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On uniquely k-determined permutations

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Abstract

Motivated by a new point of view to study occurrences of consecutive patterns in permutations, we introduce the notion of uniquely k-determined permutations. We give two criteria for a permutation to be uniquely k-determined: one in terms of the distance between two consecutive elements in a permutation, and the other one in terms of directed hamiltonian paths in the certain graphs called path-schemes. Moreover, we describe a finite set of prohibitions that gives the set of uniquely k-determined permutations. Those prohibitions make the application of the transfer matrix method possible for determining the number of uniquely k-determined permutations.

Keywords: Permutation; de Bruijn graph; Consecutive pattern; Set of prohibitions; Reconstruction

1. Introduction

A *pattern* τ is a permutation on $\{1, 2, \ldots, k\}$. An occurrence of a *consecutive pattern* τ in a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ is a word $\pi_i\pi_{i+1} \ldots \pi_{i+k-1}$ that is *order-isomorphic* to τ. For example, the permutation 253164 contains two occurrences of the pattern 132, namely 253 and 164. In this paper we deal only with consecutive patterns, which causes omitting the word "consecutive" in defining a pattern to shorten the notation.

There are several approaches in the literature to study the *distribution* and, in particular, *avoidance*, of consecutive patterns in permutations. For example, direct combinatorial considerations are used in [9]; the *method of inclusion–exclusion* is used in [7,10]; the *tree representations of permutations* are used in [5]; the *spectral theory of integral operators* on $L^2([0, 1]^k)$ is used in [4]. In this paper we suggest yet another approach to study occurrences of consecutive patterns in permutations. The approach is based on considering the *graph of patterns overlaps* defined below, which is a similar to the *de Bruijn graph* studied broadly in the literature mainly in connection with combinatorics on words and graph theory. However, we do not intend to study/develop the approach in this paper, rather using it as a possible motivation for introducing our objects of interest.

Suppose we are interested in the number of occurrences of a pattern τ of length k in a permutation π of length n. To find this number, we scan π from left to right with a "window" of length k, that is, we consider $P_i = \pi_i\pi_{i+1} \ldots \pi_{i+k-1}$ for $i = 1, 2, \ldots, n-k+1$: if we meet an occurrence of τ, we register it. Each P_i forms a pattern of length k, and the procedure of scanning π gives us a path in the *graph* $\mathscr{P}_k$ *of patterns overlaps of order* k defined as follows (graphs of

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patterns/permutations overlaps appear in [1,2,8]). The nodes of $\mathscr{P}_k$ are all $k!$ k-permutations, and there is an arc from a node $a_1a_2\ldots a_k$ to a node $b_1b_2\ldots b_k$ if and only if $a_2a_3\ldots a_k$ and $b_1b_2\ldots b_{k-1}$ form the same pattern. Thus, for any n-permutation there is a path in $\mathscr{P}_k$ of length $n-k+1$ corresponding to it. For example, if $k=3$ then to the permutation 13542 there corresponds the path $123 \to 132 \to 321$ in $\mathscr{P}_3$. It is also clear that every path in the graph corresponds to at least one permutation.

Our approach to study the distribution of a consecutive pattern τ of length k among n-permutations is to take $\mathscr{P}_k$ and to consider all paths of length $n-k+1$ passing through the node τ exactly ℓ times, where $\ell=0, 1, \ldots, n-k+1$. Then we could count the permutations corresponding to the paths. Similarly, for the "avoidance problems" that attracted much attention in the literature, we proceed as follows: given a set of patterns of length k to avoid, we remove the corresponding nodes with the corresponding arcs from $\mathscr{P}_k$, consider all the paths of certain length in the graph obtained, and then count the permutations of interest.

However, a complication with the approach is that a permutation does not need to be reconstructible uniquely from the path corresponding to it. For example, the permutation 13542 above has the same path in $\mathscr{P}_3$ corresponding to it as the permutations 23541 and 12543. Thus, different paths in $\mathscr{P}_k$ may have different contributions to the number of permutations with the required properties; in particular, some of the paths in $\mathscr{P}_k$ give exactly one permutation corresponding to them. We call such permutations *uniquely k-determined* or just *k-determined* for brevity. The study of such permutations is the main concern of the paper, and it should be considered as the first step in understanding how to use our approach to the problems described. Also, in our considerations we assume that all the nodes in $\mathscr{P}_k$ are allowed while dealing with k-determined permutations, that is, we do not prohibit any pattern.

The paper is organized as follows. In Section 2 we study the set of k-determined permutations. In particular, we give two criteria for a permutation to be k-determined: one in terms of the distance between two *consecutive elements* in a permutation, and the other one in terms of directed hamiltonian paths in the certain graphs called *path-schemes*. We use the second criteria to establish (rough) upper and lower bounds for the number of k-determined permutations. Moreover, given an integer k, we describe a finite set of prohibitions that determines the set of k-determined permutations. Those prohibitions make the application of the *transfer matrix method* [14, Theorem 4.7.2] possible for determining the number of k-determined permutations and we discuss this in Section 3. As a corollary of using the method, we get that the generating function for the number of k-determined permutations is rational. Besides, we show that there are no *crucial permutations* in the set of k-determined permutations. (Crucial objects, in the sense defined below, are natural to study in infinite sets of objects defined by prohibitions; for instance, see [6] for some results in this direction related to words.) We consider in more details the case $k=3$ in Section 3.1. Finally, in Section 4, we state several open problems for further research.

2. Uniquely k-determined permutations

2.1. Distance between consecutive elements; a criterion on k-determinability

Suppose $\pi=\pi_1\pi_2\ldots\pi_n$ is a permutation and $i<j$. The *distance* $d_\pi(\pi_i,\pi_j)=d_\pi(\pi_j,\pi_i)$ between the elements π_i and π_j is $j-i$. For example, $d_{253164}(3,6)=d_{253164}(6,3)=2$.

Theorem 1 (*First criterion on k-determinability*). *An n-permutation π is k-determined if and only if for each $1\leqslant x<n$, the distance $d_\pi(x,x+1)\leqslant k-1$.*

Proof. Suppose for an n-permutation π, $d(x,x+1)\geqslant k$ for some $1\leqslant x<n$. This means that x and $x+1$ will never be inside a window of length k while scanning consecutive elements of π. Thus, these elements are incomparable in π in the sense that switching x and $x+1$ in π will lead to another permutation π' having the same path in $\mathscr{P}_k$ as π has. So, π is not k-determined.

On the other hand, if for each $1\leqslant x<n$, the distance $d_\pi(x,x+1)\leqslant k-1$, then assuming we know where 1 is in π (which is the case, see below), we can place uniquely 2, then 3, and so on, leading to the fact that π is k-determined.

To complete the proof, suppose that there are two permutations, $\pi_1\neq\pi_2$, satisfying the element distance condition, having the same path in $\mathscr{P}_n$, and such that 1 is in position i in π_1 and 1 is in position j in π_2, $i<j$. Let $X=x_1x_2\ldots x_k$ be the pattern formed by the elements in positions from the set $Y=\{j-k+1, j-k+2, \ldots, j\}$ in the permutations. Consider now placing the elements $1, 2, \ldots,$ one-by-one, in π_1. Let t, $1\leqslant t\leqslant j-1$, be the minimum element such that

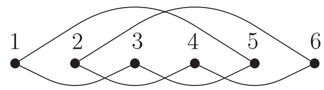

Fig. 1. The path-scheme $P(6, \{2, 4\})$.

t occupies one of the positions from Y, say s, in π_1. Because of the element distance condition, we cannot "jump over" X while placing two consecutive elements, and thus $s < j$. We get a contradiction, since because of π_1, x_s must be minimal in X while because of π_2, x_j must be minimal in X. Thus, the position of 1 is uniquely determined and we get the desired result. □

The following corollary to Theorem 1 is straightforward.

Corollary 2. *An n-permutation π is not k-determined if and only if there exists x, $1 \leqslant x < n$, such that $d_\pi(x, x+1) \geqslant k$.*

So, to determine if a given n-permutation is k-determined, all we need to do is to check the distance for $n-1$ pairs of numbers: $(1, 2), (2, 3), \ldots, (n-1, n)$. Also, the language of determined k-permutations is *factorial* in the sense that if $\pi_1\pi_2\ldots\pi_n$ is k-determined, then so is the pattern of $\pi_i\pi_{i+1}\ldots\pi_j$ for any $i \leqslant j$, which is a simple corollary to Theorem 1. Coming back to the permutation 13542 above and using Corollary 2, we see why this permutation is not 3-determined ($k = 3$): the distance $d_{13542}(2, 3) = 3 = k$.

2.2. *Directed hamiltonian paths in path-schemes; another criterion on k-determinability*

Let $V = \{1, 2, \ldots, n\}$ and M be a subset of V. A *path-scheme* $P(n, M)$ is a graph $G = (V, E)$, where the edge set E is $\{(x, y) \mid |x - y| \in M\}$. See Fig. 1 for an example of a path-scheme.

Path-schemes, also known as "*distance graphs*" [3], appear in the literature, for example, in connection with counting independent sets (see [11]). However, we will be interested in path-schemes having $M = \{1, 2, \ldots, k-1\}$ for some k (the number of independent sets for such M in case of n nodes is given by the $(n+k)$th *k-generalized Fibonacci number*). Let $\mathcal{G}_{k,n} = P(n, \{1, 2, \ldots, k-1\})$, where $k \leqslant n$. Clearly, $\mathcal{G}_{k,n}$ is a subgraph of $\mathcal{G}_{n,n}$.

Any permutation $\pi = \pi_1\pi_2\ldots\pi_n$ determines uniquely a directed hamiltonian path in $\mathcal{G}_{n,n}$ starting with π_1, then going to π_2, then to π_3 and so on. The reverse is also true: given a directed hamiltonian path in $\mathcal{G}_{n,n}$ we can easily construct the permutation corresponding to it.

The following theorem is just another formulation of Theorem 1.

Theorem 3 (*Second criterion on k-determinability*). *Let Φ be the map that sends a k-determined n-permutation π to the directed hamiltonian path in $\mathcal{G}_{n,n}$ corresponding to π^{-1}. Φ is a bijection between the set of all k-determined n-permutations and the set of all directed hamiltonian paths in $\mathcal{G}_{k,n}$.*

Theorem 3 suggests a quick checking of whether an n-permutation π is k-determined or not. One simply needs to consider $n-1$ differences of the adjacent elements in π^{-1} and check whether at least one of those differences exceeds $k-1$ or not. Moreover, one can find the number of k-determined n-permutations by listing them and checking for each of them the differences of consecutive elements in the manner described above. Using this approach, one can run a computer program to get the number of k-determined n-permutations for initial values of k and n, which we record in Table 1.

It is remarkable that the sequence corresponding to the case $k = 3$ in Table 1 appears in [13, A003274], where we learn that the inverses to the 3-determined permutations are called the *key permutations* and they appear in [12]. Another sequence appearing in Table 1 is [13, A003274]: 0, 2, 12, 72, 480, 3600, ... In our case, this is the number of n-determined $(n+1)$-permutations, $n \geqslant 1$; in [13], this is the number of $(n+1)$-permutations that have two predetermined elements non-adjacent (e.g., for $n = 2$, the permutations with say 1 and 2 non-adjacent are 132 and 231). It is clear that both of the last objects are counted by $n!(n-1)$. Indeed, to create a n-determined $(n+1)$-permutation, we take any permutation (there are $n!$ choices) and extend it to the right by one element making sure that the extension is not adjacent to the leftmost element of the permutation (there are $n-1$ possibilities; here we use Theorem 1). On the

Table 1
The initial values for the number of k-determined n-permutations

$k=2$	1, 2, 2, 2, 2, 2, 2, 2, 2, ...
$k=3$	1, 2, 6, 12, 20, 34, 56, 88, 136, ...
$k=4$	1, 2, 6, 24, 72, 180, 428, 1042, 2512, ...
$k=5$	1, 2, 6, 24, 120, 480, 1632, 5124, 15860, ...
$k=6$	1, 2, 6, 24, 120, 720, 3600, 15600, 61872, ...
$k=7$	1, 2, 6, 24, 120, 720, 5040, 30240, 159840, ...
$k=8$	1, 2, 6, 24, 120, 720, 5040, 40320, 282240, ...

other hand, to create a "good" permutation appearing in [13], we take any of $n!$ permutations, and insert one of the predetermined elements into any position not adjacent to the other predetermined element (there are $(n-1)$ choices). A bijection between the sets of permutations above is given by the following: suppose a and b are the predetermined elements in $\pi = \pi_1 \dots \pi_n$, and $\pi_i = a$ and $\pi_j = b$. We build the permutation π' corresponding to π by setting $\pi'_1 = i$, $\pi'_n = j$, and $\pi'_2 \dots \pi'_{n-1}$ is obtained from π by first removing a and b, and then, inwhat is left, by replacing i by a and j by b. For example, assuming that 2 and 4 are the determined elements, to $13\underline{4}5\underline{2}6$ there corresponds $\underline{5}1426\underline{3}$ which is a 5-determined 6-permutation.

Another application of Theorem 3 is finding lower and upper bounds for the number $A_{k,n}$ of k-determined n-permutations.

Theorem 4. *We have* $2((k-1)!)^{\lfloor n/k \rfloor} < A_{k,n} < 2(2(k-1))^n$.

Proof. According to Theorem 3, we can estimate the number of directed hamiltonian paths in $\mathscr{G}_{k,n}$ to get the desired result. This number is two times the number of (non-directed) hamiltonian paths in $\mathscr{G}_{k,n}$, which is bounded from above by $(2(k-1))^n$, since $2(k-1)$ is the maximum degree of $\mathscr{G}_{k,n}$ (for $n \geqslant 2k-1$). So, $A_{k,n} < 2(2(k-1))^n$.

To see that $A_{k,n} > 2((k-1)!)^{\lfloor n/k \rfloor}$, consider hamiltonian paths starting at node 1 and *not* going to any of the nodes i, $i \geqslant k+1$ before it goes through *all* the nodes $1, 2, \dots, k$. Going through all the first k nodes can be arranged in $(k-1)!$ different ways. After covering the first k nodes we send the path under consideration to node $k+1$, which can be done since we deal with $\mathscr{G}_{k,n}$. Then the path covers *all*, but not any other, of the $k-1$ nodes $k+2, k+3, \dots, 2k$ (this can be done in $(k-1)!$ ways) and comes to node $2k+1$, etc. That is, we subdivide the nodes of $\mathscr{G}_{k,n}$ into groups of k nodes and go through all the nodes of a group before proceeding with the nodes of the group to the right of it. The number of such paths can be estimated from below by $((k-1)!)^{\lfloor n/k \rfloor}$. Clearly, we get the desired result after multiplying the last formula by 2 (any hamiltonian path can be oriented in two ways). $\square$

Remark 5. The bounds we get in Theorem 4 are rough. We are grateful to the referee for pointing out to an improvement of the upper bound in the theorem by making it $n(2k-2)(2k-3)^{n-2}$. In fact, one can show that $n(2k-2)(2k)!(2k-3)^{n-2k-2}$ is an upper bound in Theorem 4. However, these expressions being less compact do not improve significantly the asymptotic behavior of the bound, and we keep the original upper bound in the theorem.

3. Prohibitions giving *k*-determinability

The set of k-determined n-permutations can be described by the language of prohibited patterns $\mathscr{L}'_{k,n}$ as follows. Using Theorem 1, we can describe the set of k-determined n-permutations by prohibiting patterns of the forms $xX(x+1)$ and $(x+1)Xx$, where X is a permutation on $\{1, 2, \dots, |X|+2\} - \{x, x+1\}$ ($|X|$ is the number of elements in X), the length of X is at least $k-1$, and $1 \leqslant x \leqslant 1+|X| < n$. We collect all such patterns in the set $\mathscr{L}'_{k,n}$; also, let $\mathscr{L}'_k = \bigcup_{n \geqslant 0} \mathscr{L}'_{k,n}$.

Before proceeding further, we need to justify that prohibiting the patterns from $\mathscr{L}'_k$ we indeed get *all* k-determined permutations and no other permutation. One direction is trivial: if one has a factor $yY(y+1)$ (considerations for $(y+1)Yy$ are similar and we skip them in what follows) of length $k+1$ or more in a permutation π, then π is *not* k-determined, and the pattern of this factor is of the form $xX(x+1)$ with the conditions stated above.

Conversely, we have to prove that, as soon as a factor in π has a pattern $xX(x+1)$, the permutation is *not* k-determined. Note that the factor itself may not be of the form $yY(y+1)$: for instance, take $\pi = 14532$, $k=3$; the pattern of 1453

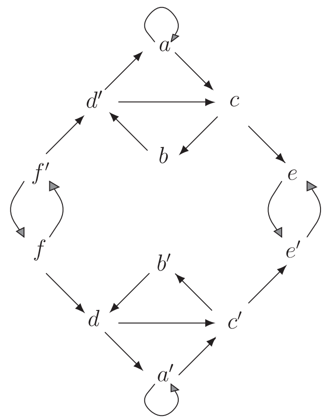

Fig. 2. Graph $\mathscr{P}_4(\mathscr{L}_3)$ (the case $k=3$).

is of the prohibited form, but 1453 itself is not of the form $yY(y+1)$. Let a factor yYz in π, $y<z$, have a pattern $xX(x+1)$ from $\mathscr{L}'_k$. Obviously, there are no t in Y such that $y<t<z$. Next, consider maximum i, $0\leqslant i\leqslant z-1$, such that all the elements $y, y+1, \ldots, y+i$ are to the left of Y in π and $y+i+1$ is to the right of Y in π (such $y+i+1$ exists because of z). Then clearly the distance between $y+i$ and $y+i+1$ is at least k and thus π is *not* k-determined due to Theorem 1.

A prohibited pattern $X=aYb$ from $\mathscr{L}'_k$, where a and b are some consecutive elements and Y is a (possibly empty) word, is called *irreducible* if the patterns of Yb and aY are not prohibited, in other words, if the patterns of Yb and aY are k-determined permutations. Let $\mathscr{L}_k$ be the set consisting *only* of irreducible prohibited patterns in $\mathscr{L}'_k$.

Theorem 6. *Let k be fixed. The number of (irreducible) prohibitions in $\mathscr{L}_k$ is finite. Moreover, the longest prohibited patterns in $\mathscr{L}_k$ are of length $2k-1$.*

Proof. Suppose that a pattern $P=xX(x+1)$ of length $2k$ or larger belongs to $\mathscr{L}_k$ (the case $P=(x+1)Xx$ can be considered in the same way). Then obviously X contains either $x-1$ or $x+2$ on the distance at least $k-1$ from either x or $x+1$. In any case, clearly we get either a prohibited pattern $P'=yY(y+1)$ or $P'=(y+1)Yy$, which is a proper factor of P. Contradiction with P being irreducible. □

Theorem 6 allows us to use the transfer matrix method to find the number of k-determined permutations. Indeed, we can consider the graph $\mathscr{P}_{2k-1}(\mathscr{L}_k)$, which is the graph $\mathscr{P}_{2k-1}$ of pattern overlaps without nodes containing prohibited patterns as factors. Then each path in the graph determines a single permutation since to be "k-determined" implies to be "$(2k-1)$-determined." Thus the number $A_{k,n}$ of k-determined n-permutation is equal to the number of paths of length $n-2k+1$ in the graph, which can be found using the transfer matrix method [14, Theorem. 4.7.2]. In particular, the method makes the following statement true.

Theorem 7. *The generating function $A_k(x)=\sum_{n\geqslant 0}A_{k,n}x^n$ for the number of k-determined permutations is rational.*

Remark 8. In fact, one can use a smaller graph, namely $\mathscr{P}_{2k-2}(\mathscr{L}_k)$, in which we mark arcs by corresponding permutations of length $2k-1$; then we remove arcs containing prohibitions and use the transfer matrix method. In this case, to an n-permutation there corresponds a path of length $n-2k+2$. See Fig. 2 for such a graph in the case $k=3$.

A permutation is called *crucial* with respect to a given set of prohibitions, if it does not contain any prohibitions, but adjoining any element to the right of it leads to a permutation containing a prohibition. In our case, an n-permutation is crucial if it is k-determined, but adjoining any element to the right of it, and thus creating an $(n+1)$-permutation, leads to a *non*-k-determined permutation.[1] If such a π exists, then the path in $\mathscr{P}_{2k-1}(\mathscr{L}_k)$ corresponding to π ends

[1] As it is mentioned in the Introduction, crucial *words* are studied, for example, in [6]. We define crucial permutations with respect to a set of prohibited patterns in a similar way. However, as Theorem 9 shows, there are no crucial permutations with respect to $\mathscr{L}_k$.

	a	c	b	e	d'	e'	f'	f	d	c'	a'	b'
a	1	1	0	0	0	0	0	0	0	0	0	0
d'	1	1	0	0	0	0	0	0	0	0	0	0
c	0	0	1	1	0	0	0	0	0	0	0	0
b	0	0	0	0	1	0	0	0	0	0	0	0
e	0	0	0	0	0	1	0	0	0	0	0	0
e'	0	0	0	1	0	0	0	0	0	0	0	0
f'	0	0	0	0	1	0	0	1	0	0	0	0
f	0	0	0	0	0	0	1	0	1	0	0	0
d	0	0	0	0	0	0	0	0	0	1	1	0
a'	0	0	0	0	0	0	0	0	0	1	1	0
c'	0	0	0	0	0	1	0	0	0	0	0	1
b'	0	0	0	0	0	0	0	0	1	0	0	0

Fig. 3. Transfer matrix corresponding to $\mathscr{P}_4(\mathscr{L}_3)$.

up in a sink. However, the following theorem shows that there are no crucial permutations with respect to the set of prohibitions $\mathscr{L}_k$, thus any path in $\mathscr{P}_{2k-1}(\mathscr{L}_k)$ can always be continued.

Theorem 9. *There do not exist crucial permutations with respect to* $\mathscr{L}_k$.

Proof. If $k = 2$ then only the monotone permutations are k-determined, and we always can extend to the right a decreasing permutation by the least element, and the increasing permutation by the largest element.

Suppose $k \geqslant 3$ and let Xx be an n-permutation avoiding $\mathscr{L}_k$, that is, Xx is k-determined. If $x = 1$ then Xx can be extended to the right by 1 without creating a prohibition; if $x = n$ then Xx can be extended to the right by $n + 1$ without creating a prohibition. Otherwise, due to Theorem 1, both $x - 1$ and $x + 1$ must be among the k leftmost elements of Xx. In particular, at least one of them, say y, is among the $k - 1$ leftmost elements of Xx. If $y = x - 1$, we extend Xx by x (the "old" x becomes $(x + 1)$); if $y = x + 1$, we extend Xx by $x + 1$ (the "old" $x + 1$ becomes $(x + 2)$). In either of the cases considered above, Theorem 1 guarantees that no prohibitions will be created. So, Xx can be extended to the right to form a k-determined $(n + 1)$-permutation, and thus Xx is not a crucial n-permutation. □

3.1. The case $k = 3$

In this subsection we take a closer look at the graph $\mathscr{P}_4(\mathscr{L}_3)$ whose paths give all 3-determined permutations (we read marked arcs of a path to form the permutation corresponding to it). It turns out that $\mathscr{P}_4(\mathscr{L}_3)$ has a nice structure (see Fig. 2).

Suppose w' denotes the *complement* to an n-permutation $w = w_1 w_2 \dots w_n$. That is, $w'_i = n - w_i + 1$ for $1 \leqslant i \leqslant n$. $\mathscr{P}_4(\mathscr{L}_3)$ has the following 12 nodes (those are all 3-determined 4-permutations): $a = 1234$, $a' = 4321$, $b = 1324$, $b' = 4231$, $c = 1243$, $c' = 4312$, $d = 3421$, $d' = 2134$, $e = 1423$, $e' = 4132$, $f = 3241$, $f' = 2314$.

In Fig. 2 we draw 20 arcs corresponding to the 20 3-determined 5-permutations. Notice that $\mathscr{P}_4(\mathscr{L}_3)$ is not strongly connected: for example, there is no directed path from c to f.

To find the generating function $A_3(x) = \sum_{n \geqslant 0} A_{3,n} x^n$ for the number of 3-determined permutations one can build a 12×12 matrix (for example, the one given by the adjacency table in Fig. 3) corresponding to $\mathscr{P}_4(\mathscr{L}_3)$ and to use the transfer matrix method to get

$$A_3(x) = \frac{1 - 2x + 2x^2 + x^3 - x^5 + x^6}{(1 - x - x^3)(1 - x)^2}$$

which is true due to the known result [13, A003274] mentioned above. Note that the largest eigenvalue of the matrix given by Fig. 3 is $1.4655\dots$, and thus $A_{3,n}$ grows like $(1.4655\dots)^n$, while the lower bound in Theorem 4 is $2^{n/3} = (1.2599\dots)^n$.

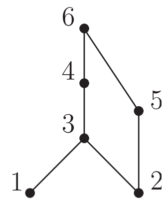

Fig. 4. The poset associated with the path $w = 134265$ in $\mathscr{P}_3$ ($k = 3$).

In general, even though finding explicit generating functions for $k \geqslant 4$ by the transfer matrix method is a rather difficult problem, one can use the method for studying the asymptotic behavior of $A_{k,n}$ by attempting to find/estimate the largest eigenvalue of the corresponding matrix.

4. Open problems

It is clear that any n-permutation is n-determined, whereas for $n \geqslant 2$ no n-permutation is 1-determined. Moreover, for any $n \geqslant 2$ there are exactly two 2-determined permutations, namely the monotone permutations. For a permutation π, we define its *index* $\mathrm{IR}(\pi)$ *of reconstructibility* to be the minimal integer k such that π is k-determined.

Problem 1. Describe the distribution of $\mathrm{IR}(\pi)$ among all n-permutations.

Problem 2. Study the set of k-determined permutations in the case when a set of nodes is removed from $\mathscr{P}_k$, that is, when some of patterns of length k are prohibited.

An n-permutation π is *m-k-determined*, $m, k \geqslant 1$, if there are exactly m (different) n-permutations having the same path in $\mathscr{P}_k$ as π has. In particular, the k-determined permutations correspond to the case $m = 1$.

Problem 3. Find the number of m-k-determined n-permutations.

Problem 3 is directly related to finding the number of *linear extensions* of a *poset*. Indeed, to any path w in $\mathscr{P}_k$ there naturally corresponds a poset $\mathscr{W}$. In particular, any factor of length k in w consists of mutually comparable elements in $\mathscr{W}$. For example, if $k = 3$ and $w = 134265$ then $\mathscr{W}$ is the poset in Fig. 4 (the elements of the poset are the w's elements; the order of the poset is the natural order on $\{1, 2, \ldots, n\}$ where two elements are comparable if and only if they lie at distance at most $k - 1$ in w).

If all the elements are comparable to each other in w, then $\mathscr{W}$ is a linear order and w gives a k-determined permutation. If $\mathscr{W}$ contains exactly one pair of incomparable elements, then w gives (two) 2-k-determined permutations. In the example in Fig. 4, there are four pairs of incomparable elements, (1,2), (1,5), (3,5), and (4,5), and this poset can be extended to a linear order in 7 different ways giving (seven) 7-3-determined permutations.

Problem 4. Which posets on n elements appear while considering paths (of length $n-k+1$) in $\mathscr{P}_k$? Give a classification of the posets (different from the classification by the number of pairs of incomparable elements).

Problem 5. How many linear extensions can a poset (associated to a path in $\mathscr{P}_k$) on n elements with t pairs of incomparable elements have?

Problem 6. Describe the structure of $\mathscr{L}_k$ (see Section 3 for definitions) that consists of irreducible prohibitions. Is there a nice way to generate $\mathscr{L}_k$? How many elements does $\mathscr{L}_k$ have?

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