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Regularity of very weak solutions for elliptic equation of divergence form ☆

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Abstract

In this paper, we study the local regularity of very weak solution $u \in L^1_{loc}(\Omega)$ of the elliptic equation $D_j(a_{ij}(x)D_i u) = 0$. Using the bootstrap argument and the difference quotient method, we obtain that if $a_{ij} \in C^{0,1}_{loc}(\Omega)$, then $u \in W^{2,p}_{loc}(\Omega)$ for any $p < \infty$.

Keywords: Very weak solution; Regularity; Linear elliptic equation; Sobolev space

1. Introduction

The simplest kind of linear elliptic equations in divergence form is

$$D_j\big(a_{ij}(x)D_i u\big) = 0, \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geqslant 2$, and the coefficients $a_{ij}(x)$ are bounded measurable functions satisfying the uniformly ellipticity condition, i.e.,

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$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{1.2}$$

with $0 < \lambda \leqslant \Lambda < \infty$. $u \in W^{1,p}_{loc}(\Omega)$ for $p \in [1,\infty)$ is called a weak solution of (1.1) over Ω if

$$\int_\Omega a_{ij} D_i u D_j \varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.3}$$

A fundamental result of E. De Giorgi [6] states that if $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of (1.1), then u is locally bounded and then locally Hölder continuous. N.G. Meyers [15] also proved that $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 2$.

J. Serrin in [18] showed by a counterexample that in general the solutions of (1.1) in $W^{1,p}_{loc}(\Omega)$ for $p \in (1,2)$ need not be locally bounded only under the assumption (1.2). He proposed a conjecture that if the coefficients a_{ij} are locally Hölder continuous, then any weak solution $u \in W^{1,1}_{loc}(\Omega)$ of (1.1) must be in $W^{1,2}_{loc}(\Omega)$. R.A. Hager and J. Ross [11] proved that the conjecture is true for the weak solutions in $W^{1,p}_{loc}(\Omega)$ for $p \in (1,2)$. In 2008, a celebrated theorem was established by H. Brezis (see [2], a full proof can be found in [1]).

Theorem 1.1. *Assume that a_{ij} are Dini continuous in Ω, and let $u \in BV_{loc}(\Omega)$ be a weak solution of* (1.1)*, then $u \in W^{1,2}_{loc}(\Omega)$.*

Here the coefficients a_{ij} are Dini continuous in Ω, i.e., $a_{ij} \in C^0(\Omega)$, and for any subdomain $\Omega' \Subset \Omega$, there exists a function φ, such that

$$\left|a_{ij}(x) - a_{ij}(y)\right| \leqslant \varphi\big(|x-y|\big), \quad x, y \in \Omega', \quad \text{where} \int_0^{\operatorname{diam}\Omega'} \frac{\varphi(r)}{r}\,dr < \infty.$$

And $u \in BV_{loc}(\Omega)$ means $u \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega'} |Du| = \sup\left\{\int_{\Omega'} u \cdot \operatorname{div}\vec{v}:\ \vec{v} \in C_0^1\big(\Omega', \mathbb{R}^N\big), |\vec{v}| \leqslant 1\right\} < \infty, \quad \forall \Omega' \Subset \Omega.$$

Theorem 1.1 confirmed completely Serrin's conjecture in the case of less smooth given coefficients and solutions, since Hölder continuity on a_{ij} were replaced by Dini continuity, and u was extended from $W^{1,1}_{loc}(\Omega)$ to $BV_{loc}(\Omega)$.

For merely continuity on a_{ij}, H. Brezis obtained the following result.

Theorem 1.2. *Assume that $a_{ij} \in C^0(\Omega)$. If $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of* (1.1) *for some $p > 1$, then $u \in W^{1,q}_{loc}(\Omega)$ for every $q < \infty$.*

There are the counterexamples to show Theorem 1.2 is not true in the cases $p = 1$ or $q = \infty$. Therefore Theorem 1.2 is optimal in some sense. For the unit ball B_1 and the continuous coefficients a_{ij}, T. Jin, V. Mazya and J.V. Schaftingen [13] constructed a weak solution

$u \in W^{1,1}_{loc}(B_1) \backslash W^{1,p}_{loc}(B_1)$ for every $p > 1$. They also gave a function $u \in W^{1,q}_{loc}(B_1) \backslash W^{1,\infty}_{loc}(B_1)$ for every $q < \infty$, satisfying (1.1).

Now we consider a very weak solution $u \in L^1_{loc}(\Omega)$ of (1.1), namely

$$\int_\Omega u D_i(a_{ij} D_j \varphi) = 0, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.4}$$

It is easy to be seen that a very weak solution in $W^{1,p}_{loc}(\Omega)$, $p \in [1, \infty)$, of (1.1) must be a usual weak solution. Because of the very weak assumptions made on the solutions it is natural that the coefficients should be interpreted as the local Lipschitz functions. H. Brezis raised a question whether any very weak solution $u \in L^1_{loc}(\Omega)$ is in $W^{1,2}_{loc}(\Omega)$, and then one can apply E. De Giorgi's theory if a_{ij} are only in $C^{0,1}_{loc}(\Omega)$. We give the positive answer and have

Theorem 1.3. *Assume that $a_{ij} \in C^{0,1}_{loc}(\Omega)$. If $u \in L^1_{loc}(\Omega)$ is a very weak solution of* (1.1)*, then $u \in W^{2,p}_{loc}(\Omega)$ for any $p \in [1, \infty)$.*

Throughout the paper, we always assume that the coefficients $a_{ij} \in C^{0,1}_{loc}(\Omega)$ are elliptic, i.e., for any subdomain $\Omega' \Subset \Omega$, there exist the constants K, λ, Λ, depending only on Ω', such that

$$\left|a_{ij}(x) - a_{ij}(y)\right| \leqslant K|x-y|, \quad \forall x, y \in \Omega',\ i, j = 1, 2, \ldots, N, \tag{1.5}$$

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega',\ \xi \in \mathbb{R}^N. \tag{1.6}$$

The very weak solution has been studied by many authors. In [3], H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa proved the existence and uniqueness theorem for a very weak solution in $L^1(\Omega)$ of the Poisson equations $\Delta u = f(x)$ with zero boundary value. They also established the estimate

$$\|u\|_{L^1(\Omega)} \leqslant \left\| f \cdot dist(x, \partial\Omega) \right\|_{L^1(\Omega)}.$$

Later, X. Cabré and Y. Martel [4] showed the very weak solution is in $L^q(\Omega)$ for any $1 \leqslant q < \frac{N}{N-2}$.

Therefore, the question of the integrability of the weak derivative of the very weak solution arises in a natural way.

Recently, J.I. Diaz and J.M. Rakotoson [8] extended the results of Brezis et al. to $Lu = f(x)$, where L is a linear second order elliptic operator with variable coefficients. They obtained if $f \cdot dist^\alpha(x, \partial\Omega) \in L^1(\Omega)$, $0 \leqslant \alpha < 1$, then Du belongs to the Lorentz space $L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$, where

$$L^{\frac{N}{N-1+\alpha},\infty}(\Omega) = \left\{ f\colon \Omega \to \mathbb{R} \text{ measurable and } \sup_{t \leqslant |\Omega|} t^{\frac{\alpha-1}{N}} \int_0^t |f|_*\, ds < \infty \right\},$$

$|f|_*(s) = \inf\{t \in \mathbb{R}\colon \text{ meas}\{|f| > t\} \leqslant s\}$ for $0 \leqslant s \leqslant |\Omega|$. In particular, for Poisson equation (Lemma 6 in [8]), if $f \in L^1(\Omega)$, then $Du \in L^q(\Omega) \subset L^{\frac{N}{N-1},\infty}(\Omega)$, where $1 \leqslant q < \frac{N}{N-1}$.

There are other results on the very weak solutions, such as [7,9,12,14,19] for semilinear elliptic equations, [16] and [20] for elliptic systems, [17] for Neumann problems.

The rest part of the paper is organized as follows: In the next section we present some preliminary facts which will be used later. Section 3 is devoted to the proof of Theorem 1.3. We obtain the theorem by using the bootstrap argument and the difference quotient method.

2. Some preliminary facts

In this section, we list some preliminary facts that will be needed in our proof.

For convenience, we abbreviate a ball with center x_0 and radius R as B_R, and then consider the Dirichlet problem

$$\begin{cases} a_{ij}(x)D_{ij}v = f(x), & x \in B_R, \\ v = 0, & x \in \partial B_R. \end{cases} \tag{2.7}$$

Lemma 2.1. *Suppose that $a_{ij} \in C^0(\overline{B_R})$ satisfy* (1.6) *in B_R and $f \in L^p(B_R)$ with $1 < p < \infty$. Then* (2.7) *exists a unique solution $v \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$ satisfying*

$$\|v\|_{W^{2,p}(B_R)} \leqslant C\|f\|_{L^p(B_R)},$$

where C depends only on N, p, λ, Λ, R and the modulus of continuity of a_{ij} on $\overline{B_R}$.

This lemma is the direct conclusion of Theorems 6.3 and 6.4 in Chapter 3 in [5].

Lemma 2.2. *Let u be a $W^{2,p}(B_R)$ solution of* (2.7) *with $1 < p < \infty$. If $a_{ij} \in C^{0,1}(\overline{B_R})$ are uniformly elliptic, and $f \in W^{1,q}(B_R)$ with $1 < q < \infty$, then $u \in W^{3,q}(B_R)$.*

This lemma is a special case of Theorem 9.19 in [10].

3. Proof of Theorem 1.3

Proof. For fixed $\Omega'' \Subset \Omega' \Subset \Omega$, let $2\delta = \min\{1, \frac{1}{2}d(\Omega'', \partial\Omega')\}$. For any $x_0 \in \Omega''$, we let $\eta(x) \in C_c^\infty(B_{2\delta}(x_0))$ be a cut-off function:

$$\eta(x) = \begin{cases} 1, & |x - x_0| \leqslant \delta, \\ 0, & |x - x_0| \geqslant 2\delta, \end{cases}$$

such that $0 \leqslant \eta(x) \leqslant 1$,

$$|D\eta| \leqslant \frac{M_1}{\delta} \quad \text{and} \quad \left|D^2\eta\right| \leqslant \frac{M_2}{\delta^2},$$

where M_1, M_2 are positive constants.

For the sake of clarity, we divide the estimate in Theorem 1.3 into three steps.

Step 1. L^p regularity.

(1) $1 < p < \frac{N}{N-1}$.

Let w_1 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_1 \in W^{2,q_1}(B_{2\delta}) \cap W_0^{1,q_1}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x)D_{ij}v_1 = w_1, & x \in B_{2\delta}, \\ v_1 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_1\|_{W^{2,q_1}(B_{2\delta})} \leqslant C\|w_1\|_{L^{q_1}(B_{2\delta})}, \tag{3.8}$$

where $q_1 \in (N, \infty)$. Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_1 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,q_1}(B_{2\delta})$, by Lemma 2.2, we have $v_1 \in W^{3,q_1}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we get $v_1 \in C^{2,\alpha}(\overline{B_{2\delta}})$.

From (1.4) and a density argument, we have

$$\int_\Omega u D_i(a_{ij} D_j v) = 0, \quad \forall v \in C_0^2(\Omega). \tag{3.9}$$

Now we choose $v = \eta^2 v_1$ in (3.9), and get

$$\begin{aligned} 0 &= \int_\Omega u D_i\big(a_{ij} D_j\big(\eta^2 v_1\big)\big) \\ &= \int_{B_{2\delta}} u D_i a_{ij} D_j\big(\eta^2 v_1\big) + u a_{ij} D_{ij}\big(\eta^2 v_1\big) \\ &= \int_{B_{2\delta}} 2\eta u v_1 D_i a_{ij} D_j \eta + u\eta^2 D_i a_{ij} D_j v_1 + 2u v_1 a_{ij} D_i \eta D_j \eta \\ &\quad + \int_{B_{2\delta}} 2\eta u v_1 a_{ij} D_{ij}\eta + 4\eta u a_{ij} D_i \eta D_j v_1 + u\eta^2 a_{ij} D_{ij} v_1. \end{aligned}$$

By the properties of the cut-off function and a_{ij}, we have

$$\begin{aligned} \left|\int_{B_{2\delta}} u\eta^2 a_{ij} D_{ij} v_1\right| &\leqslant \frac{2KM_1}{\delta}\int_{B_{2\delta}} |uv_1| + K\int_{B_{2\delta}} |uD_j v_1| + \frac{2\Lambda {M_1}^2}{\delta^2}\int_{B_{2\delta}} |uv_1| \\ &\quad + \frac{2\Lambda M_2}{\delta^2}\int_{B_{2\delta}} |uv_1| + \frac{4\Lambda M_1}{\delta}\int_{B_{2\delta}} |uD_j v_1| \\ &\leqslant C\|v_1\|_{W^{1,\infty}(B_{2\delta})}\|u\|_{L^1(B_{2\delta})}. \end{aligned}$$

By the Sobolev imbedding theorem and (3.8), we have

$$\|v_1\|_{W^{1,\infty}(B_{2\delta})} \leqslant C\|v_1\|_{W^{2,q_1}(B_{2\delta})} \leqslant C\|w_1\|_{L^{q_1}(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} u\eta^2 w_1\right| \leqslant C\|w_1\|_{L^{q_1}(B_{2\delta})}\|u\|_{L^1(B_{2\delta})}.$$

Since w_1 is an arbitrary smooth function in $\overline{B_{2\delta}}$, we conclude

$$\|\eta^2 u\|_{L^{p_1}(B_{2\delta})} \leqslant C\|u\|_{L^1(B_{2\delta})}, \tag{3.10}$$

where $p_1 := \frac{q_1}{q_1-1} \in (1, \frac{N}{N-1})$.

Now using finite covering theorem, we obtain $u \in L^p_{loc}(\Omega)$, $\forall p \in (1, \frac{N}{N-1})$.

(2) $p = \frac{N}{N-1}$.

Let w_2 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_2 \in W^{2,N}(B_{2\delta}) \cap W_0^{1,N}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x)D_{ij}v_2 = w_2, & x \in B_{2\delta}, \\ v_2 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_2\|_{W^{2,N}(B_{2\delta})} \leqslant C\|w_2\|_{L^N(B_{2\delta})}. \tag{3.11}$$

Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_2 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,N}(B_{2\delta})$, by Lemma 2.2, we have $v_2 \in W^{3,N}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we get $v_2 \in W^{2,r_2}(B_{2\delta})$, $\forall r_2 < \infty$.

From (1.4) and a density argument, we have

$$\int_\Omega u D_i(a_{ij}D_j v) = 0, \quad \forall v \in W_0^{2,l_2}(B_{2\delta}),\ l_2 \in (N,\infty). \tag{3.12}$$

Now we choose $v = \eta^4 v_2$ in (3.12), use the properties of the cut-off function and a_{ij}, and have

$$\left|\int_{B_{2\delta}} u\eta^4 a_{ij}D_{ij}v_2\right| \leqslant C\|v_2\|_{W^{1,q_1}(B_{2\delta})}\|\eta^2 u\|_{L^{p_1}(B_{2\delta})},$$

where $p_1 \in (1, \frac{N}{N-1})$, $q_1 = \frac{p_1}{p_1-1} \in (N,\infty)$.

By the Sobolev imbedding theorem and (3.11), we obtain

$$\|v_2\|_{W^{1,q_1}(B_{2\delta})} \leqslant C\|v_2\|_{W^{2,N}(B_{2\delta})} \leqslant C\|w_2\|_{L^N(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} u\eta^4 w_2\right| \leqslant C\|w_2\|_{L^N(B_{2\delta})}\|\eta^2 u\|_{L^{p_1}(B_{2\delta})}.$$

From a duality argument and (3.10), we conclude

$$\left\|\eta^4 u\right\|_{L^{\frac{N}{N-1}}(B_{2\delta})} \leqslant C\|u\|_{L^1(B_{2\delta})}. \tag{3.13}$$

Now using finite covering theorem, we have $u \in L^p_{loc}(\Omega)$, $p = \frac{N}{N-1}$.

(3) $\frac{N}{N-1} < p < \frac{N}{N-2}$.

Let w_3 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_3 \in W^{2,q_3}(B_{2\delta}) \cap W_0^{1,q_3}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x) D_{ij} v_3 = w_3, & x \in B_{2\delta}, \\ v_3 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_3\|_{W^{2,q_3}(B_{2\delta})} \leqslant C\|w_3\|_{L^{q_3}(B_{2\delta})}, \tag{3.14}$$

where $q_3 \in (\frac{N}{2}, N)$. Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_3 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,q_3}(B_{2\delta})$, by Lemma 2.2, we have $v_3 \in W^{3,q_3}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we obtain $v_3 \in W^{2,r_3}(B_{2\delta})$, $r_3 = \frac{Nq_3}{N-q_3} \in (N, \infty)$.

From (1.4) and a density argument, we have

$$\int_\Omega u D_i(a_{ij} D_j v) = 0, \quad \forall v \in W_0^{2,N}(B_{2\delta}). \tag{3.15}$$

Now we choose $v = \eta^6 v_3$ in (3.15), use the properties of the cut-off function and a_{ij}, we obtain

$$\left|\int_{B_{2\delta}} u\eta^6 a_{ij} D_{ij} v_3\right| \leqslant C\|v_3\|_{W^{1,\frac{Nq_3}{N-q_3}}(B_{2\delta})} \left\|\eta^4 u\right\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})}.$$

Recall that $q_3 \in (\frac{N}{2}, N)$. So we have $\frac{Nq_3}{Nq_3-N+q_3} \in (1, \frac{N}{N-1})$, and

$$\left\|\eta^4 u\right\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})} \leqslant \left\|\eta^4 u\right\|_{L^{\frac{N}{N-1}}(B_{2\delta})}.$$

By the Sobolev imbedding theorem and (3.14), we obtain

$$\|v_3\|_{W^{1,\frac{Nq_3}{N-q_3}}(B_{2\delta})} \leqslant C\|v_3\|_{W^{2,q_3}(B_{2\delta})} \leqslant C\|w_3\|_{L^{q_3}(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} u\eta^6 w_3\right| \leqslant C\|w_3\|_{L^{q_3}(B_{2\delta})} \left\|\eta^4 u\right\|_{L^{\frac{N}{N-1}}(B_{2\delta})}.$$

From a duality argument and (3.13), we conclude

$$\left\|\eta^6 u\right\|_{L^{\frac{q_3}{q_3-1}}(B_{2\delta})} \leqslant C\|u\|_{L^1(B_{2\delta})}. \tag{3.16}$$

Since $q_3 \in (\frac{N}{2}, N)$, we have $\frac{q_3}{q_3-1} \in (\frac{N}{N-1}, \frac{N}{N-2})$. By taking $p_3 = \frac{q_3}{q_3-1}$, it follows that $q_3 = \frac{p_3}{p_3-1}$. So we get

$$\frac{Nq_3}{Nq_3 - N + q_3} = \frac{Np_3}{N + p_3} \in \left(1, \frac{N}{N-1}\right),$$

and

$$\left\|\eta^6 u\right\|_{L^{p_3}(B_{2\delta})} \leqslant C\|u\|_{L^1(B_{2\delta})}, \tag{3.17}$$

for $\frac{N}{N-1} < p_3 < \frac{N}{N-2}$.

Now using finite covering theorem, we have $u \in L^p_{loc}(\Omega)$, $p = p_3 \in (\frac{N}{N-1}, \frac{N}{N-2})$.

(4) $p \geqslant \frac{N}{N-2}$ $(N \geqslant 3)$.

From (2) and (3), we have $\eta^6 u \in L^p(B_{2\delta})$ for any $p \in [\frac{N}{N-1}, \frac{N}{N-2})$. Likewise, for any given positive integer $k = 3, 4, \ldots, N$, we obtain that $\eta^{2(k+1)}u \in L^p(B_{2\delta})$, $\forall p \in [\frac{N}{N+1-k}, \frac{N}{N-k})$. Moreover,

$$\left\|\eta^{2(k+1)} u\right\|_{L^p(B_{2\delta})} \leqslant C\|u\|_{L^1(B_{2\delta})}, \tag{3.18}$$

for all $p \geqslant \frac{N}{N-2}$.

Now, for $p > 1$, using finite covering theorem, we have

$$\|u\|_{L^p(\Omega'')} \leqslant C\|u\|_{L^1(\Omega')},$$

where the constant C depends only on N, p, λ, Λ, K, Ω'' and Ω'.

Step 2. $W^{1,p}$ regularity.

Recall that η is the cut-off function defined at the beginning of our proof. For fixed $h < \frac{1}{3}dist(\operatorname{supp}\eta, \partial B_{2\delta})$ and $k = 1, 2, \ldots, N$,

$$\Delta_h^k u(x) = \frac{u(x + he_k) - u(x)}{h} \in L^p\big(\Omega''\big),$$

we have $|\eta\Delta_h^k u|^{p-1}\operatorname{sign}(\eta\Delta_h^k u) \in L^r(B_{2\delta})$, where $r = \frac{p}{p-1}$. According to Lemma 2.1, there must be a unique function

$$v_h \in W^{2,r}(B_{2\delta}) \cap W_0^{1,r}(B_{2\delta}),$$

such that

$$\begin{cases} a_{ij}(x)D_{ij}v_h = \left|\eta\Delta_h^k u\right|^{p-1}\operatorname{sign}(\eta\Delta_h^k u), & x \in B_{2\delta}, \\ v_h = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_h\|_{W^{2,r}(B_{2\delta})} \leqslant C\left\||\eta\Delta_h^k u|^{p-1}\right\|_{L^r(B_{2\delta})} \leqslant C\left\|\eta\Delta_h^k u\right\|^{p-1}_{L^p(B_{2\delta})}. \tag{3.19}$$

From (1.4) and a density argument, we have

$$\int_{\Omega} uD_i(a_{ij}D_j w) = 0, \quad \forall w \in W_0^{2,r}(\Omega). \tag{3.20}$$

Now we choose $w = \eta\Delta_{-h}^k v_h$ in (3.20), and get

$$\begin{aligned}
0 &= \int_{B_{2\delta}} uD_i\bigl(a_{ij}D_j\bigl(\eta\Delta_{-h}^k v_h\bigr)\bigr) \\
&= \int_{B_{2\delta}} uD_i a_{ij}D_j\eta\Delta_{-h}^k v_h + u\eta D_i a_{ij}\Delta_{-h}^k(D_j v_h) + ua_{ij}D_{ij}\eta\bigl(\Delta_{-h}^k v_h\bigr) \\
&\quad + \int_{B_{2\delta}} 2ua_{ij}D_i\eta\Delta_{-h}^k(D_j v_h) + ua_{ij}\eta\Delta_{-h}^k(D_{ij}v_h).
\end{aligned} \tag{3.21}$$

Meanwhile by the property of difference quotients, we get

$$\begin{aligned}
\int_{B_{2\delta}} ua_{ij}\eta\Delta_{-h}^k(D_{ij}v_h) = &-\int_{B_{2\delta}} \eta\Delta_h^k ua_{ij}D_{ij}v_h \\
&-\int_{B_{2\delta}} \bigl(\Delta_h^k\eta\bigr)u(x+he_k)a_{ij}(x+he_k)D_{ij}v_h \\
&-\int_{B_{2\delta}} \bigl(\Delta_h^k a_{ij}\bigr)\eta u(x+he_k)D_{ij}v_h.
\end{aligned} \tag{3.22}$$

From (3.21) and (3.22),

$$\begin{aligned}
\int_{B_{2\delta}} \eta\Delta_h^k ua_{ij}D_{ij}v_h \leqslant &\int_{B_{2\delta}} \eta u(x+he_k)\bigl(\Delta_h^k a_{ij}\bigr)D_{ij}v_h \\
&+\int_{B_{2\delta}} \bigl(\Delta_h^k\eta\bigr)u(x+he_k)a_{ij}(x+he_k)D_{ij}v_h \\
&+\int_{B_{2\delta}} \bigl|uD_i a_{ij}D_j\eta\bigl(\Delta_{-h}^k v_h\bigr)\bigr| \\
&+\int_{B_{2\delta}} \bigl|u\eta D_i a_{ij}\Delta_{-h}^k(D_j v_h)\bigr| + \bigl|ua_{ij}D_{ij}\eta\bigl(\Delta_{-h}^k v_h\bigr)\bigr| \\
&+\int_{B_{2\delta}} 2\bigl|ua_{ij}D_i\eta\Delta_{-h}^k(D_j v_h)\bigr|.
\end{aligned}$$

By the properties of the cut-off function and a_{ij}, we have

$$\int_{B_{2\delta}} \eta\Delta_h^k u a_{ij} D_{ij} v_h \leqslant K \int_{\operatorname{supp}\eta} \left|u(x+he_k) D_{ij} v_h\right| + \frac{\Lambda M_1}{\delta} \int_{\operatorname{supp}\eta} \left|u(x+he_k) D_{ij} v_h\right| + \frac{K M_1}{\delta} \int_{\operatorname{supp}\eta} \left|u\Delta_{-h}^k v_h\right| + K \int_{\operatorname{supp}\eta} \left|u\Delta_{-h}^k (D_j v_h)\right| + \frac{\Lambda M_2}{\delta^2} \int_{\operatorname{supp}\eta} \left|u\Delta_{-h}^k v_h\right| + \frac{2\Lambda M_1}{\delta} \int_{\operatorname{supp}\eta} \left|u\Delta_{-h}^k (D_j v_h)\right|.$$

By Hölder inequality, (3.19), Sobolev imbedding theorem, the property of difference quotients and Young inequality, we obtain

$$\int_{B_{2\delta}} \eta a_{ij}\Delta_h^k u D_{ij} v_h \leqslant \frac{1}{2}\left\|\eta\Delta_h^k u\right\|_{L^p(B_{2\delta})}^p + \frac{C}{\delta^{2p}}\|u\|_{L^p(B_{2\delta})}^p. \tag{3.23}$$

Meanwhile

$$\int_{B_{2\delta}} \eta a_{ij}\Delta_h^k u D_{ij} v_h = \int_{B_{2\delta}} \eta\Delta_h^k u \left|\eta\Delta_h^k u\right|^{p-1} \operatorname{sign}\left(\eta\Delta_h^k u\right) = \left\|\eta\Delta_h^k u\right\|_{L^p(B_{2\delta})}^p. \tag{3.24}$$

From (3.23) and (3.24), we have

$$\left\|\Delta_h^k u\right\|_{L^p(B_\delta)} \leqslant \frac{C}{\delta^2}\|u\|_{L^p(B_{2\delta})}.$$

Using the property of difference quotients again, we obtain $D_k u \in L^p(B_\delta)$, and

$$\|D_k u\|_{L^p(B_\delta)} \leqslant C\|u\|_{L^p(B_{2\delta})}.$$

Now, using finite covering theorem, we have

$$\|u\|_{W^{1,p}(\Omega'')} \leqslant C\|u\|_{L^p(\Omega')},$$

where C depends only on N, p, λ, Λ, K, Ω'' and Ω'.

Step 3. $W^{2,p}$ regularity.

Now, $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of (1.1), i.e.,

$$\int_\Omega a_{ij} D_i u D_j \varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

In particular,

$$\int_\Omega a_{ij} D_i u D_j(\eta\varphi) = 0, \quad \forall \varphi \in C_c^\infty(\Omega),$$

where η is the cut-off function given at the beginning of our proof. Let $v = \eta u$, then $v = 0$ on $\partial B_{2\delta}$ and

$$\int_{B_{2\delta}} a_{ij} D_i v D_j \varphi = \int_{B_{2\delta}} f \varphi, \quad \forall \varphi \in C_c^\infty(B_{2\delta}), \tag{3.25}$$

where

$$f(x) = -D_j(u a_{ij} D_i \eta) - a_{ij} D_i u D_j \eta \in L^p(B_{2\delta}).$$

Consider the Dirichlet problem

$$\begin{cases} D_j\big(a_{ij}(x) D_i w\big) = -f(x), & x \in B_{2\delta}, \\ w = 0, & x \in \partial B_{2\delta}. \end{cases} \tag{3.26}$$

Using Theorem 6.3 in [5], (3.26) exists a unique $w \in W^{2,p}(B_{2\delta}) \cap W_0^{1,p}(B_{2\delta})$ and

$$\|w\|_{W^{2,p}(B_{2\delta})} \leqslant C\|f\|_{L^p(B_{2\delta})} \leqslant C\|u\|_{W^{1,p}(B_{2\delta})}.$$

Obviously, w is also a weak solution of (3.26) in $B_{2\delta}$. By the uniqueness of the weak solution of (3.26), we conclude $v = w \in W^{2,p}(B_{2\delta})$, i.e., $\eta u \in W^{2,p}(B_{2\delta})$. Moreover

$$\|u\|_{W^{2,p}(B_\delta)} \leqslant C\|u\|_{W^{1,p}(B_{2\delta})}.$$

Now, using finite covering theorem, we obtain

$$\|u\|_{W^{2,p}(\Omega'')} \leqslant C\|u\|_{W^{1,p}(\Omega')}.$$

Finally, from Step 1 to Step 3, we conclude that

$$\|u\|_{W^{2,p}(\Omega'')} \leqslant C\|u\|_{L^1(\Omega')},$$

where C depends only on N, p, λ, Λ, K, Ω'' and Ω'. □

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