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Application of the Decomposition Method to Inversion of Matrices

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The decomposition method (G. Adomian, "Stochastic Systems," Academic Press, New York/London, 1983) is applied to inversion of matrices by approximation.

INTRODUCTION

The decomposition method [1, 2] has been applied by Adomian and his co-workers to dynamical systems which can be nonlinear and stochastic eliminating the need for a number of often nonphysical assumptions previously employed. This paper shows applicability to matrix inversion for deterministic or random matrices. In particular, it will be valuable in connection with the work of Adomian and Sibul on stochastic control theory [3].

DISCUSSION

We begin with the matrix equation $A\psi = x$ or

$$(\Lambda + \Lambda_r)\,\psi = x$$

where ψ, x are column vectors and Λ, Λ_r are matrices such that matrix products are defined. We assume A is given and we decompose A into

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$\Lambda + \Lambda_r$ in a convenient way. If A is a matrix whose elements are numbers we can let Λ be nearest integers. If A is stochastic, Λ can be an easily invertible deterministic matrix and Λ_r would contain the remainder. Formally

$$\psi = (\Lambda + \Lambda_r)^{-1} x.$$

But previous work [1] shows we can write for $|\Lambda| \neq 0$

$$(\Lambda + \Lambda_r)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\Lambda^{-1}\Lambda_r)^n \Lambda^{-1}.$$

THEOREM. $(\Lambda + \Lambda_r)^{-1}(\Lambda + \Lambda_r) = I$ *if* $(\Lambda + \Lambda_r)^{-1}$ *is defined as* $\sum_{i=0}^{\infty}(-1)^i(\Lambda^{-1}\Lambda_r)^i \Lambda^{-1}$.

Proof.

$$\begin{aligned}
&\sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^i \Lambda^{-1}(\Lambda + \Lambda_r) \\
&\quad = \sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^i \Lambda^{-1}\Lambda + \sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^i \Lambda^{-1}\Lambda_r \\
&\quad = \sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^i I + \sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^{i+1} \\
&\quad = \sum_{i=0}^{\infty} (-1)^i(\Lambda^{-1}\Lambda_r)^i I - \sum_{i=0}^{\infty} (-1)^{i+1}(\Lambda^{-1}\Lambda_r)^{i+1} \\
&\quad = I.
\end{aligned}$$

It is reasonable from the same work to expect

$$(\Lambda + \Lambda_r)^{-1} \simeq \sum_{n=0}^{N \ll \infty} (-1)^n (\Lambda^{-1}\Lambda_r)^n \Lambda^{-1}$$

i.e., we expect a good approximation in a reasonable number of terms.

The method can be used for stochastic matrices as well letting Λ be deterministic and invertible and Λ_r representing random terms. That the appropriate statistics can be obtained with no statistical separability problems was shown by Adomian and Sibul [3].

CONVERGENCE

The Λ is to be chosen with the nearest integers to the given $A = (\Lambda + \Lambda_r)$ matrix so the element of Λ_r may be quite small. Each term of Φ_n involves an additional multiplication by $\Lambda^{-1}\Lambda_r$. Note that if we factor out the Λ^{-1} to the right, each multiplication by $\Lambda^{-1}\Lambda_r$ reduces the magnitude since the elements of Λ_r are always <1 because we take nearest integers for elements of Λ. (The worst case for a particular element of Λ_r is the value 0.50 but generally the values would be smaller.) We have, consequently, an alternating series with terms of decreasing magnitude. We have shown earlier that the series does indeed represent the inverse operator.

Part II. Decomposition into Diagonal Matrices

Consider (Case I)

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = L + R$$

where

$$L = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{pmatrix}.$$

Though this will yield slower convergence (because this choice of L is farther from Λ than the nearest integers choice of Part I), the terms Λ_n^{-1} are much easier to compute in analogy to differential operators (see Adomian [1]). We could alternatively (Case II) decompose Λ into

$$L = \begin{pmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}.$$

Returning to Case I,

$$L^{-1} = \begin{pmatrix} 1/\lambda_{11} & 0 \\ 0 & 1/\lambda_{22} \end{pmatrix} \equiv \Lambda_0^{-1}$$

$$\begin{aligned}
\Lambda_1^{-1} &= (-1)(L^{-1}R)\,L^{-1} \\
&= -\begin{pmatrix} 1/\lambda_{11} & 0 \\ 0 & 1/\lambda_{22} \end{pmatrix}\begin{pmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{pmatrix}\begin{pmatrix} 1/\lambda_{11} & 0 \\ 0 & 1/\lambda_{22} \end{pmatrix} \\
&= -\begin{pmatrix} 0 & \lambda_{12}/\lambda_{11} \\ \lambda_{21}/\lambda_{22} & 0 \end{pmatrix}\begin{pmatrix} 1/\lambda_{11} & 0 \\ 0 & 1/\lambda_{22} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\lambda_{12}/\lambda_{11}\lambda_{22} \\ -\lambda_{21}/\lambda_{11}\lambda_{22} & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\Lambda_2^{-1} &= (-1)^2 (L^{-1}R)(L^{-1}R)\, L^{-1} \\
&= \begin{pmatrix} 0 & \lambda_{12}/\lambda_{11} \\ \lambda_{21}/\lambda_{22} & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_{12}/\lambda_{11}\lambda_{22} \\ \lambda_{21}/\lambda_{11}\lambda_{22} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_{12}\lambda_{21}/\lambda_{11}^2\lambda_{22} & 0 \\ 0 & \lambda_{12}\lambda_{21}/\lambda_{11}\lambda_{22}^2 \end{pmatrix} \\
\Lambda_3^{-1} &= (-1)^3 (L^{-1}R)(L^{-1}R)^2\, L^{-1} \\
&= -\begin{pmatrix} 0 & \lambda_{12}/\lambda_{11} \\ \lambda_{21}/\lambda_{22} & 0 \end{pmatrix} \begin{pmatrix} \lambda_{12}\lambda_{21}/\lambda_{11}^2\lambda_{22} & 0 \\ 0 & \lambda_{12}\lambda_{21}/\lambda_{11}\lambda_{22}^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\lambda_{12}^2\lambda_{21}/\lambda_{11}^2\lambda_{22}^2 \\ -\lambda_{12}\lambda_{21}^2/\lambda_{11}^2\lambda_{22}^2 & 0 \end{pmatrix} \\
\Lambda_4^{-1} &= (-1)^4 (L^{-1}R)(L^{-1}R)^3\, L^{-1} \\
&= \begin{pmatrix} 0 & \lambda_{12}/\lambda_{11} \\ \lambda_{21}/\lambda_{22} & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_{12}^2\lambda_{21}/\lambda_{11}^2\lambda_{22}^2 \\ \lambda_{12}\lambda_{21}^2/\lambda_{11}^2\lambda_{22}^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_{12}^2\lambda_{21}^2/\lambda_{11}^3\lambda_{22}^2 & 0 \\ 0 & \lambda_{12}^2\lambda_{21}^2/\lambda_{11}^2\lambda_{22}^3 \end{pmatrix} \\
&\vdots
\end{aligned}$$

Inspection of the Λ_n^{-1} terms shows that alternating terms are zero and the nonzero terms are successively multiplied by the multiplier $r = \lambda_{12}\lambda_{21}/\lambda_{11}\lambda_{22}$ so we have a geometric progression. The smaller this term the faster the convergence. We must have $\lambda_{12}\lambda_{21} < \lambda_{11}\lambda_{22}$. We cannot have equality. If $\lambda_{12}\lambda_{21} > \lambda_{11}\lambda_{22}$ we simply choose L and R as in Case II (the zero elements of L should be on the smallest diagonal).

EXAMPLE. $\Lambda = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$. Choose $L = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then,

$$\begin{aligned}
\Lambda_0^{-1} &= L^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \\
\Lambda_1^{-1} &= (-1)(L^{-1}R)\, L^{-1} \\
&= -\begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \\
&= -\begin{pmatrix} 0 & 1/2 \\ 1/3 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix}
\end{aligned}$$

$$\Lambda_2^{-1} = (-1)^2(L^{-1}R)(L^{-1}R)\,L^{-1}$$
$$= \begin{pmatrix} 0 & 1/2 \\ 1/3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/6 \\ 1/6 & 0 \end{pmatrix} = \begin{pmatrix} 1/12 & 0 \\ 0 & 1/18 \end{pmatrix}$$

etc.

The approximate inverse obtained from only seven terms, i.e., $\Phi_7 = \sum_{n=0}^{6} \Lambda_n^{-1}$, is given by

$$\begin{pmatrix} 0.599 & -0.198 \\ -0.198 & 0.398 \end{pmatrix} \approx \begin{pmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{pmatrix}$$

which is the *exact* inverse.

Since the multiplier r here was 1/6, each term of the geometric sequence of corresponding elements of the Λ_n^{-1} are given by λr^{n-1} where λ is the first element of the sequence. Since $\lambda r^{n-1} = \lambda/200$ corresponds to $\frac{1}{2}\%$ error, and $(1/6)^3 = 1/216$, $n = 4$ is sufficient[1] starting from Λ_0^{-1}. Because of the alternating zeros, the estimate Λ_6^{-1} is sufficient as we have found. If we began with

$$\Lambda = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

we have

$$\Lambda_0^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$

and since $r = 2/3$, $\lambda(2/3)^{n-1} = \lambda/200$ requires $n = 14$ for $< \frac{1}{2}\%$ error in the result. Because of the alternating zeros, it means calculation of approximately 26 terms to get a very good approximation to

$$\Lambda^{-1} = \begin{pmatrix} 1.5 & -0.5 \\ -2 & 1 \end{pmatrix}.$$

Since we know corresponding elements of Λ_n^{-1} are a geometric sequence, let us now find an expression for Λ^{-1}. Consider again the 2×2 matrix as a generic form:

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

[1] If error in percent is $(100/p)$, $\lambda r^{n-1} = \lambda/p$ or $n = (-\ln p/\ln r) + 1$.

provided $\lambda_{12}\lambda_{21} \neq \lambda_{11}\lambda_{12}$. Assume the Case I situation then $|\lambda_{12}\lambda_{21}| < |\lambda_{11}\lambda_{22}|$. This is equivalent to saying the absolute value of the determinant of L is greater than the absolute value of the determinant of R. Since $\lambda_{12}\lambda_{21}/\lambda_{11}\lambda_{22}$ is a multiplier r for the nonzero terms we write

$$\Lambda^{-1} = \begin{pmatrix} \sigma/\lambda_{11} & -\lambda_{12}\sigma/\lambda_{11}\lambda_{22} \\ -\lambda_{21}\sigma/\lambda_{11}\lambda_{22} & \sigma/\lambda_{22} \end{pmatrix}$$

since the first nonzero terms are multiplied by r successively. Now σ is given by

$$\begin{aligned} \sigma &= \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} (\lambda_{12}\lambda_{21}/\lambda_{11}\lambda_{22})^n \\ &= 1/[1 - (\lambda_{12}\lambda_{21})/(\lambda_{11}\lambda_{22})]. \end{aligned}$$

For Case II where $|\lambda_{11}\lambda_{22}| < |\lambda_{12}\lambda_{21}|$

$$\Lambda^{-1} = \begin{pmatrix} -\lambda_{22}\sigma/\lambda_{12}\lambda_{21} & \sigma/\lambda_{21} \\ \sigma/\lambda_{12} & -\lambda_{11}\sigma/\lambda_{12}\lambda_{21} \end{pmatrix}$$

where

$$\sigma = \sum_{n=0}^{\infty} (\lambda_{11}\lambda_{22}/\lambda_{12}\lambda_{21})^n = 1/[1 - (\lambda_{11}\lambda_{22})/(\lambda_{12}\lambda_{21})].$$

THEOREM. *The inverse of the matrix Λ given by*

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

is given by

$$\Lambda^{-1} = \begin{pmatrix} \sigma/\lambda_{11} & -\lambda_{12}\sigma/\lambda_{11}\lambda_{22} \\ -\lambda_{21}\sigma/\lambda_{11}\lambda_{22} & \sigma/\lambda_{22} \end{pmatrix}$$

where

$$\sigma = \frac{1}{1 - \dfrac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}} = \frac{\lambda_{11}\lambda_{22}}{|\Lambda|}$$

if $|\lambda_{12}\lambda_{21}| < |\lambda_{11}\lambda_{22}|$ and by

$$\Lambda^{-1} = \begin{pmatrix} -\lambda_{22}\sigma/\lambda_{12}\lambda_{21} & \sigma/\lambda_{21} \\ \sigma/\lambda_{12} & -\lambda_{11}\sigma/\lambda_{12}\lambda_{21} \end{pmatrix}$$

where

$$\sigma = \frac{1}{1 - \lambda_{11}\lambda_{22}/\lambda_{12}\lambda_{21}}$$

if $|\lambda_{11}\lambda_{22}| < |\lambda_{12}\lambda_{21}|$.

EXAMPLE—CASE I.

$$A_I = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$A_I^{-1} = \begin{pmatrix} \sigma/2 & -\sigma/6 \\ -4\sigma/6 & \sigma/3 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -2 & 1 \end{pmatrix}$$

since $r = 2/3$ and σ consequently equals $1/[1-(2/3)] = 3$.

EXAMPLE—CASE II.

$$A_{II} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$A_{II}^{-1} = \begin{pmatrix} -4/6\sigma & \sigma/3 \\ \sigma/2 & -\sigma/6 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

since $\sigma = 3$ again.

Verification.

$$A_I^{-1}A_I = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3/2 & -1/2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_{II}^{-1}A_{II} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we wish to invert

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

using the method of Part II. Identifying this as Case I,

$$A^{-1} = \begin{pmatrix} \sigma/2 & -\sigma/6 \\ -4\sigma/6 & \sigma/3 \end{pmatrix}$$

and since $\sigma = 1/(1 - \frac{2}{3}) = 3$

$$A^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ -2 & 1 \end{pmatrix}$$

which is the correct inverse as easily verified.

EXAMPLE (Case II).

$$A = \begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$$

$$\sigma = \sum_{n=0}^{\infty} \left(\frac{25}{100}\right)^n = \frac{1}{1-(1/4)} = 4/3$$

$$A^{-1} = \begin{pmatrix} \dfrac{-5(4/3)}{100} & \dfrac{(4/3)}{10} \\ \dfrac{(4/3)}{10} & \dfrac{-5(4/3)}{100} \end{pmatrix}$$

$$= \begin{pmatrix} -1/15 & 2/15 \\ 2/15 & -1/15 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}\begin{pmatrix} -1/15 & 2/15 \\ 2/15 & -1/15 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The case given by

$$A = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}$$

should be rapidly convergent. We see that

$$\sigma = 1 + 0.01 + 0.0001 + \cdots = 10/9 = 1.01$$

$$A^{-1} = \begin{pmatrix} 1.01/10 & -1.01/100 \\ -1.01/100 & 1.01/10 \end{pmatrix}$$

$$= \begin{pmatrix} 0.101 & -0.0101 \\ -0.0101 & 0.101 \end{pmatrix}$$

and could be done quickly as well by summing A_n^{-1} for a few terms.

EXAMPLE—3 × 3 MATRIX.

$$\Lambda = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} = \Lambda_0^{-1}$$

$$-L^{-1}R = (-1) \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/2 \\ -1/3 & 0 & 0 \\ 0 & -1/2 & 0 \end{pmatrix}$$

$$\begin{aligned} \Lambda_1^{-1} &= -[L^{-1}R] \cdot \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1/2 \\ -1/3 & 0 & 0 \\ 0 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1/4 \\ -1/6 & 0 & 0 \\ 0 & -1/6 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Lambda_2^{-1} &= \begin{pmatrix} 0 & 0 & -1/2 \\ -1/3 & 0 & 0 \\ 0 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1/4 \\ -1/6 & 0 & 0 \\ 0 & -1/6 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1/12 & 0 \\ 0 & 0 & 1/12 \\ 1/12 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The approximation in only three terms is $\Phi_3 = \Lambda_0^{-1} + \Lambda_1^{-1} + \Lambda_2^{-1}$ given by

$$\Phi_3 = \begin{pmatrix} 1/2 & 1/12 & -1/4 \\ -1/6 & 1/3 & 1/12 \\ 1/12 & -1/6 & -1/12 \end{pmatrix} = \begin{pmatrix} 0.50 & 0.08 & -0.25 \\ -0.17 & 0.33 & 0.08 \\ 0.08 & -0.17 & 0.58 \end{pmatrix}.$$

The correct inverse is

$$\Lambda^{-1} = \begin{pmatrix} 6/13 & 1/13 & -3/13 \\ -2/13 & 4/13 & 1/13 \\ 1/13 & -2/13 & 6/13 \end{pmatrix}$$

$$= \begin{pmatrix} 0.46 & 0.08 & -0.23 \\ -0.15 & 0.31 & 0.08 \\ 0.08 & -0.15 & 0.46 \end{pmatrix}$$

which we are approaching rapidly, i.e., with only a three-term approximation.

If we calculate the first 20 terms of Φ_n, the elements of Λ_{19}^{-1}, for example (to calculate Φ_{20}), are all less than 1×10^{-7}. Calculation of Φ_{21} yields the elements of the correct inverse within $3 \times 10^{-6}\%$.

3×3 *matrices.*

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} = L + R$$

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & 0 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & 0 \end{pmatrix}$$

$$\Lambda_0^{-1} = L^{-1} = \begin{pmatrix} 1/\lambda_{11} & 0 & 0 \\ 0 & 1/\lambda_{22} & 0 \\ 0 & 0 & 1/\lambda_{33} \end{pmatrix}$$

Assume $|\lambda_{11}\lambda_{22}\lambda_{33}| > |-\lambda_{12}(\lambda_{23}\lambda_{31}) + \lambda_{13}(\lambda_{21}\lambda_{32})|$ (Case I), i.e., $|\det L| > |\det R|$.

$$\Lambda_1^{-1} = (-L^{-1}R)\,L^{-1}$$

$$= (-1)\begin{pmatrix} 1/\lambda_{11} & 0 & 0 \\ 0 & 1/\lambda_{22} & 0 \\ 0 & 0 & 1/\lambda_{33} \end{pmatrix}\begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & 0 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & 0 \end{pmatrix}\begin{pmatrix} 1/\lambda_{11} & 0 & 0 \\ 0 & 1/\lambda_{22} & 0 \\ 0 & 0 & 1/\lambda_{33} \end{pmatrix}.$$

The product $-L^{-1}R$ is given by

$$\begin{pmatrix} 0 & -\lambda_{12}/\lambda_{11} & -\lambda_{13}/\lambda_{11} \\ -\lambda_{21}/\lambda_{22} & 0 & -\lambda_{23}/\lambda_{22} \\ -\lambda_{31}/\lambda_{33} & -\lambda_{32}/\lambda_{33} & 0 \end{pmatrix}.$$

The $-(L^{-1}R)$ matrix multiplies L^{-1} to yield Λ_1^{-1}. The same matrix multiplies Λ_1^{-1} to yield Λ_2^{-1}, etc.

Systems of matrix equations. Consider the matrix equations

$$Ax + By = f$$
$$Cx + Dy = g$$

where A, B, C, D are matrices, x, y, f, g are vectors and

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

If $\binom{x}{y} = \xi$ and $\binom{f}{g} = \eta$ and finally

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \Gamma$$

then

$$\Gamma\xi = \eta \qquad \text{or} \qquad \xi = \Gamma^{-1}\eta$$

is in our general form. Generalizing, consider

$$A_{11}x + A_{12}y + A_{13}z = G_1$$
$$A_{21}x + A_{22}y + A_{23}z = G_2$$
$$A_{31}x + A_{32}y + A_{33}z = G_3$$

x, y, z, G_1, G_2, G_3 are 3×1 matrices (vectors) and the A_{ij} are 3×3 matrices.

Let $A_{ij}=a_{ij}+\alpha_{ij}$ where a_{ij} are deterministic and α_{ij} are stochastic. Similarly $G_i=g_i+\gamma_i$. Now

$$\left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}.$$

Denoting the 9×9 matrix of A_{ij}'s by A and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

by x we can write

$$Ax=G$$

$$x=A^{-1}G.$$

Inversion of random matrices. Consider

$$A=\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}=L+R$$

$$L=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$R=\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

Choose $\|L\|>\|R\|$ a.s. (where the notation indicates absolute value of the determinants). Furthermore let $L=L_1+L_2$ where

$$L_1=\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

$$L_2=\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

If $\|L_1\|>\|L_2\|$ then $L^{-1}=\sum_{n=0}^{\infty}(-1)^n(L_1^{-1}L_2)^n L_1^{-1}$ where

$$L_1^{-1}=\begin{pmatrix} 1/a_{11} & 0 \\ 0 & 1/a_{22} \end{pmatrix}$$

hence

$$L^{-1}=\sum_{n=0}^{\infty}(-1)^n\left\{\begin{pmatrix}1/a_{11} & 0\\ 0 & 1/a_{22}\end{pmatrix}\begin{pmatrix}0 & a_{12}\\ a_{21} & 0\end{pmatrix}\right\}^n\begin{pmatrix}1/a_{11} & 0\\ 0 & 1/a_{22}\end{pmatrix}.$$

Depending on the elements of A, i.e., whether A falls into Case I or Case II, we can sum and write the inverse as a single matrix for L^{-1}. Then

$$A^{-1}=\sum_{n=0}^{\infty}(-1)^n(L^{-1}R)^n L^{-1}$$

where R contains the random elements which require averaging. It is a straightforward step to compute the various statistical measures as defined in [1] such as the expected inverse $\langle A^{-1}\rangle$ or the correlation of the inverse matrix. Adomian and Sibul [3] have already shown that statistical separability will hold on the various averagings.

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