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A Certain Family of Meromorphically Multivalent Functions

S. B. JOSHI
Department of Mathematics, Walchand College of Engineering
Sangli 416415, Maharashtra, India
joshisb@wces.ernet.in

H. M. SRIVASTAVA
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3P4, Canada
hmsri@uvvm.uvic.ca

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Abstract—By making use of a familiar analogue of the Ruscheweyh derivative as well as of the principle of subordination between two analytic functions, the authors introduce and study rather systematically a certain family of meromorphically multivalent functions in the open unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Several inclusion properties of this family are associated with an integral operator of the Bernardi-Libera-Livingston type.

Keywords—Meromorphic functions, p-valent functions, Ruscheweyh derivative, Analytic functions, Inclusion property, Integral operator, Hadamard product (or convolution), Jack's Lemma, Parseval's identity.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Let Σ_p denote the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^{k-p+1}, \qquad (p \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic *and* p-valent in the *punctured* unit disk

$$\mathcal{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathcal{U} \setminus \{0\}.$$

In terms of the Hadamard product (or convolution) of two functions, we define an analogue of the familiar Ruscheweyh derivative [1] by

$$\mathcal{D}^{n+p-1} f(z) := \frac{1}{z^p(1-z)^{n+p}} * f(z), \qquad (n > -p;\ p \in \mathbb{N};\ f \in \Sigma_p) \tag{1.2}$$

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or equivalently, by

$$\mathcal{D}^{n+p-1} f(z) = \frac{1}{z^p} \left(\frac{z^{n+2p-1} f(z)}{(n+p-1)!} \right)^{(n+p-1)}, \qquad (n > -p;\ p \in \mathbb{N};\ f \in \Sigma_p), \tag{1.3}$$

where, *and throughout this paper*, n is assumed to be an integer ($> -p$).

It follows readily from (1.1) and (1.2) (or (1.3)) that

$$\mathcal{D}^{n+p-1} f(z) = z^{-p} + \sum_{k=0}^{\infty} \delta(n,k)\, a_k\, z^{k-p+1}, \qquad (n > -p;\ p \in \mathbb{N};\ f \in \Sigma_p), \tag{1.4}$$

where $f \in \Sigma_p$ is given by (1.1) and (for convenience)

$$\delta(n,k) := \binom{n+p+k}{k+1} = \binom{n+p+k}{n+p-1}. \tag{1.5}$$

Next, with a view to recalling the principle of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is *subordinate* to $g(z)$ if there exists a function $h(z)$ analytic in $\mathcal{U}$, with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1, \tag{1.6}$$

such that

$$f(z) = g\,(h(z))\,, \qquad (z \in \mathcal{U}). \tag{1.7}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.8}$$

In particular, if $g(z)$ is univalent in $\mathcal{U}$, subordination (1.8) is equivalent to (cf. [2, p. 190])

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \tag{1.9}$$

Many interesting families of analytic and multivalent functions were considered by earlier authors in *Geometric Function Theory* (cf., e.g., [3,4]). Motivated essentially by some recent works of Uralegaddi *et al.* [5–7], we aim at investigating here various properties and characteristics of a *new* family

$$\Omega_{n,p}(A,B;\alpha), \qquad (-1 \le B < A \le 1;\ 0 \le \alpha < p)$$

of meromorphically p-valent functions in $\mathcal{U}$, which is given by the following.

DEFINITION. *A function $f \in \Sigma_p$ is said to be in the class $\Omega_{n,p}(A,B;\alpha)$ of meromorphically p-valent functions in $\mathcal{U}$ if*

$$-z^{p+1} \left(\mathcal{D}^{n+p-1} f(z)\right)' \prec \frac{p + \{pB + (p-\alpha)(A-B)\}\, z}{1+Bz}, \qquad (z \in \mathcal{U};\ -1 \le B < A \le 1;\ 0 \le \alpha < p;\ n > -p;\ p \in \mathbb{N}) \tag{1.10}$$

or equivalently, if

$$-z^{p+1} \left(\mathcal{D}^{n+p-1} f(z)\right)' = \frac{p + \{pB + (p-\alpha)(A-B)\}\, h(z)}{1+Bh(z)}, \qquad (z \in \mathcal{U};\ h \in \Lambda;\ -1 \le B < A \le 1;\ 0 \le \alpha < p;\ n > -p;\ p \in \mathbb{N}), \tag{1.11}$$

where

$$\Lambda := \{f : f \text{ analytic in } \mathcal{U},\ f(0) = 0, \text{ and } |f(z)| < 1\ (z \in \mathcal{U})\}. \tag{1.12}$$

Since

$$z\left(\mathcal{D}^{n+p-1} f(z)\right)' = (n+p)\,\mathcal{D}^{n+p} f(z) - (n+2p)\,\mathcal{D}^{n+p-1} f(z), \tag{1.13}$$

this last condition (1.11) can be rewritten in yet another equivalent form

$$z^p\left\{(n+p)\,\mathcal{D}^{n+p} f(z) - (n+2p)\,\mathcal{D}^{n+p-1} f(z)\right\} = -\frac{p + \{pB + (p-\alpha)(A-B)\}\,h(z)}{1+Bh(z)}, \tag{1.14}$$
$$(z \in \mathcal{U};\ h \in \Lambda;\ -1 \le B < A \le 1;\ 0 \le \alpha < p;\ n > -p;\ p \in \mathbb{N}).$$

It should be remarked in passing that the special class $\Omega_{n,p}(A, B;\, 0)$ was considered earlier by Uralegaddi and Somanatha [6]. Furthermore, various other subclasses of the class Σ_p, defined in terms of the *modified* Ruscheweyh derivative (1.2), were studied recently (cf., e.g., [8–10]).

Each of the following results (Lemma 1 and Lemma 2 below) will be required in our present investigation. Lemma 2, in particular, is popularly known as *Jack's Lemma.*

LEMMA 1. *A function $f \in \Sigma_p$ is in the class $\Omega_{n,p}(A, B;\, \alpha)$ if and only if*

$$\left|z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + m\right| < M, \qquad (z \in \mathcal{U}), \tag{1.15}$$

where (and in what follows)

$$m := \alpha + \frac{(p-\alpha)(1-AB)}{1-B^2} \quad \text{and} \quad M := \frac{(p-\alpha)(A-B)}{1-B^2}, \qquad (-1 < B < A \le 1). \tag{1.16}$$

PROOF. Suppose that $f \in \Omega_{n,p}(A, B;\, \alpha)$. We then find from (1.11) and (1.16) that

$$z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + m = -M\,\omega(z), \tag{1.17}$$

where

$$\omega(z) := \frac{B + h(z)}{1 + Bh(z)}, \qquad (z \in \mathcal{U}). \tag{1.18}$$

Clearly, since $h \in \Lambda$, we have

$$|\omega(z)| < 1, \qquad (z \in \mathcal{U}), \tag{1.19}$$

and the desired inequality (1.15) follows immediately.

Conversely, suppose that inequality (1.15) holds true. Then

$$\left|\frac{z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + m}{M}\right| < 1, \qquad (z \in \mathcal{U}), \tag{1.20}$$

where m and M are given by (1.16).

Setting

$$g(z) = -\frac{z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + m}{M}, \qquad (z \in \mathcal{U}) \tag{1.21}$$

and

$$\begin{aligned} h(z) &= \frac{g(z) - g(0)}{1 - g(z)\,g(0)} \\ &= -\frac{z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + p}{pB + (p-\alpha)(A-B) + B\,z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)'}, \qquad (z \in \mathcal{U}), \end{aligned} \tag{1.22}$$

it is easily seen that $h \in \Lambda$.

Upon rearranging (1.22), we arrive at condition (1.11). Hence $f \in \Omega_{n,p}(A, B;\, \alpha)$.

LEMMA 2. *(See [11,12].) Let the (nonconstant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0\, w'(z_0) = c\, w(z_0), \tag{1.23}$$

where c is a real number and $c \ge 1$.

We shall also make use of the integral operator $\mathcal{J}_{\mu,p}$ analogous to the Bernardi-Libera-Livingston integral operator (cf., e.g., [4]), which we define here by (see also [8, p. 162, equation (2.11)])

$$(\mathcal{J}_{\mu,p} f)(z) := \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)\, dt, \qquad (\mu > 0;\ p \in \mathbb{N};\ f \in \Sigma_p). \tag{1.24}$$

2. AN INCLUSION PROPERTY OF THE CLASS $\Omega_{n,p}(A,B;\,\alpha)$

Applying Lemma 1 and Lemma 2, we shall first prove the main inclusion property of the class $\Omega_{n,p}(A,B;\,\alpha)$, which is given by the following.

THEOREM 1. *For any integer $n > -p$ ($p \in \mathbb{N}$),*

$$\Omega_{n+1,p}(A,B;\,\alpha) \subset \Omega_{n,p}(A,B;\,\alpha). \tag{2.1}$$

PROOF. Let $f \in \Omega_{n+1,p}(A,B;\,\alpha)$ and suppose that

$$z^p \left\{(n+p)\,\mathcal{D}^{n+p} f(z) - (n+2p)\,\mathcal{D}^{n+p-1} f(z)\right\} = -\frac{p + \{pB + (p-\alpha)(A-B)\}\,w(z)}{1 + B\,w(z)}, \tag{2.2}$$

where the function $w(z)$ is either analytic or meromorphic in $\mathcal{U}$. Clearly, $w(0) = 0$.

Upon differentiating both sides of (2.2) with respect to z and making use of identity (1.13), we obtain

$$\begin{aligned} z^{p+1}\left(\mathcal{D}^{n+p} f(z)\right)' = &-\frac{p + \{pB + (p-\alpha)(A-B)\}\,w(z)}{1 + B\,w(z)} \\ &- \frac{(p-\alpha)(A-B)}{n+p}\left[\frac{z\,w'(z)}{\{1 + B\,w(z)\}^2}\right]. \end{aligned} \tag{2.3}$$

Therefore,

$$\begin{aligned} z^{p+1}\left(\mathcal{D}^{n+p} f(z)\right)' + m = &\frac{m - p - \{(p-m)\,B + (p-\alpha)(A-B)\}\,w(z)}{1 + B\,w(z)} \\ &- \frac{(p-\alpha)(A-B)}{n+p}\left[\frac{z\,w'(z)}{\{1 + B\,w(z)\}^2}\right], \end{aligned} \tag{2.4}$$

where m is defined by (1.16).

Now let r^* denote the distance, from the origin, of the nearest pole of $w(z)$ in $\mathcal{U}$. Then $w(z)$ is analytic in

$$|z| < r_0 = \min\{r^*, 1\}.$$

By Lemma 2, there exists a point z_0 in the closed disk

$$|z| \leq r, \qquad (0 < r < r_0)$$

such that

$$z_0\,w'(z_0) = c\,w(z_0), \qquad (c \geq 1). \tag{2.5}$$

From (2.4) and (2.5), we readily get

$$z_0^{p+1}\left(\mathcal{D}^{n+p} f(z_0)\right)' + m = \frac{N(z_0)}{D(z_0)}, \tag{2.6}$$

where, for convenience,

$$\left(\mathcal{D}^{n+p} f(z_0)\right)' := \left(\mathcal{D}^{n+p} f(z)\right)'\Big|_{z=z_0}, \tag{2.7}$$

$$\begin{aligned} N(z_0) = (n+p)(m-p) - [(n+p)\,\{2(p-m)\,B + (p-\alpha)(A-B)\} \\ -c(p-\alpha)(A-B)]\,w(z_0) - (n+p)\,\{(p-m)\,B + (p-\alpha)(A-B)\}\,B\,\{w(z_0)\}^2, \end{aligned} \tag{2.8}$$

and

$$D(z_0) = (n+p)\left[1 + 2B\, w(z_0) + B^2 \{w(z_0)\}^2\right]. \tag{2.9}$$

Suppose that it were possible to have

$$\max_{|z|=r} |w(z)| = |w(z_0)| = 1 \tag{2.10}$$

for some r $(0 < r < r_0 \leq 1)$. Then, in view of the identities

$$p - m = BM \quad \text{and} \quad pB + (p-\alpha)(A-B) - Bm = M \tag{2.11}$$

for m and M defined by (1.16), we have

$$|N(z_0)|^2 - M^2|D(z_0)|^2 = \xi + 2\eta\, \Re\{w(z_0)\}, \tag{2.12}$$

where

$$\xi = c(p-\alpha)(A-B)\left\{c(p-\alpha)(A-B) + 2(n+p)\left(1+B^2\right)\, M\right\} \tag{2.13}$$

and

$$\eta = 2c(n+p)(p-\alpha)(A-B)\, BM. \tag{2.14}$$

It is easily observed from (2.12) that

$$|N(z_0)|^2 - M^2\, |D(z_0)|^2 > 0, \tag{2.15}$$

provided that

$$\xi \pm 2\eta > 0.$$

From (2.13) and (2.14) we do find that

$$\xi + 2\eta = c(p-\alpha)(A-B)\left\{c(p-\alpha)(A-B) + 2(n+p)(1+B)^2 M\right\} > 0$$

and

$$\xi - 2\eta = c(p-\alpha)(A-B)\left\{c(p-\alpha)(A-B) + 2(n+p)(1-B)^2 M\right\} > 0.$$

Thus it follows from (2.6) and (2.15) that

$$\left| z_0^{p+1} \left(\mathcal{D}^{n+p} f(z_0)\right)' + m \right| > M,$$

which, in view of Lemma 1, contradicts our assumption that

$$f \in \Omega_{n+1,p}(A, B;\, \alpha).$$

So we cannot have $|w(z_0)| = 1$. Consequently,

$$|w(z)| \neq 1, \qquad (|z| < r_0)\,.$$

Since $w(0) = 0$, $|w(z)|$ is continuous, and $|w(z)| \neq 1$ in $|z| < r_0$, $w(z)$ cannot have a pole on $|z| = r_0$. Therefore, $w(z)$ is analytic in $\mathcal{U}$ and satisfies the inequality

$$|w(z)| < 1, \qquad (z \in \mathcal{U}).$$

It follows from (2.2) and (1.14) that

$$f \in \Omega_{n,p}(A, B;\, \alpha),$$

which evidently completes the proof of Theorem 1.

3. AN INCLUSION PROPERTY ASSOCIATED WITH THE CLASS-PRESERVING INTEGRAL OPERATOR $\mathcal{J}_{\mu-p+1,p}$ $(\Re(\mu) > p-1;\ p \in \mathbb{N})$

In this section, we shall prove an inclusion property of the class $\Omega_{n,p}(A,B;\,\alpha)$ associated with the class-preserving integral operator $\mathcal{J}_{\mu-p+1,p}$ defined by (1.24). We first state the following.

THEOREM 2. *If $f \in \Omega_{n,p}(A,B;\,\alpha)$, then the function*

$$F(z) := (\mathcal{J}_{\mu-p+1,p}\, f)(z), \qquad (\Re(\mu) > p-1;\ p \in \mathbb{N}) \tag{3.1}$$

also belongs to the same class $\Omega_{n,p}(A,B;\,\alpha)$.

PROOF. It is easily verified that the function $F(z)$ defined by (3.1) satisfies the identity

$$z\left(\mathcal{D}^{n+p-1} F(z)\right)' = (\mu-p+1)\,\mathcal{D}^{n+p-1} f(z) - (\mu+1)\,\mathcal{D}^{n+p-1} F(z). \tag{3.2}$$

Now let us suppose that

$$-z^{p+1}\left(\mathcal{D}^{n+p-1} F(z)\right)' = \frac{p + \{pB + (p-\alpha)(A-B)\}\, w(z)}{1 + B\,w(z)}, \tag{3.3}$$

where the function $w(z)$ is either analytic or meromorphic in $\mathcal{U}$. Clearly, $w(0) = 0$.

It follows readily from (3.2) and (3.3) that

$$\begin{aligned}(\mu-p+1)\,\mathcal{D}^{n+p-1} f(z) = {} & (\mu+1)\,\mathcal{D}^{n+p-1} F(z) \\ & - z^{-p}\left[\frac{p + \{pB + (p-\alpha)(A-B)\}\, w(z)}{1 + B\,w(z)}\right].\end{aligned} \tag{3.4}$$

Upon differentiating both sides of (3.4) and making use of (3.3), we obtain

$$\begin{aligned}z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' = {} & -\frac{p + \{pB + (p-\alpha)(A-B)\}\, w(z)}{1 + B\,w(z)} \\ & - \frac{(p-\alpha)(A-B)}{\mu-p+1}\left[\frac{z\,w'(z)}{\{1 + B\,w(z)\}^2}\right],\end{aligned} \tag{3.5}$$

which readily yields

$$\begin{aligned}z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + m = {} & \frac{m - p - \{(p-m)\,B + (p-\alpha)(A-B)\}\, w(z)}{1 + B\,w(z)} \\ & - \frac{(p-\alpha)(A-B)}{\mu-p+1}\left[\frac{z\,w'(z)}{\{1 + B\,w(z)\}^2}\right],\end{aligned} \tag{3.6}$$

where m is defined, as before, by (1.16).

The assertion of Theorem 2 can now be deduced from (3.6) by employing the same technique as in our proof of Theorem 1 from (2.4).

In its *special* case when

$$\mu = n + 2p - 1, \qquad (n > -p;\ p \in \mathbb{N}), \tag{3.7}$$

the class-preserving operator involved in Theorem 2 would yield yet another inclusion property contained in the following.

THEOREM 3. *The function*

$$F(z) := (\mathcal{J}_{n+p,p} f)(z), \qquad (n > -p;\ p \in \mathbb{N}) \tag{3.8}$$

is in the class $\Omega_{n+1,p}(A, B;\ \alpha)$ *if and only if*

$$f \in \Omega_{n,p}(A, B;\ \alpha).$$

PROOF. Under the special case (3.7), identity (3.2) reduces immediately to

$$z\left(\mathcal{D}^{n+p-1} F(z)\right)' = (n+p)\,\mathcal{D}^{n+p-1} f(z) - (n+2p)\,\mathcal{D}^{n+p-1} F(z), \tag{3.9}$$

where $F(z)$ is *now* defined by (3.8).

Upon expressing the first member of (3.9) by means of identity (1.13), and then comparing the corresponding right-hand side with the second member of (3.9), we get (cf. [8, p. 164, equation (2.18)])

$$\mathcal{D}^{n+p} F(z) = \mathcal{D}^{n+p-1} f(z), \tag{3.10}$$

and hence,

$$\left(\mathcal{D}^{n+p} F(z)\right)' = \left(\mathcal{D}^{n+p-1} f(z)\right)', \qquad (n > -p;\ p \in \mathbb{N}), \tag{3.11}$$

which obviously proves Theorem 3.

4. THEOREMS INVOLVING SHARP COEFFICIENT ESTIMATES AND CONVEXITY OF THE CLASS $\Omega_{n,p}(A, B;\ \alpha)$

The following result provides *sharp* coefficient estimates for functions in the class $\Omega_{n,p}(A, B;\ \alpha)$.

THEOREM 4. *Let the function $f(z)$ defined by (1.1) be in the class $\Omega_{n,p}(A, B;\ \alpha)$. Then*

$$|a_k| \le \frac{(p-\alpha)(A-B)}{(k-p+1)\,\delta(n,k)}, \qquad \left(k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ k \ne p-1;\ p \in \mathbb{N}\right), \tag{4.1}$$

where $\delta(n,k)$ is defined by (1.5).

The result is sharp with the extremal function $f(z)$ given by

$$-z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' = \frac{p + \{pB + (p-\alpha)(A-B)\}\, z^{k+1}}{1 + B\, z^{k+1}}, \qquad (k \in \mathbb{N}_0). \tag{4.2}$$

PROOF. Since $f \in \Omega_{n,p}(A, B;\ \alpha)$, condition (1.11) can be rewritten in the form

$$\left(\mathcal{D}^{n+p-1} f(z)\right)' = -z^{-p-1}\,\frac{p + \{pB + (p-\alpha)(A-B)\}\, h(z)}{1 + B\, h(z)}, \tag{4.3}$$

where the function $h(z)$ given by

$$h(z) = \sum_{j=1}^{\infty} b_j\, z^j, \qquad (z \in \mathcal{U}) \tag{4.4}$$

is in the class Λ defined by (1.12). Thus, upon writing (4.3) in the form

$$\left(\mathcal{D}^{n+p-1} f(z)\right)' + p\, z^{-p-1} = -\left[\{pB + (p-\alpha)(A-B)\}\, z^{-p-1} + B\left(\mathcal{D}^{n+p-1} f(z)\right)'\right] h(z), \tag{4.5}$$

we find from (1.4) and (4.4) that

$$\begin{aligned}\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,a_k\,z^{k-p}\\ = -\left[(p-\alpha)(A-B)\,z^{-p-1}+B\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,a_k\,z^{k-p}\right]\sum_{j=1}^{\infty}b_j\,z^j,\end{aligned}\tag{4.6}$$

where $\delta(n,k)$ is defined by (1.5).

Upon rewriting (4.6) in its *equivalent* form

$$\begin{aligned}\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,a_k\,z^{k-p} = -(p-\alpha)(A-B)\sum_{k=0}^{\infty}b_{k+1}\,z^{k-p}\\ -B\sum_{k=0}^{\infty}\left(\sum_{j=1}^{k}(k-j-p+1)\,\delta(n,k-j)\,a_{k-j}\,b_j\right)z^{k-p},\end{aligned}\tag{4.7}$$

and equating the coefficients of z^{k-p} from both sides of (4.7), we obtain

$$\begin{aligned}(k-p+1)\,\delta(n,k)\,a_k = -(p-\alpha)(A-B)\,b_{k+1}\\ -B\sum_{j=1}^{k}(k-j-p+1)\,\delta(n,k-j)\,a_{k-j}\,b_j, \qquad (k\in\mathbb{N}_0),\end{aligned}\tag{4.8}$$

where, as usual, an empty sum is to be interpreted as nil.

Formula (4.8) expresses the coefficient a_k in terms of $a_0, a_1, \ldots, a_{k-1}$ $(k \in \mathbb{N})$. Hence, for $k \in \mathbb{N}_0$, it follows from (4.6) that

$$\begin{aligned}\sum_{j=0}^{k}(j-p+1)\,\delta(n,j)\,a_j\,z^{j+1}+\sum_{j=k}^{\infty}c_j\,z^{j+2}\\ = -\left[(p-\alpha)(A-B)+B\sum_{j=0}^{k-1}(j-p+1)\,\delta(n,j)\,a_j\,z^{j+1}\right]h(z), \qquad (k\in\mathbb{N}_0)\end{aligned}\tag{4.9}$$

for some complex numbers c_j $(j = k, k+1, k+2, \ldots)$. Since $|h(z)| < 1$ $(z \in \mathcal{U})$, by applying Parseval's identity (cf. [13, p. 100]), we get

$$\begin{aligned}\sum_{j=0}^{k}(j-p+1)^2\{\delta(n,j)\}^2|a_j|^2 r^{2(j+1)}+\sum_{j=k}^{\infty}|c_j|^2 r^{2(j+2)}\\ \le (p-\alpha)^2(A-B)^2+B^2\sum_{j=0}^{k-1}(j-p+1)^2\{\delta(n,j)\}^2|a_j|^2 r^{2(j+1)}\\ \le (p-\alpha)^2(A-B)^2+B^2\sum_{j=0}^{k-1}(j-p+1)^2\{\delta(n,j)\}^2|a_j|^2, \qquad (0<r<1).\end{aligned}\tag{4.10}$$

Letting $r \to 1-$ in (4.10), we obtain the inequality

$$\begin{aligned}\sum_{j=0}^{k}(j-p+1)^2\{\delta(n,j)\}^2|a_j|^2+\sum_{j=k}^{\infty}|c_j|^2\\ \le (p-\alpha)^2(A-B)^2+B^2\sum_{j=0}^{k-1}(j-p+1)^2\{\delta(n,j)\}^2|a_j|^2,\end{aligned}\tag{4.11}$$

which may be simplified as

$$\begin{aligned}(k-p+1)^2\left\{\delta(n,k)\right\}^2|a_k|^2 &\leq (p-\alpha)^2(A-B)^2-\left(1-B^2\right)\sum_{j=0}^{k-1}(j-p+1)^2\left\{\delta(n,j)\right\}^2|a_j|^2 \\ &\leq (p-\alpha)^2(A-B)^2, \qquad (k\in\mathbb{N}_0).\end{aligned} \tag{4.12}$$

The main assertion of Theorem 4 follows immediately from (4.12).

Next we give a *sufficient* condition, in terms of the coefficients, for a function to be in the class $\Omega_{n,p}(A,B;\alpha)$ when $-1\leq B<0$.

THEOREM 5. *Let the function $f(z)$ defined by (1.1) be analytic in the punctured unit disk $\mathcal{U}^*$. Also let $-1\leq B<0$. If*

$$\sum_{k=0}^{\infty}(k-p+1)(1-B)\,\delta(n,k)\,|a_k|\leq(p-\alpha)(A-B), \tag{4.13}$$

where $\delta(n,k)$ is defined by (1.5), then

$$f\in\Omega_{n,p}(A,B;\alpha), \qquad (-1\leq B<0;\ B<A\leq 1).$$

The result is sharp with the extremal function $f(z)$ given by

$$\begin{gathered}f(z)=z^{-p}+\frac{(p-\alpha)(A-B)}{(k-p+1)(1-B)\,\delta(n,k)}\,z^{k-p+1}, \\ (k\in\mathbb{N}_0;\ k\neq p-1;\ p\in\mathbb{N};\ -1\leq B<0;\ B<A\leq 1).\end{gathered} \tag{4.14}$$

PROOF. Suppose that inequality (4.13) holds true. Then, for $z\in\mathcal{U}$, we find from (1.4) that

$$\begin{aligned}\left|z^{p+1}\left(\mathcal{D}^{n+p-1}f(z)\right)'+p\right| &= \left|\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,a_k\,z^{k+1}\right| \\ &\leq \sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,|a_k|\,r^{k+1}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\left|\{pB+(p-\alpha)(A-B)\}+B\,z^{p+1}\left(\mathcal{D}^{n+p-1}f(z)\right)'\right| \\ &\quad=\left|(p-\alpha)(A-B)+B\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,a_k\,z^{k+1}\right| \\ &\quad\geq (p-\alpha)(A-B)+B\sum_{k=0}^{\infty}(k-p+1)\,\delta(n,k)\,|a_k|\,r^{k+1},\end{aligned} \tag{4.16}$$

since $-1\leq B<0$ $(0<r<1)$. Letting $r\to 1-$ in (4.15) and (4.16), if we appropriately combine the resulting inequalities, we obtain

$$\begin{aligned}&\left|z^{p+1}\left(\mathcal{D}^{n+p-1}f(z)\right)'+p\right|-\left|\{pB+(p-\alpha)(A-B)\}+B\,z^{p+1}\left(\mathcal{D}^{n+p-1}f(z)\right)'\right| \\ &\leq\sum_{k=0}^{\infty}(k-p+1)(1-B)\,\delta(n,k)\,|a_k|-(p-\alpha)(A-B) \\ &\leq 0,\end{aligned} \tag{4.17}$$

by virtue of condition (4.13). We thus find from (4.17) that

$$\left|\frac{z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + p}{\{pB + (p-\alpha)(A-B)\} + B\, z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)'}\right| \le 1, \qquad (z \in \mathcal{U}), \tag{4.18}$$

which is easily seen to be equivalent to condition (1.11). Hence $f \in \Omega_{n,p}(A,B;\,\alpha)$ under the hypotheses of Theorem 5.

The equality in (4.13) is attained for the extremal function $f(z)$ given by (4.14), since

$$\left|\frac{z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' + p}{\{pB + (p-\alpha)(A-B)\} + B\, z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)'}\right| = 1, \qquad (z = 1), \tag{4.19}$$

where $f(z)$ is given by (4.14).

REMARK. The converse of Theorem 5 is *not* true. Consider the function $f(z)$ given by (1.1) for which the following condition holds true:

$$\begin{gathered} z^{p+1}\left(\mathcal{D}^{n+p-1} f(z)\right)' = \frac{p + \{pB + (p-\alpha)(A-B)\}\, z}{1+Bz}, \\ (z \in \mathcal{U};\ -1 \le B < 0;\ B < A \le 1;\ 0 \le \alpha < p;\ p \in \mathbb{N}). \end{gathered} \tag{4.20}$$

It is evident that $f \in \Omega_{n,p}(A,B;\,\alpha)$. Furthermore, it is easily verified for this function that

$$a_k = \frac{(p-\alpha)(A-B)(-B)^k}{(k-p+1)\,\delta(n,k)}, \qquad (k \in \mathbb{N}_0;\ k \ne p-1;\ p \in \mathbb{N}). \tag{4.21}$$

It follows from (4.21) that

$$\begin{aligned} \sum_{k=0}^{\infty}(k-p+1)(1-B)\,\delta(n,k)\,|a_k| &= (p-\alpha)(A-B)\sum_{k=0}^{\infty}(1-B)(-B)^k \\ &> (p-\alpha)(A-B), \qquad (-1 \le B < 0;\ B < A \le 1), \end{aligned} \tag{4.22}$$

which obviously contradicts condition (4.13) of Theorem 5.

Finally, we state a theorem which exhibits the fact that the class $\Omega_{n,p}(A,B;\alpha)$ is *convex*. The proof is fairly straightforward and is left as an exercise for the interested reader.

THEOREM 6. *Suppose that each of the functions $f(z)$ and $g(z)$ is in the class $\Omega_{n,p}(A,B;\,\alpha)$. Then the function $h(z)$ defined by*

$$h(z) := \mu\, f(z) + (1-\mu)\, g(z), \qquad (0 \le \mu \le 1) \tag{4.23}$$

is also in the same class $\Omega_{n,p}(A,B;\,\alpha)$.

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