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Some of My Recollections of George David Birkhoff*

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George David Birkhoff, who is widely regarded as the leading native American mathematician of his generation, died in 1944 at the age of 60. His great mathematical ability and his keen interest and vibrant enthusiasm about almost all parts of mathematics are rarely found combined in one man. He was eager to talk or correspond with anyone about almost any topic of mathematical research. Although our special interests were different, I was fortunate enough to have had close mathematical contact with him for many years. Birkhoff's advice and encouragement played a prominent part in my choosing to become a professional mathematician, so I owe this tribute to his memory.

I am much indebted to Garrett Birkhoff for his comments and suggestions during the preparation of this paper, and to Marston Morse and J. L. Walsh for their opinions expressed herein under their names.

In our conversations and correspondence, Birkhoff and I rarely discussed anything but mathematics. He talked a great deal about his own work when I was with him, particularly concerning dynamics and related subjects. The results he discussed were almost always, I think, later published. A good bit of the material discussed by us related to number theory, in which he always maintained a keen interest. He seemed primarily interested in difficult and important unsolved problems. That being the case, he often brought up during our conversation some of the well-known unsolved problems in number theory.

In 1901 I received my first letter from Birkhoff in which he referred to problems in number theory proposed in the *American Mathematical Monthly*. He also mentioned that he was much interested in Fermat's Last Theorem and had been working for a time on this problem. I answered his letter, and this began a voluminous correspondence which lasted for several years, and led to a paper which he and I published as co-authors.[1] I did not meet

* The work on this paper was done during the period of National Science Foundation Grant 19665 awarded me.

[1] On the integral divisors of $a^n - b^n$. *Ann. Math.* 5, 173-180 (1904).

him personally, however, until 1913 at a meeting of the American Mathematical Society in New York, and as the years went by, I met him in various scientific meetings in a number of different places.

Aside from Birkhoff's principal interests, I do not think he was more interested in number theory than he was in other parts of mathematics, though, as I shall bring out later in this paper, it seems probable that until 1906 geometry and number theory were his favorite subjects. After that time he seemed to become more and more interested in various other topics in applied mathematics such as relativity, quantum mechanics, and other physical theories. When at mathematical meetings, he would seek out the leading men in various parts of mathematics and converse with them at length. He was keenly interested in the advancement of nearly all parts of mathematics and was willing to work hard at contributing toward this end. One of my great regrets is that I shall never hear him talk about mathematics again.

As to the character of the mathematical work which Birkhoff published during his lifetime and the international recognition which he received, these subjects were treated in articles written by R. E. Langer, Oswald Veblen, and Marston Morse, which were reprinted in volume I of *Birkhoff's Collected Works*, published by the American Mathematical Society, New York, 1951, pages xv-lvii. (Hereafter, reference to this publication will be abbreviated as *Works*.) In view of these accounts I shall confine myself to discussing some of his mathematical interests which were not treated in much detail by these authors. But some references will be made to investigations of my own into Birkhoff's career which confirm ideas expressed in their articles. In particular, also, I hope that items have been brought out in this paper which indicate to the reader *that the passage of time has not dimmed Birkhoff's fame as a mathematician.*

His Interest in Geometry

Birkhoff's remarkable mathematical precocity was brought to my attention in one of our early conversations when he told me that he rediscovered the Lunes of Hippocrates when he was ten years old. His natural mathematical genius is further described in recollections and references which follow.

In the *American Mathematical Monthly*, Volume III, 1895, page 157, the following problem was proposed:

> "Prove that if two angle bisectors of a triangle are equal, the triangle is isosceles."

Although obviously quite elementary, this problem previously had caused great difficulty to the readers of the *Monthly*, and during the second half of the last century to a number of mathematicians in France, England, and the United States, including several leading ones. A number of the supposed demonstrations which were published were fallacious. Very curiously, one of these arguments was pronounced "thoroughly sound" by several first-class mathematicians.[2] The problem was proposed two more times in the *Monthly*—Volume V (1898), pages 108–9, and Volume VII (1900), pages 226–28. In this latter volume, page 228, third paragraph from the top, the editor of the *Monthly* (at that time B. F. Finkel) stated, "Demonstrations were again furnished by G. B. M. Zerr, P. C. Cullen, George D. Birkhoff. We also received a few demonstrations[3] that contained fallacies." *I have been unable to find an exact reference to his first submitted demonstration, but he was only 11 when the problem was first proposed in 1895, and almost certainly gave a solution before 1899, when he was 14 or 15.*

Birkhoff often talked to me about his work on the so-called "four-color map problem." I recall his telling me around 1915 that it was the only mathematical problem he had studied up to that time which had kept him awake at night. He then followed this remark with some details. It was been proved that five colors are sufficient to color any map in the sense usually defined. He worked to some extent, he said, on examining the connection with the work that had been done involving these five colors and that which had been done with four colors. He stated that some of his first results depended on the fact that five is a prime number. The four-color problem interested him throughout life, and one of his last papers, written with D. C. Lewis, treated this topic at length.[4] In this connection E. T. Bell stated (*Mathematics, Queen and Servant of Science*, McGraw-Hill, New York, 1951, p. 152):

> "As a personal reminiscence I recall that G. D. Birkhoff said shortly before his death that in spite of all his efforts, one of which I witnessed

[2] ARCHIBALD HENDERSON, A classic problem in Euclidean geometry. *Elisha Mitchell Sci. Soc.* **53**, 246-281 (1937), *dedicated to Birkhoff* (p. 246). The author discussed the history of the problem, several of the erroneous proofs, and a number of apparently accurate demonstrations of his own. In particular, he stated that of the nine different purported solutions which appeared in the *Monthly*, in the papers cited only two of the supposed demonstrations were free from flaws. Coxeter, in his book entitled *Introduction to Geometry* (Wiley, New York, 1961), p. 16, Problem 4, has indicated a simple indirect proof of the theorem.

[3] Though it is conceivable that Birkhoff made an error in his work in one or both of these solutions, I doubt this very much. I never found any error in any discussion, either written or verbal, that he had with me. Also, he was quick to observe errors made by others. For example, I think it was in 1913 that I found, in working with the Fermat quotient $((a^{p-1} - 1)/p$, with p prime and a prime to p), what seemed to be interesting new results, and I promptly wrote to Birkhoff hoping he would share my satisfaction over this. However, within a couple of days I received a reply from him in which he pointed out an error which vitiated all my work that I had just communicated to him.

[4] Chromatic polynomials. *Am. Math. Soc. Trans.* **60**, 355-451, (1946).

> in 1911, to crack the four-color problem wide open, he had not even scratched it."

I do not think such a situation is unusual with mathematicians in their later days. Many individuals who have spent most of their lives on mathematical research will often regret deeply that certain problems on which they had spent a great deal of time were never solved by them.

One day around 1920 I was discussing with Marston Morse, who had been a student of Birkhoff, some characteristics which we had particularly noted in him. One thing Morse said was that he had a most extraordinary physical intuition. When recently I reminded Morse about this episode he wrote me a letter which included the following statement:

> "I agree with you that this extraordinary physical intuition was largely geometrical. I believe that more geometrical intuition in physicists today would help in their theories. In particular I believe that they will eventually discover that quantum mechanics in its qualitative aspects is an application of critical point theory."

Again in connection with geometry, Veblen stated (*Works*, Vol. I, p. xix), "Although Birkhoff's most notable successes were in the geometrical aspects of dynamics, he did not neglect, nor was he deficient in power over the analytic formalism." Birkhoff used geometry extensively also in his work on aesthetics. He considered the aesthetic qualities in polygonal forms, ornaments, and vases.

However, Birkhoff wrote several papers in which geometry alone was employed, in particular, a paper entitled "A Proof of Poincaré's Geometric Theorem." Indeed, this is one of the two articles for which he is perhaps best known. He also wrote a book on geometry and several other articles on this subject which I do not think are quite as well known as they should be; hence they are listed below.

1. "Basic Geometry" (with R. Beatley). Scott, Foresman, New York, 1940.
2. A new approach to elementary geometry, (with R. Beatley). *In* "The Fifth Yearbook of the National Council of Teachers of Mathematics," 1930.
3. A set of postulates for plane geometry, based on scale and protractor. *Ann. Math.* ser. 2, **33**, 329-345 (1932). (Also, see: Note on a preceding paper. *Ann. Math.* **33**, no. 4, 788 (1932).)

In connection with the first two of the above references, the postulates which Birkhoff uses for his system assume as known the notion of real numbers and measurement. That being the case, he gave also postulates to cover operations with real numbers (*Basic Geometry*, pages 284-288). However, he does not define the term "number." He states that this concept of number acquires meaning from the postulates governing its use.

Morse states (*Works*, Vol. I, p. 388) that reports on the use of this book in practice are favorable: "In any case the introduction of new ideas from the pen of a man as eminent as Birkhoff should be a great stimulus to the subject of elementary geometry." Concerning the first part of Morse's statement I would say that based on what I know of experiences with students and high school teachers here, we would agree. Concerning the second statement of Morse, I am glad to say that it was quite a stimulus to me. Further, Birkhoff's formulation underlies the approach to elementary geometry advocated today by the School Mathematics Study Group.[5]

His Interest in Number Theory

Considering again the problems to which Birkhoff referred in some of his first letters to me, in particular he worked at a problem in Diophantine Analysis which had been proposed (this result was originally stated by Euler) in the *Monthly* as follows:

> "Prove that it is impossible to find integral values for x, y, and z, none zero, such that the relation $x^2y + xz^2 = y^2z$ is satisfied."

Then in a later letter to me (1902) he not only proved this theorem but gave a proof of the following extension of it:

> If m and n are positive integers not both even, $x^my^n + y^mz^n + z^mx^n = 0$ has nonzero integral solutions if and only if $u^t + v^t + w^t = 0$ has such solutions, where $t = m^2 - mn + n^2$.

The proof is not easy, and I had not seen this result before, so I suggested that he write a paper including his demonstration of this theorem and try to have it published. As far as I know, he never did this. Later, however (1908), A. Hurwitz published a proof of the theorem, and this result excited considerable attention among mathematicians. The statement of the theorem obviously indicates a relation of the Fermat problem to it.

In the *Bulletin of the American Mathematical Society*, Volume 19, 1913, pages 233-236, R. D. Carmichael, who was the first mathematician to take his Ph. D. degree under Birkhoff, published an article entitled "Note on Fermat's Last Theorem." This article contained a proof of the following statement:

"If p is an odd prime and the equation

$$x^p + y^p + z^p = 0$$

[5] "Mathematics for High School Geometry (Part 1)," Preface. School Math. Group, Yale University, 1960; "Mathematics for High School Geometry with Coordinates (Part 1)," Preface. School Math. Group, Stanford University, 1961.

has a solution in integers x, y, z each of which is prime to p, then there exists a positive integer s, less than $(p-1)/2$, such that

$$(s+1)^{p^2} \equiv s^{p^2} \bmod p^3.\text{''} \tag{1}$$

It seems that Birkhoff read this article, and then advised Carmichael that the utility of the criteria made in the above statement was limited to the cases $p=3$ and $p=6n-1$. Consequently, Carmichael published a paper in the *Bulletin* entitled "Second Note on Fermat's Last Theorem," Volume 19, pp. 402-403 (1913). In it he gave Birkhoff's argument, showing that the latter's statement was correct. During this period in his work I happen to know that Birkhoff studied critically several published articles in number theory.

In his proof of the theorem that Fermat's Last Theorem is true for regular primes, Kummer initiated the proof by expressing $x^n + y^n$ as the product of n factors involving nth roots of unity where n is prime and $x^n + y^n = z^n$. Concerning this, Birkhoff remarked to me that this method seemed to him to be a natural one, that is, the introduction of the theory of the nth roots of unity into the argument. As far as I know, up to the present time the Fermat Theorem has not been proved for any exponent greater than 3 except by use of a method which involves that just mentioned.

In 1915 the writer published an article (*Bulletin of the American Mathematical Society*, Volume 22, pages 61-68) which begins with the following statement:

"In 1903, Professor G. D. Birkhoff communicated to me the following theorem:

If p is a prime integer and a is a positive integer prime to p, then there is at least one and not more than two pairs (x, y) such that

$$a \equiv \pm x/y \pmod{p} \tag{2}$$

where x and y are integers prime to each other and $0 < x < \sqrt{p}$, $0 < y < \sqrt{p}$."

At the same time, he also sent a proof which agrees with one later given by Vinogradov. This proof is far simpler than the one I gave in the article mentioned, but the latter includes an algorithm for determining directly the values of x and y in (2). Birkhoff, however, was more closely involved in this paper of mine than is indicated by the above. I recall now that he made the suggestion to me that perhaps another proof of his theorem could be found by using some results of Minkowski. Following up this discussion, the writer found that the result was a special case of Minkowski's classic theorem which reads as follows:

"If

$$f_1 = a_{11}u_1 + \cdots + a_{1m}u_m,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f_m = a_{m1}u_1 + \cdots + a_{mm}u_m,$$

are m linear homogeneous forms in $u_1, u_2, \cdots, u_m$ with arbitrary real coefficients $a_{11}, \cdots a_{mm}$ of determinant Δ, then it is always possible to select integers for $u_1, u_2, \cdots, u_m$ so that

$$|f_i| \leqslant \sqrt[m]{\Delta}(i = 1, 2, \cdots, m)."$$

Birkhoff's statement follows when we consider the relation involving f_1 and set $m = 2$ and the a's integers. So I should have given him more credit in my paper than I did. I note also from the other proofs of his statement that p is not necessarily prime in (1). The result found by Birkhoff with p an arbitrary positive integer is now generally referred to as Thue's Theorem, but Minkowski's result was published as far back as 1896, which seems to antedate anything written on the topic by Thue.

Aside from the article "On the Integral Divisors of $a^n - b^n$" (1904) which Birkhoff co-authored with me, his only other mathematical publication on number theory appears to have been "Note on Certain Quadratic Number Systems for Which Factorization Is Unique," *American Mathematical Monthly*, vol. 13, pp. 156-159, Aug.-Sept. 1906.

We now follow the above by noting some remarks and published statements which Birkhoff made concerning number theory in general.

One day in conversation with Birkhoff I remarked that it seemed to me I was specializing too much in the theory of algebraic numbers, to my regret. He emphatically replied, "You must be a specialist to get anywhere in the theory of algebraic numbers."

Birkhoff read a paper of mine on number theory in 1915, and in one of our discussions about it he asked me why I had used an unusual device to prove the principal theorem in the paper. I explained to him that number theorists had to do this kind of thing often; in fact, the most successful number theorists seem to have a bag of tricks, the use of which may lead them to new results. Birkhoff was puzzled—he seemed to think that these things should (or could) be done more naturally. This conversation, however, took place before Birkhoff had had much experience with number theory. In this subject often many complicated and indirect proofs of a certain result are published, and perhaps only after many years will a demonstration which appears natural be produced. This situation was well expressed by Gauss.[6]

[6] Quoted by E. T. Bell, "Mathematics, Queen and Servant of Science," p. 222. McGraw-Hill, New York, 1951.

In 1927 Edmund Landau of Göttingen published a work in three volumes entitled *Vorlesungen über Zahlentheorie* (Hirzel, Leipzig). *Although Birkhoff was not primarily a number theorist, Landau requested him to review it.*[7] The review includes the statement:

> "These three excellently printed and arranged volumes form an addition of the highest importance to the literature of the theory of numbers. With them, the reader familiar with the basic elements of the theory of functions of a real and complex variable, can follow many of the astonishing recent advances in this fascinating field."

Birkhoff, in conversation with me one day, noted that most of classical number theory really consisted of consideration of conditional equations or inequalities involving rational numbers and particularly integers. I think this will be obvious to any number theorist as soon as it is pointed out to him, but he might not think of it otherwise.

In about 1947 a mathematician told me that at a meeting he had attended some years before, Birkhoff, who was also present, showed considerable interest in an article on number theory that was given at the meeting and made a number of comments concerning it which also interested some of the other persons present. He was asked why he had not spent more time with that subject during his career. To this question he replied that the leading unsolved problems in number theory seemed to be of extraordinary difficulty, and it would take too much of his time, which he had to devote to other subjects, to get anywhere with them. This is similar to a statement which he made to the writer to the effect that he himself did not carry on for too long his research in a subject which, after some consideration by him, did not "open up" and yield substantial results. As time went on, however, it appeared that he was not entirely consistent in this, as he must have spent an inordinate amount of time on the four-color map problem.

To return again to Birkhoff's remarks concerning why he did not pursue number theory as his main interest, I am definitely of the opinion that if he had done so for a number of years, he would have obtained important new results in the subject. I am reminded of a saying I heard made several times, mainly by Continental mathematicians, that often a promising young mathematician will start with his main interest number theory, but in his later days he will have gradually changed it to analysis, or perhaps even applied mathematics. As stated before, I would say that prior to 1906 Birkhoff's main interests were probably geometry and number theory.

Although number theory was not Birkhoff's main interest, later, in 1929,

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[7] *Works*, Vol. III, pp. 275-277, reprinted from *Bull. Am. Math. Soc.* **35**, 401-403 (1929).

if I recall correctly, it was at his suggestion that G. H. Hardy was asked to address the American Mathematical Society, the title of his lecture being "An Introduction to the Theory of Numbers" (published in the *Bulletin*, Nov.-Dec. issue, 1929, pp. 778-818). Hardy was giving a course of lectures at Princeton at that time.

His high regard for number theory is again expressed in his review of Hardy and Wright's *Introduction to the Theory of Numbers* (Oxford University Press, 1938):

> "It is much to be hoped that other mathematical works having the appeal of the book by Hardy and Wright will soon be written; and that a much wider public than at present will come to realize how through such works the highest artistic and intellectual enjoyment may be obtained, only to be compared with that to be derived from literature, art and music."[8]

His Attitude Toward Abstract Algebra

When I first met Birkhoff he seemed to have little, if any, interest in abstract algebra, which was beginning to be developed in those times. As I was interested in the subject, I would sometimes bring up some points in it, but I found usually that he made no reply to may remarks. However, later he must have changed his mind about such matters, as he was co-author with Garrett Birkhoff of an article on such a subject.[9] Also, he spoke of various things in abstract algebra in several of his reviews of mathematical books, which indicated he had developed a high regard for the subject. This is in line with the statements made concerning the development of algebra in his article in *AMS Semicentennial Publications*, Volume II (1938), "Fifty Years of American Mathematics," pp. 270-315.

Garrett Birkhoff made the following statement in a letter to me: "My father's question about abstract algebra was this: How much did it contribute to the solution of difficult problems which were meaningful independently of the new language invented? He did not admire generalizations for their own sake, but only if they helped to solve concrete problems—though parts of his own work on dynamics (e.g., his theory of "central motions") might seem abstract to some."[10] Morse, upon reading this, wrote me,

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[8] *Works*, Vol. III, p. 661.

[9] Distributive postulates for systems like Boolean algebras. *Am. Math. Soc. Trans.* **60**, 3-11 (1946).

[10] Garrett Birkhoff made a further statement also concerning this matter as follows: "I think my father was suspicious of *verbal* transformations of known results when no important new extension or application was involved. He felt the central thing was to provide *algorithms* for obtaining new results. However, genuinely new concepts, such as his own abstract concept of a 'central motion' he certainly considered important."

> "I confirm his son's statement that Birkhoff approved of abstractions that added something directly connected with concrete problems in mathematics. An example of this was the fact that the highest praise which he ever gave me was on my *Abstract* Variational Theory (Gautier-Villars). It provided the mechanism in *advance* for solving for the first time the problem of the existence of unstable solutions of the minimal surface problem."

Concerning the statement of Garrett Birkhoff, "He did not admire generalizations for their own sake, but only if they helped to solve concrete problems," I think there is a time element involved here which should be considered.[11]

His Recognitions and Honors

These have been enumerated at length by Langer,[12] but I should like to elaborate a bit on three of these he received during his lifetime.

Birkhoff's versatility in mathematics always amazed me. An example of this is the address which he gave, by invitation, in 1938, to the American Mathematical Society at the 50th Anniversary of the Society in New York, entitled "Fifty Years of American Mathematics." By including a background of discussion concerning work prior to 1888, in effect he covered, often in considerable detail, the whole range of American mathematical research starting in 1864 with the work of Benjamin Peirce.

At the International Mathematical Congress in Bologna, Italy, in 1929, Birkhoff was scheduled to give a lecture entitled "Quelques éléments mathématiques de l'art." The committee having charge of the Congress made the nice gesture of arranging matters so that his lecture was given in the Palazzo Vecchio in Florence in lieu of Bologna, the former city being the art center of Italy.

When Birkhoff gave a lecture at the International Mathematical Congress in Oslo, the Crown Prince of Norway attended, in a special red chair in the front row.

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[11] Since I began to study mathematics I have noted many cases where a first-class mathematician would develop new abstract ideas in algebra and publish the same, some of which were, in effect, *generalizations* of known concrete results. Yet he would not indicate he had in mind any application of these ideas to any mathematical subject other than algebra itself. *Later, however, and sometimes many years later*, another first-class mathematician would publish an article in which the results and/or methods described in the previously-published paper would be applied with success to a concrete problem. I have not checked all Birkhoff's papers, but I would not be surprised to learn that some of these abstract ideas in his published papers, either before or after his death, have been applied by other mathematicians to concrete problems.

[12] *Works*, Vol. I, pp. xiii, xiv.

As to posthumous honors, we shall mention the following items.

Richard Bellman published in 1961 *A Collection of Modern Mathematical Classics* (Analysis). This consists of *verbatim* copies of thirteen papers written by mathematicians who did most of their research during the first half of the present century. Included in this set is a paper by Birkhoff entitled "Proof of the Ergodic Theorem," and another paper which he co-authored with O. D. Kellogg entitled "Invariant Points in Function Space." *These are the only papers written by native American mathematicians which were included in the collection*, and Birkhoff also has the distinction of being the only mathematician who has two separate articles appearing, one which he wrote alone and the other with a co-author.

Several years after Birkhoff's death, an issue of the *Encyclopaedia Britannica* appeared containing a biography of him written by R. E. Langer, and this was included in all succeeding issues of the *Britannica* until 1960 when the biography was replaced with another more complete one by J. L. Walsh. Among those native American research mathematicians who did most of their mathematical work during the first half of the present century, Oswald Veblen was the only other such mathematician so honored. Also, I do not know of any other native American mathematician who has had his collected papers published. In Birkhoff's case this happened in the year 1950.

In 1956 James R. Newman published his work entitled *The World of Mathematics*, in four volumes (Simon and Schuster, New York). The fourth volume contains two articles by Birkhoff, "Mathematics of Aesthetics" and "A Mathematician's Approach to Ethics." As an introduction to these, Newman discusses some of the main features of Birkhoff's mathematical career. This set of books has sold over 100,000 copies.

Another honor has been paid Birkhoff by Dorothea Frances (Canfield) Fisher, who has included in her book *American Portraits* (Henry Holt, 1946, pages 183–4) both a sketch of his career and a portrait.

At a meeting inaugurating the 1950 International Congress of Mathematicians at Harvard, Oswald Veblen, who had been appointed President of the Congress, in the beginning of his opening speech stated his great regret that Birkhoff was not there to open the Congress as originally planned. Birkhoff had been appointed President of the Congress which was to have been held in 1940, but this had been postponed due to war conditions.

Miscellaneous Opinions of Birkhoff and Others

Marston Morse stated that Poincaré was Birkhoff's real teacher.[13] The latter often talked about Poincaré's work in conversation with me. He also

[13] *Works*, Vol. I, p. xxiii, reprinted from *Bull. Am. Math. Soc.* **52**, Pt. 1, 357-391 (1946).

was particularly enthusiastic about some of the work of Levi-Civita, and of course in his articles and reviews of books, as well as in talks with me, he expressed high opinions of the work of a number of other mathematicians. He spoke highly of Hardy, and one day when we were discussing Hardy and Ramanujan on partitions problems he said, "Well, when you can start with a formula you really have something."

When Birkhoff talked to me about his work in mathematics he generally betrayed a certain confidence that he would be able to develop said work farther and obtain results which would give him much personal satisfaction; however, there were a few exceptions. I recall that at one Annual Meeting of the American Mathematical Society in New York he was very much discouraged with the progress he was making in a particular line of investigation, so much so that I was unable to console him. However, a year later when I saw him again in New York, his discouragement was gone; he had, a few months before, discovered his Ergodic Theorem! (*Proc. Natl. Acad. Sci.* **17**, 656–60, (1931)).

In a letter to me Morse wrote that "He [Birkhoff] made one remarkable statement connecting the general with the particular. He said 'If I had presented to me the six principal ideas of Gauss in concrete form I could have independently developed the general theory which contained these concrete exemplifications.' " (Morse then stated that this is not an exact quotation but is substantially correct.) He added, "I have never found myself disagreeing with Birkhoff's principal ideas on the nature of mathematics. I have often felt the need of more abstraction and precise definition."

Birkhoff had a high opinion of young American mathematicians. I recall attending a meeting of the Council of the American Mathematical Society about the year 1935 at which Birkhoff was present and at which the matter of appointing lecturers sponsored by the Society to give lectures at various universities in the United States during a particular year was brought up. During this discussion Birkhoff stated that in this opinion we had a number of brilliant young mathematicians in the United States who were perfectly capable of giving excellent lectures, and that we did not need to appoint too many foreign lecturers for this honor. These ideas of his are connected with what Veblen said in his biographical memoir[14] of Birkhoff concerning "American Mathematics."

Birkhoff's research and teaching ability may have been somewhat responsible for the fact that six of his students, who received their Ph. D. degrees under him between 1917 and 1932, became members of the National Academy of Sciences. They are Morrey, Morse, Slepian, Stone, Walsh, and Whitney.

[14] *Works*, Vol. I, p. xx.

I do not known of any mathematician, other than he, who has had as many of his doctoral students admitted to the Academy.

Principally Birkhoff and I talked about mathematics, but I recall that sometimes Birkhoff would relax, and our conversations were not all about mathematics. I particularly remember that at a mathematical meeting in New Orleans he and I took an interesting boat trip around the harbor. I recall another day in 1928 when I met Birkhoff and his wife while strolling through the streets in Vienna, and we discussed our mutual opinion that Paris and Vienna were the most beautiful large cities we had seen, and in particular we remarked that both cities seemed to have been designed in large part with the idea of comfort of the inhabitants. They also said that Budapest was a beautiful city and recommended that I should go there (which I regret not having done). I recall also that during one of our conversations Birkhoff expressed his great relief that he had finished writing a book on dynamics upon which he had been working for several years prior to 1928.

J. L. Walsh, who was quite intimate with the Birkhoff family, stated in a letter to me that Marjorie (Mrs. G. D. Birkhoff) did so much to stimulate and encourage her husband in his scientific work that I think it should be mentioned in this paper.

Birkhoff seemed to be a man who was untiring in his mathematical work. I recall that he once remarked to me that doing first-class creative mathematical research does not always depend on one's physical condition. He said he found that sometimes he was able to obtain new results that pleased him even when physically he was not up to par, although he did not indicate at all that the latter situation existed often. One of Birkhoff's senior colleagues at Harvard, J. L. Walsh, remarked to me that Birkhoff probably hastened his death by never sparing himself in his work at Harvard.[15]

[15] The latter statement is supported by the remark of Veblen on page xxi of Vol. I of *Works.*