The cohomology structure of string algebras

Juan Carlos Bustamante

Département de Mathématiques, Université de Sherbrooke, 2500 Boulevard de l’Université, Sherbrooke, Que., Canada J1K 2R1

Received 20 October 2004; received in revised form 3 May 2005
Available online 1 August 2005
Communicated by C. Kassel

Abstract

We show that the graded commutative ring structure of the Hochschild cohomology $HH^*(A)$ is trivial in case $A$ is a triangular quadratic string algebra. Moreover, in case $A$ is gentle, the Lie algebra structure on $HH^*(A)$ is also trivial.

© 2005 Elsevier B.V. All rights reserved.

MSC: 16E40

0. Introduction

Let $k$ be a field, and $A$ be a finite dimensional $k$-algebra. In this situation, the Hochschild cohomology groups $HH^i(A, M)$ with coefficients in some $A$–$A$–$A$-bimodule $M$ can be identified with the groups $Ext^i_{A−A}(A, M)$. In case $A_A = A_A$, we simply write $HH^i(A)$. The sum $HH^*(A) = \bigoplus_{i \geq 0} HH^i(A)$ is a graded commutative ring under the Yoneda product, which coincides with a cup-product $\cup$. Beside this, there is another product, namely the bracket product $[−, −]$ which makes of $HH^*(A)$ a graded Lie algebra. Moreover, these two structures are related so that $HH^*(A)$ is in fact a Gerstenhaber algebra [9].

In general, very little information is known about these structures. As Green and Solberg point out in [11], “the ring structure of $HH^*(A)$ has often been observed to be trivial”. Moreover, Cibils [7] showed that if $A = kQ/F^2$, where $F$ is the two sided ideal of $kQ$ generated by the arrows, and $Q$ is not an oriented cycle, then the ring structure of $HH^*(A)$ is trivial. Although “one knows that for many self injective rings there are nonzero products in

E-mail address: juan.carlos.bustamante@usherbrooke.ca.
HH\(^*(A)\)”, there are rather few known examples of algebras having finite global dimension such that the ring structure of HH\(^*(A)\) is not trivial. This leads us to state the following:

**Conjecture.** Let \(A = kQ/I\) be a monomial triangular algebra. Then the ring structure of HH\(^*(A)\) is trivial.

In case \(A = kQ/I\) is a monomial algebra, one has a precise description of a minimal resolution of \(A\). The \(A\)-\(A\)-projective bimodules appearing in this resolution are described in terms of paths of \(Q\) (see [3]). Green et al. [10] use this resolution to achieve a deep study of Hochschild cohomology ring, modulo nilpotent elements of a monomial algebra in [10]. In the general case, this description leads to hard combinatorial computations. However, in case \(A\) is a monomial quadratic algebra, the minimal resolution is particularly easy to handle. Among monomial algebras, the class of string algebras, which are always tame, is particularly well understood, at least from the representation theoretic point of view [5]. We study the cohomology structure of triangular quadratic string algebras.

After recalling some notions concerning the products in HH\(^*(A)\) in Section 0, we establish some technical preliminary results in Section 1. Finally, Section 2 is devoted to state and show the main result of this work, which proves the conjecture below in the case of triangular quadratic string algebras.

**Theorem 3.1.** Let \(A = kQ/I\) be a triangular quadratic string algebra, \(n > 0\) and \(m > 0\). Then HH\(^n(A)\) \(\cup\) HH\(^m(A)\) = 0.

In addition, we obtain a similar result concerning the Lie algebra structure for a particular case of string algebras, the so-called gentle algebras.

**Theorem 3.2.** Let \(A = kQ/I\) be a triangular gentle algebra, \(n > 1\) and \(m > 1\). Then \([HH^n(A), HH^m(A)]\) = 0.

1. **Preliminaries**

1.1. **Algebras.** Let \(Q = (Q_0, Q_1, s, t)\) be a finite quiver (see [4]), and \(k\) be a commutative field. We consider algebras of the form \(kQ/I\), with \(I\) an admissible ideal of the path algebra \(kQ\). This includes, for instance, all basic connected finite dimensional algebras over algebraically closed fields (see [4]).

While we briefly recall some particular concepts concerning bound quivers and algebras, we refer the reader to [4], for instance, for unexplained notions. The composition of two arrows \(x_1 : s(z_1) \rightarrow t(z_1)\), and \(x_2 : s(z_2) \rightarrow t(z_2)\) such that \(s(z_2) = t(z_1)\) is the path \(z_1z_2\), and will be denoted by \(z_1z_2 : s(z_1) \rightarrow t(z_2)\). If \(Q\) has no oriented cycles, then \(A\) is said to be a triangular algebra. A two sided ideal \(I \subseteq kQ\) generated by a set of paths in \(Q\) is said to be monomial. Moreover if the generators of an admissible ideal \(I\) are linear combinations paths of length 2, then \(I\) is said to be quadratic. An algebra \(A = kQ/I\) is said to be monomial, or quadratic, depending on whether \(I\) is monomial or quadratic. In the remaining part of this paper, all algebras are triangular, quadratic and monomial.
Among monomial algebras, the so-called string algebras [5] are particularly well understood, at least from the representation theoretic point of view. Recall from [5] that an algebra \( A = kQ/I \) is said to be a string algebra if \((Q, I)\) satisfies the following conditions:

(S1) \( I \) is a monomial ideal,
(S2) each vertex of \( Q \) is the source and the target of at most two arrows, and
(S3) for an arrow \( \alpha \) in \( Q \) there exists at most one arrow \( \beta \) and at most one arrow \( \gamma \) such that \( \alpha \beta \notin I \) and \( \alpha \gamma \notin I \).

A string algebra \( A = kQ/I \) is called a gentle algebra [2] if in addition \((Q, I)\) satisfies:

(G1) For an arrow \( \alpha \) in \( Q \) there exists at most one arrow \( \beta \) and at most one arrow \( \gamma \) such that \( \alpha \beta \in I \) and \( \alpha \gamma \in I \),
(G2) \( I \) is quadratic.

1.2. Hochschild cohomology. Given an algebra \( A \) over a field \( k \), the Hochschild cohomology groups of \( A \) with coefficients in some \( A \)-\( A \)-bimodule \( M \), denoted by \( \text{HH}^i(A, M) \), are the groups \( \text{Ext}^i_{A\text{e}}(A, M) \) where \( A\text{e} = A \otimes_k A^{op} \) is the enveloping algebra of \( A \). In case \( M \) is the \( A \)-\( A \)-bimodule \( A A \), we simply denote them by \( \text{HH}^i(A) \). We refer the reader to [12], for instance, for general results about Hochschild (co)-homology of algebras.

The lower cohomology groups, that is those for \( 0 \leq i \leq 2 \), have interpretations and give information about, for instance, the simple-connectedness or the rigidity properties of \( A \) (see [1,8], for instance).

Beside this, the sum \( \text{HH}^*\ A = \bigsqcup_{i \geq 0} \text{HH}^i\ A \) has an additional structure given by two products, namely the cup-product \( \cup \), and the bracket \([−, −]\), which are defined in [9] at the cochain level. The definitions are given in Section 1.5.

1.3. Resolutions. From [6], we have a projective resolution of \( A \) over \( A\text{e} \) which is smaller than the standard Bar resolution. Let \( E \) be the (semi-simple) subalgebra of \( A \) generated by \( Q_0 \). In the remaining part of this note, tensor products will be taken over \( E \), unless it is explicitly otherwise stated. Note that, as \( E \)-\( E \)-bimodules, we have \( A \simeq E \oplus \text{rad} A \). Let \( \text{rad} A \otimes^n \) denote the \( n \)th tensor power of \( \text{rad} A \) with itself. With these notations, one has a projective resolution of \( A \) as \( A \)-\( A \)-bimodule, which we denote by \( \mathcal{H}^\bullet_{\text{rad}}\ A \)

\[
\cdots \rightarrow A \otimes \text{rad} A \otimes^n \otimes A \xrightarrow{b_{n-1}} A \otimes \text{rad} A \otimes^{n-1} \otimes A \xrightarrow{b_{n-2}} \cdots \rightarrow A \otimes \text{rad} A \otimes A \xrightarrow{b_0} A \otimes A \xrightarrow{\varepsilon} A \rightarrow 0
\]

where \( \varepsilon \) is the multiplication of \( A \), and

\[
b_{n-1}(1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1) = r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes 1 + \sum_{j=1}^{n-1} (-1)^j 1 \otimes r_1 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes r_n \otimes 1 + (-1)^n 1 \otimes r_1 \otimes \cdots \otimes r_n.
\]
Remark. Note that since the ideal \( I \) is assumed to be monomial, rad \( A \) is generated, as \( E-E \)-bimodule, by classes \( \overline{p} = p + I \) of paths of \( Q \) of length greater or equal than 1. Moreover, since the tensor products are taken over \( E \), then \( A \otimes \text{rad} A^{\otimes n} \otimes A \) is generated, as \( A - A \)-bimodule, by elements of the form \( 1 \otimes \overline{p}_1 \otimes \cdots \otimes \overline{p}_n \otimes 1 \) where \( p_i \) are paths of \( Q \) such that the ending point of \( p_i \) is the starting point of \( p_{i+1} \), for each \( i \) such that \( 1 \leq i < n \).

Keeping in mind that \( I \) is quadratic, the minimal resolution of Bardzell [3] has the following description: Let \( \Gamma_0 = Q_0 \), \( \Gamma_1 = Q_1 \), and for \( n \geq 2 \) let \( \Gamma_n = \{ z_1 z_2 \cdots z_n \mid z_i z_{i+1} \in I, \text{ for } 1 \leq i < n \} \). For \( n \geq 0 \), we let \( k\Gamma_n \) be the \( E-E \)-bimodule generated by \( \Gamma_n \). With these notations, we have a minimal projective resolution of \( A \) as \( A-A \)-module, which we denote by \( \mathcal{K}_{\text{min}}^*(A) \):

\[
\cdots \longrightarrow A \otimes k\Gamma_n \otimes A \xrightarrow{\delta_{n-1}} A \otimes k\Gamma_{n-1} \otimes A \xrightarrow{\delta_{n-2}} \cdots \xrightarrow{\delta_0} A \otimes k\Gamma_1 \otimes A \xrightarrow{\epsilon} A \longrightarrow 0,
\]

where, again, \( \epsilon \) is the composition of the isomorphism \( A \otimes k\Gamma_0 \otimes A \simeq A \otimes A \) with the multiplication of \( A \), and, given \( 1 \otimes z_1 \cdots z_n \otimes 1 \in A \otimes k\Gamma_n \otimes A \) we have

\[
\delta_{n-1}(1 \otimes z_1 \cdots z_n \otimes 1) = z_1 \otimes z_2 \cdots z_n \otimes 1 + (-1)^n 1 \otimes z_1 \cdots z_{n-1} \otimes z_n.
\]

In order to compute the Hochschild cohomology groups of \( A \), we apply the functor \( \text{Hom}_{A^e}(\cdot, A) \) to the resolutions \( \mathcal{K}_{\text{min}}^*(A) \) and \( \mathcal{K}_{\text{rad}}^*(A) \). To avoid cumbersome notations we will write \( \mathcal{E}_{\text{rad}}^n(A, A) \) instead of \( \text{Hom}_{A^e}(A \otimes \text{rad} A^{\otimes n} \otimes A, A) \) which in addition we identify to \( \text{Hom}_{E^e}(\text{rad} A^{\otimes n}, A) \). In a similar way we define \( \mathcal{E}_{\text{min}}^n(A, A) \) and, moreover, the obtained differentials will be denoted by \( b^* \) and \( \delta^* \).

The products in the cohomology \( \text{HH}^n(A) \) are induced by products defined using the Bar resolution of \( A \) (see [9]). Keeping in mind that our algebras are assumed to be triangular (so that the elements of \( \mathcal{E}_{\text{rad}}^n(A, A) \) take values in \( \text{rad} A \)), these products are easily carried to \( \mathcal{K}_{\text{rad}}^*(A) \) (see Section 1.5). However, the spaces involved in that resolution are still “too big” to work with. We wish to carry these products from \( \mathcal{K}_{\text{rad}}^*(A) \) to \( \mathcal{K}_{\text{min}}^*(A) \). In order to do so, we need explicit morphisms \( \mu_* : \mathcal{K}_{\text{min}}^*(A) \longrightarrow \mathcal{K}_{\text{rad}}^*(A) \), and \( \omega_* : \mathcal{K}_{\text{rad}}^*(A) \longrightarrow \mathcal{K}_{\text{min}}^*(A) \) (compare with [13] p. 736).

Define a morphism of \( A-A \)-bimodules \( \mu_n : A \otimes k\Gamma_n \otimes A \longrightarrow A \otimes \text{rad} A^{\otimes n} \otimes A \) by the rule \( \mu_n(1 \otimes z_1 \cdots z_n \otimes 1) = 1 \otimes z_1 \otimes \cdots \otimes z_n \otimes 1 \). A straightforward computation shows that \( \mu_* = (\mu_n)_{n \geq 0} \) is a morphism of complexes, and that each \( \mu_n \) splits.

On the other hand, an element \( 1 \otimes \overline{p}_1 \otimes \cdots \otimes \overline{p}_n \otimes 1 \) of \( A \otimes \text{rad} A^{\otimes n} \otimes A \), can always be written as \( 1 \otimes \overline{p}_1 z_1 \otimes \overline{p}_2 \otimes \cdots \otimes \overline{z}_n \overline{p}_n \otimes 1 \) with \( z_1, z_n \) arrows in \( Q \). Define a map \( \omega_n : A \otimes \text{rad} A^{\otimes n} \otimes A \longrightarrow A \otimes k\Gamma_n \otimes A \) by the rule

\[
\omega_n(1 \otimes \overline{p}_1 z_1 \otimes \cdots \otimes \overline{z}_n \overline{p}_n \otimes 1) = \begin{cases} \overline{p}_1 z_1 p_2 \cdots p_{n-1} z_n \otimes \overline{p}_n & \text{if } z_1 p_2 \cdots p_{n-1} z_n \in \Gamma_n, \\ 0 & \text{otherwise.} \end{cases}
\]

Again, \( \omega_* = (\omega_n)_{n \geq 0} \) defines a morphism of complexes, which is an inverse for \( \mu_* \). Summarizing what precedes, we obtain the following lemma.
1.4. Lemma. With the above notations,

(a) \( \mu_\bullet : \mathcal{K}_\text{\text{min}}^\bullet (A) \rightarrow \mathcal{K}_\text{\text{rad}}^\bullet (A) \) and \( \omega_\bullet : \mathcal{K}_\text{\text{rad}}^\bullet (A) \rightarrow \mathcal{K}_\text{\text{min}}^\bullet (A) \) are morphisms of complexes and

(b) \( \omega_\bullet \mu_\bullet = \text{id} \), and thus, since the involved complexes have the same homology, these morphisms are quasi-isomorphisms.

1.5. Products in cohomology. Following [9], given \( f \in \mathcal{C}^n_\text{\text{rad}}(A, A) \), \( g \in \mathcal{C}^m_\text{\text{rad}}(A, A) \), and \( i \in \{1, \ldots, n\} \), define the element \( f \circ_i g \in \mathcal{C}^{n+m-1}_\text{\text{rad}}(A, A) \) by the rule

\[
\overline{f \circ_i g} = \overline{f} \circ \overline{g} - (-1)^{(i-1)(m-1)} \overline{g} \circ \overline{f}.
\]

The composition product \( f \circ g \) is then defined as \( \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g \). Let \( f \circ g = 0 \) in case \( n = 0 \), and define the bracket

\[
[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f.
\]

On the other hand, the cup-product \( f \cup g \in \mathcal{C}^{n+m}_\text{\text{rad}}(A, A) \) is defined by the rule

\[
(f \cup g)(r_1 \otimes \cdots \otimes r_{n+m}) = f(r_1 \otimes \cdots \otimes r_i) g(r_{i+1} \otimes \cdots \otimes r_{n+m}).
\]

Recall that a Gerstenhaber algebra is a graded \( k \)-vector space \( A \) endowed with a product which makes \( A \) into a graded commutative algebra, and a bracket \([-, -]\) of degree \(-1\) that makes \( A \) into a graded Lie algebra, and such that \([x, yz] = [x, y]z + (-1)^{|x||y|}y[x, z]\), that is, a graded analogous of a Poisson algebra.

The cup product \( \cup \) and the bracket \([- , -]\) define products (still denoted \( \cup \) and \([ , , , \) in the Hochschild cohomology \( \bigoplus_{i \geq 0} \operatorname{HH}^i(A) \), which becomes then a Gerstenhaber algebra (see [9]).

Using \( \mu_\bullet \) and \( \omega_\bullet \) we can define analogous products in \( \mathcal{C}^\bullet_\text{\text{min}}(A, A) \). We study these products later.

2. Preparatory lemmata

Recall that all our algebras are assumed to be triangular, monomial and quadratic. The spaces \( \mathcal{C}^n_\text{\text{min}}(A, A) = \operatorname{Hom}_E(k \Gamma_n, A) \) have natural bases \( \mathcal{B}^n \) that we identify to the sets \( \{(x_1 \cdots x_n, \mathcal{B})| x_1 \cdots x_n \in \Gamma_n, \ p \text{ is a path with } s(p) = s(x_1), \ t(p) = t(x_n), \ p \notin I\} \). More precisely the map of \( E \)-\( E \)-bimodules \( f : k \Gamma_n \rightarrow A \) corresponding to \((x_1 \cdots x_n, \mathcal{B})\) is defined by

\[
f(\overline{\gamma_1 \cdots \gamma_n}) = \begin{cases} \overline{p} & \text{if } \gamma_1 \cdots \gamma_n = x_1 \cdots x_n, \\ 0 & \text{otherwise}. \end{cases}
\]

Given a path \( p \) in \( Q \), denote by \((p)\) the two sided ideal of \( kQ \) generated by \( p \). We distinguish three different kinds of basis elements in \( \mathcal{C}^n_\text{\text{min}}(A, A) \), which yield three subspaces \( \mathcal{C}^n_\text{\text{-}}, \mathcal{C}^n_\text{\text{+}}, \mathcal{C}^n_\text{\text{b}} \),
1. $C_n$ is generated by elements of the form $(x_1 \cdots x_n, \overline{x_1 p})$,
2. $C_0$ is generated by elements of the form $(x_1 \cdots x_n, \overline{w})$ such that $w \notin \langle x_1 \rangle$, $w \notin \langle x_n \rangle$, and
3. $C_+^{\infty}$ is generated by elements of the form $(x_1 \cdots x_n, q \overline{x_n})$ such that $q \notin \langle x_1 \rangle$.

Clearly, as vector spaces we have $C_n^{\min} (A, A) = C_n^{-} \sqcup C_0^{-} \sqcup C_+^{\infty}$. Moreover, let $B_−$, $B_0$, and $B_+$ be the natural bases of $C_n$, $C_0$, and $C_+^{\infty}$.

Consider an element $f = (x_1 \cdots x_n, \overline{p}) \in B_n$. First of all, since $x_1 x_2 \in I$, $p \notin \langle x_2 \rangle$, and, since $A$ is assumed to be triangular, $p \notin \langle x_1 \rangle$. The following figure illustrates this situation:

$$
\begin{array}{c}
\text{e}_0 \xrightarrow{x_1} \text{e}_1 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} \text{e}_n \xrightarrow{x_n} \text{e}_{n+1} \\
\text{p}
\end{array}
$$

Now, define $f_+: k \Gamma_n \rightarrow A$ in the following way:

$$
f_+(\gamma_1 \cdots \gamma_n) = \begin{cases} (-1)^{n+1} \overline{p x_{n+1}} & \text{if } \gamma_1 \cdots \gamma_n = x_2 \cdots x_{n+1} \in \Gamma_n, \\ 0 & \text{otherwise.} \end{cases}
$$

Analogously, given $g = (x_1 \cdots x_n, \overline{q x_n}) \in B_+$, we define $g_-: k \Gamma_n \rightarrow A$ by

$$
g_-(\gamma_1 \cdots \gamma_n) = \begin{cases} \overline{x_0 q} & \text{if } \gamma_1 \cdots \gamma_n = x_0 \cdots x_{n-1} \in \Gamma_n, \\ 0 & \text{otherwise.} \end{cases}
$$

It is easily seen that $f_+ \in C_n^{-}$, and $g_- \in C_n^{-}$.

2.1. Lemma. Let $A = kQ/I$ be a string triangular algebra, $f \in B_-$, and $g \in B_+$. 

(a) $f - f_+ \in \text{Im} \delta^{n-1}$, and
(b) $g - g_- \in \text{Im} \delta^{n-1}$.

Proof. We only prove statement (a). Let $h = (x_2 \cdots x_n, \overline{p}) \in C_{n-1}^{\min} (A, A)$. We show that $f_+ = f - \delta^{n-1} h$. Indeed

$$
\delta^{n-1} h(x_1 \cdots x_n) = \overline{x_1 h(x_2 \cdots x_n)} + (-1)^n h(x_1 \cdots x_{n-1}) \overline{x_n} = \overline{x_1 p} = f(x_1 \cdots x_n).
$$

Thus $(f - \delta^{n-1} h)(x_1 \cdots x_n) = 0 = f_+(x_1 \cdots x_n)$.

Now let $p = \beta p'$ with $\beta$ an arrow. Since $x_1 p \notin I, x_1 \beta \notin I$. But $A$ is a string algebra, thus condition (S3) ensures that for every arrow $\beta'$ such that $t(\beta') = t(x_1')$ we have $\beta' \in I$, thus $\beta' p \in I$. 
Thus, again, \((f - \delta^{n-1}h)(x'_1 \cdots x_n) = 0 = f_+(x'_1 \cdots x_n)\). On the other hand, assume there exists an arrow \(x_{n+1}\) such that \(x_2 \cdots x_{n+1} \in \Gamma_n\). Then we have
\[
\delta^{n-1}h(x_2 \cdots x_{n+1}) = \overline{x}_{n+1}h(x_3 \cdots x_{n+1}) + (-1)^n h(x_2 \cdots x_n) = (-1)^n p\overline{x}_{n+1}
\]
and hence
\[
(f - \delta^{n-1}h)(x_2 \cdots x_{n+1}) = 0 - (-1)^n p\overline{x}_{n+1} = f_+(x_2 \cdots x_{n+1}).
\]
Finally, note that \(f, f_+, h\) vanish on any other element \(\gamma_1 \cdots \gamma_n\) of \(\Gamma_n\). \(\square\)

According to the decomposition \(\mathcal{C}_{\min}^n (A, A) = \mathcal{B}_n^0 \sqcup \mathcal{B}_n^+ \sqcup \mathcal{B}_n^-\), given an element \(\phi \in \mathcal{C}_{\min}^n (A, A)\) one can write \(\phi = f + h + g\), where \(f \in \mathcal{B}_n^-, h \in \mathcal{B}_n^0\), and \(g \in \mathcal{B}_n^+\). Define then
\[
\phi_\leq = f_+ + h + g \quad \text{and} \quad \phi_\geq = f + h + g_+.
\]

**Remark.** It follows from the preceding lemma that \(\phi - \phi_\leq\) and \(\phi - \phi_\geq\) belong to \(\text{Im} \, \delta^{n-1}\). This will be useful later.

**2.2. Lemma.** Let \(A\) be a triangular string algebra, \(\phi \in \mathcal{C}_{\min}^n (A, A)\), and \(x_1 \cdots x_n \in \Gamma_n\). Then

(a) \(\phi_\leq (x_1 \cdots x_n) = (W + Q x_n) + I\) where \(W\) is a linear combination of paths none of which belongs to \(\langle x_1 \rangle\) nor to \(\langle x_n \rangle\), and \(Q\) is a linear combination of paths none of which belongs to \(\langle x_1 \rangle\), and

(b) \(\phi_\geq (x_1 \cdots x_n) = (x_1 P + W') + I\), where \(W'\) is a linear combination of paths none of which belongs to \(\langle x_1 \rangle\) nor to \(\langle x_n \rangle\), and \(P\) is a linear combination of paths none of which belongs to \(\langle x_n \rangle\).

**Proof.** By construction we have \(\phi_\leq \in \mathcal{B}_n^0 \sqcup \mathcal{B}_n^+\), and \(\phi_\geq \in \mathcal{B}_n^- \sqcup \mathcal{B}_n^0\). \(\square\)
2.3. Lemma. Let \( \phi \in \text{Ker } \delta^n \), and \( \alpha_1 \cdots \alpha_n \in \Gamma_n \) such that \( \phi(\alpha_1 \cdots \alpha_n) \neq 0 \).

(a) If there exists \( \alpha_{n+1} \) such that \( \alpha_1 \cdots \alpha_{n+1} \in \Gamma_{n+1} \) then \( \phi \leq (\alpha_1 \cdots \alpha_n)\alpha_{n+1} = 0 \).

(b) If there exists \( \alpha_0 \) such that \( \alpha_0 \cdots \alpha_n \in \Gamma_{n+1} \) then \( \exists \phi \geq (\alpha_1 \cdots \alpha_n) = 0 \).

Proof. We only prove (a). Let \( \phi \in \text{Ker } \delta^n \). Then, since \( \phi - \phi \leq \in\text{Im } \delta^{n-1} \), we have \( \phi \leq \in \text{Ker } \delta^n \). Thus

\[
0 = \delta^n \phi \leq (\alpha_1 \cdots \alpha_n)\alpha_{n+1}
\]

\[
= \exists \alpha_1 \phi \leq (\alpha_2 \cdots \alpha_{n+1}) + (-1)^{n+1}\phi \leq (\alpha_1 \cdots \alpha_n)\alpha_{n+1}.
\]

The statement follows from the fact that paths are linearly independent, and from the preceding lemma. \( \square \)

3. The cohomology structure

The morphisms \( \mu_* : \mathcal{H}_\text{min}^* (A) \rightarrow \mathcal{H}_\text{rad}^* (A) \), and \( \omega_* : \mathcal{H}_\text{rad}^* (A) \rightarrow \mathcal{H}_\text{min}^* (A) \) allow us to carry the products defined in \( \mathcal{C}_\text{rad}^* (A, A) \) to \( \mathcal{C}_\text{min}^* (A, A) \). In this way we obtain a cup product and a bracket, which we still denote \( \cup \) and \([ - , - ]\). More precisely, applying the functor \( \text{Hom}_A (-, A) \) and making the identifications of Section 1.5, we obtain morphisms of complexes \( \mu_* : \mathcal{C}_\text{min}^* (A, A) \rightarrow \mathcal{C}_\text{rad}^* (A, A) \), and \( \omega_* : \mathcal{C}_\text{rad}^* (A, A) \rightarrow \mathcal{C}_\text{min}^* (A, A) \). Given \( f \in \mathcal{C}_\text{min}^* (A, A) \), \( g \in \mathcal{C}_\text{min}^* (A, A) \), define \( f \cup g \in \mathcal{C}_\text{min}^* (A, A) \) as

\[
f \cup g = \mu^{n+m} (\omega^n f \cup \omega^m g).
\]

Thus, given an element \( \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m \in \Gamma_{n+m} \), we have

\[
f \cup g (\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m) = f (\alpha_1 \cdots \alpha_n) g (\beta_1 \cdots \beta_m).
\]

As usual, we have \( \delta^{n+m} (f \cup g) = \delta^n f \cup g + (-1)^m f \cup \delta^m g \), so the product \( \cup \) defined in \( \mathcal{C}_\text{min}^* (A, A) \) induces a product at the cohomology level. In fact, the latter coincides with the cup-product of Section 1.5 and the Yoneda product. In what follows, we work with the product \( \cup \) defined in \( \mathcal{C}_\text{min}^* (A, A) \).

3.1. Theorem. Let \( A = kQ/I \) be a triangular quadratic string algebra, \( n > 0 \) and \( m > 0 \). Then \( \text{HH}^n (A) \cup \text{HH}^m (A) = 0 \).

Proof. Let \( \overline{\phi} \in \text{HH}^n (A) \), and \( \overline{\psi} \in \text{HH}^m (A) \). We will show that \( \overline{\phi \leq \cup \overline{\psi} \geq 0} = 0 \) in \( \text{HH}^{n+m} (A) \). Assume the contrary is true. In particular there exist an element \( \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m \in \Gamma_{n+m} \) such that

\[
\phi \leq (\alpha_1 \cdots \alpha_n) \psi \geq (\beta_1 \cdots \beta_m) \neq 0.
\]

With the notations of Lemma 2.2, this is equivalent to

\[
(W + Qz_0)(W' + \beta_1 P) \notin I. \quad (1)
\]
Now, since \( x_n \beta_1 \in I \), Lemma 2.3 gives
\[
(W + Q x_n) \beta_1 \in I \quad \text{and} \quad x_1 (W' + \beta_1 P) \in I.
\]

Thus, Eq. (1) gives \( WW' \notin I \). Let \( w \) and \( w' \) be paths appearing in \( W \) and \( W' \) respectively, such that \( ww' \notin I \). Moreover, write \( w = u \gamma, \ w' = \gamma' u' \) with \( \gamma, \gamma' \) arrows of \( Q \). Since \( ww' \notin I \), we have that \( \gamma \gamma' \notin I \). But then, since \( (Q, I) \) satisfies (S3), this implies \( x_n \gamma' \in I \), so that \( x_1 \cdots x_n \gamma' \in I_{n+1} \) and, again, Lemma 2.3 gives \( (W + Q x_n) \gamma' \in I \) so that \( W \gamma' \in I \), and, in particular \( w \gamma' \in I \), a contradiction. \( \square \)

**Remark.** At this point, it is important to note that even if Lemmas 2.1, 2.2, and 2.3 were stated assuming that \( A = kQ/I \) is a string triangular algebra, the condition (S2) in the definition of string algebras has never been used. Thus, the preceding theorem holds for every monomial quadratic, triangular algebra \( A = kQ/I \) such that \( (Q, I) \) satisfies (S3). This includes, for instance, the gentle algebras.

We now turn our attention to the Lie algebra structure \( HH^*(A) \).

Given \( \phi \in \mathcal{C}_n^{\min}(A, A), \ \psi \in \mathcal{C}_m^{\min}(A, A), \) and \( i \in \{1, \ldots, n\} \) we define \( \phi \circ_i \psi \in \mathcal{C}_{n+m}^{n+m-1}(A, A) \) as
\[
\phi \circ_i \psi = \mu^{n+m-1}((\phi \circ_i) \psi)
\]
and from this, the bracket \([\cdot, \cdot]\) is defined as in Section 1.5. In particular, and this is the crucial point for what follows, given \( x_1 \cdots x_{n+m-1} \in I_{n+m-1} \) we have
\[
\phi \circ_i \psi(x_1 \cdots x_{n+m-1}) = \begin{cases} 
\phi(x_1 \cdots x_{i-1} \psi(x_i \cdots x_{i+m-1}) x_{i+m} \cdots x_{n+m-1}) & \text{if} \ \ x_1 \cdots x_{i-1} \psi(x_i \cdots x_{i+m-1}) x_{i+m} \cdots x_{n+m-1} \in I_n, \\
0 & \text{otherwise}.
\end{cases}
\]

Again, one can verify that \( \delta^{n+m-1} [f, g] = (-1)^{m-1} [\delta^n f, g] + [f, \delta^m g] \), so that \([\cdot, \cdot]\) induces a bracket at the cohomology level, which we still denote by \([\cdot, \cdot]\) (note that, by construction, this bracket coincides with that of Section 1.5).

This leads us to the following result:

**3.2. Theorem.** Let \( A = kQ/I \) be a triangular gentle algebra. Then, for \( n > 1 \) and \( m > 1 \), we have \( [HH^n(A), HH^m(A)] = 0 \).

**Proof.** We show that, in fact, under the hypothesis, the products \( \circ_i \) are equal to zero at the cochain level. This follows immediately from the discussion above. Indeed, with those notations, if \( x_1 \cdots x_{n+m-1} \in I_{n+m-1} \) we have, in particular that \( x_{i-1} x_i \in I \). But \( A \) being gentle, there is no other arrow \( \beta \) in \( Q \) with \( s(\beta) = t(x_{i-1}) \) such that \( x_{i-1} \beta \in I \). \( \square \)

The following example shows that the previous result cannot be extended to string algebras which are not gentle.
3.3. Example. Let $A = kQ/I$ where $Q$ is the quiver

![Quiver Diagram]

and $I$ is the ideal generated by paths of length 2. This is a string algebra which is not gentle. From Theorem 3.1 in [7] we get

$$\dim_k \text{HH}^i(A) = \begin{cases} 1 & \text{if } i \in \{0, 2, 3, 4\}, \\ 2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Keeping in mind the identifications of Section 1, the generators of $\text{HH}^i(A)$ are the elements corresponding to $(\beta, \overline{\beta})$, and $(\gamma, \overline{\gamma})$ for $i = 1$; $(x_2x_3, \beta)$ for $i = 2$; $(x_1\beta x_4, \overline{\gamma})$, for $i = 3$; and $(x_1x_2x_3x_4, \overline{\gamma})$ for $i = 4$. A straightforward computation shows that $[\text{HH}^n(A), \text{HH}^n(A)] = \text{HH}^{n+m-1}(A)$.

The first Hochschild cohomology group of an algebra $A$ is by its own right a Lie algebra. In case the algebra $A$ is monomial, this structure has been studied in [13]. The following result gives information about the role played by $\text{HH}^1(A)$ in the whole Lie algebra $\text{HH}^*(A)$ in our context.

3.4. Proposition. Let $A = kQ/I$ be a triangular monomial and quadratic algebra. Then $[\text{HH}^n(A), \text{HH}^1(A)] = \text{HH}^n(A)$, whenever $n > 1$.

Proof. In fact, for every $f \in C^n_{\text{min}}(A, A)$ there exists $g \in C^1_{\text{min}}(A, A)$ such that $[f, g] = f$, and $\overline{g} \neq 0$ in $\text{HH}^1(A)$. Clearly it is enough to consider an arbitrary basis element $f \in C^n_{\text{min}}(A, A)$. Let $f$ be such an element, corresponding to $(x_1 \cdots x_n, \overline{p})$. Define $g \in C^1_{\text{min}}(A, A)$ by

$$g(\gamma) = \begin{cases} x_1 & \text{if } \gamma = x_1, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $f \circ g = f \circ_1 g$, and, since we assume $A$ triangular and $n > 1$, that $g \circ f = 0$ so that $[f, g] = f$. Moreover, direct computations show that $\overline{g} \neq 0$ in $\text{HH}^1(A)$. \(\square\)

Acknowledgements

The author gratefully thanks Professor E.N. Marcos, for several interesting discussions and comments, as well as Professors E.N. Marcos and I. Assem for a careful reading of preliminary versions of this work. This work was done while the author had post-doctoral fellowship at the University of São Paulo, Brazil. He gratefully acknowledges the University for hospitality during his stay there, as well as financial support from FAPESP.
References