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Walks and regular integral graphs

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Abstract

We establish a useful correspondence between the closed walks in regular graphs and the walks in infinite regular trees, which, after counting the walks of a given length between vertices at a given distance in an infinite regular tree, provides a lower bound on the number of closed walks in regular graphs. This lower bound is then applied to reduce the number of the feasible spectra of the 4-regular bipartite integral graphs by more than a half.

Next, we give the details of the exhaustive computer search on all 4-regular bipartite graphs with up to 24 vertices, which yields a total of 47 integral graphs.

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1. Introduction

A graph is *integral* if all the eigenvalues of its adjacency matrix are integers. The search for the integral graphs was initiated by Harary and Schwenk [15] more than 30 years ago. At the Open Problem Session of the Aveiro Workshop on Graph Spectra, held in Aveiro, Portugal, from April 10–12, 2006, Horst Sachs asked to renew interest in integral graphs motivated by the algebraic property that they are exactly the graphs whose characteristic polynomial is fully factorable over the field **Q** of rationals.

Either way, ever since their invention the integral graphs were hardly more than a curiosity. Their first application has been found only recently in enabling the perfect transfer of arbitrary states in quantum spin networks [9, Eq. (19)]. The underlying undirected graphs in such networks enable the perfect transfer if and only if for every quadruple $(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ of the adjacency eigenvalues it holds that

$$\frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_l} \in \mathbf{Q}.$$

A simple argument then shows that such graph must either be integral or have eigenvalues that are integer multiples of $\sqrt{c}$, where c is a square-free natural number.

Over the years, the efforts in the search for integral graphs were roughly divided in two main directions:

- to find all integral graphs (always finitely many) among regular/nonregular, bipartite/nonbipartite graphs with given maximum vertex degree Δ, or
- to find (usually infinitely many) integral graphs in classes of graphs with more specific structure.

The typical procedure in the first direction consists of finding the feasible spectra first and then determining all graphs having such spectra. So far, the cases of integral graphs with $\Delta \leqslant 3$ [12,10,8,17] and the nonregular, nonbipartite, integral graphs with $\Delta = 4$ [16,18] are fully resolved. Partial results exist in the problems of determining the nonregular, bipartite, integral graphs with $\Delta = 4$ [5–7] and the 4-regular integral graphs [14,20–22]. For further details, as well as the overview of results on integral graphs in more specific classes of graphs, we refer the reader to the surveys [2,23].

We are interested here in advancing further the study of 4-regular bipartite integral graphs. The feasible spectra of such graphs were determined in [14] and the complete list, containing 1888 entries, may be found in [21]. Earlier we have shown the nonexistence of graphs with certain number of these spectra [20]. In Section 2 we further decrease this number down to 828, by using a new lower bound on the number of closed walks in regular graphs. This bound is obtained by showing that the closed walks in an r-regular graph are in one-to-one correspondence to certain walks in an infinite r-regular tree $\mathscr{B}_r$, and counting walks in $\mathscr{B}_r$ turns out to be simpler and more elegant than trying to count the walks in original graph. The complete list of the remaining feasible spectra is given in Appendix A.

Further, the modern computer software and hardware enabled us to perform the exhaustive search for integral graphs among all 4-regular bipartite graphs with up to 24 vertices (note that there are 317,579,563 such graphs with 24 vertices). In Section 3 we describe the method and present the results of this computer search. The graphs found here extend the list of known

4-regular integral graphs from [14], and we hope they can help better understand the true nature of integral graphs.

2. Walking across infinite regular trees

In Section 2.1 we establish a correspondence between the closed walks in an r-regular graph G and the walks in an infinite r-regular tree $\mathscr{B}_r$. Walks in $\mathscr{B}_r$ turn out to be easy to count, which is the topic of Section 2.2. Then in Section 2.3 we apply these results to provide a lower bound on the number of closed walks in 4-regular bipartite integral graphs, which will show that the majority of spectra from [14,21] are not feasible.

2.1. Treelike closed walks

Let us consider a special kind of closed walks in simple graphs which imitate walking across a tree.

Definition 1. A closed walk consisting of a single vertex w_0 of a simple graph G is *treelike*. A closed walk $W = w_0 w_1 \cdots w_k$, $w_k = w_0$, $k \geqslant 2$, is *treelike* if there exists $j > 0$ such that $w_0 = w_j$, $w_1 = w_{j-1}$ and the closed walks $W' = w_1 \cdots w_{j-1}$ and $W'' = w_j \cdots w_k$ are both treelike.

In other words, a closed walk is treelike if on its way back it always returns through the edges traversed earlier. Every edge appears an even number of times in a treelike closed walk, and it may contain circuits, as far as it returns back through them.

The inspiration for the term *treelike* becomes evident from the following:

Lemma 1. *Every closed walk in a tree is treelike.*

Proof. Proof is by induction on the length of a closed walk. Note that trees are bipartite, and thus, all closed walks must be of even length.

If a closed walk W in a tree T has length 0 and thus, consists of a single vertex, then it is treelike by definition.

Suppose now that the closed walks of length at most $2k - 2$, $k \geqslant 1$, are treelike. Let $W = w_0 w_1 \cdots w_{2k}$ be a closed walk of length $2k$ in T. As every edge of T is a bridge, denote by T_0 the component of $T - w_0 w_1$ which contains w_0 and by T_1 the component of $T - w_0 w_1$ which contains w_1. At the start, the walk W goes from w_0 to w_1 and so it passes from T_0 to T_1. The only way to return to T_0 is by using the same edge $w_0 w_1$, and so for the smallest $j > 0$ such that $w_j = w_0$ it holds that $w_{j-1} = w_1$. The closed walks $W' = w_1 \cdots w_{j-1}$ and $W'' = w_j \cdots w_k$ are of length at most $2k - 2$, and they are treelike by the induction hypothesis. $\square$

Next, we show that for every vertex of an r-regular graph the number of treelike closed walks with length $2k$ originating from that vertex is constant. Let $\mathscr{W}(G, u, 2k)$ be the set of closed walks of length $2k$ in graph G originating from a vertex u, and let $\mathscr{T}\mathscr{W}(G, u, 2k) \subseteq \mathscr{W}(G, u, 2k)$ be the subset of treelike closed walks.

Theorem 1. *Let u be a vertex of an r-regular graph G and let v be a vertex of an infinite, r-regular tree $\mathscr{B}_r$. For each $k \geqslant 0$, it holds that*

$$|\mathscr{T}\mathscr{W}(G, u, 2k)| = |\mathscr{W}(\mathscr{B}_r, v, 2k)|.$$

Proof. Let us construct inductively a homomorphism $f\colon V(\mathscr{B}_r) \mapsto V(G)$. Let $S(\mathscr{B}_r, v, d)$ be the sphere of $\mathscr{B}_r$ with center at v and radius d. At each step $d \geqslant 0$, we will map vertices of $S(\mathscr{B}_r, v, d)$ to the vertices of G.

For $d = 0$, we simply put $f(v) = u$.

Suppose that the map f has already been constructed for the vertices of $\cup_{i=0}^{d} S(\mathscr{B}_r, v, i)$, $d \geqslant 0$. Each vertex $w \in S(\mathscr{B}_r, v, d)$ has exactly $r - 1$ neighbors $w_1, w_2, \ldots, w_{r-1} \in S(\mathscr{B}_r, v, d + 1)$ and exactly one neighbor $w_0 \in S(\mathscr{B}_r, v, d - 1)$. In order to define f on $S(\mathscr{B}_r, v, d + 1)$ it suffices to choose arbitrary bijection from $\{w_1, w_2, \ldots, w_{r-1}\}$ to $S(G, f(w), 1) \setminus \{f(w_0)\}$ for each $w \in S(\mathscr{B}_r, v, d)$.

The mapping f is well-defined since each vertex in $S(\mathscr{B}_r, v, d + 1)$ has a unique neighbor in $S(\mathscr{B}_r, v, d)$. It is homomorphic by its construction, since the neighbors of any vertex $u \in V(\mathscr{B}_r)$ are mapped to the neighbors of $f(u)$ in G.

The mapping f induces a mapping $F\colon \mathscr{W}(\mathscr{B}_r, v, 2k) \mapsto \mathscr{W}(G, u, 2k)$ by

$$F(w_0 w_1 \cdots w_{2k}) = f(w_0) f(w_1) \cdots f(w_{2k}).$$

From Definition 1, it can be easily shown by induction that the homomorphic image of a treelike closed walk is again treelike. Thus, from Lemma 1 it holds that

$$F(\mathscr{W}(\mathscr{B}_r, v, 2k)) \subseteq \mathscr{T}\mathscr{W}(G, u, 2k).$$

The mapping F is injective: let $W' = w'_0 w'_1 \cdots w'_{2k}$ and $W'' = w''_0 w''_1 \cdots w''_{2k}$ be distinct walks in $\mathscr{W}(\mathscr{B}_r, v, 2k)$ and let $j > 0$ be the smallest index such that $w'_j \neq w''_j$. The vertices w'_j and w''_j are distinct neighbors of $w'_{j-1} = w''_{j-1}$ in $\mathscr{B}_r$ and by the construction of f, it holds that $f(w'_j) \neq f(w''_j)$. Thus, it holds that $F(W') \neq F(W'')$.

Next, note that given any walk $U = u_0 u_1 \cdots u_t$ in G with $u_0 = u$, there exists a unique walk $F^*(U) = v_0 v_1 \cdots v_t$ in $\mathscr{B}_r$ with $v_0 = v$, such that

$$f(v_i) = u_i, \quad i = 0, 1, \ldots, t.$$

The walk $F^*(U)$ may be constructed iteratively as follows: whenever the part $v_0 \cdots v_{i-1}$ of $F^*(U)$, $i \geqslant 1$, is constructed, the vertex v_i is determined as the unique neighbor of v_{i-1} which is mapped by f to $u_i \in S(G, u_{i-1}, 1)$.

If $U \in \mathscr{W}(G, u, 2k)$ then $F^*(U)$ is not necessarily a closed walk in $\mathscr{B}_r$. Rather, the walk $F^*(U)$ ends in another vertex of $\mathscr{B}_r$ that is also mapped to u_0 by f.

However, if $U \in \mathscr{T}\mathscr{W}(G, u, 2k)$ then $F^*(U) \in \mathscr{W}(\mathscr{B}_r, v, 2k)$, which can be proved by induction on k.

For $k = 0$, the walk U consists of a single vertex u and the walk $F^*(U)$ consists of a single vertex v, so that $F^*(U) \in \mathscr{W}(\mathscr{B}_r, v, 0)$.

Suppose now that the statement is proved for all treelike closed walks of length less than $2k$, for some $k \geqslant 1$, and let $U \in \mathscr{T}\mathscr{W}(G, u, 2k)$. By Definition 1, there exists index $j > 0$ such that $u_j = u_0$, $u_{j-1} = u_1$ and the walks $U' = u_1 \cdots u_{j-1}$ and $U'' = u_j \cdots u_{2k}$ are both treelike. Let v_1 be the unique neighbor of v_0 which is mapped to u_1 by f. By the induction hypothesis, $F^*(U')$

is a closed walk originating from v_1 and $F^*(U'')$ is a closed walk originating from v_0 in $\mathscr{B}_r$. By the iterative construction of $F^*(U)$, it follows that the walk $F^*(U)$ consists of:

- walking an edge v_0v_1,
- followed by the walk $F^*(U')$,
- walking back the edge v_1v_0,
- followed by the walk $F^*(U'')$.

Altogether, it holds that $F^*(U) \in \mathscr{W}(\mathscr{B}_r, v, 2k)$.

Thus, F is an injective mapping from $\mathscr{W}(\mathscr{B}_r, v, 2k)$ to $\mathscr{T}\mathscr{W}(G, u, 2k)$ and F^* is an injective mapping from $\mathscr{T}\mathscr{W}(G, u, 2k)$ to $\mathscr{W}(\mathscr{B}_r, v, 2k)$, showing that indeed it holds that

$$|\mathscr{W}(\mathscr{B}_r, v, 2k)| = |\mathscr{T}\mathscr{W}(G, u, 2k)|. \qquad \square$$

As evident from the previous proof, for any closed walk W of G starting at u, the walk $F^*(W)$ will start at v and end in some vertex v' with $f(v') = u$. This way, we may distinguish the closed walks of G starting at u by the end vertices of $F^*(W)$: those walks W for which $F^*(W)$ ends in v are treelike closed walks, while the remaining walks contain some cycles.

Now, let C be a cycle of G, let u be a vertex of C and let W be a closed walk starting at u and going once around C. Then the walk $F^*(W)$ starts at v in $\mathscr{B}_r$, while it ends at a vertex v' at distance $|C|$ from v such that $f(v') = u$. If W goes twice around C in the same direction, then $F^*(W)$ will pass through v' and end at yet another vertex v'' at distance $2|C|$ from v with $f(v'') = u$. By letting W to go around C infinitely many times in both directions, we may see that $F^*(W)$ represents an infinite path $\mathscr{P}_C$ in $\mathscr{B}_r$, which exactly corresponds to all vertices of $\mathscr{B}_r$ that are mapped to the vertices of C by f.

Thus, we may now say (or define) that a closed walk W in G *contains* C if, after removing all of the treelike closed subwalks, the walk $F^*(W)$ utilizes some part of the path $\mathscr{P}_C$. Note that W need not necessarily start at a vertex of C.

2.2. *Counting walks in an infinite regular tree*

Here we give a recurrence for the number $w_r(d, l)$ of walks of length l between the vertices at distance d in $\mathscr{B}_r$. Since $\mathscr{B}_r$ is a bipartite graph, it holds that $w_r(d, l) = 0$ whenever d and l have different parity. Moreover, as $\mathscr{B}_r$ is vertex-transitive, we may suppose that the walks counted start from v.

We deal first with the closed walks, i.e., we count $w_r(0, 2k) = |\mathscr{W}(\mathscr{B}_r, v, 2k)|$. As an auxiliary graph, let $\mathscr{B}'_r$ be an infinite tree in which a central vertex v has degree $r-1$, while all other vertices have degree r, and let $w'_r(0, 2k) = |\mathscr{W}(\mathscr{B}'_r, v, 2k)|$.

Now, let $W = w_0w_1 \cdots w_{2k} \in \mathscr{W}(\mathscr{B}_r, v, 2k)$. It holds that $w_0 = w_{2k} = v$ and let $j > 0$ be the smallest index such that $w_j = w_0$ (it may be that $j = 2k$; note that j must be even, as $\mathscr{B}_r$ is bipartite). Since W is a treelike closed walk, it holds that $w_{j-1} = w_1$. The closed walk $W' = w_1 \cdots w_{j-1}$ does not contain w_0 and it may be considered as a walk in $\mathscr{W}(\mathscr{B}'_r, w_1, j-2)$. On the other hand, the closed walk $W'' = w_j \cdots w_{2k}$ belongs to $\mathscr{W}(\mathscr{B}_r, v, 2k-j)$.

Having in mind that the vertex w_1 for a walk W may be chosen in r ways, we may see from the above argument that it must hold that

$$w_r(0, 2k) = r \sum_{j=2,4,\ldots,2k} w'_r(0, j-2)\, w_r(0, 2k-j).$$

Table 1
The number of walks of length $d+2k$, $0 \leqslant k \leqslant 6$, between vertices at distance d, $d \leqslant 20$, in an infinite 4-regular tree

d	d	$d+2$	$d+4$	$d+6$	$d+8$	$d+10$	$d+12$
0	1	4	28	232	2092	19,864	195,352
1	1	7	58	523	4966	48,838	492,724
2	1	10	97	958	9658	99,124	1,032,673
3	1	13	145	1564	16,762	179,983	1,941,505
4	1	16	202	2368	26,953	302,944	3,387,646
5	1	19	268	3397	40,987	482,047	5,589,904
6	1	22	343	4678	59,701	734,086	8,826,460
7	1	25	427	6238	84,013	1,078,852	13,444,588
8	1	28	520	8104	114,922	1,539,376	19,871,104
9	1	31	622	10,303	153,508	2,142,172	28,623,544
10	1	34	733	12,862	200,932	2,917,480	40,322,071
11	1	37	853	15,808	258,436	3,899,509	55,702,111
12	1	40	982	19,168	327,343	5,126,680	75,627,718
13	1	43	1120	22,969	409,057	6,641,869	101,105,668
14	1	46	1267	27,238	505,063	8,492,650	133,300,282
15	1	49	1423	32,002	616,927	10,731,538	173,548,978
16	1	52	1588	37,288	746,296	13,416,232	223,378,552
17	1	55	1762	43,123	894,898	16,609,858	284,522,188
18	1	58	1945	49,534	1,064,542	20,381,212	358,937,197
19	1	61	2137	56,548	1,257,118	24,805,003	448,823,485
20	1	64	2338	64,192	1,474,597	29,962,096	556,642,750

Similarly, by considering walks in $\mathscr{W}(\mathscr{B}'_r, v, 2k)$ we get that

$$w'_r(0,2k) = (r-1)\sum_{j=2,4,\ldots,2k} w'_r(0,j-2)w'_r(0,2k-j).$$

Combining this with $w'_r(0,0) = w_r(0,0) = 1$, the sequences $w_r(0,2k)$ and $w'_r(0,2k)$ become determined for every $k \geqslant 0$. According to [19], the generating function for the sequence $w_r(0,2k)$, $k \geqslant 0$, is

$$\frac{2(r-1)}{r-2+r\sqrt{1-4(r-1)x}}.$$

In case $r=4$, the first few numbers $w_4(0,2k)$ are shown in the first row of Table 1.

In case $d>0$, the recurrence relation is quite simple:

$$w_r(d,l) = w_r(d-1,l-1) + (r-1)w_r(d+1,l-1).$$

It is obtained by noting that in the walk $W = w_0w_1\cdots w_{l-1}w_l$, $w_0 = v$, the vertex w_{l-1} is either a unique neighbor of w_l at distance $d-1$ from v or one of its $r-1$ neighbors at distance $d+1$ from v. Thus, the above recurrence enables to calculate the table of numbers $w_r(d,l)$ column-by-column. (Have in mind here that $w_r(d,l) = 0$ for $d > l$.) In case $r=4$, some nonzero entries are shown in Table 1.

2.3. Application to spectra of integral graphs

Let us count the number of closed walks of length l in an r-regular graph G that contain a particular cycle C. Such a closed walk W need not start at a vertex of C—it is only necessary that $F^*(W)$ traverses part of $\mathscr{P}_C$ in $\mathscr{B}_r$. Let $k|C|$ be the total distance between each entry and the next exit point of $F^*(W)$ on $\mathscr{P}_C$. In other words, k is the number of copies of C that W contains. Further, let d be the distance from the start vertex of W to C in G. Then, the distance in $\mathscr{B}_r$ between the first and the last vertex of $F^*(W)$ is equal to $2d + k|C|$, and so the number of such walks is equal to $w_r(2d + k|C|, l)$.

It has further to be taken into account that the cycle C may be traversed in two directions, that there are $|C|$ choices for the entry point of $F^*(W)$ on $\mathscr{P}_C$, and that there are $(r-2)(r-1)^{d-1}$ vertices at distance $d, d \geqslant 1$, from $\mathscr{P}_C$ in $\mathscr{B}_r$. In total, we may conclude that the number of closed walks of length l containing C is equal to

$$2|C|\left(\sum_{k|C|\leqslant l} w_r(k|C|, l) + \sum_{2d+k|C|\leqslant l} (r-2)(r-1)^{d-1} w_r(2d + k|C|, l)\right)$$

(where the first summand above corresponds to $d = 0$, and the second to $d \geqslant 1$).

In particular, for $r = 4$, $l = 8$ and $|C| = 4$ we get

$$2 \cdot 4 \cdot (w_4(4, 8) + w_4(8, 8) + 2w_4(6, 8) + 6w_4(8, 8)) = 2024$$

closed walks containing a given quadrangle in a 4-regular graph, while for $|C| = 6$ we get

$$2 \cdot 6 \cdot (w_4(6, 8) + 2w_4(8, 8)) = 288$$

closed walks containing a given hexagon in the same graph. Thus, we have the following:

Claim 1. *Let G be a 4-regular graph with p vertices, q quadrangles and h hexagons. Then the number of closed walks of length 8 is at least $2092p + 2024q + 288h$.*

The closed walks of length 8 not counted by the above claim include those that contain an octagon and those that contain two distinct quadrangles having a common vertex.

Let us now apply this knowledge on walks in $\mathscr{B}_r$ to the main topic of our manuscript. So, let G be a connected 4-regular bipartite integral graph. Regular bipartite graphs have the same number of vertices in each part so that we may assume that G has $p = 2n$ vertices. Further, the spectrum of adjacency matrix of a bipartite graph is symmetric with respect to zero. Thus, using superscripts to represent multiplicities, the spectrum of G may be written in the form

$$\{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}, \quad x, y, z, w \geqslant 0.$$

Further, let q and h denote the numbers of quadrangles and hexagons in G. The number of closed walks of length k in G is equal to the sum of the kth power of eigenvalues of G (see, e.g., [11]), and for $k = 0, 2, 4, 6$, we get the following equalities

$$\frac{1}{2}\sum_i \lambda_i^0 = 1 + x + y + z + w = n,$$

$$\frac{1}{2}\sum_i \lambda_i^2 = 16 + 9x + 4y + z = 4n,$$

$$\frac{1}{2}\sum_i \lambda_i^4 = 256 + 81x + 16y + z = 28n + 4q,$$

$$\frac{1}{2}\sum_i \lambda_i^6 = 4096 + 729x + 64y + z = 232n + 72q + 6h.$$

These equalities were used in [14,21] to determine 1888 feasible spectra of the 4-regular, bipartite, integral graphs.

From Claim 1 we get the following corollary

$$\frac{1}{2}\sum_i \lambda_i^8 = 65536 + 6561x + 256y + z \geqslant 2092n + 1012q + 144h.$$

This inequality is not satisfied by 1026 spectra from [21]. After taking into account earlier nonexistence results from [20], it follows that there are 828 remaining feasible spectra. Their complete list is given in Appendix A. The largest of these spectra has 560 vertices, and actually there are only 12 spectra with more than 360 vertices. Although this is still far from the reach of the exhaustive search by modern computers, it is certainly a considerable improvement over the previous upper bound of 5040 on the number of vertices.

3. The computer search

The search for the 4-regular integral graphs was also one of the topics in Ph.D. thesis of one of us [21]. Among other things, the thesis contains the characterization of all connected 4-regular integral graphs that do not contain ± 3 in the spectrum, which was obtained by using interactively the system GRAPH [13] and theoretical arguments. This particular study lasted for six months in 1997. Less than ten years after, both computer hardware and software have advanced enough to enable the exhaustive search for integral graphs among all connected 4-regular bipartite graphs with at most 24 vertices in shorter time (not too much shorter, actually :-). The details of the search are given in this section. In total, we found 47 integral graphs. The longest task, the search on 24 vertices, lasted for 2900 hours, split among 11 PCs working simultaneously on different pieces of the search. Moreover, the search unearthed two graphs that should have been found already in [21]. Further, we tried to decompose each of these bipartite graphs into a direct product of a nonbipartite graph and K_2, which yielded 14 additional nonbipartite integral graphs (however, these 14 graphs are already known from [1,3,4]).

3.1. The method

Different software packages and programs were used for this task. In short, connected 4-regular bipartite graphs were generated by **genbg**, their spectra calculated and integral graphs among them selected by **integrality**, and finally their drawings were produced by **neato**.

The package **nauty** by Brendan D. McKay, available from

```
http://cs.anu.edu.au/~bdm/nauty
```

contains the program **genbg** for generating bipartite graphs. Besides specifying the number of vertices, the program has a number of options to specify the minimum and the maximum number of edges, as well as the minimum and the maximum vertex degrees. In particular, all connected 4-regular bipartite graphs with ⟨**n**⟩ vertices in each part can be generated with a command

```
genbg -c -d4 -D4 ⟨n⟩ ⟨n⟩
```

The graphs are then sent to standard output in **g6** format, whose description may be found at

```
http://cs.anu.edu.au/~bdm/data/formats.txt
```

From there, the graphs were piped to the program **integrality**, which calculates the spectrum of each graph and for each integral graph it finds, it writes the graph data to a file **4regint.adj** and the edge list to a separate file in **dot** format. For eigenvalue calculation, we have used procedure **eigens**, written in C by Stephen L. Moshier implementing algorithm of J. von Neumann, and available from

`http://www.koders.com/c/fidE84C20FBE5E04676208CAF6014705B6A43A15792.aspx`

(Instead of typing the address, it is easier to go to `http://www.koders.com/` and search the site for **eigens**.) The source code of **integrality** may be obtained upon request.

After the integral graphs have been found, the program **neato** from the package **GraphViz**, available from

`http://www.graphviz.org`

uses generated **dot** files to produce drawings of integral graphs. The actual command that produced drawings for this manuscript, telling to the program that the nodes should not overlap and that the edges should not be drawn across vertices, was

```
for %%g in (*.dot) do
  neato -Tps -Goverlap=scale -Gsplines=true %%g -o %%g.eps
```

For the decomposition of a connected bipartite graph into a direct product of a connected nonbipartite graph and K_2, we wrote a simple program based primarily on the following lemma of Schwenk [17].

Lemma 2. *A bipartite graph $B = (U, V, E)$ may be decomposed into a direct product $G \times K_2$ if and only if there exists a bijection $f : U \mapsto V$ such that for each $u, v \in V$ the vertex u is not adjacent to $f(u)$ and if the vertex u is adjacent to $f(v)$ then v is adjacent to $f(u)$.*

3.2. The results

For each feasible number of vertices up to 24, Table 2 contains the total number of connected 4-regular bipartite graphs, the number of integral graphs among them and the time in seconds needed to run the search. There is no row for 22 vertices, as there are no feasible spectra with 22 vertices (see Appendix A).

One PC with AMD Athlon XP 1700+ processor was used for the search on up to 20 vertices. For the search on 24 vertices, generation of graphs with **genbg** was divided into 250 parts which were distributed among eleven PCs each having AMD Athlon XP 1700+ processor.

Table 2
The total and the numbers of integral graphs

No. of vertices	Total	Integral	Time (s)
8	1	1	<1
10	1	1	<1
12	4	2	<1
14	16	1	<1
16	193	3	<1
18	3,528	7	11
20	121,785	11	499
24	317,579,563	21	10,441,176

The present search also corrects some results of [21]: for each of the spectra $\{4, 2^4, 0^6, -2^4, -4\}$ and $\{4, 2^5, 1^4, -1^4, -2^5, -4\}$, the search found an additional graph that failed to be found in [21].

The drawings and spectra of the 47 bipartite integral graphs found are given in Appendix B. The drawings and spectra of the 14 nonbipartite integral graphs found by decomposing these 47 bipartite graphs are given in Appendix C.

3.3. The comments

It is interesting to note the increasing cardinalities of the sets of cospectral bipartite integral graphs:

- there is a set of three cospectral graphs on 16 vertices;
- there is a set of four cospectral graphs on 18 vertices;
- there is a set of seven cospectral graphs on 20 vertices;
- there is a set of eleven cospectral graphs on 24 vertices.

Thus, it is plausible that there are larger sets of cospectral 4-regular bipartite integral graphs on more than 24 vertices.

The symmetry, in particular the six-fold symmetry, is evident in a number of graphs, but other kinds of symmetry may also be observed.

Another interesting observation is that all the bipartite integral graphs found turn out to be Hamiltonian. We have no explanation for this.

4. Conclusion

In its own way, the second part of this paper testifies of the enormous advance that the computer hardware and mathematical software have undergone in just one decade. While the graphs obtained here settle most of the spectra studied in [21], one should still have in mind that the largest integral graph found in [21] by theoretical arguments had 32 vertices. The exhaustive search on this order is still out of reach of modern computers. (Maybe for just another ten years?)

Nevertheless, due to the extremely small percentage of integral graphs among connected 4-regular bipartite graphs, as well as the large sets of cospectral graphs, it would certainly be more prospective to create a nontrivial algorithm to construct all graphs with a given spectrum in order to find integral graphs on the larger number of vertices. This, however, is not an easy task.

Appendix A

This appendix contains all 828 remaining feasible spectra of the connected 4-regular bipartite integral graphs. Each spectrum is represented as $nxyzwqh$, denoting, respectively, the number of vertices in each bipartite class, the multiplicities of eigenvalues 3, 2 and 1, one half of the multiplicity of eigenvalue 0, as well as the numbers of quadrangles and hexagons in a graph. The plus sign in front denotes those spectra for which all graphs having that spectra are known.

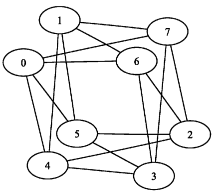
+ 4 0 0 0 3 36 96
+ 5 0 0 4 0 30 130
+ 6 0 1 4 0 27 138
+ 6 0 2 0 3 30 112
+ 7 1 0 3 2 36 102
+ 8 0 4 0 3 24 128
+ 8 1 1 3 2 33 110
+ 9 0 4 4 0 18 162
16 3 4 5 3 30 112
16 3 5 1 6 33 86
16 4 1 8 2 39 94
16 4 2 4 5 42 68
16 4 3 0 8 45 42
18 1 11 3 2 3 190
18 2 8 6 1 12 172
18 2 9 2 4 15 146
18 3 5 9 0 21 154
18 3 6 5 3 24 128
18 3 7 1 6 27 102
18 4 3 8 2 33 110
18 4 4 4 5 36 84
18 5 1 7 4 45 66
18 5 2 3 7 48 40
20 2 10 6 1 6 188
20 2 11 2 4 9 162
20 3 8 5 3 18 144
20 3 9 1 6 21 118
20 4 5 8 2 27 126
20 4 6 4 5 30 100
20 5 2 11 1 36 108
20 5 3 7 4 39 82
20 5 4 3 7 42 56
20 6 1 6 6 51 38
20 6 2 2 9 54 12
21 2 11 6 1 3 196
21 2 12 2 4 6 170
21 3 10 1 6 18 126
21 3 8 9 0 12 178
21 3 9 5 3 15 152
21 4 6 8 2 24 134
21 4 7 4 5 27 108
21 4 8 0 8 30 82
21 5 3 11 1 33 116
21 5 4 7 4 36 90
21 5 5 3 7 39 64
21 6 0 14 0 42 98
21 6 1 10 3 45 72
21 6 2 6 6 48 46
21 6 3 2 9 51 20
24 3 12 5 3 6 176
24 3 13 1 6 9 150
24 4 10 4 5 18 132
24 4 11 0 8 21 106
24 4 9 8 2 15 158
24 5 6 11 1 24 140
24 5 7 7 4 27 114
24 5 8 3 7 30 88
24 6 4 10 3 36 96
24 6 5 6 6 39 70
24 6 6 2 9 42 44
24 7 1 13 2 45 78
24 7 2 9 5 48 52
24 7 3 5 8 51 26
24 7 4 1 11 54 0
28 4 13 8 2 3 190
28 4 14 4 5 6 164
28 4 15 0 8 9 138
28 5 10 11 1 12 172
28 5 11 7 4 15 146
28 5 12 3 7 18 120
28 6 10 2 9 30 76
28 6 8 10 3 24 128
28 6 9 6 6 27 102
28 7 5 13 2 33 110
28 7 6 9 5 36 84
28 7 7 5 8 39 58
28 7 8 1 11 42 32
28 8 2 16 1 42 92
28 8 3 12 4 45 66
28 8 4 8 7 48 40
28 8 5 4 10 51 14
28 9 1 11 6 57 22
30 4 16 4 5 0 180
30 5 12 11 1 6 188
30 5 13 7 4 9 162
30 5 14 3 7 12 136
30 6 10 10 3 18 144
30 6 11 6 6 21 118
30 6 12 2 9 24 92
30 6 9 14 0 15 170
30 7 10 1 11 36 48
30 7 7 13 2 27 126
+ 9 1 2 3 2 30 118
+ 10 0 5 4 0 15 170
+ 10 1 3 3 2 27 126
+ 10 2 0 6 1 36 108
+ 10 2 1 2 4 39 82
+ 12 0 8 0 3 12 160
+ 12 1 5 3 2 21 142
+ 12 2 2 6 1 30 124
30 9 2 15 3 48 64
30 9 3 11 6 51 38
30 9 4 7 9 54 12
35 10 4 18 2 42 86
35 10 5 14 5 45 60
35 10 6 10 8 48 34
35 10 7 6 11 51 8
35 11 1 21 1 51 68
35 11 2 17 4 54 42
35 11 3 13 7 57 16
35 6 14 14 0 0 210
35 6 15 10 3 3 184
35 6 16 6 6 6 158
35 6 17 2 9 9 132
35 7 12 13 2 12 166
35 7 13 9 5 15 140
35 7 14 5 8 18 114
35 7 15 1 11 21 88
35 8 10 12 4 24 122
35 8 11 8 7 27 96
35 8 12 4 10 30 70
35 8 9 16 1 21 148
35 9 10 3 12 42 26
35 9 6 19 0 30 130
35 9 7 15 3 33 104
35 9 8 11 6 36 78
35 9 9 7 9 39 52
36 10 5 18 2 39 94
36 10 6 14 5 42 68
36 10 7 10 8 45 42
36 10 8 6 11 48 16
36 11 2 21 1 48 76
36 11 3 17 4 51 50
36 11 4 13 7 54 24
36 12 1 16 6 63 6
36 6 16 10 3 0 192
36 6 17 6 6 3 166
36 6 18 2 9 6 140
36 7 13 13 2 9 174
36 7 14 9 5 12 148
36 7 15 5 8 15 122
36 7 16 1 11 18 96
36 8 10 16 1 18 156
36 8 11 12 4 21 130
36 8 12 8 7 24 104
36 8 13 4 10 27 78
36 9 10 7 9 36 60
36 9 11 3 12 39 34
36 9 8 15 3 30 112
36 9 9 11 6 33 86
40 10 10 14 5 30 100
40 10 11 10 8 33 74
40 10 12 6 11 36 48
40 10 13 2 14 39 22
40 10 9 18 2 27 126
40 11 10 5 13 48 4
40 11 7 17 4 39 82
40 11 8 13 7 42 56
40 11 9 9 10 45 30
40 12 4 20 3 48 64
40 12 5 16 6 51 38
40 12 6 12 9 54 12
40 13 1 23 2 57 46
40 13 2 19 5 60 20
40 7 18 9 5 0 180
40 7 19 5 8 3 154
40 7 20 1 11 6 128
40 8 15 12 4 9 162
40 8 16 8 7 12 136
40 8 17 4 10 15 110
40 9 12 15 3 18 144
40 9 13 11 6 21 118
40 9 14 7 9 24 92
40 9 15 3 12 27 66
42 10 11 18 2 21 142
42 10 12 14 5 24 116
42 10 13 10 8 27 90
42 10 14 6 11 30 64
42 10 15 2 14 33 38
42 11 10 13 7 36 72
42 11 11 9 10 39 46
42 11 12 5 13 42 20
42 11 8 21 1 30 124
42 11 9 17 4 33 98
+ 12 2 3 2 4 33 98
+ 12 3 0 5 3 42 80
+ 12 3 1 1 6 45 54
14 1 7 3 2 15 158
14 2 4 6 1 24 140
14 2 5 2 4 27 114
14 3 1 9 0 33 122
14 3 2 5 3 36 96
42 13 5 15 8 57 10
42 14 1 22 4 63 18
42 7 22 1 11 0 144
42 8 16 16 1 0 204
42 8 17 12 4 3 178
42 8 18 8 7 6 152
42 8 19 4 10 9 126
42 8 20 0 13 12 100
42 9 13 19 0 9 186
42 9 14 15 3 12 160
42 9 15 11 6 15 134
42 9 16 7 9 18 108
42 9 17 3 12 21 82
45 10 14 18 2 12 166
45 10 15 14 5 15 140
45 10 16 10 8 18 114
45 10 17 6 11 21 88
45 10 18 2 14 24 62
45 11 11 21 1 21 148
45 11 12 17 4 24 122
45 11 13 13 7 27 96
45 11 14 9 10 30 70
45 11 15 5 13 33 44
45 11 16 1 16 36 18
45 12 10 16 6 36 78
45 12 11 12 9 39 52
45 12 12 8 12 42 26
45 12 13 4 15 45 0
45 12 8 24 0 30 130
45 12 9 20 3 33 104
45 13 6 23 2 42 86
45 13 7 19 5 45 60
45 13 8 15 8 48 34
45 13 9 11 11 51 8
45 14 3 26 1 51 68
45 14 4 22 4 54 42
45 14 5 18 7 57 16
45 15 1 25 3 63 24
45 8 22 4 10 0 150
45 9 16 19 0 0 210
45 9 17 15 3 3 184
45 9 18 11 6 6 158
45 9 19 7 9 9 132
45 9 20 3 12 12 106
48 10 17 18 2 3 190
48 10 18 14 5 6 164
48 10 19 10 8 9 138
48 10 20 6 11 12 112
48 10 21 2 14 15 86
48 11 14 21 1 12 172
48 11 15 17 4 15 146
48 11 16 13 7 18 120
48 11 17 9 10 21 94
48 11 18 5 13 24 68
48 11 19 1 16 27 42
48 12 12 20 3 24 128
48 12 13 16 6 27 102
48 12 14 12 9 30 76
48 12 15 8 12 33 50
48 12 16 4 15 36 24
48 13 10 19 5 36 84
48 13 11 15 8 39 58
48 13 12 11 11 42 32
48 13 13 7 14 45 6
48 13 9 23 2 33 110
48 14 6 26 1 42 92
48 14 7 22 4 45 66
48 14 8 18 7 48 40
48 14 9 14 10 51 14
48 15 4 25 3 54 48
48 15 5 21 6 57 22
48 16 1 28 2 63 30
48 16 2 24 5 66 4
48 9 22 7 9 0 156
48 9 23 3 12 3 130
56 11 26 5 13 0 132
56 11 27 1 16 3 106
56 12 20 20 3 0 192
56 12 21 16 6 3 166
56 12 22 12 9 6 140
56 12 23 8 12 9 114
56 12 24 4 15 12 88
56 12 25 0 18 15 62
56 13 17 23 2 9 174
14 3 3 1 6 39 70
14 4 0 4 5 48 52
14 4 1 0 8 51 26
15 0 10 4 0 0 210
15 1 8 3 2 12 166
15 2 5 6 1 21 148
15 2 6 2 4 24 122
15 3 2 9 0 30 130
56 14 16 18 7 24 104
56 14 17 14 10 27 78
56 14 18 10 13 30 52
56 14 19 6 16 33 26
56 14 20 2 19 36 0
56 15 12 25 3 30 112
56 15 13 21 6 33 86
56 15 14 17 9 36 60
56 15 15 13 12 39 34
56 15 16 9 15 42 8
56 16 10 24 5 42 68
56 16 11 20 8 45 42
56 16 12 16 11 48 16
56 16 9 28 2 39 94
56 17 6 31 1 48 76
56 17 7 27 4 51 50
56 17 8 23 7 54 24
56 18 4 30 3 60 32
56 18 5 26 6 63 6
56 19 1 33 2 69 14
60 12 28 4 15 0 120
60 13 22 19 5 0 180
60 13 23 15 8 3 154
60 13 24 11 11 6 128
60 13 25 7 14 9 102
60 13 26 3 17 12 76
60 14 19 22 4 9 162
60 14 20 18 7 12 136
60 14 21 14 10 15 110
60 14 22 10 13 18 84
60 14 23 6 16 21 58
60 14 24 2 19 24 32
60 15 16 25 3 18 144
60 15 17 21 6 21 118
60 15 18 17 9 24 92
60 15 19 13 12 27 66
60 15 20 9 15 30 40
60 15 21 5 18 33 14
60 16 13 28 2 27 126
60 16 14 24 5 30 100
60 16 15 20 8 33 74
60 16 16 16 11 36 48
60 16 17 12 14 39 22
60 17 11 27 4 39 82
60 17 12 23 7 42 56
60 17 13 19 10 45 30
60 17 14 15 13 48 4
60 18 10 22 9 54 12
60 18 8 30 3 48 64
60 18 9 26 6 51 38
60 19 5 33 2 57 46
60 19 6 29 5 60 20
60 20 3 32 4 69 2
63 13 28 7 14 0 126
63 13 29 3 17 3 100
63 14 22 22 4 0 186
63 14 23 18 7 3 160
63 14 24 14 10 6 134
63 14 25 10 13 9 108
63 14 26 6 16 12 82
63 14 27 2 19 15 56
63 15 18 29 0 6 194
63 15 19 25 3 9 168
63 15 20 21 6 12 142
63 15 21 17 9 15 116
63 15 22 13 12 18 90
63 15 23 9 15 21 64
63 15 24 5 18 24 38
63 15 25 1 21 27 12
63 16 16 28 2 18 150
63 16 17 24 5 21 124
63 16 18 20 8 24 98
63 16 19 16 11 27 72
63 16 20 12 14 30 46
63 16 21 8 17 33 20
63 17 13 31 1 27 132
63 17 14 27 4 30 106
63 17 15 23 7 33 80
63 17 16 19 10 36 54
63 17 17 15 13 39 28
63 17 18 11 16 42 2
63 18 10 34 0 36 114
63 18 11 30 3 39 88
63 18 12 26 6 42 62
15 3 3 5 3 33 104
15 3 4 1 6 36 78
15 4 0 8 2 42 86
15 4 1 4 5 45 60
16 0 12 0 3 0 192
16 1 9 3 2 9 174
16 2 6 6 1 18 156
16 2 7 2 4 21 130
63 20 7 28 7 63 0
63 21 2 39 0 66 34
63 21 3 35 3 69 8
70 15 30 9 15 0 120
70 15 31 5 18 3 94
70 15 32 1 21 6 68
70 16 24 24 5 0 180
70 16 25 20 8 3 154
70 16 26 16 11 6 128
70 16 27 12 14 9 102
70 16 28 8 17 12 76
70 16 29 4 20 15 50
70 17 20 31 1 6 188
70 17 21 27 4 9 162
70 17 22 23 7 12 136
70 17 23 19 10 15 110
70 17 24 15 13 18 84
70 17 25 11 16 21 58
70 17 26 7 19 24 32
70 17 27 3 22 27 6
70 18 17 34 0 15 170
70 18 18 30 3 18 144
70 18 19 26 6 21 118
70 18 20 22 9 24 92
70 18 21 18 12 27 66
70 18 22 14 15 30 40
70 18 23 10 18 33 14
70 19 15 33 2 27 126
70 19 16 29 5 30 100
70 19 17 25 8 33 74
70 19 18 21 11 36 48
70 19 19 17 14 39 22
70 20 12 36 1 36 108
70 20 13 32 4 39 82
70 20 14 28 7 42 56
70 20 15 24 10 45 30
70 20 16 20 13 48 4
70 21 10 35 3 48 64
70 21 11 31 6 51 38
70 21 12 27 9 54 12
70 21 9 39 0 45 90
70 22 7 38 2 57 46
70 22 8 34 5 60 20
70 23 4 41 1 66 28
70 23 5 37 4 69 2
72 16 28 16 11 0 144
72 16 29 12 14 3 118
72 16 30 8 17 6 92
72 16 31 4 20 9 66
72 17 23 27 4 3 178
72 17 24 23 7 6 152
72 17 25 19 10 9 126
72 17 26 15 13 12 100
72 17 27 11 16 15 74
72 17 28 7 19 18 48
72 17 29 3 22 21 22
72 18 20 30 3 12 160
72 18 21 26 6 15 134
72 18 22 22 9 18 108
72 18 23 18 12 21 82
72 18 24 14 15 24 56
72 18 25 10 18 27 30
72 18 26 6 21 30 4
72 19 18 29 5 24 116
72 19 19 25 8 27 90
72 19 20 21 11 30 64
72 19 21 17 14 33 38
72 19 22 13 17 36 12
72 20 15 32 4 33 98
72 20 16 28 7 36 72
72 20 17 24 10 39 46
72 20 18 20 13 42 20
72 21 12 35 3 42 80
72 21 13 31 6 45 54
72 21 14 27 9 48 28
72 21 15 23 12 51 2
72 22 10 34 5 54 36
72 22 11 30 8 57 10
72 23 7 37 4 63 18
72 24 4 40 3 72 0
80 18 32 14 15 0 120
80 18 33 10 18 3 94
80 18 34 6 21 6 68
80 18 35 2 24 9 42

30 7 8 9 5 30 100
30 7 9 5 8 33 74
30 8 4 16 1 36 108
30 8 5 12 4 39 82
30 8 6 8 7 42 56
30 8 7 4 10 45 30
30 9 1 19 0 45 90
80 20 23 32 4 9 162
80 20 24 28 7 12 136
80 20 25 24 10 15 110
80 20 26 20 13 18 84
80 20 27 16 16 21 58
80 20 28 12 19 24 32
80 20 29 8 22 27 6
80 21 21 31 6 21 118
80 21 22 27 9 24 92
80 21 23 23 12 27 66
80 21 24 19 15 30 40
80 21 25 15 18 33 14
80 22 18 34 5 30 100
80 22 19 30 8 33 74
80 22 20 26 11 36 48
80 22 21 22 14 39 22
80 23 15 37 4 39 82
80 23 16 33 7 42 56
80 23 17 29 10 45 30
80 23 18 25 13 48 4
80 24 13 36 6 51 38
80 24 14 32 9 54 12
80 25 10 39 5 60 20
80 26 7 42 4 69 2
84 19 34 13 17 0 108
84 19 35 9 20 3 82
84 19 36 5 23 6 56
84 19 37 1 26 9 30
84 20 28 28 7 0 168
84 20 29 24 10 3 142
84 20 30 20 13 6 116
84 20 31 16 16 9 90
84 20 32 12 19 12 64
84 20 33 8 22 15 38
84 20 34 4 25 18 12
84 21 24 35 3 6 176
84 21 25 31 6 9 150
84 21 26 27 9 12 124
84 21 27 23 12 15 98
84 21 28 19 15 18 72
84 21 29 15 18 21 46
84 21 30 11 21 24 20
84 22 22 34 5 18 132
84 22 23 30 8 21 106
84 22 24 26 11 24 80
84 22 25 22 14 27 54
84 22 26 18 17 30 28
84 22 27 14 20 33 2
84 23 19 37 4 27 114
84 23 20 33 7 30 88
84 23 21 29 10 33 62
84 23 22 25 13 36 36
84 23 23 21 16 39 10
84 24 16 40 3 36 96
84 24 17 36 6 39 70
84 24 18 32 9 42 44
84 24 19 28 12 45 18
84 25 14 39 5 48 52
84 25 15 35 8 51 26
84 25 16 31 11 54 0
84 26 11 42 4 57 34
84 26 12 38 7 60 8
90 20 40 4 25 0 60
90 21 34 19 15 0 120
90 21 35 15 18 3 94
90 21 36 11 21 6 68
90 21 37 7 24 9 42
42 12 5 24 0 39 106
42 12 6 20 3 42 80
42 12 7 16 6 45 54
42 12 8 12 9 48 28
42 12 9 8 12 51 2
42 13 3 23 2 51 62
42 13 4 19 5 54 36
90 21 38 3 27 12 16
90 22 28 34 5 0 180
90 22 29 30 8 3 154
90 22 30 26 11 6 128
90 22 31 22 14 9 102
90 22 32 18 17 12 76
90 22 33 14 20 15 50
90 22 34 10 23 18 24
90 23 24 41 1 6 188
90 23 25 37 4 9 162
90 23 26 33 7 12 136
90 23 27 29 10 15 110
90 23 28 25 13 18 84
90 23 29 21 16 21 58
90 23 30 17 19 24 32
90 23 31 13 22 27 6
90 24 21 44 0 15 170
90 24 22 40 3 18 144
90 24 23 36 6 21 118
90 24 24 32 9 24 92
90 24 25 28 12 27 66
90 24 26 24 15 30 40
90 24 27 20 18 33 14
90 25 19 43 2 27 126
90 25 20 39 5 30 100
90 25 21 35 8 33 74
90 25 22 31 11 36 48
90 25 23 27 14 39 22
90 26 16 46 1 36 108
90 26 17 42 4 39 82
90 26 18 38 7 42 56
90 26 19 34 10 45 30
90 26 20 30 13 48 4
90 27 14 45 3 48 64
90 27 15 41 6 51 38
90 27 16 37 9 54 12
90 28 12 44 5 60 20
105 25 40 19 20 0 90
105 25 41 15 23 3 64
105 25 42 11 26 6 38
105 26 34 34 10 0 150
105 26 35 30 13 3 124
105 26 36 26 16 6 98
105 26 37 22 19 9 72
105 26 38 18 22 12 46
105 26 39 14 25 15 20
105 27 28 49 0 0 210
105 27 29 45 3 3 184
105 27 30 41 6 6 158
105 27 31 37 9 9 132
105 27 32 33 12 12 106
105 27 33 29 15 15 80
105 27 34 25 18 18 54
105 27 35 21 21 21 28
105 27 36 17 24 24 2
105 28 26 48 2 12 166
105 28 27 44 5 15 140
105 28 28 40 8 18 114
105 28 29 36 11 21 88
105 28 30 32 14 24 62
105 28 31 28 17 27 36
105 28 32 24 20 30 10
105 29 24 47 4 24 122
105 29 25 43 7 27 96
105 29 26 39 10 30 70
105 29 27 35 13 33 44
105 29 28 31 16 36 18
56 13 18 19 5 12 148
56 13 19 15 8 15 122
56 13 20 11 11 18 96
56 13 21 7 14 21 70
56 13 22 3 17 24 44
56 14 14 26 1 18 156
56 14 15 22 4 21 130
105 30 22 46 6 36 78
105 30 23 42 9 39 52
105 30 24 38 12 42 26
105 30 25 34 15 45 0
105 31 19 49 5 45 60
105 31 20 45 8 48 34
105 31 21 41 11 51 8
105 32 17 48 7 57 16
112 26 48 6 31 0 24
112 27 42 21 21 0 84
112 27 43 17 24 3 58
112 27 44 13 27 6 32
112 27 45 9 30 9 6
112 28 36 36 11 0 144
112 28 37 32 14 3 118
112 28 38 28 17 6 92
112 28 39 24 20 9 66
112 28 40 20 23 12 40
112 28 41 16 26 15 14
112 29 31 47 4 3 178
112 29 32 43 7 6 152
112 29 33 39 10 9 126
112 29 34 35 13 12 100
112 29 35 31 16 15 74
112 29 36 27 19 18 48
112 29 37 23 22 21 22
112 30 29 46 6 15 134
112 30 30 42 9 18 108
112 30 31 38 12 21 82
112 30 32 34 15 24 56
112 30 33 30 18 27 30
112 30 34 26 21 30 4
112 31 26 49 5 24 116
112 31 27 45 8 27 90
112 31 28 41 11 30 64
112 31 29 37 14 33 38
112 31 30 33 17 36 12
112 32 24 48 7 36 72
112 32 25 44 10 39 46
112 32 26 40 13 42 20
112 33 22 47 9 48 28
112 33 23 43 12 51 2
112 34 19 50 8 57 10
120 28 52 4 35 0 0
120 29 46 19 25 0 60
120 29 47 15 28 3 34
120 29 48 11 31 6 8
120 30 40 34 15 0 120
120 30 41 30 18 3 94
120 30 42 26 21 6 68
120 30 43 22 24 9 42
120 30 44 18 27 12 16
120 31 34 49 5 0 180
120 31 35 45 8 3 154
120 31 36 41 11 6 128
120 31 37 37 14 9 102
120 31 38 33 17 12 76
120 31 39 29 20 15 50
120 31 40 25 23 18 24
120 32 31 52 4 9 162
120 32 32 48 7 12 136
120 32 33 44 10 15 110
120 32 34 40 13 18 84
120 32 35 36 16 21 58
120 32 36 32 19 24 32
120 32 37 28 22 27 6
120 33 29 51 6 21 118
63 18 13 22 9 45 36
63 18 14 18 12 48 10
63 19 10 25 8 54 18
63 19 8 33 2 48 70
63 19 9 29 5 51 44
63 20 5 36 1 57 52
63 20 6 32 4 60 26
120 33 30 47 9 24 92
120 33 31 43 12 27 66
120 33 32 39 15 30 40
120 33 33 35 18 33 14
120 34 27 50 8 33 74
120 34 28 46 11 36 48
120 34 29 42 14 39 22
120 35 25 49 10 45 30
120 35 26 45 13 48 4
120 36 22 52 9 54 12
126 30 52 10 33 0 12
126 31 46 25 23 0 72
126 31 47 21 26 3 46
126 31 48 17 29 6 20
126 32 40 40 13 0 132
126 32 41 36 16 3 106
126 32 42 32 19 6 80
126 32 43 28 22 9 54
126 32 44 24 25 12 28
126 32 45 20 28 15 2
126 33 35 51 6 3 166
126 33 36 47 9 6 140
126 33 37 43 12 9 114
126 33 38 39 15 12 88
126 33 39 35 18 15 62
126 33 40 31 21 18 36
126 33 41 27 24 21 10
126 34 33 50 8 15 122
126 34 34 46 11 18 96
126 34 35 42 14 21 70
126 34 36 38 17 24 44
126 34 37 34 20 27 18
126 35 30 53 7 24 104
126 35 31 49 10 27 78
126 35 32 45 13 30 52
126 35 33 41 16 33 26
126 35 34 37 19 36 0
126 36 28 52 9 36 60
126 36 29 48 12 39 34
126 36 30 44 15 42 8
126 37 26 51 11 48 16
140 34 56 14 35 0 0
140 35 50 29 25 0 60
140 35 51 25 28 3 34
140 35 52 21 31 6 8
140 36 44 44 15 0 120
140 36 45 40 18 3 94
140 36 46 36 21 6 68
140 36 47 32 24 9 42
140 36 48 28 27 12 16
140 37 39 55 8 3 154
140 37 40 51 11 6 128
140 37 41 47 14 9 102
140 37 42 43 17 12 76
140 37 43 39 20 15 50
140 37 44 35 23 18 24
140 38 37 54 10 15 110
140 38 38 50 13 18 84
140 38 39 46 16 21 58
140 38 40 42 19 24 32
140 38 41 38 22 27 6
140 39 35 53 12 27 66
140 39 36 49 15 30 40
140 39 37 45 18 33 14
140 40 32 56 11 36 48
140 40 33 52 14 39 22
140 41 30 55 13 48 4
80 19 26 29 5 0 180
80 19 27 25 8 3 154
80 19 28 21 11 6 128
80 19 29 17 14 9 102
80 19 30 13 17 12 76
80 19 31 9 20 15 50
80 19 32 5 23 18 24
144 36 52 28 27 0 48
144 36 53 24 30 3 22
144 37 46 43 17 0 108
144 37 47 39 20 3 82
144 37 48 35 23 6 56
144 37 49 31 26 9 30
144 37 50 27 29 12 4
144 38 41 54 10 3 142
144 38 42 50 13 6 116
144 38 43 46 16 9 90
144 38 44 42 19 12 64
144 38 45 38 22 15 38
144 38 46 34 25 18 12
144 39 39 53 12 15 98
144 39 40 49 15 18 72
144 39 41 45 18 21 46
144 39 42 41 21 24 20
144 40 36 56 11 24 80
144 40 37 52 14 27 54
144 40 38 48 17 30 28
144 40 39 44 20 33 2
144 41 34 55 13 36 36
144 41 35 51 16 39 10
168 43 58 37 29 0 36
168 43 59 33 32 3 10
168 44 52 52 19 0 96
168 44 53 48 22 3 70
168 44 54 44 25 6 44
168 44 55 40 28 9 18
168 45 48 59 15 6 104
168 45 49 55 18 9 78
168 45 50 51 21 12 52
168 45 51 47 24 15 26
168 45 52 43 27 18 0
168 46 46 58 17 18 60
168 46 47 54 20 21 34
168 46 48 50 23 24 8
168 47 43 61 16 27 42
168 47 44 57 19 30 16
180 46 64 34 35 0 0
180 47 58 49 25 0 60
180 47 59 45 28 3 34
180 47 60 41 31 6 8
180 48 52 64 15 0 120
180 48 53 60 18 3 94
180 48 54 56 21 6 68
180 48 55 52 24 9 42
180 48 56 48 27 12 16
180 49 50 63 17 12 76
180 49 51 59 20 15 50
180 49 52 55 23 18 24
180 50 48 62 19 24 32
180 50 49 58 22 27 6
210 55 70 49 35 0 0
210 56 64 64 25 0 60
210 56 65 60 28 3 34
210 56 66 56 31 6 8
210 57 61 67 24 9 42
210 57 62 63 27 12 16
210 58 58 70 23 18 24
240 64 76 64 35 0 0
240 65 71 75 28 3 34
240 65 72 71 31 6 8
252 68 76 76 31 0 24
280 76 84 84 35 0 0

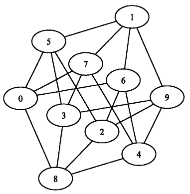
Appendix B. The bipartite integral graphs

This appendix contains the drawings and spectra of all 47 connected, 4-regular, integral, bipartite graphs with up to 24 vertices, shown in Tables 3–8. Since the spectrum of a bipartite graph

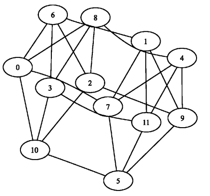
Table 3
Graphs with 8, 10 and 12 vertices



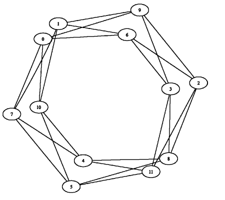
$B_1, \{4, 0^6, -4\}$



$B_2, \{4, 1^4 \ldots\}$

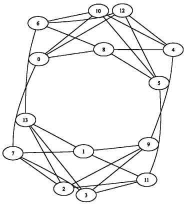


$B_3, \{4, 2, 1^4 \ldots\}$

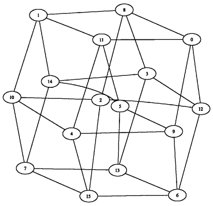


$B_4, \{4, 2^2, 0^6 \ldots\}$

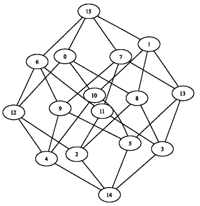
Table 4
Graphs with 14 and 16 vertices



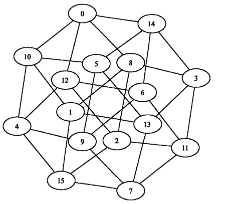
$B_5, \{4, 3, 1^3, 0^4 \ldots\}$



$B_6, \{4, 2^4, 0^6 \ldots\}$

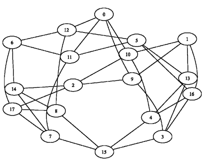


$B_7, \{4, 2^4, 0^6 \ldots\}$

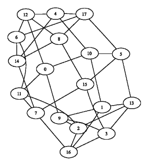


$B_8, \{4, 2^4, 0^6 \ldots\}$

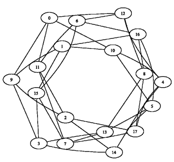
Table 5
Graphs with 18 vertices



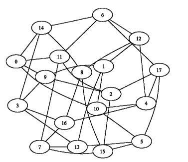
$B_9, \{4, 3, 2^2, 1^3, 0^4 \ldots\}$



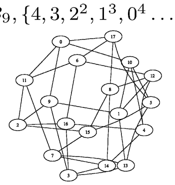
$B_{10}, \{4,3,2^2,1^3,0^4\ldots\}$



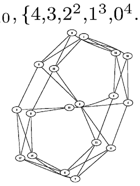
$B_{11}, \{4, 2^4, 1^4 \ldots\}$



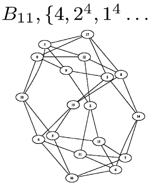
$B_{12}, \{4, 2^4, 1^4 \ldots\}$



$B_{13}, \{4, 2^4, 1^4 \ldots\}$



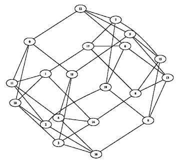
$B_{14}, \{4,3,2^2,1^3,0^4\ldots\}$



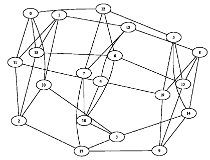
$B_{15}, \{4,3,2^2,1^3,0^4\ldots\}$

is symmetric with respect to 0, we show the nonnegative part of the spectrum only below each graph drawing.

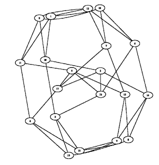
Table 6
Graphs with 20 vertices



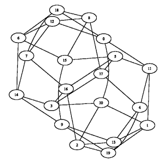
$B_{16}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



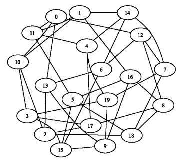
$B_{17}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



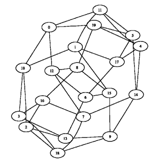
$B_{18}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



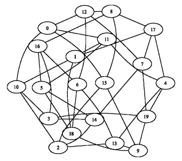
$B_{19}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



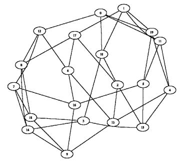
$B_{20}, \{4, 2^5, 1^4 \ldots\}$



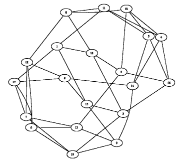
$B_{21}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



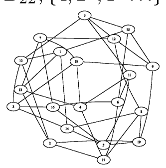
$B_{22}, \{4, 2^5, 1^4 \ldots\}$



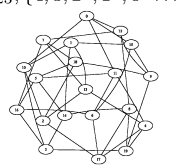
$B_{23}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



$B_{24}, \{4, 3, 2^3, 1^3, 0^4 \ldots\}$



$B_{25}, \{4, 2^5, 1^4 \ldots\}$



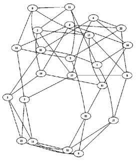
$B_{26}, \{4, 2^5, 1^4 \ldots\}$

Appendix C. The nonbipartite integral graphs

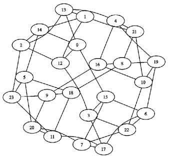
This appendix contains the drawings and spectra of all 14 connected, 4-regular, integral, nonbipartite graphs obtained by decomposing the graphs from Appendix B into a direct product of a nonbipartite graph and K_2. The decompositions are as follows:

$$B_2 \cong D_1 \times K_2,$$
$$B_4 \cong D_2 \times K_2,$$
$$B_5 \cong D_3 \times K_2,$$
$$B_8 \cong D_4 \times K_2,$$
$$B_{10} \cong D_5 \times K_2,$$
$$B_{11} \cong D_6 \times K_2 \cong D_7 \times K_2,$$
$$B_{14} \cong D_8 \times K_2,$$
$$B_{40} \cong D_9 \times K_2 \cong D_{10} \times K_2,$$
$$B_{43} \cong D_{11} \times K_2 \cong D_{12} \times K_2 \cong D_{13} \times K_2 \cong D_{14} \times K_2.$$

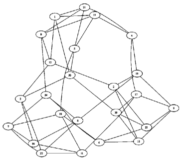
Table 7
Graphs with 24 vertices



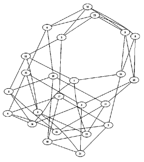
$B_{27}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



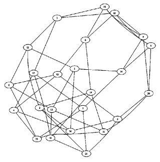
$B_{28}, \{4, 3^3, 1^5, 0^6 \ldots\}$



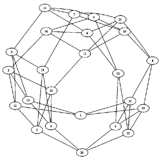
$B_{29}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



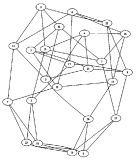
$B_{30}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



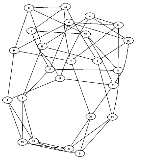
$B_{31}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



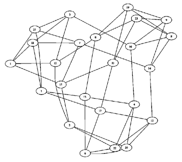
$B_{32}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



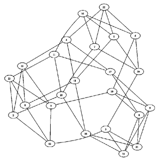
$B_{33}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



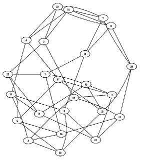
$B_{34}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



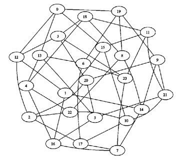
$B_{35}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



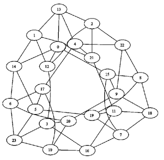
$B_{36}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



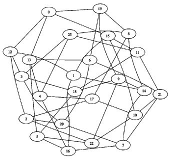
$B_{37}, \{4, 3, 2^5, 1^3, 0^4 \ldots\}$



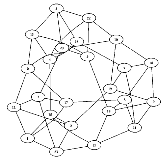
$B_{38}, \{4, 2^8, 0^6 \ldots\}$



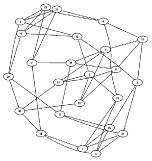
$B_{39}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



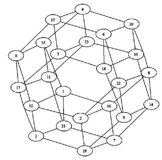
$B_{40}, \{4, 2^8, 0^6 \ldots\}$



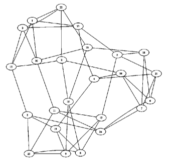
$B_{41}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



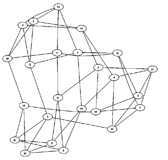
$B_{42}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



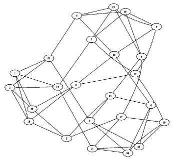
$B_{43}, \{4, 3^2, 2^2, 1^6, 0^2 \ldots\}$



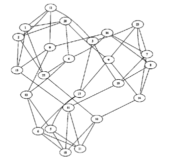
$B_{44}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$



$B_{45}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$

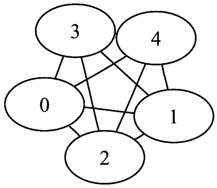


$B_{46}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$

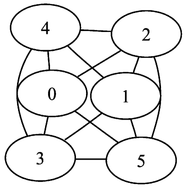


$B_{47}, \{4, 3^2, 2^3, 1^2, 0^8 \ldots\}$

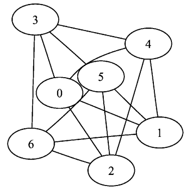
Table 8
The nonbipartite graphs



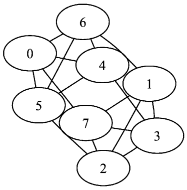
$D_1, \{4, -1^4\}$



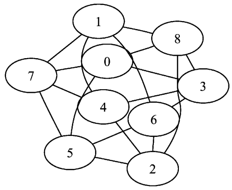
$D_2, \{4, 0^3, -2^2\}$



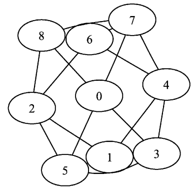
$D_3, \{4, 1, 0^2, -1^2, -3\}$



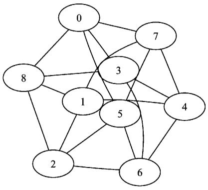
$D_4, \{4, 2, 0^3, -2^3\}$



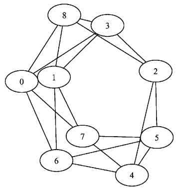
$D_5, \{4, 1^3, 0^2, -2^2, -3\}$



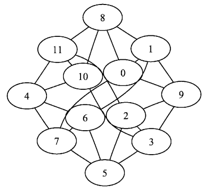
$D_6, \{4, 2, 1^2, -1^2, -2^3\}$



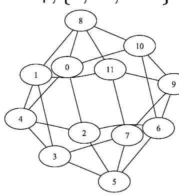
$D_7, \{4, 1^4, -2^4\}$



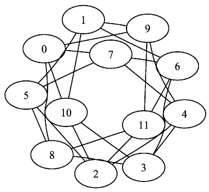
$D_8, \{4,2,1,0^2,-1^2,-2,-3\}$



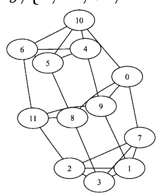
$D_9, \{4, 2^3, 0^3, -2^5\}$



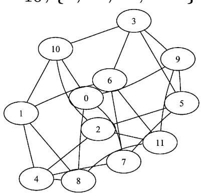
$D_{10}, \{4, 2^3, 0^3, -2^5\}$



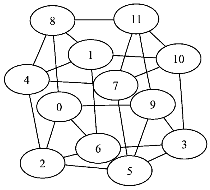
$D_{11}, \{4, 1^6, 0, -2^2, -3^2\}$



$D_{12}, \{4,3,1^3,0,-1^3,-2^2,-3\}$



$D_{13}, \{4,2,1^4,0,-1^2,-2,-3^2\}$



$D_{14}, \{4,2^2,1^2,0,-1^4,-3^2\}$

References

[1] K. Balińska, D. Cvetković, M. Lepović, S. Simić, There are exactly 150 connected integral graphs up to 10 vertices, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat., 10 (1999) 95–105.
[2] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, A survey on integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 13 (2002) 42–65.
[3] K.T. Balińska, M. Kupczyk, S.K. Simić, K.T. Zwierzyński, On generating all integral graphs on 11 vertices, Computer Science Center Report No. 469, Technical University of Poznań, 1999/2000, pp. 1–53.

[4] K.T. Balińska, M. Kupczyk, S.K. Simić, K.T. Zwierzyński, On generating all integral graphs on 12 vertices, Computer Science Center Report No. 482, Technical University of Poznań, 2001, pp. 1–36.
[5] K.T. Balińska, S.K. Simić, Some remarks on integral graphs with maximum degree four, Novi Sad J. Math. 31 (1) (2001) 19–25.
[6] K.T. Balińska, S.K. Simić, The nonregular, bipartite, integral graphs with maximum degree four, Discrete Math. 236 (2001) 13–24.
[7] K.T. Balińska, S.K. Simić, K.T. Zwierzyński, Which non-regular, bipartite, integral graphs with maximum degree four do not have ± 1 as eigenvalues? Discrete Math. 286 (2004) 15–24.
[8] F.C. Bussemaker, D. Cvetković, There are exactly 13 connected, cubic, integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz. 544–576 (1976) 43–48.
[9] M. Christandl, N. Datta, T.C. Dorlas, A. Ekert, A. Kay, A.J. Landahl, Perfect transfer of arbitrary states in quantum spin networks, v2, 22 Nov. 2004. <arXiv:quant-ph/0411020>.
[10] D. Cvetković, Cubic integral graphs, Univ. Beograd, Publ Elektrotehn. Fak. Ser. Mat. Fiz. 498–541 (1975) 107–113.
[11] D. Cvetković, M. Doob, H. Sachs, Spectra of graphs – Theory and Application, Deutscher Verlag der Wissenschaften, Academic Press, Berlin–New York, 1980; second ed. 1982; third ed., Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.
[12] D. Cvetković, I. Gutman, N. Trinajstić, Conjugated molecules having integral graph spectra, Chem. Phys. Lett. 29 (1974) 65–68.
[13] D. Cvetković, S. Simić, Graph theoretical results obtained by the support of the expert system "Graph"—an extended survey, in: S. Fajtlowicz, P.W. Fowler, P. Hansen, M.F. Janowitz, F.S. Roberts (Eds.), Graphs and Discovery, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 69, AMS, Rhode Island, 2005, pp. 39–70.
[14] D. Cvetković, S. Simić, D. Stevanović, 4-regular integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 9 (1998) 89–102.
[15] F. Harary, A.J. Schwenk, Which graphs have integral spectra? in: R. Bari, F. Haray (Eds.), Graphs and Combinatorics, Springer-Verlag, Berlin, 1974, pp. 45–51.
[16] Z. Radosavljević, S. Simić, There are just thirteen connected nonregular nonbipartite integral graphs having maximum vertex degree four, in: Proc. Sixth Yugoslav Seminar on Graph Theory (Dubrovnik 1985), Univ. Novi Sad, 1986, pp. 183–187.
[17] A.J. Schwenk, Exactly thirteen connected cubic graphs have integral spectra, in: Y. Alavi, D. Lick (Eds.), Proceedings of the International Graph Theory Conference at Kalamazoo, May 1976, Lecture Notes in Mathematics, vol. 642, Springer-Verlag, Berlin, 1978, pp. 516–533.
[18] S. Simić, Z. Radosavljević, The nonregular, nonbipartite, integral graphs with maximum degree four, J. Comb. Inf. Syst. Sci. 20 (1995) 9–26.
[19] N.J.A. Sloane, Online Encyclopedia of Integer Sequences, the sequences A089022 and A035610 from <http://www.research.att.com/~njas/sequences/>.
[20] D. Stevanović, Nonexistence of some 4-regular integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 10 (1999) 81–86.
[21] D. Stevanović, Some compositions of graphs and integral graphs (in Serbian), Ph.D. thesis, University of Niš, 2000.
[22] D. Stevanović, 4-regular integral graphs avoiding ± 3 in the spectrum, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 14 (2003) 99–110.
[23] L. Wang, A survey of results on integral trees and integral graphs, Department of Applied Mathematics, Faculty of EEMCS, University of Twente, Memorandum No. 1783 (2005), 1–22. Available online at <http://www.math.utwente.nl/publications/2005/1763.pdf>.