The aim of this paper is to introduce new tools for studying the following two important and difficult problems in $\mathbb{R}^3$: (1) The Minkowski problem (to prescribe the Gauss curvature) for hedgehogs (i.e., for Minkowski differences of convex bodies); (2) The search for Sturm–Hurwitz type theorems (relating number of zeros to expansions in spherical harmonics). First, (1) we give a brief survey of hedgehog theory and a short introduction to these problems; (2) we recall briefly the main results already obtained (one of which is a counter-example to a conjecture of A.D. Alexandrov) and we explain why new tools are necessary for going further. Finally, we introduce a new notion of index for studying hedgehogs and we give first geometrical applications.

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0. General introduction

The set $\mathcal{K}^{n+1}$ of convex bodies of $(n + 1)$-Euclidean vector space $\mathbb{R}^{n+1}$ is usually equipped with Minkowski addition and multiplication by non-negative real numbers which are respectively defined by:

(i) $\forall (K, L) \in (\mathcal{K}^{n+1})^2$, $K + L = \{u + v | u \in K, v \in L\}$;
(ii) $\forall \lambda \in \mathbb{R}_+, \forall K \in \mathcal{K}^{n+1}$, $\lambda \cdot K = \{\lambda u | u \in K\}$.

Of course, $(\mathcal{K}^{n+1}, +, \cdot)$ does not constitute a vector space since we cannot subtract convex bodies in $\mathcal{K}^{n+1}$. Now, in the same way as we construct the group of integers from the set of natural numbers, we can construct the real vector space $(\mathcal{H}^{n+1}, +, \cdot)$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$ from $(\mathcal{K}^{n+1}, +, \cdot)$. Moreover, we can: 1. Consider each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$, called a hedgehog; 2. Extend the mixed volume $V : (\mathcal{K}^{n+1})^{n+1} \rightarrow \mathbb{R}$ to a symmetric $(n + 1)$-linear form on $\mathcal{H}^{n+1}$. Thus, the development of hedgehog theory can be seen as an attempt to extend certain parts of the Brunn–Minkowski theory to...
by first constructing ‘hyperbolic polytopal hedgehogs’, and

Conjecture (H) (1917)

Conjecture (C) 13

Conjecture (H)

If a Fourier series has at least as many zeros as its first nonvanishing harmonics. It has many

= operator, respectively the sum and the product of the eigenvalues of the Hessian of \(h\) (resp. with positive Gauss curvature) whose principal curvatures \(k_1\) and \(k_2\) satisfy the following inequality

\[(k_1 - c) (k_2 - c) \leq 0,\]

with some constant \(c > 0\), then \(S\) must be a sphere of radius \(1/c\).

Since the problem is to compare \(S\) with a sphere \(\Sigma\) of radius \(1/c\), the author had the idea to consider the hedgehog \(\mathcal{H} = S - \Sigma\) and to split \(S\) into the sum \(\Sigma + \mathcal{H}\). This approach led to the following reformulation of Conjecture (C):

Conjecture (H). If \(\mathcal{H}\) is a hedgehog of \(\mathbb{R}^3\) with a \(C^2\) support function whose curvature function – the product of the principal radii of curvature – is non-positive all over the unit sphere \(S^2\), then \(\mathcal{H}\) is (reduced to) a single point.

Formulations (C) and (H) are equivalent. In particular, if \(\mathcal{H}\) is any counter-example to (H) and \(\Sigma\) any sphere with a large enough radius, then \(S = \Sigma + \mathcal{H}\) is a counter-example to (C). Having produced an explicit counter-example to (H), the author thus disproved Conjecture (C) [11]. Later, Panina produced new counter-examples to Conjecture (H) by first constructing ‘hyperbolic polytopal hedgehogs’, and then using smoothening techniques [19].

Let us illustrate the second principle by two important problems, the first of which is the Minkowski problem for hedgehogs. The classical Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of \(\mathbb{R}^{n+1}\) whose Gauss curvature is prescribed on \(S^n\) as a function of the normal. For \(C^2\)-hypersurfaces (i.e., \(C^2\)-hypersurfaces with positive Gauss curvature), this well-known problem is equivalent to the question of solutions of certain Monge–Ampère equations of elliptic type. Minkowski proved [17] that: If \(K\) is a continuous positive function on \(S^n\) of \(\mathbb{R}^{n+1}\) satisfying the following integral condition

\[
\int_{S^n} \frac{u}{K(u)} \, d\sigma(u) = 0, \tag{1}
\]

where \(\sigma\) is the spherical Lebesgue measure on \(S^n\), then \(K\) is the Gauss curvature (in the sense of Gauss’ definition) of a unique (up to translation) closed convex hypersurface \(\mathcal{H}\). The strong solution is due to Pogorelov [20] and Cheng and Yau [3] who proved independently that if \(K\) is of class \(C^m\) on \(S^n\), \((m \geq 3)\), then the support function of \(\mathcal{H}\) is of class \(C^{m+1,\alpha}\) for every \(\alpha = 0, 1\). This Minkowski problem has a natural extension to hedgehogs (i.e. to Minkowski differences of closed convex hypersurfaces). For non-convex ones, this extension is equivalent to the question of solutions of certain Monge–Ampère PDE’s of non-elliptic type (for which there was no global result). This geometrization enabled the author to give examples of Monge–Ampère PDE’s of mixed type with no solution [13] (resp. with non-unique solutions [12]) on \(S^2\). Besides, the falsity of Conjecture (H) can be stated as follows (which disproves a conjecture of Koutroufiotis and Nirenberg) [5]:

There exists a nonlinear function \(f : \mathbb{R}^3 \to \mathbb{R}\) whose restriction to \(S^2\), say \(h\), is \(C^2\) and satisfies the inequality \(h^2 + h\Delta_2 h + \Delta_{22} h \leq 0\), where \(\Delta_2\) is the spherical Laplacian and \(\Delta_{22}\) the Monge–Ampère operator, respectively the sum and the product of the eigenvalues of the Hessian of \(h = f|_{S^2}\).

The Sturm–Hurwitz theorem states that any continuous periodic real function expandable in a Fourier series has at least as many zeros as its first nonvanishing harmonics. It has many
geometrical consequences such as the 4-vertex theorem (e.g., [22]). The second problem is the search for Sturm–Hurwitz type theorems (particularly in higher dimensions). In the case of $C^2$-functions, the author gave a geometrical interpretation and a new proof of the Sturm–Hurwitz theorem by considering plane $N$-hedgehogs, ($N \in \mathbb{N}^+$) [14]. A plane $N$-hedgehog is defined as the envelope of a family of cooriented lines having exactly $N$ cooriented support lines with a given normal vector. Plane 1-hedgehogs are just plane hedgehogs.

**Which notion of index for studying hedgehogs in $\mathbb{R}^3$?**

As recalled, hedgehogs have given first interesting results for both problems. In order to go further, it is necessary to introduce new tools in dimensions greater than 2. In the first results, an essential role was played by the following relationship between the winding number $i_h(x)$ of an $N$-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ around a point $x \in \mathbb{R}^2 - \mathcal{H}_h$ and the number of cooriented support lines of $\mathcal{H}_h$ through $x$ (i.e. of zeros of $h_x : [0, 2N\pi] \rightarrow \mathbb{R}, \theta \mapsto h(x, u(\theta))$, where $u(\theta) = (\cos \theta, \sin \theta)$):

$$i_h(x) = N - \frac{1}{2} n_h(x),$$

where $h$ is the support function of $\mathcal{H}_h$. Given any hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, we can still define the index $i_h(x)$ of a point $x \in \mathbb{R}^3 - \mathcal{H}$ with respect to $\mathcal{H}_h$ (e.g. as an algebraic intersection number). But, as we shall see, it can no longer play the same role. For instance if $\mathcal{H}_h \subset \mathbb{R}^3$ is projective (i.e., if $h$ is antisymmetric), it gives no information either on $\mathcal{H}_h$ or on zeros of $h_x(u) = h(u) - (x, u)$ , $(u \in S^2)$. This paper introduces an $i_h$-index for hedgehogs of $\mathbb{R}^3$ that can in certain respects play the role the $i_h$-index does in $\mathbb{R}^2$. This index induces a series of new notions (of interior, algebraic volume, etc.) which allow us to study the geometry of hedgehogs of $\mathbb{R}^3$ (including projective ones). Besides, it also induces a natural notion of transverse orientation (which may change on certain curves of self-intersection) involved in the multiplicity of the solutions to the Minkowski problem.

**1. Generalities on hedgehogs**

As we have said, hedgehog theory consists of: 1. Considering the Brunn–Minkowski theory in the space $\mathcal{H}^{n+1}$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$; 2. Constructing geometrically any formal difference $K - L$ of convex bodies $K, L \in \mathcal{K}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$. In the case of convex hypersurfaces (or convex bodies) of class $C^2_+$ (i.e., of $C^2$-hypersurfaces with positive Gauss curvature), this can be done easily. As shown on Fig. 1, we can subtract such hypersurfaces by subtracting the points corresponding to a same outer normal to obtain a (possibly singular) hypersurface that we shall call a hedgehog. Let us recall how such a hedgehog can be defined.

As is well known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : S^n \rightarrow \mathbb{R}$, $u \mapsto \sup \{ \langle x, u \rangle | x \in K \}$, ($h_K(u)$ is the signed distance from the origin to the support hyperplane with normal $u$). In particular, every closed convex hypersurface of class $C^2_+$ is determined by its support function $h$ (which must be of class $C^2$ [21, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u \rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the following two equations

$$\begin{align*}
\langle x, u \rangle &= h(u) \\
\langle x, . \rangle &= dh_u(.),
\end{align*}$$

of which the second is obtained from the first by performing a partial differentiation with respect to $u$. From the first equation, the orthogonal projection of $x$ onto the line spanned by $u$ is $h(u)u$ and from the second one its orthogonal projection onto $u^\perp$ is the gradient of $h$ at $u$. Therefore, for each $u \in S^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

Now, the envelope $\mathcal{H}_h$ is in fact well defined for any $C^2$-function $h$ on $S^n$ (even if $h$ is not the support function of a convex body). Its parametrization $x_h : S^n \rightarrow \mathcal{H}_h, u \mapsto x_h(u)$ can be interpreted as the inverse of its Gauss map, in the sense that at each regular point $x_h(u)$, $u$ is normal to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function $h$. If $h$ is only $C^1$ then $\mathcal{H}_h$ is still defined but it is not necessarily a difference of convex bodies and can be a fractal [10].
**Gauss curvature of hedgehogs with a $C^2$ support function**

Let us begin by describing briefly hedgehogs with a $C^2$ support function. As we saw, such a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ may be singular. As $x_h : \mathbb{S}^n \to \mathcal{H}_h$ can be regarded as the inverse of the Gauss map, its Gauss curvature $K_h$ is given by $1$ over the determinant of the tangent map $T_u x_h: \forall u \in \mathbb{S}^n$, $K_h(u) = 1/|T_u x_h|$. Therefore, singularities are exactly the points where $K_h$ becomes infinite.

An important point for our study is that the so-called ‘curvature function’ $R_h := 1/K_h$ is well defined and continuous all over the unit sphere, including at the singular points, so that the Minkowski problem arises naturally for hedgehogs.

From an analytical point of view, we get exactly the same formulas as in the convex case. In particular, the curvature function can be given by

$$R_h(u) = \det \left[ H_{ij}(u) + h(u) \delta_{ij} \right],$$

where $\delta_{ij}$ are the Kronecker symbols and $(H_{ij}(u))$ the Hessian of $h$ at $u$ with respect to an orthonormal frame on the unit sphere $\mathbb{S}^n$.

**Orientation**

The hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ will be regarded as the oriented (possibly singular) hypersurface $x_h(\mathbb{S}^n)$ image of $\mathbb{S}^n$, equipped with its canonical orientation, under the map $x_h : \mathbb{S}^n \to \mathcal{H}_h \subset \mathbb{R}^{n+1}$. If $K_h(u) > 0$ (resp. $K_h(u) < 0$), the orientation of the tangent space $T_u \mathbb{S}^n$ is preserved (resp. reversed) by the tangent map $T_u x_h : T_u \mathbb{S}^n \to T_{x_h(u)} \mathcal{H}_h = T_{x_h(u)} \mathbb{S}^n$.

**Example of projective hedgehogs**

Concerning the spherical image of the classical models of the real projective plane in $\mathbb{R}^3$, such as the Boy surface or the Roman surface, Hilbert and Cohn-Vossen have written in *Geometry and the imagination*: “Unfortunately, the way in which it is distributed over the unit sphere has not yet been studied”. For projective hedgehogs $\mathcal{H}_h \subset \mathbb{R}^{n+1}$, i.e. for hedgehogs with an antisymmetric support function $h$, each pair of antipodal points on $\mathbb{S}^n$ corresponds to one and the same point on $\mathcal{H}_h$. So not too singular projective hedgehogs $\mathcal{H}_h \subset \mathbb{R}^3$ can be regarded as models of the real projective plane $\mathbb{R} \mathbb{P}^2$ whose Gauss map is a bijection from the model onto $\mathbb{R} \mathbb{P}^2$. Here is, for instance, a hedgehog version of the Roman surface: $\mathcal{H}_h$, where $h(x, y, z) = x(x^2 - 3y^2) + 2z^3, (x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. This model is represented on Fig. 2. As the Roman surface, it has a threefold axis of symmetry and three lines of self-intersection whose end points are singular points of the same topological type as Whitney umbrellas without the handle.

**Generic singularities**

Hedgehogs with a smooth support function have only Legendre singularities. Their generic singularities are cusp points in $\mathbb{R}^2$, cuspidal edges and swallowtails in $\mathbb{R}^3$. Swallowtails are the cusp points of cuspidal edges and we can distinguish two types of swallowtails (negative or positive) according to the sign of the Gauss curvature on the tail (see Fig. 3).

**General hedgehogs as differences of arbitrary convex bodies**

General hedgehogs are defined inductively as collections of lower-dimensional ‘support hedgehogs’. See [16] for precise definitions and details.

Fig. 4 represents a polygonal hedgehog obtained by subtracting squares.
2. Minkowski problem — Sturm–Hurwitz theorem

2.1. The Minkowski problem

The main results on the Minkowski problem have been summarized in the introduction. Now, let us consider its extension to hedgehogs. In this section, ‘hedgehog’ will mean ‘hedgehog with a $C^2$ support function’. As noticed in Section 1, the curvature function $R_h := 1/K_h$ of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is well defined and continuous all over $S^n$, including at the singular points, so that the Minkowski problem arises naturally for hedgehogs.

What can we expect for hedgehogs? For $n = 1$, the curvature function is a linear function of the support function so that the problem is simple even for general hedgehogs [16]. In higher dimensions the problem is very difficult and we shall only consider the case $n = 2$. From (2), the curvature function $R_h := 1/K_h$ of $\mathcal{H}_h \subset \mathbb{R}^3$ is then given by $R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h$, where $\Delta_2$ is the spherical Laplacian and $\Delta_{22}$ the Monge–Ampère operator (respectively the sum and the product of the eigenvalues $\lambda_1, \lambda_2$ of the Hessian of $h$). So, the equation we are dealing with is the following

$$h^2 + h\Delta_2 h + \Delta_{22} h = 1/K.$$
Its type is given by the sign of $1/K$. Thus, the classical Minkowski problem boils down to the study of Monge–Ampère equations of elliptic type since closed convex hypersurfaces of class $C^2$ have a positive Gauss curvature. But for non-convex hedgehogs (which must have hyperbolic regions), we have to deal with Monge–Ampère equations of mixed type on $S^2$ (a class of equations for which there is no global result but only local ones by Lin [7] and Zuily [23]).

What (necessary and sufficient) conditions must a continuous function on $S^2$ satisfy to be the curvature function of a hedgehog? Of course, integral condition (1) is still necessary. It simply expresses that any hedgehog of $\mathbb{R}^3$ is a closed surface. But it is no longer sufficient: for instance $-1$ satisfies this condition and cannot be the curvature function of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ since there is no compact surface with negative Gauss curvature in $\mathbb{R}^3$. Can the curvature function of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ be non-positive all over $S^2$? As recalled in the introduction, the answer is positive, which disproves Alexandrov’s uniqueness Conjecture (C). However it is negative in the analytic case since Alexandrov (1966) [2] and Münzner (1967) [18] proved that Conjecture (C) is true for analytic surfaces.

The crucial fact is the existence of a (noncompact) cross-cap hedgehog whose curvature function is defined and non-positive on $S^2$ minus a semigreat circle. By fitting 4 cross-caps together with a central part, the author constructed a closed surface to which he gave an appropriate saddle form to obtain a non-trivial hedgehog whose curvature function is non-positive all over $S^2$ [11]. Such a hedgehog is a counter-example to Conjecture (H) since it is not reduced to a point. By adding a large enough sphere to it, we get a counter-example to Conjecture (C).

The notion of index $i_h(x)$ of a point $x$ with respect to a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ played an important role in the way the author studied Conjecture (H) through orthogonal projection techniques adapted to hedgehogs [11, Theorem 1].

Discrete version

The Minkowski problem has a discrete version which can be extended to polytopal hedgehogs (i.e., to differences of convex polytopes). In [15], the author presented a discretization of his counter-example to (H), composed of a central part Fig. 5(a) and 4 discrete cross-caps Fig. 5(b). Its representation on $S^2$ is shown on Fig. 5(c). For each one of its face, the $i_h$-index is everywhere non-positive. A polytopal hedgehog satisfying this property is said to be hyperbolic.

Monge–Ampère equations of mixed type

Here are examples of Monge–Ampère equations of mixed type with no solution. For every fixed $v \in S^2$, the smooth function $R(u) = 1 - 2 \langle u, v \rangle^2$ satisfies integral condition (1) but is not a curvature function on $S^2$ [13]. The proof makes use of orthogonal projection techniques adapted to hedgehogs.

Now here is a non-trivial example of an equation with non-unique solutions: these two non-isometric hedgehogs of $\mathbb{R}^3$ have a smooth (but not analytic) support function and the same curvature function $R \in C(S^2; \mathbb{R})$ : $\mathcal{H}_f$ and $\mathcal{H}_g$, where $f(u) = \exp(-1/z^2)$ and $g(u) = \text{sign}(z)f(u)$, $u = (x, y, z) \in S^2 \subset \mathbb{R}^3$ and $z \neq 0$. Of course, if $f \in C^2(S^2; \mathbb{R})$ is a solution of the Monge–Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = R$, where $R \in C(S^2; \mathbb{R})$ satisfies the integral condition

$$\int_{S^2} uR(u) \, d\sigma(u) = 0,$$

then $g = -f$ also is. But then $f$ and $g$ correspond to isometric hedgehogs. If these hedgehogs bound a convex body, one of them will be transversally oriented towards the interior and the other towards the exterior.
2.2. The Sturm–Hurwitz theorem

Another important problem is the search for Sturm–Hurwitz type theorems. The Sturm–Hurwitz theorem states that any continuous real function of the form

$$h(\theta) = \sum_{n=-\infty}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

for some sequences of real numbers \((a_n)\) and \((b_n)\), has at least as many zeros as its first nonvanishing harmonics: \(# \{\theta \in [0, 2\pi] : h(\theta) = 0\} \geq 2N\).

For \(C^2\)-functions, we can give a geometrical interpretation and a geometrical proof by considering the \(2N\pi\)-periodic function \(h(\theta/N)\) as the support function of an \('N-hedgehog'\) \(\mathcal{H}_h \subset \mathbb{R}^2\) that is, of the envelope of a family of cooriented lines having exactly \(N\) cooriented support lines with a given normal \(u \in \mathbb{S}^1\) [14]; \(N\) is just the number of full rotations of the coorienting normal vector. Fig. 6(a) shows a projective hedgehog and Fig. 6(b) a 3-hedgehog. In the case of \(C^2\)-functions, the Sturm–Hurwitz theorem can be stated in terms of hedgehogs [14]:

**Hedgehog version of the theorem.** If \(\mathcal{H}_h \subset \mathbb{R}^2\) is an \(N\)-hedgehog such that

$$h(N\theta) = \sum_{n=-\infty}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

for some sequences of real numbers \((a_n)\) and \((b_n)\), then \(\mathcal{H}_h\) has no ‘positive area’ (that is, \(i_h(x) \leq 0\) for all \(x \in \mathbb{R}^2 - \mathcal{H}_h\)).

3. Usefulness and limitations of the usual index

The above hedgehog version of the Sturm–Hurwitz theorem is based on the following relationship between the index \(i_h(x)\) of \(x\) with respect to \(\mathcal{H}_h\) and the number of zeros of \(h_x(\theta) = h(\theta) - \langle x, u(\theta) \rangle\), where \(u(\theta) = (\cos \theta, \sin \theta)\) [9,14].

**Theorem** ([14]). For every \(N\)-hedgehog \(\mathcal{H}_h \subset \mathbb{R}^2\) with a \(C^2\) support function, we have:

$$\forall x \in \mathbb{R}^2 - \mathcal{H}_h, \quad i_h(x) = N - \frac{1}{2} n_h(x),$$

where \(n_h(x)\) is the number of cooriented support lines through \(x\) (i.e. the number of zeros of \(h_x : [0, 2N\pi] \rightarrow \mathbb{R}, \theta \mapsto h(\theta) - \langle x, u(\theta) \rangle\), where \(u(\theta) = (\cos \theta, \sin \theta)\)). Note that relationship (3) allows us to define \(i_h(x) \in \mathbb{Z} \cup \{-\infty\}\) for any \(x \in \mathbb{R}^2\).

The geometrical proof given in [14] consists in proving the hedgehog version using the two following key points: 1. The evolute of \(\mathcal{H}_h \subset \mathbb{R}^2\) is the \(N\)-hedgehog with support function \((\partial h)(\theta) = h'(\theta - \frac{\pi}{2})\); 2. For every \(x \in \mathbb{R}^2\), \(i_{\partial h}(x) \leq i_h(x)\).
**Conclusion.** This notion of index with respect to an $N$-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ and its relationship with the number of zeros of $h$, played an essential role in the way the author: 1. Geometrized the Sturm–Hurwitz theorem and gave a proof of it [14]; 2. Studied Conjecture (H) through orthogonal projection techniques [11].

What about the index in higher dimensions?

Given a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$, $(n \geq 1)$, the $i_h$-index of $x \in \mathbb{R}^{n+1} - \mathcal{H}_h$ with respect to $\mathcal{H}_h$ can be defined as the degree of the map

$$U_{(h,x)} : \mathbb{S}^n \to \mathbb{S}^n, \quad u \mapsto \frac{x_h(u) - x}{\|x_h(u) - x\|},$$

and interpreted as the algebraic intersection number of an oriented half-line with origin $x$ with the hypersurface $\mathcal{H}_h$ equipped with its transverse orientation (number independent of the oriented half-line for an open dense set of directions).

**Remark.** Many notions from the theory of convex bodies carry over to hedgehogs, and quite a number of classical results find their counterparts with, of course, a few adaptations. In particular, areas and volumes have to be replaced by their algebraic versions, which can take negative values. For example, the (algebraic $(n + 1)$-dimensional) volume of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ can be defined by

$$V(h) := \int_{\mathbb{R}^{n+1} - \mathcal{H}_h} i_h(x) d\lambda(x),$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$, and it satisfies

$$V(h) = \frac{1}{n+1} \int_{\mathbb{S}^n} h(u) R_h(u) d\sigma(u),$$

where $R_h$ is the curvature function and $\sigma$ the spherical Lebesgue measure on $\mathbb{S}^n$. See [8] for Alexandrov–Fenchel type inequalities for hedgehogs.

The $i_h$-index remains natural in dimension 3 but it is no longer relevant for studying our two problems. To understand it, consider the case of projective hedgehogs $\mathcal{H}_h \subset \mathbb{R}^3$ that is, the case where $h$ is antisymmetric. As $i_h(x)$ can be regarded as the algebraic intersection number of almost every oriented half-line with origin $x$ with $\mathcal{H}_h$ equipped with its transverse orientation, the map $x \mapsto i_h(x)$ is then identically equal to 0 on $\mathbb{R}^3 - \mathcal{H}_h$. So, it gives no information either on $\mathcal{H}_h$ or on zeros of the function $h_x(u) = h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$.

**Index of $\mathcal{H}_h \subset \mathbb{R}^3$ at a point $x$ and sign of $h_x(u) = h(u) - \langle x, u \rangle$**

Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog whose support function $h$ is of class $C^2$ on $\mathbb{S}^2$. For every $x \in \mathbb{R}^3$, define $h_x \in C^2(\mathbb{S}^2; \mathbb{R})$ by $h_x(u) := h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$: $h_x(u)$ may be interpreted as the signed distance from $x$ to the support hyperplane cooriented by $u$. It is such that: $\forall u \in \mathbb{S}^2, x_h(u) = h_x(u) + (\nabla h_x)(u) = x_h(u) - x$. Thus, for every $x \in \mathbb{R}^3 - \mathcal{H}_h$, $(\nabla h_x)(u) \neq 0$ whenever $h_x(u) = 0$.

**Remark.** For every $x \in \mathbb{R}^3 - \mathcal{H}_h$, the set $h_x^{-1}(\{0\})$ consists of a finite number of disjoint simple smooth closed curves of $\mathbb{S}^2$ on which $h_x$ changes sign cleanly.

**Theorem 1.** Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with support function $h \in C^2(\mathbb{S}^2; \mathbb{R})$. For every $x \in \mathbb{R}^3$, define $h_x \in C^2(\mathbb{S}^2; \mathbb{R})$ by $h_x(u) := h(u) - \langle x, u \rangle$, $(u \in \mathbb{S}^2)$. $h_x(u)$ may be interpreted as the signed distance from $x$ to the support hyperplane cooriented by $u$. We have: $\forall x \in \mathbb{R}^3 - \mathcal{H}_h$,

$$i_h(x) = r^-_h(x) - r^+_h(x),$$

where $r^-_h(x)$ (resp. $r^+_h(x)$) denotes the number of connected components of $\mathbb{S}^2 - h_x^{-1}(\{0\})$ on which $h_x$ is negative (resp. positive).
Lemma 2 follows from the map $x \mapsto h(x)$ that it essentially follows from the fact that the parametrization $h_x$ of $\mathcal{H}_h$ at which the support plane passes through $x$. Note that:

$$c_h(x) = r_h^-(x) + r_h^+(x) - 1.$$  

The proof is based on the two following lemmas.

**Lemma 1.** The map $x \mapsto i_h(x) - (r_h^+(x) - r_h^-(x))$ is constant on $\mathbb{R}^3 - \mathcal{H}_h$.

**Proof of Lemma 1.** The first step consists in proving that the map $x \mapsto i_h(x) - (r_h^+(x) - r_h^-(x))$ is constant on each connected component $\Omega$ of $\mathbb{R}^3 - \mathcal{H}_h$ by noticing that $x \mapsto r_h^-(x), x \mapsto r_h^+(x)$ and thus $x \mapsto c_h(x)$ are constant on $\Omega$. The second consists in proving that $x \mapsto i_h(x) - (r_h^+(x) - r_h^-(x))$ remains constant as $x$ crosses $\mathcal{H}_h$ transversally at a regular point $m$, distinguishing the cases of an elliptic – respectively hyperbolic – $m$. As $x$ crosses $\mathcal{H}_h$ transversally at a simple elliptic point $m = x_h(u)$ from locally convex to locally concave side, we have to distinguish two cases: (i) If $u$ is pointing towards the locally concave side then $i_h(x)$ decreases by one unit whereas $r_h^+(x)$ increases by one unit and $r_h^+(x)$ remains constant; (ii) If $u$ is pointing towards the locally convex side then $i_h(x)$ decreases by one unit whereas $r_h^+(x)$ remains constant. As $x$ crosses $\mathcal{H}_h$ transversally at a simple hyperbolic point $m = x_h(u)$ in the direction of $-u$, which is the unit normal at $m$ since $m$ is hyperbolic, then $i_h(x)$ decreases by one unit and there are exactly two possibilities: (i) If $c_h(x)$ increases by one unit then $r_h^+(x)$ increases by one unit and $r_h^-(x)$ remains constant; (ii) If $c_h(x)$ decreases by one unit then $r_h^+(x)$ decreases by one unit and $r_h^-(x)$ remains constant. The proof is similar at multiple regular points (as for instance in the projective case). □

**Lemma 2.** If the Euclidean norm of $x$ is sufficiently large, then $c_h(x) = 1$.

**Proof of Lemma 2.** It essentially follows from the fact that the parametrization $x_h : \mathbb{S}^2 \to \mathcal{H}_h$ can be interpreted as the inverse of the Gauss map of $\mathcal{H}_h$. □

**Lemma 2** implies that $r_h^- = r_h^+ = 1$ when the Euclidean norm of $x$ is sufficiently large. Thus Theorem 1 follows from Lemma 1. □

4. New notion of index in $\mathbb{R}^3$ and applications

Now, here is a more appropriate notion of index for studying hedgehogs of $\mathbb{R}^3$.

**Definition.** Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with support function $h \in C^2(\mathbb{S}^2; \mathbb{R})$. For every $x \in \mathbb{R}^3$, define $h_x \in C^2(\mathbb{S}^2; \mathbb{R})$ by $h_x(u) := h(u) - (x, u), \ (u \in \mathbb{S}^2)$ : $h_x(u)$ may be interpreted as the signed distance from $x$ to the support hyperplane cooriented by $u$. For every $x \in \mathbb{R}^3 - \mathcal{H}_h$, define the $j_h$-index of $x$ with respect to $\mathcal{H}_h$ by:

$$j_h(x) := 1 - c_h(x),$$

where $c_h(x)$ denotes the number of connected components of $h_x^{-1}(\{0\}) \subset \mathbb{S}^2$, that is the number of spherical curves corresponding to points of $\mathcal{H}_h$ at which the support plane passes through $x$.

In certain respects, this $j_h$-index can play in $\mathbb{R}^3$ the same role as the $i_h$-index does in $\mathbb{R}^2$ (compare the definition of $j_h(x)$ with the relationship between the $i_h$-index of $x$ with respect to $\mathcal{H}_h \subset \mathbb{R}^2$ and the number of zeros of the function $h_x(u) = h(u) - (x, u), \ (u \in \mathbb{S}^1)$).

Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a hedgehog with a $C^2$-support function. When the Euclidean norm of $x \in \mathbb{R}^3$ is sufficiently large, $c_h(x)$ must be equal to 1 (see Lemma 2) and thus $j_h(x)$ to 0. In other words, the map $x \mapsto j_h(x)$ is identically equal to 0 on the unbounded connected component of $\mathbb{R}^3 - \mathcal{H}_h$. Note that we may have $j_h(x) = 0$ on a bounded connected component of $\mathbb{R}^3 - \mathcal{H}_h$.

**Remark.** The value of $j_h(x)$ must obviously decrease as $x$ crosses $\mathcal{H}_h$ transversally at an elliptic point from locally convex to locally concave side.
Additional definitions. Here are some additional definitions to describe the geometry of hedgehogs of $\mathbb{R}^3$. The interior (resp. the exterior) of $\mathcal{H}_h \subset \mathbb{R}^3$ relative to its $j_h$-index, or $j_h$-interior (resp. $j_h$-exterior) of $\mathcal{H}_h$, will be defined by:

$$
\mathcal{F}_h = \{ x \in \mathbb{R}^3 : \mathcal{H}_h \cap \{ x \} \neq 0 \} \\
(\text{resp. } \mathcal{E}_h = \{ x \in \mathbb{R}^3 : \mathcal{H}_h \cap \{ x \} = 0 \}).
$$

Recall that the interior (resp. exterior) of $\mathcal{H}_h$ relative to the $i_h$-index is usually defined by

$$
\mathcal{I}_h = \{ x \in \mathbb{R}^3 : \mathcal{H}_h \cap \{ x \} \neq 0 \} \\
(\text{resp. } \mathcal{E}_h = \{ x \in \mathbb{R}^3 : \mathcal{H}_h \cap \{ x \} = 0 \}).
$$

For all $x \in \mathbb{R}^3 - \mathcal{H}_h$, $j_h(x) = 1 - c_h(x) = 0$ implies $i_h(x) = r_h^+(x) - r_h^-(x) = 0$. Therefore $\mathcal{I}_h \subset \mathcal{F}_h$. This inclusion may be strict as shown by the example of non-trivial projective hedgehogs of $\mathbb{R}^3$ (see geometrical applications below): indeed, for such a hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, we have $\mathcal{I}_h = \emptyset$ and $\mathcal{F}_h \neq \emptyset$.

Recall that we defined the convex interior of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ as the following convex subset of $\mathbb{R}^2$ [9]: $\mathcal{C}_h = \{ x \in \mathbb{R}^2 : \mathcal{H}_h \cap \{ x \} = 1 \}$. Similarly:

**Proposition and definition.** Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$. The following subset of $\mathbb{R}^3$

$$
\mathcal{C}_h = \{ x \in \mathbb{R}^3 : \mathcal{H}_h \cap \{ x \} = 1 \},
$$

is convex. We say that $\mathcal{C}_h$ is the convex interior of $\mathcal{H}_h$.

**Convexity of $\mathcal{C}_h$.** Suppose without loss of generality that the integral of $h$ over $\mathbb{S}^2$ is non-negative. From the definition of the $j_h$-index, $\mathcal{C}_h$ is the set of all the points $x \in \mathbb{R}^3$ for which $h_x(u) = h(u) - \langle x, u \rangle$ has no zero on $\mathbb{S}^2$. As $h_x$ is a continuous function on $\mathbb{S}^2$, this last condition implies that $h_x$ is positive on $\mathbb{S}^2$ (it cannot be negative since its integral over $\mathbb{S}^2$ is equal to the one of $h$). Thus, $\mathcal{C}_h$ can be written in the form

$$
\mathcal{C}_h = \bigcap_{u \in \mathbb{S}^2} E_h^-(u),
$$

where $E_h^-(u)$ is the open halfspace with equation $\langle x, u \rangle < h(u)$. Therefore, $\mathcal{C}_h$ is a convex subset of $\mathbb{R}^3$ as intersection of convex subsets of $\mathbb{R}^3$. □

This new notion of index also implies a new notion of (algebraic) volume. The volume of $\mathcal{H}_h$ relative to its $j_h$-index, or $j_h$-volume of $\mathcal{H}_h$, will be defined by:

$$
V_{j_h}(h) := \int_{\mathbb{R}^3 - \mathcal{H}_h} j_h(x)d\lambda(x),
$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^3$.

**Case of polytopal hedgehogs**

We can naturally extend the definition of the $j_h$-index to hedgehogs of $\mathbb{R}^3$ whose support function is not of class $C^2$ on $\mathbb{S}^2$. In particular, we can define it for any polytopal hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ (i.e., for any difference $P - Q$ of two convex polytopes of $\mathbb{R}^3$) and the conclusion of Theorem 1 still holds for such a hedgehog.

**Examples of geometrical applications**

As an example of application, let us consider some geometrical consequences for projective hedgehogs of $\mathbb{R}^3$. By convention, we shall say that $x = x_h(u)$ is a simple point of a projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$ if $-u$ and $u$ are the only two elements of $\mathbb{S}^2$ that are mapped to $x$ by the parametrization $x_h : \mathbb{S}^2 \to \mathcal{H}_h$.

**Theorem 2.** Let $\mathcal{H}_h \subset \mathbb{R}^3$ be a projective hedgehog whose (antisymmetric) support function is of class $C^2$ on $\mathbb{S}^2$. The following properties are satisfied:

(i) For every $x \in \mathbb{R}^3 - \mathcal{H}_h$, we have $j_h(x) = 1 - c_h(x) \leq 0$. In particular, the $j_h$-volume of $\mathcal{H}_h$ is non-positive: $V_{j_h}(h) \leq 0$;
(ii) Let \( x_h(u) \) be a simple elliptic point of \( \mathcal{H}_h \) adherent to the \( j_h \)-exterior. Then \( \mathcal{H}_h \) turns its convexity towards its \( j_h \)-interior at \( x_h(u) \) (in other words, there exists a neighbor of \( x_h(u) \) in \( \mathbb{R}^3 \) in which the support plane with equation \( (x, u) = h(u) \) does not intersect the \( j_h \)-exterior of \( \mathcal{H}_h \));

(iii) \( \mathcal{H}_h \) lies in the convex hull of its singularities;

(iv) The \( j_h \)-volume of \( \mathcal{H}_h \) is negative if \( \mathcal{H}_h \) is not reduced to a single point.

**Proof of Theorem 2.** Property (i). Since \( h \) is antisymmetric (and non-identically equal to zero) on \( \mathbb{S}^2 \), it must change sign on \( \mathbb{S}^2 \), so that \( c_h(x) \geq 1 \).

Property (ii). From (i), as \( x \) crosses \( \mathcal{H}_h \) transversally at \( x_h(u) \) in the direction of its \( j_h \)-interior, \( j_h(x) \) must decrease from 0 to \(-2\) (knowing that the \( j_h \)-index of a projective hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) takes its values in \( 2\mathbb{Z} \) since the parametrization \( x_h \) describes the surface twice). In other words, \( x \) is then crossing \( \mathcal{H}_h \) transversally at \( x_h(u) \) from locally convex to locally concave side.

Property (iii) is an immediate consequence of property (ii).

Property (iv). A non-trivial projective hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) must have elliptic points (see [11]) so that its \( j_h \)-index cannot be identically equal to 0 on \( \mathbb{R}^3 - \mathcal{H}_h \). \( \square \)

**Remarks.** 1. Property (iii) already appeared in [9]. Beware of plane representations of projective hedgehogs. They may be deceptive regarding singularities. For instance, when considering Fig. 2, Property (iii) seems to be not satisfied by our projective hedgehog version \( \mathcal{H}_h \) of the Roman surface. But in fact, the apparent contour of \( \mathcal{H}_h \) is entirely composed of singular points of \( \mathcal{H}_h \).

2. Properties (i)-(iv) have to be compared with the corresponding properties of plane projective hedgehogs (for which, of course, \( i_h \) is replacing \( j_h \)) [9].

3. It is not difficult to check that properties (i)-(iv) still hold for any hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) whose support function \( h \) satisfies

\[
\int_{\mathbb{S}^2} h(u) \, d\sigma(u) = 0,
\]

where \( \sigma \) denotes the spherical Lebesgue measure on \( \mathbb{S}^2 \). Let \( \mathcal{H}_h \subset \mathbb{R}^3 \) be such a hedgehog and assume that all its singularities are generic, \( h \in C^\infty(\mathbb{S}^2; \mathbb{R}) \). Then no negative swallowtail of \( \mathcal{H}_h \) is able to be seen from its \( j_h \)-exterior \( F_h \). In other words, if a point \( x_h(u) \) is a negative swallowtail of \( \mathcal{H}_h \) belonging to the closure of \( F_h \) then, near this point, the hyperbolic region to which it corresponds lies in the complement of \( F_h \).

4. Let us mention this problem raised by Langevin, Levitt and Rosenberg in [6]:

*Does there exist a projective hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) whose singular locus is reduced to one (or several) immersed cuspidal edge(s) (without any swallowtail)?*

The example of projective hedgehogs thus shows that the \( j_h \)-index is more appropriate for studying the geometry of hedgehogs in \( \mathbb{R}^3 \).

**Transverse orientation relative to the \( j_h \)-index and \( \epsilon_h \) functions**

This new notion of index induces a natural notion of transverse orientation which may change on certain curves of self-intersection. For any hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) with a \( C^2 \)-support function, this orientation is defined as follows: at each simple regular point \( x_h(u) \) of \( \mathcal{H}_h \), orient the normal line in the direction of the decrease of \( j_h(x) \). We then define \( \epsilon_h(u) \in \{-1, 1\} \) in order that

\[
\nu_h(u) = \epsilon_h(u) \text{ sign } [1/K_h(u)] \, u
\]

be the corresponding unit normal at \( x_h(u) \), where \( K_h(u) \) is the Gauss curvature of \( \mathcal{H}_h \) at \( x_h(u) \). If \( x_h(u) \) is not a simple regular point of \( \mathcal{H}_h \), let \( \epsilon_h(u) = 0 \). The sign of \( \epsilon_h(u) \) simply indicates if \( \nu_h(u) \) correspond to the usual transverse orientation of the hypersurface \( \mathcal{H}_h = x_h(\mathbb{S}^n) \) or not.

*Unless otherwise stated, from now on ‘transverse orientation’ will mean ‘transverse orientation relative to the \( j_h \)-index’.*

**Case of convex (resp. projective) hedgehogs**

**Remark.** For any hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) with a \( C^2 \)-support function, consider the hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) with support function \( \hat{h}(-u) = -h(u) \), \( (u \in \mathbb{S}^2) \). These two hedgehogs \( \mathcal{H}_h \) and \( \mathcal{H}_h \) have the same
geometrical realization: \( \forall u \in \mathbb{S}^2, x_h^\epsilon(u) = x_h(u) \). For every \( u \in \mathbb{S}^2 \), the support hyperplane of \( \mathcal{H}_h \) cooriented by \( u \) is the support hyperplane of \( \mathcal{H}_h^\epsilon \) cooriented by \( -u \), so that: \( \forall u \in \mathbb{S}^2, \epsilon_h^\epsilon(-u) = -\epsilon_h(u) \). In the case of a nonsingular hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \), which must bound a convex body \( K, \mathcal{H}_h \) is transversally oriented towards the exterior of \( K \). Moreover, in this case, for any interior point \( x \) of \( K \), we have: \( \epsilon_h = \text{sign}(h_x), \) where \( h_x(u) := h(u) - \langle x, u \rangle, (u \in \mathbb{S}^2) \). In the case of a projective hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \), we have \( \tilde{h} = h \) and thus: \( \forall u \in \mathbb{S}^2, \epsilon_h(-u) = -\epsilon_h(u) \).

**Changes of transverse orientation on a hedgehog of \( \mathbb{R}^3 \)**

It follows that non-trivial projective hedgehogs of \( \mathbb{R}^3 \) necessary present changes of transverse orientation on certain curves of self-intersection. Let us consider the example of our projective hedgehog version \( \mathcal{H}_h \) of the Roman surface, which is represented in Fig. 2. At any simple regular point \( x_h(u) \) of \( \mathcal{H}_h \), the unit normal \( v_h(u) \) points towards the \( j^\epsilon \)-interior, which is composed of the bounded components of \( \mathbb{R}^3 - \mathcal{H}_h \). On these components, \( j^\epsilon \) is everywhere equal to \(-2\) and the corresponding transverse orientation of \( \mathcal{H}_h \) changes on the three curves of self-intersection.

**Integral condition**

The \( j^\epsilon \)-volume of a hedgehog \( \mathcal{H}_h \subset \mathbb{R}^3 \) can be given by:

\[
V_{\mathcal{H}}(h) = \int_{\mathbb{S}^2} \epsilon_{h}^\epsilon(u) h(u) \frac{u}{K_h(u)} d\sigma(u),
\]

where \( \sigma \) is the spherical Lebesgue measure on \( \mathbb{S}^2 \) and \( K_h \) the Gauss curvature of \( \mathcal{H}_h \). From the translation invariance of this volume, we deduce the following relationship (which has to be compared with integral condition (1)):

**Proposition.** Let \( \mathcal{H}_h \subset \mathbb{R}^3 \) be a hedgehog with a \( C^2 \) support function. Then, we have:

\[
\int_{\mathbb{S}^2} \epsilon_{h}^\epsilon(u) \frac{u}{K_h(u)} d\sigma(u) = 0,
\]

where \( \sigma \) is the spherical Lebesgue measure on \( \mathbb{S}^2 \) and \( K_h \) the Gauss curvature.

**Proof.** For every \( x \in \mathbb{R}^3 \), consider the hedgehog with support function \( h_x(u) := h(u) - \langle x, u \rangle, (u \in \mathbb{S}^2) \). For all \( x \in \mathbb{R}^3 \), we have \( x_h(u) = x_h(u) - x \) and in particular \( \mathcal{H}_{h_x} = \mathcal{H}_h - \{x\} \). Therefore, we have:

\[
K_{h_x} = K_h, \quad \epsilon_{h_x} = \epsilon_h \quad \text{and} \quad V_{\mathcal{H}}(h_x) = V_{\mathcal{H}}(h).
\]

Using these equalities for every \( x \in \mathbb{R}^3 \), we obtain immediately:

\[
\forall x \in \mathbb{R}^3, \quad \int_{\mathbb{S}^2} \langle x, u \rangle \epsilon_{h}^\epsilon(u) \frac{u}{K_h(u)} d\sigma(u) = 0,
\]

that is,

\[
\left\langle x, \int_{\mathbb{S}^2} \epsilon_{h}^\epsilon(u) \frac{u}{K_h(u)} d\sigma(u) = 0 \right\rangle = 0,
\]

which achieves the proof. \( \Box \)

**On \( \epsilon_h \) functions and the non-uniqueness in the Minkowski problem**

As we have said, the following two non-isometric hedgehogs of \( \mathbb{R}^3 \) have the same curvature function \( R \in C(\mathbb{S}^2; \mathbb{R}) : \mathcal{H}_f \) and \( \mathcal{H}_g \), where \( f(u) = \exp(-1/z^2) \) and \( g(u) = \text{sign}(z)f(u), u = (x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3 \) and \( z \neq 0 \) (see Fig. 7). Note that \( \mathcal{H}_f \) is centrally symmetric whereas \( \mathcal{H}_g \) is projective. It is interesting to notice that \( \mathcal{H}_f \) and \( \mathcal{H}_g \) correspond to different \( \epsilon_h \) functions. More precisely, \( \epsilon_f(u) = -1 \) and \( \epsilon_g(u) = -\text{sign}(z) \) for all \( u = (x, y, z) \in \mathbb{S}^2 \) such that \( z \neq 0 \).

Similarly, if hedgehogs \( \mathcal{H}_f \) and \( \mathcal{H}_g \) are bounding the same centrally symmetric convex body \( K \subset \mathbb{R}^3 \) but equipped with opposite (usual) transverse orientations, then they have the same curvature function but opposite \( \epsilon_h \) functions. These examples suggest that a study of the multiplicity of solutions in the Minkowski problem for hedgehogs should take into account these \( \epsilon_h \) functions.
Fig. 7. Non-uniqueness in the Minkowski problem.

References