|_{L^q}\left(\|I\|_{L^p(W\times W)} + \left\|\mathrm{div}_\mu(Z)\right\|_{L^p}\right).$$

Now the inequality (2.1) and Proposition 2.3 lead to the result. □

By the expression (2.20), for cylindrical v and Z, we have

$$\lim_{\varepsilon\to 0} B_\varepsilon^1(x) = -\frac{1}{\sqrt{2}}\int_W \tilde v(x,y)\left(\int_0^1 \frac{1}{\sqrt{2s}} I(x,y)\,ds\right)d\mu(y) = -v(x)\int_W I(x,y)\,d\mu(y).$$

Since

$$\int_W I(x,y)\,d\mu(y) = \sum_{i=1}^N \left(D_{h_i} f_i(x) - f_i(x)h_i(x)\right) = -\mathrm{div}_\mu(Z)(x),$$

therefore

$$\lim_{\varepsilon\to 0} B_\varepsilon^1(x) = v\,\mathrm{div}_\mu(Z)(x).$$

Again by (2.19), we have $\lim_{\varepsilon\to 0} B_\varepsilon(v,Z)(x) = 0$. Hence the dominated convergence theorem gives

$$\lim_{\varepsilon\to 0}\left\|B_\varepsilon(v,Z)\right\|_{L^r} = 0.$$

Theorem 2.7. *Assume $p,q,r \geqslant 1$ with $1 < p \leqslant 2$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then for any $v \in L^q(W)$ and $Z \in \mathbb{D}_1^p(W,H)$, we have $B_\varepsilon(v,Z) \in L^r(W)$ and*

$$\left\|B_\varepsilon(v,Z)\right\|_{L^r} \leqslant C_p\|v\|_{L^q}\|Z\|_{\mathbb{D}_1^p}. \tag{2.21}$$

Moreover,

$$\lim_{\varepsilon\to 0}\left\|B_\varepsilon(v,Z)\right\|_{L^r} = 0. \tag{2.22}$$

Proof. First we fix some $Z \in \mathrm{Cylin}(W,H)$. For any $v \in L^q(W)$, there exists a sequence $\{v_n\colon n \geqslant 1\} \subset \mathrm{Cylin}(W)$ such that $\lim_{n\to+\infty}\|v_n - v\|_{L^q} = 0$. By the linearity of $v \mapsto B_\varepsilon(v,Z)$ and Theorem 2.6, we know that $\{B_\varepsilon(v_n,Z)\colon n \geqslant 1\}$ is a Cauchy sequence in $L^r(W)$. We denote by $B_\varepsilon(v,Z)$ its limit in $L^r(W)$, and we have

$$\left\|B_\varepsilon(v,Z)\right\|_{L^r} = \lim_{n\to+\infty}\left\|B_\varepsilon(v_n,Z)\right\|_{L^r} \leqslant \liminf_{n\to+\infty} C_p\|v_n\|_{L^q}\|Z\|_{\mathbb{D}_1^p} = C_p\|v\|_{L^q}\|Z\|_{\mathbb{D}_1^p},$$

hence (2.21) follows. On the other hand, for every $n \geqslant 1$,

$$\begin{aligned}\left\| B_\varepsilon(v, Z)\right\|_{L^r} &\leqslant \left\| B_\varepsilon(v_n, Z)\right\|_{L^r} + \left\| B_\varepsilon(v - v_n, Z)\right\|_{L^r} \\ &\leqslant \left\| B_\varepsilon(v_n, Z)\right\|_{L^r} + C_p \|v - v_n\|_{L^q} \|Z\|_{\mathbb{D}_1^p}.\end{aligned}$$

Letting $\varepsilon \to 0$ and by the above discussion, we get

$$\limsup_{\varepsilon \to 0} \left\| B_\varepsilon(v, Z)\right\|_{L^r} \leqslant C_p \|v - v_n\|_{L^q} \|Z\|_{\mathbb{D}_1^p}.$$

Now we conclude (2.22) in this case by letting $n \to +\infty$.

For general $Z \in \mathbb{D}_1^p(W, H)$, using the linearity of $Z \mapsto B_\varepsilon(v, Z)$ and the analogous argument works. □

3. Transport equations on the Wiener space

Let $Z \in \mathbb{D}_1^p(W, H)$ be a vector field on W and $c \in L^p(W)$. In the sequel, we always assume $1 < p \leqslant 2$ and denote by $D_Z v$ or $Z \cdot \nabla v$ the directional derivatives. Let $T > 0$ be given.

Definition 3.1. We say that $u_\cdot \in L^\infty([0, T], L^q(W))$ solves the transport equation

$$\frac{du_t}{dt} + Z \cdot \nabla u_t + c u_t = 0, \quad u|_{t=0} = u_0, \tag{3.1}$$

if for any $\alpha \in C_c^\infty([0, T))$ and $F \in \mathrm{Cylin}(W)$,

$$\int_0^T \int_W \left[-\alpha'(t) F u_t + \alpha(t) D_Z^* F u_t + \alpha(t) F c u_t\right] d\mu\, dt = \int_W \alpha(0) F u_0\, d\mu, \tag{3.2}$$

where $D_Z^* F = -Z \cdot \nabla F + \mathrm{div}_\mu(Z) F$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Note that (3.2) holds for all $\alpha \in C_c^1([0, T))$. In what follows, we will discuss the existence and uniqueness of solutions to (3.1). First we prove the uniqueness under suitable conditions. Let $u_t \in L^q(W)$ be a solution to (3.1), we consider $u_t^\varepsilon := P_\varepsilon u_t$, where P_ε is the Ornstein–Uhlenbeck semigroup on W. For a given $F \in \mathrm{Cylin}(W)$ and $\alpha \in C_c^\infty([0, T))$, we have by Proposition 2.1(3),

$$\int_0^T \int_W \alpha'(t) F u_t^\varepsilon\, d\mu\, dt = \int_0^T \int_W \alpha'(t) u_t P_\varepsilon F\, d\mu\, dt.$$

We know from Proposition 2.1(1) that $P_\varepsilon F$ is also cylindrical, therefore by (3.2), this last term is equal to

$$\int_0^T \int_W \alpha(t) D_Z^*(P_\varepsilon F) u_t\, d\mu\, dt + \int_0^T \int_W \alpha(t) c u_t P_\varepsilon F\, d\mu\, dt - \int_W \alpha(0) u_0 P_\varepsilon F\, d\mu. \tag{3.3}$$

Let B_ε be defined in Theorem 2.6 and by (2.21), $B_\varepsilon(u, Z)$ is a well-defined function for $u \in L^q$ and $Z \in \mathbb{D}_1^p$. Therefore

$$\int_0^T \int_W \alpha(t) D_Z^*(P_\varepsilon F) u_t \, d\mu \, dt$$

$$= \int_0^T \int_W \alpha(t) u_t \big(P_\varepsilon\big(D_Z^* F\big) + D_Z^*(P_\varepsilon F) - P_\varepsilon\big(D_Z^* F\big)\big) \, d\mu \, dt$$

$$= \int_0^T \int_W \alpha(t) u_t^\varepsilon D_Z^* F \, d\mu \, dt - \int_0^T \int_W \alpha(t) F B_\varepsilon(u_t, Z) \, d\mu \, dt. \tag{3.4}$$

The second term in (3.3) is equal to

$$\int_0^T \int_W \alpha(t) F c P_\varepsilon u_t \, d\mu \, dt + \int_0^T \int_W \alpha(t) F \tilde{r}_\varepsilon(t) \, d\mu \, dt,$$

where

$$\tilde{r}_\varepsilon(t) = P_\varepsilon(c u_t) - c P_\varepsilon u_t.$$

Combining this with (3.3) and (3.4), we obtain

$$\int_0^T \int_W \big[-\alpha'(t) F u_t^\varepsilon + \alpha(t) u_t^\varepsilon D_Z^* F + \alpha(t) F c u_t^\varepsilon\big] \, d\mu \, dt$$

$$= \int_0^T \int_W \alpha(t) F \big[B_\varepsilon(u_t, Z) - \tilde{r}(t)\big] \, d\mu \, dt + \int_W \alpha(0) F P_\varepsilon u_0 \, d\mu.$$

Therefore u_t^ε satisfies the transport equation

$$\frac{du_t^\varepsilon}{dt} + Z \cdot \nabla u_t^\varepsilon + c u_t^\varepsilon = B_\varepsilon(u_t, Z) - \tilde{r}_\varepsilon(t) \tag{3.5}$$

with the initial condition $P_\varepsilon u_0$.

Proposition 3.2. *We have*

$$\lim_{\varepsilon \to 0} \int_0^T \int_W \big(\big|B_\varepsilon(u_t, Z)\big| + \big|\tilde{r}_\varepsilon(t)\big|\big) \, d\mu \, dt = 0. \tag{3.6}$$

Proof. For $t \in [0, T]$, applying Theorem 2.7 with $r = 1$, we have $\lim_{\varepsilon \to 0} \|B_\varepsilon(u_t, Z)\|_{L^1} = 0$ and

$$\big\|B_\varepsilon(u_t, Z)\big\|_{L^1} \leqslant C_p \|u_t\|_{L^q} \|Z\|_{\mathbb{D}_1^p} \leqslant C_p \Big(\sup_{t \in [0,T]} \|u_t\|_{L^q} \Big) \|Z\|_{\mathbb{D}_1^p}.$$

Hence Lebesgue's dominated convergence theorem leads to

$$\lim_{\varepsilon \to 0} \int_0^T \int_W \big|B_\varepsilon(u_t, Z)\big| \, d\mu \, dt = 0.$$

Moreover, by Proposition 2.1(2), for each fixed $t \in [0, T]$, $P_\varepsilon(cu_t)$ converges to cu_t in L^1 as $\varepsilon \to 0$, therefore $\tilde{r}_\varepsilon(t) \to 0$ in $L^1(W)$. But

$$\left\|\tilde{r}_\varepsilon(t)\right\|_{L^1(W)} \leqslant 2\|c\|_{L^p} \cdot \Big(\sup_{t\in[0,T]} \|u_t\|_{L^q}\Big),$$

again by Lebesgue's dominated convergence theorem, $\lim_{\varepsilon\to 0} \int_0^T \int_W |\tilde{r}_\varepsilon(t)|\, d\mu\, dt = 0$. □

Theorem 3.3 *(Uniqueness). Let $Z \in \mathbb{D}_1^p(W, H)$ with $1 < p \leqslant 2$. Assume that*

$$\int_W \exp \lambda_1 \left|\mathrm{div}_\mu(Z)\right| d\mu < +\infty, \qquad \int_W \exp \lambda_2 |c|\, d\mu < +\infty \quad \textit{for some } \lambda_1, \lambda_2 > 0, \tag{3.7}$$

then the transport equation (3.1) *admits at most one solution $u \in L^\infty([0, T], L^q(W))$ for the given $u_0 \in L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. By linearity, we assume that $u_0 = 0$. Let $u_t^\varepsilon = P_\varepsilon u_t$. We have by Proposition 2.1(4),

$$\left\|Z \cdot \nabla u_t^\varepsilon\right\|_{L^1} \leqslant \|Z\|_{L^p} \left\|\nabla u_t^\varepsilon\right\|_{L^q} \leqslant C_{q,\varepsilon} \|Z\|_{L^p} \|u_t\|_{L^q}.$$

Therefore by (3.5), for μ-a.s. $w \in W$, $t \to u_t^\varepsilon$ is absolutely continuous. Then for $\beta \in C_b^1(\mathbb{R})$, $\frac{d}{dt}\beta(u_t^\varepsilon) = \beta'(u_t^\varepsilon) \frac{d}{dt} u_t^\varepsilon$ and $Z \cdot \nabla(\beta(u_t^\varepsilon)) = \beta'(u_t^\varepsilon) Z \cdot \nabla u_t^\varepsilon$. Thus

$$\begin{aligned} \frac{d}{dt}\beta\left(u_t^\varepsilon\right) + Z \cdot \nabla\left(\beta\left(u_t^\varepsilon\right)\right) &= \beta'\left(u_t^\varepsilon\right)\left(\frac{d}{dt}u_t^\varepsilon + Z \cdot \nabla u_t^\varepsilon\right) \\ &= \beta'\left(u_t^\varepsilon\right)\left(-cu_t^\varepsilon + B_\varepsilon(u_t, Z) - \tilde{r}_\varepsilon(t)\right), \end{aligned}$$

or

$$\frac{d}{dt}\beta\left(u_t^\varepsilon\right) + Z \cdot \nabla\left(\beta\left(u_t^\varepsilon\right)\right) + \beta'\left(u_t^\varepsilon\right)cu_t^\varepsilon = \beta'\left(u_t^\varepsilon\right)\left(B_\varepsilon(u_t, Z) - \tilde{r}_\varepsilon(t)\right). \tag{3.8}$$

Letting $\varepsilon \to 0$ in (3.8) gives

$$\frac{d}{dt}\beta(u_t) + Z \cdot \nabla\beta(u_t) + \beta'(u_t)cu_t = 0. \tag{3.9}$$

For $F \in \mathrm{Cylin}(W)$, we have

$$\frac{d}{dt}\int_W F\beta(u_t)\, d\mu = \int_W D_Z F\beta(u_t)\, d\mu - \int_W \mathrm{div}_\mu(Z) F\beta(u_t)\, d\mu - \int_W cF\beta'(u_t)u_t\, d\mu. \tag{3.10}$$

Taking $F = 1$ in (3.10), we get

$$\frac{d}{dt}\int_W \beta(u_t)\, d\mu = -\int_W \mathrm{div}_\mu(Z)\beta(u_t)\, d\mu - \int_W c\beta'(u_t)u_t\, d\mu. \tag{3.11}$$

After a smoothing procedure as in [10], we can take $\beta(s) = (|s| \wedge M)^q$ for $M > 0$ fixed. In this case, $\beta'(s) = 0$ for $|s| > M$, and $|\beta'(s)| = q|s|^{q-1}$ for $|s| < M$; therefore $|\beta'(u_t)| \cdot |u_t| \leqslant q\beta(u_t)$. Following [7], for $R > 0$, we define

$$\Sigma_R = \left\{x \in W;\ \left|\mathrm{div}_\mu(Z)\right| + q|c| \geqslant R\right\}.$$

Then for $\lambda > 0$, $\mu(\Sigma_R) \leqslant e^{-\lambda R} \int_W e^{\lambda(|\mathrm{div}_\mu(Z)|+q|c|)}\,d\mu$ which is finite by condition (3.7) for

$$\lambda \leqslant \lambda_0 := \min(\lambda_1/2, \lambda_2/2).$$

Denote by $C^2_{\lambda_0,q} = \int_W e^{\lambda_0(|\mathrm{div}_\mu(Z)|+q|c|)}\,d\mu$. We have by Cauchy–Schwarz inequality,

$$\int_{\Sigma_R} \big(\big|\mathrm{div}_\mu(Z)\big| + q|c|\big)\beta(u_t)\,d\mu \leqslant \|\beta\|_\infty \big(\big\|\big|\mathrm{div}_\mu(Z)\big| + q|c|\big\|_{L^2}\big) e^{-\lambda_0 R/2} C_{\lambda_0,q}.$$

According to (3.11), we get for some constant $C_{\lambda_0,M}$ independent of R,

$$\frac{d}{dt}\int_W \beta(u_t)\,d\mu \leqslant R\int_W \beta(u_t)\,d\mu + C_{\lambda_0,M} e^{-\lambda_0 R/2}.$$

By Gronwall lemma, we have for $t \in [0,T]$,

$$\int_W \beta(u_t)\,d\mu \leqslant C_{\lambda,M} e^{-\lambda_0 R/2} e^{Rt},$$

which tends to 0 as $R \to +\infty$, for

$$t \leqslant T_0 < \lambda_0/2.$$

It follows that for $t \in [0, T_0]$,

$$|u_t| \wedge M = 0 \quad \text{a.s.}$$

Letting $M \to +\infty$, we see that $u_t = 0$ for $t \in [0, T_0]$. Now shifting the time, $u_t = 0$ for all $t \in [0, T]$. □

Now we are turning our attention to prove the existence of solutions to (3.1) in $L^\infty([0,T], L^q(W))$. For the sake of simplicity, we consider the condition,

$$\int_W e^{\lambda(|\mathrm{div}_\mu(Z)|+|Z|_H)}\,d\mu < +\infty \quad \text{for any } \lambda > 1. \tag{3.12}$$

We will first construct the solutions to

$$\frac{du_t}{dt} + Z\cdot\nabla u_t = c. \tag{3.13}$$

As what has been done in the above (see (3.8)), for $\beta \in C^1_b(\mathbb{R})$,

$$\frac{d}{dt}\beta\big(u_t^\varepsilon\big) + Z\cdot\nabla\beta\big(u_t^\varepsilon\big) = \beta'\big(u_t^\varepsilon\big)\big(P_\varepsilon c + B_\varepsilon(u_t, Z)\big).$$

Letting $\varepsilon \to 0$, we have

$$\frac{d}{dt}\beta(u_t) + Z\cdot\nabla\beta(u_t) = \beta'(u_t)c. \tag{3.14}$$

In order to get the bound on $\|u_t\|_{L^q}$, we proceed in two steps:
(1) Let $u_t^{(n)}$ solve

$$\frac{du_t^{(n)}}{dt} + Z_n\cdot\nabla^{V_n} u_t^{(n)} = c^{(n)}, \tag{3.15}$$

where ∇^{V_n} is the gradient operator on V_n, Z_n is defined in (2.3) and $c^{(n)} = \mathbb{E}^{V_n}(c)$. Under the conditions on Z, we have $Z_n \in \mathbb{D}_1^p(V_n, V_n)$.

(2) For $\eta \in C_b^\infty(V_n)$, $B \in C_b^\infty(V_n, V_n)$ and $\hat{u}_0 \in C_b^\infty(V_n)$, the transport equation

$$\frac{d\hat{u}_t}{dt} + B \cdot \nabla \hat{u}_t = \eta \tag{3.16}$$

admits the unique smooth solution $\hat{u}_t$ given by

$$\hat{u}_t = \hat{u}_0\big(X_t^{-1}\big) + \int_0^t \eta\big(X_{t-s}^{-1}\big)\, ds, \tag{3.17}$$

where $(X_t)_{t\in\mathbb{R}}$ is the flow of diffeomorphisms associated to B:

$$\frac{dX_t}{dt} = B(X_t), \quad X|_{t=0} = x \in V_n.$$

It is known that $(X_t)_*\gamma_n = K_t\gamma_n$ and for any $q > 1, t \in [0, T]$ (see [9,13]):

$$K_t = \exp\left(\int_0^t \mathrm{div}_{\gamma_n}(B)(X_{-s})\, ds\right), \qquad \|K_t\|_{L^q}^q \leqslant \int_{V_n} \exp\left(\frac{q^2 T}{q-1}\big|\mathrm{div}_{\gamma_n}(B)\big|\right) d\gamma_n. \tag{3.18}$$

To fix ideas, we consider in the sequel $r > 1$ and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \bar{q} > q. \tag{3.19}$$

We have $\| \int_0^t \eta(X_{t-s}^{-1})\, ds\|_{L^q} \leqslant \int_0^t \|\eta(X_{t-s}^{-1})\|_{L^q}\, ds$ and by (3.18),

$$\int_{V_n} \big|\eta\big(X_{t-s}^{-1}\big)\big|^q\, d\gamma_n \leqslant \left(\int_{V_n} |\eta|^{\bar{q}}\, d\gamma_n\right)^{q/\bar{q}} \cdot \|K_{-t+s}\|_{\bar{q}/(\bar{q}-q)}.$$

It follows that

$$\big\|\eta\big(X_{t-s}^{-1}\big)\big\|_{L^q} \leqslant \|\eta\|_{L^{\bar{q}}} \left[\int_{V_n} \exp\left(\frac{\bar{q}^2 T}{q(\bar{q}-q)}\big|\mathrm{div}_{\gamma_n}(B)\big|\right) d\gamma_n\right]^{(\bar{q}-q)/q\bar{q}}.$$

Combining this with (3.17), we get

$$\|\hat{u}_t\|_{L^q} \leqslant C_T\big(\|\hat{u}_0\|_{L^{\bar{q}}} + \|\eta\|_{L^{\bar{q}}}\big)\left[\int_{V_n} \exp\left(\frac{\bar{q}^2 T}{q(\bar{q}-q)}\big|\mathrm{div}_{\gamma_n}(B)\big|\right) d\gamma_n\right]^{(\bar{q}-q)/q\bar{q}}, \tag{3.20}$$

where C_T is a constant independent of n.

Next we will choose a sequence of $B_m \in C_b^\infty(V_n, V_n)$ which converges to Z_n in $\mathbb{D}_1^p(V_n)$ and $\eta_m \in C_b^\infty(V_n)$ to $c^{(n)}$ in L^p. By the following inequality

$$\big\|\mathrm{div}_{\gamma_n}(B)\big\|_{L^p} \leqslant C_p\|B\|_{\mathbb{D}_1^p},$$

we see that $\mathrm{div}_{\gamma_n}(B_m)$ converges to $\mathrm{div}_{\gamma_n}(Z_n)$ in $L^p(V_n)$ as $m \to +\infty$. However we need the upper bound for the exponential integrability of $\mathrm{div}_{\gamma_n}(B_m)$. To reach our aim, instead we will consider the Ornstein–Uhlenbeck semigroup P_ε on V_n. For $B \in \mathbb{D}_1^p(V_n, V_n)$, we consider $P_\varepsilon B$. Then

$$\operatorname{div}_{\gamma_n}(P_\varepsilon B) = e^\varepsilon P_\varepsilon \operatorname{div}_{\gamma_n}(B). \tag{3.21}$$

Note that $P_\varepsilon B \in C^\infty(V_n, V_n)$, but not necessarily bounded with their derivatives. For $N \geqslant 1$, consider the cut-off function $\varphi_N \in C_c^\infty(V_n)$ such that $0 \leqslant \varphi_N \leqslant 1$ and for $|x| \leqslant N$, $\varphi_N(x) = 1$; for $|x| > N + 2$, $\varphi_N(x) = 0$ and $\|\nabla \varphi_N\|_\infty \leqslant 1$. Then $(P_\varepsilon B)\varphi_N \in C_b^\infty(V_n, V_n)$. We have

$$\begin{aligned}\operatorname{div}_{\gamma_n}\big((P_\varepsilon B)\varphi_N\big) &= \varphi_N \operatorname{div}_{\gamma_n}(P_\varepsilon B) - \langle \nabla\varphi_N, P_\varepsilon B\rangle \\ &= e^\varepsilon \varphi_N P_\varepsilon \operatorname{div}_{\gamma_n}(B) - \langle \nabla\varphi_N, P_\varepsilon B\rangle,\end{aligned} \tag{3.22}$$

and also

$$\big|\operatorname{div}_{\gamma_n}\big((P_\varepsilon B)\varphi_N\big)\big| \leqslant e^\varepsilon P_\varepsilon \big|\operatorname{div}_{\gamma_n}(B)\big| + P_\varepsilon |B|. \tag{3.23}$$

By Jensen inequality and the invariance of γ_n under P_ε, we have

$$\int_{V_n} \exp\big(2e^\varepsilon P_\varepsilon \big|\operatorname{div}_{\gamma_n}(B)\big|\big)\, d\gamma_n \leqslant \int_{V_n} \exp\big(2e^\varepsilon \big|\operatorname{div}_{\gamma_n}(B)\big|\big)\, d\gamma_n.$$

Now using (3.23) and Cauchy–Schwarz inequality, we have the bound

$$\int_{V_n} e^{\lambda|\operatorname{div}_{\gamma_n}((P_\varepsilon B)\varphi_N)|}\, d\gamma_n \leqslant \bigg(\int_{V_n} \exp\big(2e^\varepsilon \lambda \big|\operatorname{div}_{\gamma_n}(B)\big|\big)\, d\gamma_n\bigg)^{1/2} \bigg(\int_{V_n} e^{2\lambda|B|}\, d\gamma_n\bigg)^{1/2}. \tag{3.24}$$

Due to (3.24), we assume for a moment that B is in the exponential class. We will take a sequence $\varepsilon_m \to 0$ and $N_m \to +\infty$. It is obvious that as $m \to +\infty$, $(P_{\varepsilon_m} B)\varphi_{N_m} \to B$ in L^p and by (3.22), $\operatorname{div}_{\gamma_n}((P_{\varepsilon_m} B)\varphi_{N_m}) \to \operatorname{div}_{\gamma_n}(B)$ in L^p. Set $B_m = (P_{\varepsilon_m} B)\varphi_{N_m}$. Applying (3.20) to B_m and we denote by $\hat{u}_t^m$ the associated solution. We have

$$\|\hat{u}_t^m\|_{L^q} \leqslant C_T\big(\|\hat{u}_0^m\|_{L^{\bar q}} + \|\eta_m\|_{L^{\bar q}}\big)\bigg[\int_{V_n} \exp\bigg(\frac{\bar q^2 T}{q(\bar q - q)}\big|\operatorname{div}_{\gamma_n}(B_m)\big|\bigg)\, d\gamma_n\bigg]^{(\bar q - q)/q\bar q}. \tag{3.25}$$

Combining with (3.24), we see that $\{\hat{u}^m;\ m \geqslant 1\}$ is bounded in $L^q([0,T] \times V_n)$; therefore up to a subsequence, $\hat{u}^m$ converges to $u^{(n)} \in L^q([0,T] \times V_n)$ weakly as $m \to +\infty$:

$$\int_0^T \int_{V_n} u_t^{(n)} \psi(t,x)\, d\gamma_n(x)\, dt = \lim_{m\to+\infty} \int_0^T \int_{V_n} \hat{u}_t^m \psi(t,x)\, d\gamma_n\, dt, \quad \psi \in L^{q'}\big([0,T] \times V_n\big),$$

where q' is the conjugate number of q: $\frac{1}{q} + \frac{1}{q'} = 1$. Since $\operatorname{div}_{\gamma_n}(B_m)$ converges to $\operatorname{div}_{\gamma_n}(B)$ in L^p, again up to a subsequence, $\operatorname{div}_{\gamma_n}(B_m)$ converges to $\operatorname{div}_{\gamma_n}(B)$ a.s. Using the bound in (3.24), as $m \to +\infty$,

$$\int_{V_n} \exp\bigg(\frac{\bar q^2 T}{q(\bar q - q)}\big|\operatorname{div}_{\gamma_n}(B_m)\big|\bigg)\, d\gamma_n \to \int_{V_n} \exp\bigg(\frac{\bar q^2 T}{q(\bar q - q)}\big|\operatorname{div}_{\gamma_n}(B)\big|\bigg)\, d\gamma_n.$$

Denote this last term by $d_n^{q\bar{q}/(\bar{q}-q)}$. By (3.25),

$$\limsup_{m\to+\infty}\|\hat{u}_t^m\|_{L^q}\leqslant C_T\big(\|u_0^{(n)}\|_{L^{\bar{q}}}+\|c^{(n)}\|_{L^{\bar{q}}}\big)\,d_n.$$

Now for $\psi\in C_b([0,T]\times V_n)$,

$$\left|\int_0^T\int_{V_n}\psi(t,x)u_t^{(n)}(x)\,d\gamma_n(x)\,dt\right|\leqslant C_T\left(\int_0^T\|\psi(t,\cdot)\|_{q'}\,dt\right)\big(\|u_0^{(n)}\|_{L^{\bar{q}}}+\|c^{(n)}\|_{L^{\bar{q}}}\big)\,d_n.$$

It follows that for a.s. $t\in[0,T]$, $u_t^{(n)}\in L^q(V_n)$ and $\|u_t^{(n)}\|_{L^q}\leqslant C_T(\|u_0^{(n)}\|_{L^{\bar{q}}}+\|c^{(n)}\|_{L^{\bar{q}}})\,d_n$. Replacing B by Z_n, we have explicitly

$$\|u_t^{(n)}\|_{L^q}\leqslant C_T\big(\|u_0^{(n)}\|_{L^{\bar{q}}}+\|c^{(n)}\|_{L^{\bar{q}}}\big)\left[\int_{V_n}\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}\big|\mathrm{div}_{\gamma_n}(Z_n)\big|\right)d\gamma_n\right]^{(\bar{q}-q)/q\bar{q}}. \tag{3.26}$$

Using the relation (2.4) and Jensen inequality, we have

$$\int_{V_n}\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}\big|\mathrm{div}_{\gamma_n}(Z_n)\big|\right)d\gamma_n\leqslant\int_W\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}\big|\mathrm{div}_{\mu}(Z)\big|\right)d\mu.$$

Define $\tilde{u}_t^{(n)}=u_t^{(n)}\circ\pi_n$. According to (3.26), $\{\tilde{u}^{(n)};\ n\geqslant1\}$ is bounded in $L^q([0,T]\times W)$. Let $u\in L^q([0,T]\times W)$ be a weak limit.

Theorem 3.4. *Let $Z\in\mathbb{D}_1^p(W,H)$ be a vector field and*

$$\int_W e^{\lambda(|Z|_H+|\mathrm{div}_\mu(Z)|)}\,d\mu<+\infty\quad\text{for each }\lambda>1. \tag{3.27}$$

Let $c\in L^{\bar{q}}(W)$. Then the transport equation (3.13) *has a solution $u\in L^\infty([0,T],L^q(W))$ with $u_0\in L^{\bar{q}}(W)$ given, and the following estimate holds*

$$\|u_t\|_{L^q}\leqslant C_T\big(\|u_0\|_{L^{\bar{q}}}+\|c\|_{L^{\bar{q}}}\big)\left[\int_W\exp\left(\frac{\bar{q}^2T}{q(\bar{q}-q)}\big|\mathrm{div}_\mu(Z)\big|\right)d\mu\right]^{(\bar{q}-q)/q\bar{q}}. \tag{3.28}$$

Proof. Fix n_0 and consider $f\in C_b^\infty(V_{n_0})$. For $n\geqslant n_0$, we have $f\circ\pi_n=f\circ\pi_{n_0}$ and $\nabla^{V_n}f\circ\pi_n=\nabla^{V_{n_0}}f\circ\pi_{n_0}$. So

$$D_{Z_n}f\circ\pi_n=\big\langle\nabla^{V_n}f\circ\pi_n,Z_n\circ\pi_n\big\rangle_H=\big\langle\nabla^{V_{n_0}}f\circ\pi_{n_0},Z_n\circ\pi_n\big\rangle.$$

Since $\nabla^{V_{n_0}}f\circ\pi_{n_0}$ is in the dual space W^*, we know that $D_{Z_n}f\circ\pi_n$ converges to $\langle\nabla^{V_{n_0}}f\circ\pi_{n_0},Z\rangle$ in all $L^r(W)$. Let $F=f\circ\pi_n$. Then

$$\int_0^T\int_W\alpha'(t)F\tilde{u}_t^{(n)}\,d\mu\,dt=\int_0^T\int_{V_n}\alpha'(t)fu_t^{(n)}\,d\gamma_n\,dt$$

$$= \int_0^T \int_{V_n} \alpha(t)\big(-D_{Z_n} f + \mathrm{div}_{\gamma_n}(Z_n) f\big) u_t^{(n)}\, d\gamma_n\, dt$$

$$- \int_0^T \int_{V_n} \alpha(t) f c^{(n)}\, d\gamma_n\, dt - \int_{V_n} \alpha(0) u_0^{(n)} f\, d\gamma_n$$

$$= \int_0^T \int_W \alpha(t)\big(-D_{Z_n} f \circ \pi_n + \mathrm{div}_{\gamma_n}(Z_n) \circ \pi_n F\big) \tilde{u}_t^{(n)}\, d\mu\, dt$$

$$- \int_0^T \int_W \alpha(t) F c^{(n)} \circ \pi_n\, d\mu\, dt - \int_W \alpha(0) \tilde{u}_0^{(n)} F\, d\mu.$$

Letting $n \to +\infty$, the above equality leads to

$$\int_0^T \int_W \alpha'(t) F u_t\, d\mu\, dt = \int_0^T \int_W \alpha(t) D_Z^* F u_t\, d\mu\, dt - \int_0^T \int_W \alpha(t) F c\, d\mu\, dt - \int_W \alpha(0) u_0 F\, d\mu.$$

In other words, u_t solves (3.13). □

Theorem 3.5. *Assume that the hypotheses in Theorem* 3.4 *hold and furthermore*

$$\int_W e^{\lambda(|u_0|+|c|)}\, d\mu < +\infty \quad \text{for each } \lambda > 1. \tag{3.29}$$

Then the solution constructed in Theorem 3.4 *is in the exponential class*: *for any* $\lambda > 0$,

$$\sup_{t\in[0,T]} \left(\int_W e^{\lambda |u_t|}\, d\mu \right) < +\infty. \tag{3.30}$$

Proof. Again going back to (3.17), we will estimate $\int_{V_n} e^{\lambda |\hat{u}_t|}\, d\gamma_n$. First by Jensen's inequality,

$$\exp\left(\lambda \int_0^t \big|\eta\big(X_{t-s}^{-1}\big)\big|\, ds \right) \leqslant \int_0^t e^{\lambda t |\eta(X_{t-s}^{-1})|}\, \frac{ds}{t}.$$

Using the estimate (3.18),

$$\int_{V_n} e^{\lambda t |\eta(X_{t-s}^{-1})|}\, d\gamma_n = \int_{V_n} e^{\lambda t |\eta|} K_{-t+s}\, d\gamma_n \leqslant \left(\int_{V_n} e^{2\lambda T |\eta|}\, d\gamma_n \right)^{1/2} \left(\int_{V_n} e^{4T |\mathrm{div}_{\gamma_n}(B)|}\, d\gamma_n \right)^{1/2}.$$

Combining the above two inequalities, we get

$$\int_{V_n} \exp\left(\lambda \int_0^t \big|\eta\big(X_{t-s}^{-1}\big)\big|\, ds \right) d\gamma_n \leqslant \left(\int_{V_n} e^{2\lambda T |\eta|}\, d\gamma_n \right)^{1/2} \left(\int_{V_n} e^{4T |\mathrm{div}_{\gamma_n}(B)|}\, d\gamma_n \right)^{1/2}.$$

It is obvious that

$$\int_{V_n} e^{\lambda|\hat{u}_0(X_t^{-1})|}\,d\gamma_n \leqslant \left(\int_{V_n} e^{2\lambda|\hat{u}_0|}\,d\gamma_n\right)^{1/2}\left(\int_{V_n} e^{4T|\mathrm{div}_{\gamma_n}(B)|}\,d\gamma_n\right)^{1/2}.$$

Now $e^{\lambda|\hat{u}_t|} \leqslant e^{\lambda|\hat{u}_0(X_t^{-1})|}\cdot \exp(\lambda\int_0^t |\eta(X_{t-s}^{-1})|\,ds)$, therefore

$$\int_{V_n} e^{\lambda|\hat{u}_t|}\,d\gamma_n \leqslant \left(\int_{V_n} e^{4\lambda|\hat{u}_0|}\,d\gamma_n\right)^{1/4}\left(\int_{V_n} e^{4\lambda T|\eta|}\,d\gamma_n\right)^{1/4}\left(\int_{V_n} e^{4T|\mathrm{div}_{\gamma_n}(B)|}\,d\gamma_n\right)^{1/2}. \tag{3.31}$$

At the final step of the construction of u_t in Theorem 3.4, we have

$$\int_W e^{\lambda|\tilde{u}_t^{(n)}|}\,d\mu \leqslant \left(\int_W e^{4\lambda|u_0|}\,d\mu\right)^{1/4}\left(\int_W e^{4\lambda T|c|}\,d\mu\right)^{1/4}\left(\int_W e^{4T|\mathrm{div}_\mu(Z)|}\,d\mu\right)^{1/2}. \tag{3.32}$$

Since $\tilde{u}_\cdot^{(n)}$ converges weakly to $u_\cdot$ in $L^q([0,T]\times W)$, and using the fact that the weak closure of a convex subset in L^q is identical to its strong closure, so $u_\cdot$ is a limit in L^q of a sequence of function U_n satisfying (3.32); up to a subsequence, U_n converges to $u_\cdot$ a.s.; therefore (3.32) holds for u:

$$\int_W e^{\lambda|u_t|}\,d\mu \leqslant \left(\int_W e^{4\lambda|u_0|}\,d\mu\right)^{1/4}\left(\int_W e^{4\lambda T|c|}\,d\mu\right)^{1/4}\left(\int_W e^{4T|\mathrm{div}_\mu(Z)|}\,d\mu\right)^{1/2}. \tag{3.33}$$

We get the result. □

Now let u_t be a solution to

$$\frac{du_t}{dt} + Z\cdot\nabla u_t = c, \quad u_0 = 0,$$

where c is supposed to be in the exponential class; let w_t be a solution to

$$\frac{dw_t}{dt} + Z\cdot\nabla w_t = 0, \quad w_0 = u_0 \in L^{\bar{q}}(W).$$

For $\varepsilon > 0$, consider $u_t^\varepsilon = P_\varepsilon u_t$, $w_t^\varepsilon = P_\varepsilon w_t$. We define $\tilde{w}_t^\varepsilon = e^{-u_t^\varepsilon} w_t^\varepsilon$. Then

$$\begin{aligned}
&\frac{d\tilde{w}_t^\varepsilon}{dt} = -e^{-u_t^\varepsilon} w_t^\varepsilon \frac{du_t^\varepsilon}{dt} + e^{-u_t^\varepsilon}\frac{dw_t^\varepsilon}{dt},\\
&Z\cdot\nabla\tilde{w}_t^\varepsilon = -e^{-u_t^\varepsilon} w_t^\varepsilon Z\cdot\nabla u_t^\varepsilon + e^{-u_t^\varepsilon} Z\cdot\nabla w_t^\varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d\tilde{w}_t^\varepsilon}{dt} + Z\cdot\tilde{\nabla} w_t^\varepsilon &= -e^{-u_t^\varepsilon} w_t^\varepsilon\left(\frac{du_t^\varepsilon}{dt} + Z\cdot\nabla u_t^\varepsilon\right) + e^{-u_t^\varepsilon}\left(\frac{dw_t^\varepsilon}{dt} + Z\cdot\nabla w_t^\varepsilon\right)\\
&= -e^{-u_t^\varepsilon} w_t^\varepsilon\big(P_\varepsilon c + B_\varepsilon(u_t, Z)\big) + e^{-u_t^\varepsilon} B_\varepsilon(w_t, Z).
\end{aligned} \tag{3.34}$$

Since u_t is exponentially integrable, we deduce from Theorem 2.7 that $B_\varepsilon(u_t, Z) \in L^{p-}(W)$. On the other hand, by the proof of Theorem 3.4 and $w_0 \in L^{\bar{q}}(W)$, we have $w_t \in L^{\bar{q}-}(W)$. Moreover, it follows from Theorem 3.5 that for each $\lambda > 0$, $e^{-u_t^\varepsilon} \in L^\lambda(W)$, therefore Theorem 2.7 and (3.19) implies, as $\varepsilon \to 0$,

$$e^{-u_t^\varepsilon} w_t^\varepsilon B_\varepsilon(u_t, Z) \to 0 \quad \text{in } L^1.$$

In the same way, $\lim_{\varepsilon\to 0}\|e^{-u_t^\varepsilon}B_\varepsilon(w_t,Z)\|_{L^1}=0$. Letting $\varepsilon\to 0$ in (3.34) gives

$$\frac{d\tilde{w}_t}{dt}+Z\cdot\tilde{w}_t=-c\tilde{w}_t,$$

or $\tilde{w}_t=e^{-u_t}w_t$ solves the transport equation (3.1) with u_0 as the initial function. We get the main result of this section:

Theorem 3.6. *Let Z be a vector field on W such that $Z\in\mathbb{D}_1^p(W,H)$ $(1<p\leqslant 2)$. Assume that c, $\mathrm{div}_\mu(Z)$ and $|Z|_H$ are in the exponential class. Then the transport equation* (3.1) *admits a unique solution $u\in L^\infty([0,T],L^q(W))$ with $u_0\in L^{\bar{q}}(W)$, $\bar{q}>q$.*

Remark 3.7. If $u_0\geqslant 0$, then the unique solution to (3.1) $u_t\geqslant 0$; if $0\leqslant u_0\leqslant M$ and $0\leqslant c\leqslant M$, then $0\leqslant u_t\leqslant (T+1)M$.

4. Flows of quasi-invariant maps

In this section, we restrict ourselves to the classical Wiener space $X=C_0([0,1],\mathbb{R}^d)$, and we will construct the flow on X associated to a vector field via the transport equations. Our starting point is the following result in finite dimension, a version given by Cipriano and Cruzeiro [7, p. 186].

Theorem 4.1. *Let $B\in\mathbb{D}_1^p(\mathbb{R}^N,\gamma_N)$ such that*

$$\int_{\mathbb{R}^N} e^{\lambda_0(|B|+|\mathrm{div}_{\gamma_N}(B)|)}\,d\gamma_N<+\infty \quad \textit{for some } \lambda_0>0.$$

Then there is a unique flow of maps $(U_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^N$ such that for a.s. $x\in\mathbb{R}^N$,

$$U_t(x)=x+\int_0^t B\big(U_s(x)\big)\,ds,\quad t\geqslant 0, \tag{4.1}$$

and $(U_t)_*\gamma_N=K_t\gamma_N$ with

$$K_t(x)=\exp\left(\int_0^t \mathrm{div}_{\gamma_N}(B)\big(U_{-s}(x)\big)\,ds\right). \tag{4.2}$$

In the case of flow of diffeomorphisms on $\mathbb{R}^N$ associated to a smooth vector field B, by taking the derivative on

$$U_{t+s}=U_t\circ U_s,$$

with respect to s and at $s=0$, we get

$$\frac{dU_t(x)}{dt}=U_t'(x)B(x).$$

For any linear map $\ell:\mathbb{R}^N\to\mathbb{R}$, $u_t:=\ell(U_t)$ satisfies the transport equation

$$\frac{du_t}{dt}-B\cdot\nabla u_t=0,\quad u_0=\ell. \tag{4.3}$$

By the method of smoothing, for vector field B on $\mathbb{R}^N$ satisfying the conditions in Theorem 4.1, Eq. (4.3) still holds, but in the sense of Section 3 (see [7]).

Now for $n \geqslant 1$ and $x \in X$, we define $\pi_n(x) \in X$ by

$$\pi_n(x)(t) = x\big(k2^{-n}\big) + 2^n\big(t - k2^{-n}\big)\big(x\big((k+1)2^{-n}\big) - x\big(k2^{-n}\big)\big),\quad t \in \big[k2^{-n}, (k+1)2^{-n}\big]. \tag{4.4}$$

We have $\|\pi_n(x)\|_\infty = \sup_{1\leqslant k\leqslant 2^n}\{|x(k2^{-n})|\} \leqslant \|x\|_\infty$ and $\lim_{n\to\infty}\|\pi_n(x) - x\|_\infty = 0$ for each $x \in X$. We still denote by

$$V_n = \pi_n(X). \tag{4.5}$$

The space V_n is of dimension $2^n d$. Let $\{h_0, h_{k,n};\ n \geqslant 1,\ k < 2^n \text{ odd}\}$ be the family of Haar functions defined on the interval [0, 1] by $\dot h_0(t) = 1$ and

$$\dot h_{k,n}(t) = \begin{cases} (\sqrt{2})^{n-1} & \text{for } t \in [(k-1)2^{-n}, k2^{-n}),\\ -(\sqrt{2})^{n-1} & \text{for } t \in [k2^{-n}, (k+1)2^{-n}),\\ 0 & \text{otherwise.}\end{cases}$$

The family $\{h_0\varepsilon_\alpha, h_{k,n}\varepsilon_\alpha;\ n \geqslant 1,\ k < 2^n \text{ odd},\ \alpha = 1,\dots,d\}$ constitutes an orthonormal basis of the Cameron–Martin space $H = \{h \in X;\ \int_0^1 |\dot h_s|^2\, ds < +\infty\}$, where $\{\varepsilon_1,\dots,\varepsilon_d\}$ is the canonical basis of $\mathbb{R}^d$. The following result is known (for a proof, we refer to [1]):

Proposition 4.2. *The space V_n is spanned by $\{h_0\varepsilon_\alpha, h_{k,m}\varepsilon_\alpha;\ 1 \leqslant m \leqslant n,\ k < 2^m \text{ odd},\ \alpha = 1,\dots,d\}$. More precisely*

$$\pi_n(x) = \sum_{\alpha=1}^d \left\{\langle h_0\varepsilon_\alpha, x\rangle h_0\varepsilon_\alpha + \sum_{m=1}^n \sum_{k<2^m \text{ odd}} \langle h_{k,m}\varepsilon_\alpha, x\rangle h_{k,m}\varepsilon_\alpha\right\}. \tag{4.6}$$

From now on, we assume the vector field $Z : X \to H$ satisfies $Z \in \mathbb{D}_1^p(X, H)$ with $1 < p \leqslant 2$ and

$$\int_X e^{\lambda_0(|Z|_H + |\mathrm{div}_\mu(Z)|)}\, d\mu < +\infty \quad \text{for some } \lambda_0 > 0. \tag{4.7}$$

For each $n \geqslant 1$, let $Z_n : V_n \to V_n$ be defined as in Section 2, then Z_n fulfils the conditions in Theorem 4.1. Denote by $U_t^n : V_n \to V_n$ the flow associated to Z_n. Then there exists a small $T_0 > 0$ such that the density K_t^n of $(U_t^n)_*\gamma_n$ with respect to γ_n satisfies the estimate:

$$\|K_t^n\|_{L^q(\gamma_n)}^q \leqslant \int_{V_n} \exp\left(\frac{q^2 T_0}{q-1}\big|\mathrm{div}_{\gamma_n}(Z_n)\big|\right) d\gamma_n, \quad t \in [0, T_0]. \tag{4.8}$$

Define $\tilde U_t^n : X \to X$ by

$$\tilde U_t^n(x) = U_t^n\big(\pi_n(x)\big) + x - \pi_n(x). \tag{4.9}$$

Then $\tilde K_t^n := K_t^n \circ \pi_n$ is the density of $(\tilde U_t^n)_*\mu$ with respect to μ and by (4.8) and (2.4), the following uniform estimate holds

$$\|\tilde K_t^n\|_{L^q(X)}^q \leqslant \int_X \exp\left(\frac{q^2 T_0}{q-1}\big|\mathrm{div}_\mu(Z)\big|\right) d\mu, \quad t \in [0, T_0]. \tag{4.10}$$

For the sake of simplicity, we write our construction for $d = 1$. For $m \leqslant n$, $k < 2^m$ odd fixed, we set

$$u_t^n(z) = \langle h_{k,m}, U_t^n(z)\rangle_{V_n}, \qquad \tilde{u}_t^n = u_t^n \circ \pi_n.$$

By the above discussion, u_t^n solves the transport equation

$$\int\limits_{[0,T]\times V_n} \left[-\alpha'(t)\varphi u_t^n - \alpha(t) D_{Z_n}^* \varphi u_t^n\right] d\gamma_n\, dt = \int\limits_{V_n} \alpha(0)\langle h_{k,m}, z\rangle \varphi\, d\gamma_n(z), \tag{4.11}$$

for $\alpha \in C_c^\infty([0,T))$ and $\varphi \in C_b^1(V_n)$. For n big enough such that $h_{k,m} \in V_n$, we have

$$\langle h_{k,m}, \tilde{U}_t^n(x)\rangle = \langle h_{k,m}, U_t^n(\pi_n(x))\rangle.$$

Using the flow property for U_t^n and (4.8) for $q = 2$, we have for any $q \geqslant 1$,

$$\begin{aligned}\int\limits_X \left|\langle h_{k,m}, \tilde{U}_{t+T_0}^n(x)\rangle\right|^q d\mu(x) &= \int\limits_{V_n} \left|\langle h_{k,m}, U_t^n(x)\rangle\right|^q K_{T_0}^n\, d\gamma_n \\ &\leqslant \left(\int\limits_X \left|\langle h_{k,m}, \tilde{U}_t^n(x)\rangle\right|^{2q} d\mu\right)^{1/2} \left(\int\limits_X e^{4T_0|\mathrm{div}_\mu(Z)|}\, d\mu\right)^{1/2},\end{aligned}$$

where we have taken $T_0 < \lambda_0/4$. Let $A = \int_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\, d\mu$. By induction, for any $t \in [0,T]$,

$$\int\limits_X \left|\langle h_{k,m}, \tilde{U}_t^n(x)\rangle\right|^q d\mu(x) \leqslant \left(\int\limits_X \left|\langle h_{k,m}, x\rangle\right|^{q2^N} d\mu(x)\right)^{2^{-N}} \cdot A^{2^{-N}+\cdots+2^{-1}},$$

where $NT_0 \geqslant T$. So there is a constant $C_{\lambda_0,T} > 0$ such that

$$\int\limits_X \left|\langle h_{k,m}, \tilde{U}_t^n(x)\rangle\right|^q d\mu(x) \leqslant C_{\lambda_0,T} \int\limits_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\, d\mu, \quad t \in [0,T]. \tag{4.12}$$

Let q be the conjugate number of p: $q = p/(p-1)$. Taking successively q and $2q$ in (4.12), we see that $\{\tilde{u}^n;\ n \geqslant 1\}$ and $\{(\tilde{u}^n)^2;\ n \geqslant 1\}$ are bounded in $L^q([0,T]\times X)$. Up to a subsequence,

$$\tilde{u}^n \to v_{k,m} \quad \text{and} \quad (\tilde{u}^n)^2 \to w_{k,m} \quad \text{weakly as } n \to +\infty.$$

Morally $\{v_{k,m};\ m \geqslant 1,\ k < 2^m \text{ odd}\}$ will be the coefficients under the Haar basis of the flow $U_t : X \to X$, that we will construct; but the above convergence is too weak to be useful. In what follows, we will use the results obtained in Section 3 to reinforce the above convergence. Fix n_0. For $\varphi \in C_b^1(V_{n_0})$ and $n \geqslant n_0$, $(D_{Z_n}\varphi)\circ\pi_n = \langle \nabla^{V_{n_0}}\varphi \circ \pi_{n_0}, Z_n \circ \pi_n\rangle$. Rewriting (4.11) in the form

$$\begin{aligned}\int\limits_{[0,T]\times X} &\Big[-\alpha'(t)(\varphi\circ\pi_n)\tilde{u}_t^n - \alpha(t)\tilde{u}_t^n\big(-\langle \nabla^{V_{n_0}}\varphi\circ\pi_{n_0}, Z_n\circ\pi_n\rangle \\ &+ (\varphi\circ\pi_n)\mathbb{E}^{V_n}\big(\mathrm{div}_\mu(Z)\big)\big)\Big]\, d\mu\, dt = \int\limits_X \alpha(0)\tilde{u}_0\varphi\circ\pi_n\, d\mu.\end{aligned} \tag{4.13}$$

Letting $n \to +\infty$ in (4.13) gives

$$\int_{[0,T]\times X} \left[-\alpha'(t)F v_{k,m} - \alpha(t)\big(-D_Z F + F\,\mathrm{div}_\mu(Z)\big)v_{k,m}\right] d\mu\, dt = \int_X \alpha(0)\tilde{u}_0 F\, d\mu,$$

where we set $F = \varphi \circ \pi_{n_0}$. In other words, $v_{k,m}$ solves the transport equation on X,

$$\frac{dv_{k,m}}{dt} - Z \cdot \nabla v_{k,m} = 0, \quad v_{k,m}|_{t=0} = \langle h_{k,m}, x\rangle. \tag{4.14}$$

Now applying the results of Section 3 (see (3.9)), for $\beta(s) = (|s| \wedge M)^2$, $\beta(v_{k,m})$ satisfies also (4.14) with the initial condition $\beta(\langle h_{k,m}, x\rangle)$:

$$\int_{[0,T]\times X} \left[-\alpha'(t)F\big(|v_{k,m}| \wedge M\big)^2 - \alpha(t)\big(-D_Z F + F\,\mathrm{div}_\mu(Z)\big)\big(|v_{k,m}| \wedge M\big)^2\right] d\mu\, dt$$
$$= \int_X \alpha(0)\big(\big|\langle h_{k,m}, x\rangle\big| \wedge M\big)^2 F\, d\mu.$$

Letting $M \to +\infty$, we see that $v_{k,m}^2$ solves the transport equation (4.14) with the initial condition $\langle h_{k,m}, x\rangle^2$. Now replacing $\tilde{u}_t^n$ by $(\tilde{u}_t^n)^2$ in the above procedure of limit, we see that $w_{k,m}$ satisfies the same transport equation as $v_{k,m}^2$ with the same initial condition. By Theorem 3.3, we have $w_{k,m} = v_{k,m}^2$. Therefore $\tilde{u}^n \to v_{k,m}$ and $(\tilde{u}^n)^2 \to v_{k,m}^2$ weakly as $n \to +\infty$, from which we deduce that

$$\lim_{n\to+\infty} \int_{[0,T]\times X} \left|\langle h_{k,m}, \tilde{U}_t^n\rangle - v_{k,m}\right|^2 d\mu\, dt = 0. \tag{4.15}$$

By the method of extracting diagonal subsequence, we may assume that the relation (4.15) holds for any $m \geqslant 1$ and $k < 2^m$ odd. In this way, we obtain the coordinates $\{v_0, v_{k,m};\ m \geqslant 1, k < 2^m \text{ odd}\}$.

Proposition 4.3. *The random series*

$$v_0(t,x)h_0 + \sum_{m\geqslant 1}\sum_{k<2^m \text{ odd}} v_{k,m}(t,x)h_{k,m} \tag{4.16}$$

converges in X a.s. for $(t,x) \in [0,T] \times X$.

Proof. We follow the idea in Levy's construction of the Brownian motion. Set

$$e_m(t,x) = \Big\| \sum_{k<2^m \text{ odd}} v_{k,m}(t,x)h_{k,m} \Big\|_\infty.$$

Since the supports of $\{h_{k,m};\ k < 2^m \text{ odd}\}$ are disjoint, we have

$$e_m(t,x) \leqslant \Big(\max_{k<2^m \text{ odd}} \big|v_{k,m}(t,x)\big|\Big) \cdot \|h_{k,m}\|_\infty = 2^{-(m+1)/2}\Big(\max_{k<2^m \text{ odd}} \big|v_{k,m}(t,x)\big|\Big).$$

Fix $\theta > 1$, we have

$$\big\{(t,x);\ e_m(t,x) \geqslant 2\theta\sqrt{2^{-m}\log 2^m}\big\} \subset \Big\{2^{-(m+1)/2} \max_{k<2^m \text{ odd}} \big|v_{k,m}(t,x)\big| \geqslant 2\theta\sqrt{2^{-m}\log 2^m}\Big\}$$
$$\subset \bigcup_{k<2^m \text{ odd}} \big\{|v_{k,m}| \geqslant 2\theta\sqrt{2m\log 2}\big\}.$$

We have

$$\big\{|v_{k,m}| \geqslant 2\theta\sqrt{2m\log 2}\big\}$$
$$\subset \big\{\big|v_{k,m} - \langle h_{k,m}, \tilde{U}_t^n\rangle\big| \geqslant \theta\sqrt{2m\log 2}\big\} \cup \big\{\big|\langle h_{k,m}, \tilde{U}_t^n\rangle\big| \geqslant \theta\sqrt{2m\log 2}\,\big\}.$$

Let λ be the normalized Lebesgue measure on $[0,T]$. Then for n big enough and as what done for (4.12),

$$(\lambda\times\mu)\big(\big\{\big|\langle h_{k,m}, \tilde{U}_t^n\rangle\big| \geqslant \theta\sqrt{2m\log 2}\,\big\}\big) \leqslant C_{\lambda_0,T}2^{-\theta^2 m/2^N},$$

where $NT_0 \geqslant T$ and $C_{\lambda_0,T}$ is a constant independent of m,k,n. Let

$$a_n = (\lambda\times\mu)\big(\big\{\big|v_{k,m} - \langle h_{k,m}, \tilde{U}_t^n\rangle\big| \geqslant \theta\sqrt{2m\log 2}\,\big\}\big).$$

By (4.15), $a_n \to 0$ as $n\to+\infty$. We have

$$(\lambda\times\mu)\big(\big\{|v_{k,m}| \geqslant 2\theta\sqrt{2m\log 2}\,\big\}\big) \leqslant a_n + C_{\lambda_0,T}2^{-\theta^2 m/2^N}.$$

Then letting $n\to+\infty$, we get $(\lambda\times\mu)(\{|v_{k,m}| \geqslant 2\theta\sqrt{2m\log 2}\}) \leqslant C_{q,T}2^{-\frac{\theta^2}{q}m}$. Therefore

$$(\lambda\times\mu)\big(\big\{e_m(t,x) \geqslant 2\theta\sqrt{2^{-m}\log 2^m}\,\big\}\big) \leqslant C_{\lambda_0,T}2^{m-1}2^{-\theta^2 m/2^N} \leqslant C_{\lambda_0,T}2^{(1-\frac{\theta^2}{2^N})m}.$$

Taking $\theta^2 > 2^N$, the following series

$$\sum_{m\geqslant 1}(\lambda\times\mu)\big(\big\{e_m(t,x) \geqslant 2\theta\sqrt{2^{-m}\log 2^m}\,\big\}\big) \leqslant C_{\lambda_0,T}\sum_{m\geqslant 1}2^{(1-\frac{\theta^2}{2^N})m} < +\infty.$$

By Borel–Cantelli lemma, for $(\lambda\times\mu)$-a.s. (t,x),

$$e_m(t,x) \leqslant 2\theta\sqrt{2^{-m}\log 2^m} \quad \text{for } m \text{ big enough},$$

so we have proved the convergence in X of the series (4.16). □

We will denote

$$U_t(x) = v_0(t,x)h_0 + \sum_{m\geqslant 1}\sum_{k<2^m \text{ odd}} v_{k,m}(t,x)h_{k,m}. \tag{4.17}$$

Next we will prove that $U_t : X \to X$ leaves the Wiener measure quasi-invariant and establish the explicit expression for the density K_t of $(U_t)_*\mu$ with respect to μ. But first of all, we prove

Proposition 4.4. *Let $\tilde{K}_t^n$ be the density of $(\tilde{U}_t^n)_*\mu$ with respect to the Wiener measure μ. Then for $t\in[0,T]$,*

$$\int_X \tilde{K}_t^n\big|\log\tilde{K}_t^n\big|\,d\mu \leqslant T\cdot\big\|\mathrm{div}_\mu(Z)\big\|_{L^{2^N}}\int_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\,d\mu, \tag{4.18}$$

where N is such that $NT_0 \geqslant T$ and T_0 appeared in (4.8) *such that $4T_0 \leqslant \lambda_0$.*

Proof. Using the explicit expression for the density K_t^n of $(U_t^n)_*\gamma_n$ with respect to γ_n, we have

$$\log K_t^n\big(U_t^n(x)\big) = \int_0^t \mathrm{div}_{\gamma_n}(Z_n)\big(U_{t-s}^n(x)\big)\,ds.$$

Then

$$\int_{V_n} K_t^n \left|\log K_t^n\right| d\gamma_n \leqslant \int_0^t \left(\int_{V_n} \left|\mathrm{div}_{\gamma_n}(Z_n)\left(U_{t-s}^n\right)\right| d\gamma_n \right) ds. \tag{4.19}$$

Note that by (4.8),

$$\left\| K_{T_0}^n \right\|_{L^2}^2 \leqslant \int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu.$$

Then by property of flow

$$\int_{V_n} \left|\mathrm{div}_{\gamma_n}(Z_n)\left(U_{t+T_0}^n\right)\right| d\gamma_n \leqslant \left\|\mathrm{div}_{\gamma_n}(Z_n)\left(U_t^n\right)\right\|_{L^2} \cdot A^{1/2},$$

where $A = \int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu$. By induction, we get for any $t \in [0, T]$,

$$\int_{V_n} \left|\mathrm{div}_{\gamma_n}(Z_n)\left(U_t^n\right)\right| d\gamma_n \leqslant \left\|\mathrm{div}_{\gamma_n}(Z_n)\right\|_{L^{2N}} \cdot A \leqslant \left\|\mathrm{div}_\mu(Z)\right\|_{L^{2N}} \cdot A.$$

Now combining with (4.19), we get (4.18). □

Proposition 4.5. *The law of $x \to \pi_m(U_t(x))$ admits a density k_t^m with respect to γ_m on V_m. Moreover*

$$\int_{V_m} k_t^m \log k_t^m \, d\gamma_m \leqslant T \cdot \left\|\mathrm{div}_\mu(Z)\right\|_{L^{2N}} \int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu. \tag{4.20}$$

Proof. Let $n \geqslant m$. Then $\pi_m(\tilde{U}_t^n)$ admits the density $k_t^{m,n}$ with respect to γ_m, given by

$$k_t^{m,n}(z) = \int_{Y_m} \tilde{K}_t^n(z, y)\, d\mu_{Y_m}(y), \tag{4.21}$$

where we use the decomposition of the Wiener space: $X = V_m \oplus Y_m$ and $\mu = \gamma_m \times \mu_{Y_m}$. By the Jensen inequality, we have

$$\int_{V_m} k_t^{m,n} \log k_t^{m,n} \, d\gamma_m \leqslant \int_X \tilde{K}_t^n \log \tilde{K}_t^n \, d\mu, \tag{4.22}$$

which is bounded, using (4.18), by

$$T \cdot \left\|\mathrm{div}_\mu(Z)\right\|_{L^{2N}} \int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu.$$

It follows that the family $\{k_\cdot^{m,n};\ n \geqslant m\}$ is weakly compact in $L^1([0, T] \times V_m)$. Up to a subsequence, $k_\cdot^{m,n}$ converges weakly to $k_\cdot^m \in L^1([0, T] \times V_m)$ as $n \to +\infty$. Note that the subset $\mathcal{C}$ of $u \in L^1([0, T] \times V_m)$ such that $u_t \geqslant 0$ and

$$\int_{V_m} u_t \log u_t \, d\gamma_m \leqslant T \cdot \left\|\mathrm{div}_\mu(Z)\right\|_{L^{2N}} \int_X e^{\lambda_0 |\mathrm{div}_\mu(Z)|}\, d\mu$$

is convex. Since the weak closure is identical to the strong one for $\mathcal{C}$, there is a sequence of function $u_n \in \mathcal{C}$ which converges to k^m in L^1. Up to a subsequence, u_n converges to k^m almost surely. Now the Fatou lemma yields (4.20).

By (4.15), there is a subsequence such that for a.s. $(t,x) \in [0,T] \times X$, as $n \to +\infty$,

$$\pi_m\big(\tilde{U}_t^n(x)\big) \to v_0(t,x)h_0 + \sum_{\ell \leqslant m} \sum_{k<2^\ell \text{ odd}} v_{k,\ell}(t,x)h_{k,\ell} = \pi_m\big(U_t(x)\big).$$

Let $\varphi \in C_c(V_m)$ and $\alpha \in C([0,T])$, we have

$$\begin{aligned}\int_0^T \int_X \alpha\varphi\big(\pi_m(U_t)\big)\,d\mu\,dt &= \lim_{n\to+\infty} \int_0^T \int_X \alpha\varphi\big(\pi_m\big(\tilde{U}_t^n\big)\big)\,d\mu\,dt \\ &= \lim_{n\to+\infty} \int_0^T \int_{V_m} \alpha\varphi k_t^{m,n}\,d\gamma_m\,dt.\end{aligned}$$

Therefore

$$\int_{[0,T]} \alpha\,dt \int_X \varphi\big(\pi_m(U_t)\big)\,d\mu = \lim_{n\to+\infty} \int_{[0,T]\times V_m} \alpha\varphi k_t^{m,n}\,d\gamma_m\,dt = \int_0^T \alpha\,dt \int_{V_m} \varphi k_t^m\,d\gamma_m.$$

It follows that for a.s. (dependent of $\varphi \in C_c(V_m)$) $t \in [0,T]$, we have $\int_X \varphi(\pi_m(U_t))\,d\mu = \int_{V_m} \varphi k_t^m\,d\gamma_m$. Using the separability of $C_c(V_m)$, we see that for a.s. $t \in [0,T]$ and all $\varphi \in C_c(V_m)$, it holds

$$\int_X \varphi\big(\pi_m(U_t)\big)\,d\mu = \int_{V_m} \varphi k_t^m\,d\gamma_m.$$

It follows that

$$\big(\pi_m(U_t)\big)_*\mu = k_t^m\gamma_m.$$

The proof is complete. □

Theorem 4.6. *The density K_t of $(U_t)_*\mu$ with respect to μ exists and satisfies*

$$\int_X K_t \log K_t\,d\mu \leqslant T \cdot \big\|\mathrm{div}_\mu(Z)\big\|_{L^{2N}} \int_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\,d\mu. \tag{4.23}$$

Proof. Set $K_t^{(m)} = k_t^m \circ \pi_m$. Then the family $\{K_t^{(m)};\ m \geqslant 1\}$ is consistent in the sense that

$$\int_X f \circ \pi_m K_t^{(n)}\,d\mu = \int_X f \circ \pi_m K_t^{(m)}\,d\mu, \quad n \geqslant m,\ f \in C_b(V_m). \tag{4.24}$$

By (4.20), for each $t \in [0,T]$, the family $\{K_t^{(n)};\ n \geqslant 1\}$ satisfies

$$\int_X K_t^{(n)} \log K_t^{(n)}\,d\mu \leqslant T \cdot \big\|\mathrm{div}_\mu(Z)\big\|_{L^{2N}} \int_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\,d\mu. \tag{4.25}$$

Therefore up to a subsequence, $K_t^{(n_k)}$ converges weakly in L^1 to $K_t \in L^1 \log L^1$; but by the consistence, for $f \in C_b(V_m)$, the sequence itself converges

$$\int_X (f \circ \pi_m) K_t \, d\mu = \lim_{n \to +\infty} \int_X (f \circ \pi_m) K_t^{(n)} \, d\mu = \int_X f \circ \pi_m(U_t) \, d\mu. \tag{4.26}$$

As m is arbitrary, we conclude that $(U_t)_* \mu = K_t \mu$ and by (4.20), we get (4.23). Now again by (4.25), it holds that

$$\lim_{n \to +\infty} \left| \int_X g K_t^{(n)} \, d\mu - \int_X g K_t \, d\mu \right| = 0 \quad \text{for all } g \in L^\infty(X, \mu),$$

therefore K_t is the weak limit of the sequence $K_t^{(n)} \in L^1(X)$. So $t \to K_t \in L^1(X)$ is weakly measurable, but Pettis theorem [17] says that $t \to K_t \in L^1(X)$ is strongly measurable. □

Proposition 4.7. *For any $q > 1$ and almost all $t < \frac{(q-1)\lambda_0}{q^2}$,*

$$\|K_t\|_{L^q}^q \leqslant \int_X \exp\bigl(\lambda_0 \bigl|\mathrm{div}_\mu(Z)\bigr|\bigr) \, d\mu. \tag{4.27}$$

Proof. Take $T_0 = \frac{(q-1)\lambda_0}{q^2}$. Let $k_t^{m,n}$ be defined in (4.21). By (4.10), for $t \in [0, T_0]$,

$$\text{(i)} \qquad \int_{V_m} \bigl|k_t^{m,n}\bigr|^q \, d\gamma_m \leqslant \int_X e^{\frac{q^2 T_0}{q-1} |\mathrm{div}_\mu(Z)|} \, d\mu.$$

Then $k_t^{m,n}$ converges weakly in $L^{q'}([0, T_0] \times V_m)$ to k_t^m: for any $\alpha \in C_b([0, T_0] \times V_m)$,

$$\lim_{n \to +\infty} \int_{[0,T_0] \times V_m} \alpha(t, z) k_t^{m,n}(z) \, d\gamma_m(z) \, dt = \int_{[0,T_0] \times V_m} \alpha(t, z) k_t^m(z) \, d\gamma_m(z) \, dt,$$

where $q' = q/(q-1)$. According to (i), we have

$$\left| \int_{[0,T_0] \times V_m} \alpha(t, z) k_t^m(z) \, d\gamma_m(z) \, dt \right| \leqslant \left(\int_0^{T_0} \bigl\|\alpha(t, \cdot)\bigr\|_{L^{q'}} \, dt \right) \left(\int_X e^{\frac{q^2 T_0}{q-1} |\mathrm{div}_\mu(Z)|} \, d\mu \right)^{1/q}.$$

It follows that for almost all $t \in [0, T_0]$,

$$\bigl\|k_t^m\bigr\|_{L^q(V_m)}^q \leqslant \int_X e^{\frac{q^2 T_0}{q-1} |\mathrm{div}_\mu(Z)|} \, d\mu. \tag{4.28}$$

Let $K_t^{(m)} = k_t^{(m)} \circ \pi_m$. We saw that for almost all $t \in [0, T]$, $K_t^{(m)}$ converges weakly in $L^1(X)$ to K_t. Combining with (4.28), we get

$$\|K_t\|_{L^q(X)}^q \leqslant \int_X e^{\frac{q^2 T_0}{q-1} |\mathrm{div}_\mu(Z)|} \, d\mu, \tag{4.29}$$

which is nothing but (4.27). □

Proposition 4.8. *It holds that*

$$\int_{[0,T]\times X} \|U_t(x)\|_\infty \, d\mu(x)\, dt < +\infty.$$

Proof. By Young inequality, $\int_X \|U_t(x)\|_\infty \, d\mu(x) = \int_X \|x\|_\infty K_t(x)\, d\mu(x)$ is dominated by

$$\int_X e^{\|x\|_\infty}\, d\mu(x) + \int_X K_t(x)\log K_t(x)\, d\mu(x),$$

which is finite, due to (4.23). □

Proposition 4.9. *The sequence $(\tilde U^n_\cdot)_{n\geqslant 1}$ converges to $U_\cdot$ in $L^1([0,T]\times X; X)$.*

Proof. Fix $m \geqslant 1$ and consider $n \geqslant m$. We have

$$\|\tilde U^n_t(x) - \pi_m(\tilde U^n_t(x))\|_\infty \leqslant \sum_{\ell=m+1}^{n} \Big\| \sum_{k<2^\ell \text{ odd}} \langle h_{k,\ell}, \tilde U^n_t(x)\rangle h_{k,\ell}\Big\|_\infty + \|x - \pi_n(x)\|_\infty.$$

Let $e^n_\ell(t,x) = \|\sum_{k<2^\ell \text{ odd}} \langle h_{k,\ell}, \tilde U^n_t(x)\rangle h_{k,\ell}\|_\infty$, which is smaller than

$$2^{-(\ell+1)/2} \max_{k<2^\ell \text{ odd}} \big|\langle h_{k,\ell}, \tilde U^n_t(x)\rangle\big|.$$

Let $\xi_\ell(y) = \max_{k<2^\ell \text{ odd}} |\langle h_{k,\ell}, y\rangle|$. Proceeding as above, for a fixed $N \geqslant T/T_0$, we have

$$\int_X \xi_\ell(\tilde U^n_t)\, d\mu \leqslant \|\xi_\ell\|_{L^{2^N}} \cdot \int_X e^{\lambda_0|\mathrm{div}_\mu(Z)|}\, d\mu. \tag{4.30}$$

Using Lemma 4.10 below, for any $k > 2^N$, there is a constant $C_{N,k} > 0$ such that

$$\|\xi_\ell\|_{L^{2^N}} \leqslant C_{N,k} 2^{\ell/2k}.$$

Hence for some constant $C_{\lambda_0,k} > 0$,

$$\int_X e^n_\ell \, d\mu \leqslant C_{\lambda_0,k} 2^{-(1-\frac{1}{k})\ell/2}.$$

Let $\varepsilon > 0$. Thus for m big enough and $n \geqslant m$,

$$\int_{[0,T]\times X} \|\tilde U^n_t - \pi_m(\tilde U^n_t)\|_\infty \, d\mu\, dt \leqslant \varepsilon + T\int_X \|x - \pi_n(x)\|_\infty \, d\mu(x).$$

On the other hand for such a fixed m, due to (4.15) and the definition of U_t,

$$\lim_{n\to+\infty} \int_{[0,T]\times X} \|\pi_m(\tilde U^n_t) - \pi_m(U_t)\|_\infty \, d\mu\, dt = 0.$$

Now note that

$$\|\tilde U^n_t - U_t\|_\infty \leqslant \|\tilde U^n_t - \pi_m(\tilde U^n_t)\|_\infty + \|\pi_m(\tilde U^n_t) - \pi_m(U_t)\|_\infty + \|\pi_m(U_t) - U_t\|_\infty,$$

our result follows. □

Lemma 4.10. *Let $\{Y_1, \dots, Y_n\}$ be a family of standard Gaussian random variables and set $\eta_n = \max_{1\leqslant i\leqslant n} |Y_i|$. Then for any $p \geqslant 2$ and $k > p$, there exists $C_{p,k} > 0$ such that*

$$\mathbb{E}\big(\eta_n^p\big) \leqslant C_{p,k} n^{p/k}. \tag{4.31}$$

Proof. We have

$$\mathbb{E}\big(\eta_n^p\big) = \int_0^\infty p s^{p-1} P(\eta_n > s)\, ds.$$

Let $\delta \geqslant 1$ be arbitrary and split the integral $\int_0^\infty p s^{p-1} P(\eta_n > s)\, ds$ into two parts $\int_0^\delta + \int_\delta^\infty$. We have

$$\int_\delta^\infty p s^{p-1} P(\eta_n > s)\, ds \leqslant \sum_{i=1}^n \int_\delta^\infty p s^{p-1} P\big(|Y_i| > s\big)\, ds \leqslant 2pn \int_\delta^\infty s^{p-1} e^{-s^2/2}\, ds.$$

Let $k > p$ be given, then there exists $c_k > 0$ such that $e^{-s^2/2} \leqslant c_k s^{-k}$ for $s \geqslant 1$. Then

$$\int_\delta^{+\infty} s^{p-1} e^{-s^2/2}\, ds \leqslant c_k \int_\delta^{+\infty} s^{p-k-1}\, ds = \frac{c_k}{k-p} \delta^{p-k}.$$

Then $\mathbb{E}(\eta_n^p) \leqslant \delta^p + 2pn \frac{c_k}{k-p} \delta^{p-k}$. Let $f(\delta)$ be the function defined by the preceding expression. We have

$$f'(\delta) = p\delta^{p-1}\big(1 - 2c_k n \delta^{-k}\big).$$

$f'(\delta) = 0$ implies that $\delta = (2c_k)^{1/k} n^{1/k}$ and

$$f(\delta) = (2c_k)^{p/k} \frac{p+k-1}{k-1} \cdot n^{p/k}.$$

The estimate (4.31) follows. □

In what follows, we will take $q = 2$ and fix $T_0 = \frac{\lambda_0}{4}$ and consider the subsequence such that $\tilde{U}^n$ converges to U almost everywhere.

Proposition 4.11. *Uniformly with respect to $[0, T_0]$, as $n \to +\infty$,*

$$\int_0^t Z_n \circ \pi_n\big(\tilde{U}_s^n\big)\, ds \to \int_0^t Z(U_s)\, ds \quad \text{in all } L^q(X, \mu).$$

Proof. We have $U_t^n(z) = z + \int_0^t Z_n(U_s^n(z))\, ds$. Then for μ-a.s. $x \in X$,

$$U_t^n\big(\pi_n(x)\big) = \pi_n(x) + \int_0^t Z_n\big(U_s^n\big(\pi_n(x)\big)\big)\, ds,$$

or $\tilde{U}_t^n(x)$ solves $\tilde{U}_t^n(x) = x + \int_0^t Z_n \circ \pi_n(\tilde{U}_s^n(x))\,ds$. For each $t \in [0, T_0]$, we have

$$\left\| \int_0^t Z_n \circ \pi_n\big(\tilde{U}_s^n\big)\,ds - \int_0^t Z(U_s)\,ds \right\|_H$$
$$\leqslant \int_0^t \big\| Z_n \circ \pi_n\big(\tilde{U}_s^n\big) - Z\big(\tilde{U}_s^n\big)\big\|_H\,ds + \int_0^t \big\| Z\big(\tilde{U}_s^n\big) - Z(U_s)\big\|_H\,ds.$$

Let $q > 1$ and denote by $\mathbb{E}$ the integration with respect to μ on X. Then there is a constant $C_{q,T_0} > 0$ such that

$$\mathbb{E}\left[\sup_{t \leqslant T_0} \left\| \int_0^t Z_n \circ \pi_n\big(\tilde{U}_s^n\big)\,ds - \int_0^t Z(U_s)\,ds \right\|_H^q\right]$$
$$\leqslant C_{q,T_0}\left\{ \int_0^{T_0} \mathbb{E}\big[\big\| Z_n \circ \pi_n\big(\tilde{U}_s^n\big) - Z\big(\tilde{U}_s^n\big)\big\|_H^q\big]\,ds + \int_0^{T_0} \mathbb{E}\big[\big\| Z\big(\tilde{U}_s^n\big) - Z(U_s)\big\|_H^q\big]\,ds \right\}$$
$$= J_n^1 + J_n^2, \quad \text{respectively.}$$

We have $\mathbb{E}[\| Z_n \circ \pi_n(\tilde{U}_s^n) - Z(\tilde{U}_s^n)\|_H^q] = \mathbb{E}[\| Z_n \circ \pi_n - Z\|_H^q \tilde{K}_s^n]$. Using (4.10) for $q = 2$, there exists a constant $C_{\lambda_0,T_0} > 0$ such that

$$J_n^1 \leqslant C_{\lambda_0,T_0} \| Z_n \circ \pi_n - Z\|_{L^{2q}},$$

which tends to 0, as $n \to +\infty$ due to (2.3). For estimating J_n^2, we pick $\zeta \in C_b(X, H)$ such that $\|\zeta - Z\|_{L^{2q}}^q \leqslant \varepsilon$. We have

$$\lim_{n \to +\infty} \mathbb{E} \int_0^{T_0} \big\| \zeta\big(\tilde{U}_s^n\big) - \zeta(U_s)\big\|_H^q\,ds = 0.$$

By the choice of ζ and according to (4.29),

$$\mathbb{E} \int_0^{T_0} \big\| \zeta(U_s) - Z(U_s)\big\|_H^q\,ds = \int_0^{T_0} \mathbb{E}\big(\|\zeta - Z\|_H^q K_s\big)\,ds \leqslant C_{\lambda_0,T_0}\varepsilon$$

and by (4.10),

$$\mathbb{E} \int_0^{T_0} \big\| \zeta\big(\tilde{U}_s^n\big) - Z\big(\tilde{U}_s^n\big)\big\|_H^q\,ds \leqslant C_{\lambda_0,T_0}\varepsilon.$$

It follows that for some constant $C_{q,\lambda_0,T_0} > 0$,

$$\mathbb{E} \int_0^{T_0} \big\| Z\big(\tilde{U}_s^n\big) - Z(U_s)\big\|_H^q\,ds \leqslant C_{q,\lambda_0,T_0}\left(2\varepsilon + \mathbb{E} \int_0^{T_0} \big\| \zeta\big(\tilde{U}_s^n\big) - \zeta(U_s)\big\|_H^q\,ds\right).$$

Therefore $\lim_{n \to +\infty} \mathbb{E} \int_0^{T_0} \| Z(\tilde{U}_s^n) - Z(U_s)\|_H^q\,ds = 0$. We complete the proof. □

Now letting $n \to +\infty$ in

$$\tilde{U}_t^n(x) = x + \int_0^t Z_n \circ \pi_n\big(\tilde{U}_s^n(x)\big)\,ds,$$

we see that $U_\cdot(x) = x + \int_0^\cdot Z(U_s(x))\,ds$ holds in $L^1([0, T_0] \times X)$. Now we redefine

$$\tilde{U}_t(x) = x + \int_0^t Z\big(U_s(x)\big)\,ds.$$

Then for $t \in [0, T_0]$,

$$\tilde{U}_t(x) = x + \int_0^t Z\big(\tilde{U}_s(x)\big)\,ds. \tag{4.32}$$

Now by considering $-Z$, $\tilde{U}_{-t}$ solves

$$\tilde{U}_{-t}(x) = x - \int_0^t Z\big(\tilde{U}_{-s}(x)\big)\,ds.$$

Proposition 4.12. *For each $t \in [0, T_0]$, the density K_t of $(\tilde{U}_t)_*\mu$ with respect to μ admits the explicit expression*

$$K_t(x) = \exp\left(\int_0^t \operatorname{div}_\mu(Z)(\tilde{U}_{-s})\,ds\right) \tag{4.33}$$

and

$$\|K_t\|_{L^q(X)}^q \leqslant \int_X e^{\frac{q^2 T_0}{q-1}|\operatorname{div}_\mu(Z)|}\,d\mu.$$

Proof. Similarly as for Z, we have, for any $q > 1$,

$$\lim_{n\to+\infty} \mathbb{E}\int_0^{T_0} \big|\operatorname{div}_{\gamma_n}(Z_n) \circ \pi_n\big(\tilde{U}_{-s}^n\big) - \operatorname{div}_\mu(Z)(U_{-s})\big|^q\,ds = 0.$$

Up to a subsequence, $\int_0^t \operatorname{div}_{\gamma_n}(Z_n) \circ \pi_n(\tilde{U}_{-s}^n)\,ds$ converges to $\int_0^t \operatorname{div}_\mu(Z)(U_{-s})\,ds$ a.s. Now $\tilde{K}_t^n$ admits the expression

$$\tilde{K}_t^n = \exp\left(\int_0^t \operatorname{div}_{\gamma_n}(Z_n) \circ \pi_n\big(\tilde{U}_{-s}^n\big)\,ds\right),$$

which converges a.s. to

$$\exp\left(\int_0^t \operatorname{div}_\mu(Z)(U_{-s})\,ds\right).$$

Therefore for each $t \in [0, T_0]$, for any $F \in C_b(X)$,

$$\int_X F(U_t)\,d\mu = \lim_{n\to+\infty} \int_X F\big(\tilde{U}_t^n\big)\,d\mu = \lim_{n\to+\infty} \int_X F\tilde{K}_t^n\,d\mu = \int_X FK_t\,d\mu, \tag{4.34}$$

the passage of the last limit is guaranteed by (4.18). □

Definition 4.13. For $t \in [0, T_0]$, we define

$$\tilde{U}_{t+T_0}(x) = \tilde{U}_t\big(\tilde{U}_{T_0}(x)\big).$$

Replacing x by $\tilde{U}_{T_0}(x)$ in (4.32), we have

$$\begin{aligned}
\tilde{U}_{t+T_0}(x) &= \tilde{U}_{T_0}(x) + \int_0^t Z\big(\tilde{U}_s\big(\tilde{U}_{T_0}(x)\big)\big)\,ds \\
&= \tilde{U}_{T_0}(x) + \int_0^t Z\big(\tilde{U}_{s+T_0}(x)\big)\,ds \\
&= \tilde{U}_{T_0}(x) + \int_{T_0}^{t+T_0} Z\big(\tilde{U}_s(x)\big)\,ds \\
&= x + \int_0^{t+T_0} Z\big(\tilde{U}_s(x)\big)\,ds.
\end{aligned}$$

In such a way, we redefine $\{\tilde{U}_t;\ t \in [0, 2T_0]\}$ which satisfies (4.32) for all $t \in [0, 2T_0]$ and so on.

Proposition 4.14. *For any $q > 1$,*

$$\lim_{m\to+\infty} \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \big\|\tilde{U}_t^n - \tilde{U}_t\big\|_\infty^q\Big] = 0. \tag{4.35}$$

Proof. We have first, for $t \in [0, T_0]$,

$$\tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t(\tilde{U}_{T_0}) = \tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t\big(\tilde{U}_{T_0}^n\big) + \tilde{U}_t\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t(\tilde{U}_{T_0}).$$

Then there exists a constant $C_q > 0$ such that

$$\begin{aligned}
&\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t(\tilde{U}_{T_0})\big\|_\infty^q \\
&\quad\leqslant C_q\Big\{\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t\big(\tilde{U}_{T_0}^n\big)\big\|_\infty^q + \sup_{t\leqslant T_0} \big\|\tilde{U}_t\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t(\tilde{U}_{T_0})\big\|_\infty^q\Big\}.
\end{aligned} \tag{4.36}$$

We have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t\big(\tilde{U}_{T_0}^n\big)\big\|_\infty^q\Big] &= \mathbb{E}\Big[\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n - \tilde{U}_t\big\|_\infty^q K_{T_0}^n\Big] \\
&\leqslant C_{\lambda_0,T_0} \cdot \Big(\mathbb{E}\Big[\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n - \tilde{U}_t\big\|_\infty^{2q}\Big]\Big)^{1/2} \to 0
\end{aligned}$$

as $n \to +\infty$, due to Proposition 4.11. For estimating the second term in (4.36), we remark that

$$\tilde{U}_\cdot : X \to C\big([0, T_0], X\big)$$

satisfies $\mathbb{E}[\sup_{t\leqslant T_0} \|\tilde{U}_t\|_\infty^{2q}] < +\infty$ due to (4.32). Note that the estimate (4.27) holds for T_0. Therefore we can proceed as for estimating J_n^2 in the proof of Proposition 4.11. Finally, we get

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{t\leqslant T_0} \big\|\tilde{U}_t^n\big(\tilde{U}_{T_0}^n\big) - \tilde{U}_t(\tilde{U}_{T_0})\big\|_\infty^q\Big] = 0.$$

Note that $\tilde{U}_{t+s}^n = \tilde{U}_t^n \circ \tilde{U}_t^n$. Then

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{t\leqslant T_0} \big\|\tilde{U}_{t+T_0}^n - \tilde{U}_{t+T_0}\big\|_\infty^q\Big] = 0.$$

Combining this with Proposition 4.11, we get

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{t\leqslant 2T_0} \big\|\tilde{U}_t^n - \tilde{U}_t\big\|_\infty^q\Big] = 0.$$

Now considering $\tilde{U}_t^n(\tilde{U}_{T_0}^n) - \tilde{U}_t(\tilde{U}_{T_0})$ for $t \in [0, 2T_0]$, we get the result

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{t\leqslant 3T_0} \big\|\tilde{U}_t^n - \tilde{U}_t\big\|_\infty^q\Big] = 0.$$

In this way, we obtain finally (4.35). □

Theorem 4.15. *Let $Z \in \mathbb{D}_1^p(X, H)$ with $1 < p \leqslant 2$. Assume that*

$$\mathbb{E}\big(e^{\lambda_0(|Z|_H + |\mathrm{div}_\mu(Z)|)}\big) < +\infty \quad \textit{for some } \lambda_0 > 0.$$

Then there exists a unique flow of maps $(\tilde{U}_t)_{t\in[-T,T]}$ such that $(\tilde{U}_t)_*\gamma = K_t\gamma$ with K_t given in (4.33) *and solves*

$$\tilde{U}_t(x) = x + \int_0^t Z(\tilde{U}_s)\, ds.$$

Proof. We prove only the uniqueness. Let $(Y_t)_{t\in[-T,T]}$ be another flow of maps enjoying the same properties as for $(\tilde{U}_t)_{t\in[-T,T]}$. Let $\ell : X \to \mathbb{R}$ be a linear map and set $u_t = \ell(Y_t)$. For any $\alpha \in C_c^1([0, T))$ and $\varphi \in C_b^1(X)$, consider

$$\text{(i)} \qquad \Delta_\eta = \int_{[0,T]\times X} \frac{\alpha(t+\eta) - \alpha(t)}{\eta} \varphi(x) u_t(x)\, d\mu\, dt.$$

For $\eta > 0$ small enough, $\int_0^T \alpha(t+\eta)\varphi(x)u_t(x)\, dt = \int_\eta^T \alpha(t)\varphi(x)u_{t-\eta}(x)\, dt$. Then

$$\text{(ii)} \qquad \Delta_\eta = \int_\eta^T \int_X \alpha(t)\varphi \frac{u_{t-\eta} - u_t}{\eta}\, d\mu\, dt - \frac{1}{\eta}\int_0^\eta \Big(\int_X \alpha(t)\varphi u_t\, d\mu\Big) dt.$$

We have

$$\text{(iii)} \qquad \int_X \varphi(x)\frac{u_{t-\eta}-u_t}{\eta}\,d\mu = \int_X \frac{\varphi(Y_\eta)K_{-\eta}-\varphi}{\eta}u_t\,d\mu.$$

Note that

$$\lim_{\eta\to 0}\int_X \frac{\varphi(Y_\eta)K_{-\eta}-\varphi}{\eta}u_t\,d\mu = \int_X \big(\nabla\varphi\cdot Z-\varphi\,\mathrm{div}_\mu(Z)\big)u_t\,d\mu.$$

Letting $\eta\to 0$ in (i)–(iii), we get

$$\int_{[0,T]\times X} \big(-\alpha'(t)\varphi(x)u_t(x)-\alpha(t)D_Z^*\varphi(x)u_t(x)\big)\,d\mu(x)\,dt = \int_X \alpha(0)\varphi(x)u_0(x)\,d\mu(x).$$

Therefore u_t solves the transport equation (4.14) with initial data ℓ. Now by Theorem 3.3, we get $\ell(Y_t)=\ell(\tilde{U}_t)$ for $t\in[0,T]$ for each linear form. Hence $Y_t=\tilde{U}_t$ for $t\in[0,T]$. □

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