Pergamon

Computers Math. Applic. Vol. 36, No. 2, pp. 71–86, 1998

0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00118-7

Interactive Fuzzy Programming for Multilevel Linear Programming Problems

M. SAKAWA AND I. NISHIZAKI
Department of Industrial and Systems Engineering
Faculty of Engineering, Hiroshima University
1-4-1 Kagamiyama, Higashi-Hiroshima, 739 Japan
<sakawa><nisizaki>@msl.sys.hiroshima-u.ac.jp

Y. UEMURA
Products Projects Development Department, Juken Sangyou Co., Ltd.
1-1 Mokuzaikouminami, Hatsukaichi, Hiroshima, 738 Japan

(Received and accepted October 1997)

Abstract—This paper presents interactive fuzzy programming for multilevel linear programming problems. In fuzzy programming for multilevel linear programming problems, recently developed by Lai *et al.*, since the fuzzy goals are determined for both an objective function and decision variables at the upper level, undesirable solutions are produced when these fuzzy goals are inconsistent. In order to overcome such problems, after eliminating the fuzzy goals for decision variables, interactive fuzzy programming for multilevel linear programming problems is presented. In our interactive method, after determining the fuzzy goals of the decision makers at all levels, a satisfactory solution is derived efficiently by updating the satisfactory degrees of decision makers at the upper level with considerations of overall satisfactory balance among all levels. Illustrative numerical examples for two-level and for three-level linear programming problems are provided to demonstrate the feasibility of the proposed method.

Keywords—Multilevel linear programming problem, Fuzzy programming, Fuzzy goals, Interactive methods.

1. INTRODUCTION

Two-level programming problems, in which a Decision Maker (DM) at the upper level makes a decision subject to an optimization problem for a DM at the lower level, admit of two interpretations. They depend on whether there is a cooperative relationship among DMs or not.

Consider a decision problem in a decentralized firm as an example of a decision problem with cooperative DMs. Top management, an executive board, or headquarters interests itself in overall management policy such as long-term corporate growth or market share. In contrast, operation divisions of the firm are concerned with coordination of daily activities. After headquarters chooses a strategy in accordance with the overall management policy, each division determines a goal to be achieved and optimizes the goal, not fully understanding the strategy chosen by the headquarters.

As an example of a decision problem without cooperative DMs, consider the Stackelberg duopoly: Firm 1 and Firm 2 supply homogeneous goods to a market. Suppose Firm 1 dom-

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX

inates Firm 2 in the market, and consequently Firm 1 first determines a level of supply and then Firm 2 decides its level of supply after it realizes Firm 1's level of supply.

There is essentially cooperative relationship between the DM at the upper level and the DM at the lower level in the former problem while each DM does not have a motivation to cooperate each other in the latter problem.

As the former's mathematical programming problem, we can model such a problem as a single-objective large scale mathematical programming problems used the decomposition method or a multiobjective programming problem with objective functions of all levels. The two-level programming formulation is intend to supplement decomposition approach, not supplant it [1]. However, the formulation is noteworthy as a mathematical programming problem with a hierarchical structure.

Studies on the latter have been seen in the literature on game theory. Such a situation is modeled as a Stackelberg game, in which there are two players, and one player determines his/her strategy and thereafter the other player decides his/her strategy [2]. Each player completely knows objective functions and constraints of an opponent and himself/herself, and the DM at the upper level (leader) first specifies his/her strategy and then the DM at the lower level (follower) specifies his/her strategy so as to optimize his/her objectives with full knowledge of decision of the DM at the upper level.

According to the rule, the DM at the upper level also specifies his/her strategy so as to optimize his/her own objective. Then a solution defined as the above-mentioned procedure is called a Stackelberg strategy (solution). The Stackelberg strategy has been employed as a solution concept when decision problems are modeled as two-level programming problems whether there is a cooperative relationship between the DMs or not. Even if objective functions of both DMs and common constraint functions are linear, it is known that this problem is a nonconvex programming problem with special structure. In general, a Stackelberg solution does not satisfy Pareto optimality because of its noncooperative nature.

Computational methods for a Stackelberg solution are classified roughly into three categories: the vertex enumeration approach based on a characteristic that an extreme point of a set of best responses of the DM at the lower level is also an extreme point of a set of the common constraints, the Kuhn-Tucker approach in which the upper level's problem with constraints involved optimality conditions of the lower level's problem is solved, and the penalty function approach which adds a penalty term to the upper level's objective function so as to satisfy optimality of the lower level's problem.

The K^{th} best method proposed by Bialas and Karwan [1] is one of vertex enumeration approaches. The solution search procedure of the method starts from a point which is an optimal solution to the problem of the upper level and checks whether it is also an optimal solution to the problem of the lower level or not. If the first point is not a Stackelberg solution, the procedure continues to examine the second best solution to the problem of the upper level and so on. The Kuhn-Tucker method is used by Bialas and Karwan [1] in their parametric complementary pivot algorithm. Bard and Falk [3] replaces the complementarity constraint (complementary slackness condition) with a separable representation and applies a general branch and bound algorithm. Bard [4] formulates a two-level programming problem as an equivalent semi-infinite problem and develops his grid search algorithm through a parametric linear program technique. Ünlü [5] proposes an algorithm based on bicriteria programming by using the result of [4]. White and Anandalingam [6] develops an approach to two-level programming using a duality gap-penalty function format.

For obtaining the Stackelberg solution to a multilevel linear programming problem, Bard [7] and Wen and Bialas [8] propose algorithms for three-level problems. Bard [7] formulates a normal nonlinear programming problem by using the Kuhn-Tucker conditions for the problems of the third level and the second level, and proposes a cutting plane algorithm employing a vertex search procedure to solve a three-level linear programming problem. Wen and Bialas [8] develop a hybrid algorithm to solve a three-level linear programming problem. The algorithm adopts the K^{th} best algorithm to generate the K^{th} best extreme point and the complementary pivot algorithm to check feasibility.

Recently, Lai [9] and Shih, Lai and Lee [10] have proposed a solution concept, which is different from the concept of a Stackelberg solution, for problems such that decisions of DMs in all the levels are sequential and all of the DMs essentially cooperate with each other. Their method is based on an idea that the DM at the lower level optimizes his/her objective function, taking a goal or preference of the upper level into consideration. DMs elicit membership functions of fuzzy goals for their objective functions, and especially, the DM at the upper level also specifies those of fuzzy goals for his/her decision variables. The DM at the lower level solves a fuzzy programming problem with a constraint on a satisfactory degree of the DM at the upper level. Unfortunately, there is a possibility that their method leads a final solution to an undesirable one because of inconsistency between the fuzzy goals of the objective function and the decision variables.

In this paper, we present interactive fuzzy programming for multilevel linear programming problems. In order to overcome the problem in the methods of Lai *et al.*, after eliminating the fuzzy goals for decision variables, multilevel linear programming problems is formulated. In our interactive method, after determining the fuzzy goals of the DM at all levels, a satisfactory solution is derived efficiently by updating the satisfactory degrees of the DMs at the upper level with considerations of overall satisfactory balance among all levels. Illustrative numerical examples for two-level and three-level linear programming problems are provided to demonstrate the feasibility of the proposed method.

2. INTERACTIVE FUZZY PROGRAMMING FOR TWO-LEVEL LINEAR PROGRAMMING PROBLEMS

A two-level linear programming problem for obtaining a Stackelberg solution is formulated as:

$$\begin{aligned}
\underset{\mathbf{x}_1}{\text{minimize}} \quad & z_1(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{c}_{11}\mathbf{x}_1 + \mathbf{c}_{12}\mathbf{x}_2, \\
& \text{where } \mathbf{x}_2 \text{ solves} \\
\underset{\mathbf{x}_2}{\text{minimize}} \quad & z_2(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{c}_{21}\mathbf{x}_1 + \mathbf{c}_{22}\mathbf{x}_2, \\
\text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \le \mathbf{b}, \\
& \mathbf{x}_1 \ge \mathbf{0},\ \mathbf{x}_2 \ge \mathbf{0},
\end{aligned} \tag{1}$$

where $\mathbf{x}_i$, $i = 1, 2$ is an n_i-dimensional decision variable, $\mathbf{c}_{i1}$, $i = 1, 2$ is an n_1-dimensional constant row vector, $\mathbf{c}_{i2}$, $i = 1, 2$ is an n_2-dimensional constant row vector, $\mathbf{b}$ is an m-dimensional constant column vector, and A_i, $i = 1, 2$ is an $m \times n_i$ constant matrix. For the sake of simplicity, we use the following notations: $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathrm{R}^{n_1+n_2}$, $\mathbf{c} = \left[\begin{smallmatrix}\mathbf{c}_{11} & \mathbf{c}_{12} \\ \mathbf{c}_{21} & \mathbf{c}_{22}\end{smallmatrix}\right]$, and $\mathbf{A} = [A_1\ A_2]$, and let DM1 denote the DM at the upper level and DM2 denote the DM at the lower level. In the two-level linear programming problem (1), $z_1(\mathbf{x}_1, \mathbf{x}_2)$ and $z_2(\mathbf{x}_1, \mathbf{x}_2)$, respectively, represent objective functions of the upper and the lower levels and $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, represent decision variables of the upper and the lower levels.

In contrast to the above formulation, in this paper, DM1 specifies a fuzzy goal and a minimal satisfactory level and evaluates a solution proposed by DM2, and DM2 solves an optimization problem, referring to the fuzzy goal and the minimal satisfactory level of DM1. Thus, a two-level linear programming problem dealt with in this paper is formally represented as:

$$\begin{aligned}
\underset{\text{Level 1}}{\text{minimize}} \quad & z_1(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{c}_{11}\mathbf{x}_1 + \mathbf{c}_{12}\mathbf{x}_2, \\
\underset{\text{Level 2}}{\text{minimize}} \quad & z_2(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{c}_{21}\mathbf{x}_1 + \mathbf{c}_{22}\mathbf{x}_2, \\
\text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \le \mathbf{b}, \\
& \mathbf{x}_1 \ge \mathbf{0},\ \mathbf{x}_2 \ge \mathbf{0},
\end{aligned} \tag{2}$$

where two objective functions z_1 and z_2 are those of the upper and the lower levels, respectively, and $\text{minimize}_{\text{Level 1}}$ and $\text{minimize}_{\text{Level 2}}$ mean that the DMs at the upper and the lower levels are minimizers for their objective functions.

It is natural that DMs have fuzzy goals for their objective functions when they take fuzziness of human judgments into consideration. For each of the objective functions $z_i(\mathbf{x})$, $i = 1, 2$ of (2), assume that the DMs have fuzzy goals such as "the objective function $z_i(\mathbf{x})$ should be substantially less than or equal to some value p_i."

The individual minimum

$$z_i^{\min} = \min_{\mathbf{x}\in X} z_i(\mathbf{x}), \qquad i = 1, 2 \tag{3}$$

and the individual maximum

$$z^{\max_i} = \max_{\mathbf{x}\in X} z_i(\mathbf{x}), \qquad i = 1, 2 \tag{4}$$

of the objective functions are referred to when the DMs elicit membership functions prescribing the fuzzy goals for the objective functions $z_i(\mathbf{x})$, $i = 1, 2$. The DMs determine the membership functions $\mu_i(z_i(\mathbf{x}))$, which are strictly monotone decreasing for $z_i(\mathbf{x})$, consulting the variation ratio of degree of satisfaction in the interval between the individual minimum (3) and the individual maximum (4). The domain of the membership function is the interval $[z_i^{\min}, z_i^{\max}]$, $i = 1, 2$, and the DM specifies the value z_i^0 of the objective function for which the degree of satisfaction is 0 and the value z_i^1 of the objective function for which the degree of satisfaction is 1. For the value undesired (larger) than z_i^0, it is defined that $\mu_i(z_i(\mathbf{x})) = 0$, and for the value desired (smaller) than z_i^1, it is defined that $\mu_i(z_i(\mathbf{x})) = 1$.

For the sake of simplicity, in this paper, we adopt a linear membership function, which characterizes the fuzzy goal of the DM at each level. The corresponding linear membership function $\mu_i(z_i)$ is defined as:

$$\mu_i(z_i(\mathbf{x})) = \begin{cases} 0, & z_i(\mathbf{x}) > z_i^0, \\ \dfrac{z_i(\mathbf{x}) - z_i^0}{z_i^1 - z_i^0}, & z_i^1 < z_i(\mathbf{x}) \le z_i^0, \\ 1, & z_i(\mathbf{x}) \le z_i^1, \end{cases} \tag{5}$$

where z_i^0 and z_i^1 denote the value of the objective function $z_i(x)$ such that the degree of membership function is 0 and 1, respectively, and it is assumed that the DMs subjectively assess z_i^0 and z_i^1.

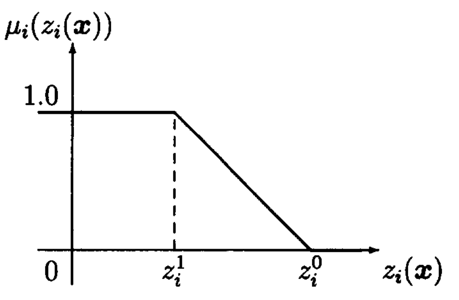

Figure 1. Linear membership function.

Suppose that applying the way suggested by Zimmermann [11], the DMs specify z_i^0 and z_i^1 in the following way. That is, using the individual minimum

$$z_i^{\min} = z_i\left(\mathbf{x}^{io}\right) = \min\left\{z_i(\mathbf{x}) \mid A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \le \mathbf{b},\ \mathbf{x}_1 \ge \mathbf{0},\ \mathbf{x}_2 \ge \mathbf{0}\right\} \tag{6}$$

together with

$$z_i^{\mathrm{m}} = z_i\left(\mathbf{x}^{jo}\right), \qquad i = 1, 2, \quad j = \begin{cases} 1, & \text{if } i = 2, \\ 2, & \text{if } i = 1, \end{cases} \tag{7}$$

the DMs determine the linear membership functions as in (5) by choosing $z_i^1 = z_i^{\min}$, $z_i^0 = z_i^{\mathrm{m}}$, $i = 1, 2$.

After eliciting the membership functions, DM1 subjectively specifies a minimal satisfactory level $\hat{\delta} \in [0,1]$ for his/her membership function $\mu_1(z_1(\mathbf{x}))$. Then, DM2 maximize his/her membership function subject to the condition that DM2's membership function $\mu_1(z_1(\mathbf{x}))$ is larger than or equal to $\hat{\delta}$ under the given constraints, that is, DM2 solves the following problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mu_2(z_2(\mathbf{x})), \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \leq \mathbf{b}, \\ & \mu_1(z_1(\mathbf{x})) \geq \hat{\delta}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{8}$$

Constraints on fuzzy goals for decision variables are eliminated in our formulation (8) while they are involved in the formulations by Lai *et al.* [9,10].

If an optimal solution to problem (8) exists, it follows that DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified by DM1's own self. However, the larger the minimal satisfactory level is assessed, the smaller DM2's satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger and it is feared that overall satisfactory balance between both levels cannot maintain.

To take account of overall satisfactory balance between both levels, DM1 needs to compromise with DM2 on DM1's minimal satisfactory level. To do so, a satisfactory degree of both DMs is defined as

$$\lambda = \min(\mu_1(z_1(\mathbf{x})), \mu_2(z_2(\mathbf{x}))), \tag{9}$$

and the following problem is substituted for problem (8):

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \leq \mathbf{b}, \\ & \mu_1(z_1(\mathbf{x})) \geq \hat{\delta} \geq \lambda, \\ & \mu_2(z_2(\mathbf{x})) \geq \lambda, \\ & \lambda \in [0,1], \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{10}$$

For problem (10), consider the auxiliary problem

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \leq \mathbf{b}, \\ & \mu_1(z_1(\mathbf{x})) \geq \lambda, \\ & \mu_2(z_2(\mathbf{x})) \geq \lambda, \\ & \lambda \in [0,1], \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{11}$$

By solving problem (11), we obtain a solution maximizing a smaller satisfactory degree between those of both DMs.

If an optimal solution $\mathbf{x}^*$ to problem (11) satisfies the condition that $\mu_1(z_1(\mathbf{x}^*)) \geq \hat{\delta}$, it follows that DM1 obtains a satisfactory solution. However, the solution $\mathbf{x}^*$ does not always satisfy the condition. Then the ratio of satisfactory degree between both levels

$$\Delta = \frac{\mu_2(z_2(\mathbf{x}^*))}{\mu_1(z_1(\mathbf{x}^*))}, \tag{12}$$

which is defined by Lai [9], is useful. If $\Delta > 1$, i.e., $\mu_2(z_2(\mathbf{x}^*)) > \mu_1(z_1(\mathbf{x}^*))$, then DM1 updates the minimal satisfactory level $\hat{\delta}$ by increasing the value $\hat{\delta}$. Receiving the updated level $\hat{\delta}'$, DM2 solves problem (8) with $\hat{\delta}'$, and then DM1 obtains a larger satisfactory degree and DM2 accepts a smaller satisfactory degree. Conversely, if $\Delta < 1$, i.e., $\mu_2(z_2(\mathbf{x}^*)) < \mu_1(z_1(\mathbf{x}^*))$, then DM1 updates the minimal satisfactory level $\hat{\delta}$ by decreasing the value $\hat{\delta}$, and DM1 obtains a smaller satisfactory degree and DM2 accepts a larger satisfactory degree.

At an iteration ℓ, let $\mu_1(z_1^\ell)$, $\mu_2(z_2^\ell)$ and λ^ℓ denote DM1's and DM2's satisfactory degrees and a satisfactory degree of both levels, respectively, and let $\Delta^\ell = \mu_2(z_2^\ell)/\mu_1(z_1^\ell)$. When DM1 is proposed a solution by DM2 and the following two conditions are satisfied, DM1 concludes the solution as a satisfactory solution and the interactive process terminates.

Termination Conditions of the Interactive Process for Two-Level Linear Programming Problems

(1) DM1's satisfactory degree is larger than or equal to the minimal satisfactory level $\hat{\delta}$ specified by DM1, i.e., $\mu_1(z_1^\ell) \geq \hat{\delta}$.
(2) The ratio Δ^ℓ of satisfactory degrees is in the closed interval between its lower and its upper bounds specified by DM1.

Condition (1) means DM1's required condition for solutions proposed by DM2. Condition (2) is provided in order to keep overall satisfactory balance between both levels.

Unless the conditions are satisfied simultaneously, DM1 needs to update his/her minimal satisfactory level $\hat{\delta}$.

Procedure for Updating the Minimal Satisfactory Level $\hat{\delta}$

(1) If condition (1) is not satisfied, then DM1 decreases the minimal satisfactory level $\hat{\delta}$.
(2) If the ratio Δ^ℓ exceeds its upper bound, then DM1 increases the minimal satisfactory level $\hat{\delta}$. Conversely, if the ratio Δ^ℓ is below its lower bound, then DM1 decreases the minimal satisfactory level $\hat{\delta}$.

Let $\hat{\delta}'$ denote the updated minimal satisfactory level. DM2 solves the following maximization problem with the updated minimal satisfactory level $\hat{\delta}'$.

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mu_2(z_2(\mathbf{x})), \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \leq \mathbf{b}, \\ & \mu_1(z_1(\mathbf{x})) \geq \hat{\delta}', \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{13}$$

The above-mentioned algorithm is summarized as follows.

Algorithm of the Interactive Fuzzy Programming for Solving Two-Level Linear Programming Problems

STEP 1. DM1 elicits the membership function $\mu_1(z_1)$ of the fuzzy goal of DM1, and specifies the minimal satisfactory level $\hat{\delta}$ and the lower and the upper bounds of the ratio of satisfactory degrees Δ^1.

STEP 2. DM2 elicits the membership function $\mu_2(z_2)$ of the fuzzy goal of DM2.

STEP 3. DM2 solves the auxiliary problem (11), and then proposes a solution (z_1^ℓ, z_2^ℓ) to problem (11), λ^ℓ, $\mu_1(z_1^\ell)$, $\mu_2(z_2^\ell)$, and Δ^ℓ to DM1.

STEP 4. If the solution proposed by DM2 to DM1 satisfies the termination conditions, DM1 concludes the solution as a satisfactory solution and the algorithm stops.

STEP 5. DM2 updates the minimal satisfactory level $\hat{\delta}$ in accordance with the procedure of updating minimal satisfactory level.

STEP 6. DM2 solves problem (13) and proposes an obtained solution to DM1. Return to Step 4.

3. INTERACTIVE FUZZY PROGRAMMING FOR MULTILEVEL LINEAR PROGRAMMING PROBLEMS

We extend the interactive fuzzy programming for two-level linear programming problems to that for multilevel problems, i.e., t-level linear programming problems.

Let $\mathbf{x} = (\mathbf{x}_1, \dots, x_t) \in \mathrm{R}^{n_1+\cdots+n_t}$,

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_{11} & \cdots & c_{1t} \\ & \cdots\cdots & \\ c_{t1} & \cdots & c_{tt} \end{bmatrix},$$

and $\mathbf{A} = [A_1 \ \cdots \ A_t]$. In a way similar to two-level problems, we can formulate a t-level linear programming problem as:

$$\begin{aligned} \underset{\text{Level 1}}{\text{minimize}} \quad & z_1(\mathbf{x}) = \mathbf{c}_{11}\mathbf{x}_1 + \cdots + c_{1t}x_t, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ \underset{\text{Level t}}{\text{minimize}} \quad & z_t(\mathbf{x}) = c_{t1}\mathbf{x}_1 + \cdots + c_{tt}x_t, \\ \text{subject to} \quad & A_1\mathbf{x}_1 + \cdots + A_t x_t \le \mathbf{b}, \\ & \mathbf{x}_1 \ge \mathbf{0}, \dots, x_t \ge \mathbf{0}. \end{aligned} \tag{14}$$

It is assumed that a DM at each level elicits a linear membership function of his/her fuzzy goal $\mu_i(z_i(\mathbf{x}))$, $i = 1, \dots, t$, and that all the DMs except for at the lowest level determine minimal satisfactory levels $\hat{\delta}_i \in [0, 1]$, $i = 1, \dots, t-1$.

In a multilevel case, a problem corresponding to problem (8) for a two-level problem can be extended to:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mu_t(z_t(\mathbf{x})), \\ \text{subject to} \quad & A_1x_1 + \cdots + A_t x_t \le \mathbf{b}, \\ & \mu_i(z_i(\mathbf{x})) \ge \hat{\delta}_i, \quad i = 1, \dots, t-1, \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned} \tag{15}$$

Defining a satisfactory degree of DMs at all the levels as

$$\lambda = \min(\mu_1(z_1(\mathbf{x})), \dots, \mu_t(z_t(\mathbf{x}))), \tag{16}$$

we substitute the following problem for problem (15).

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & A_1x_1 + \cdots + A_t x_t \le \mathbf{b}, \\ & \mu_i(z_i(\mathbf{x})) \ge \hat{\delta}_i \ge \lambda, \quad i = 1, \dots, t-1, \\ & \mu_t(z_t(\mathbf{x})) \ge \lambda, \\ & \lambda \in [0, 1], \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned} \tag{17}$$

To solve problem (17), we formulate the auxiliary problem

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & A_1x_1 + \cdots + A_t x_t \le \mathbf{b}, \\ & \mu_i(z_i(\mathbf{x})) \ge \lambda, \quad i = 1, \dots, t, \\ & \lambda \in [0, 1], \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned} \tag{18}$$

A ratio of satisfactory degrees is also extended to

$$\Delta_i = \frac{\mu_{i+1}(z_{i+1}(\mathbf{x}^*))}{\mu_i(z_i(\mathbf{x}^*))}, \qquad i = 1, \ldots, t-1. \tag{19}$$

At an iteration ℓ, let $\mu_i(z_i^\ell)$, $i = 1, \ldots, t$ and λ^ℓ denote a satisfactory degrees of the DM (DMi) at the i^{th} level, $i = 1, \ldots, t$, and a satisfactory degree of all the levels, respectively, and let $\Delta_i^\ell = \mu_{i+1}(z_{i+1}^\ell)/\mu_i(z_i^\ell)^\ell$ denote a ratio of satisfactory degrees of the i^{th} and the $(i+1)^{\text{th}}$ levels. For all $i = 1, \ldots, t-1$, the DMi is proposed a solution by the DM$(i+1)$. Then the DMs at all the levels except for the t^{th} level obtain the satisfactory solutions and the interactive process terminates if the following two conditions are satisfied.

Termination Conditions of the Interactive Process for Multilevel Linear Programming Problems

(1) For all $i = 1, \ldots, t-1$, DMi's satisfactory degree is larger than or equal to the minimal satisfactory level $\hat{\delta}_i$ specified by DMi, i.e., $\mu_i(z_i^\ell) \geq \hat{\delta}_i$, $i = 1, \ldots, t-1$.
(2) For all $i = 1, \ldots, t-1$, the ratio Δ_i^ℓ of satisfactory degrees is in the closed interval between its lower and its upper bounds specified by DMi.

Suppose that the DMs from at the $(q+1)^{\text{th}}$ level to at the $(t-1)^{\text{th}}$ level, i.e., DM$(q+1)$, DM$(q+2)$,..., and DM$(t-1)$, satisfy the proposed solution but DMq does not satisfy it. Then DMq, DM$(q+1)$,..., and DM$(t-1)$ need to update their minimal satisfactory levels $\hat{\delta}_i$, $i = q$, $q+1, \ldots, t-1$. Giving the DM at the upper level serious consideration, the DM at the lower level should update his/her the minimal satisfactory level.

Procedure for Updating the Minimal Satisfactory Level $\hat{\delta}$

(1) A DM at a level in which condition (1) is not satisfied decreases the minimal satisfactory level $\hat{\delta}_i$.
(2) If the ratio Δ_i^ℓ exceeds its upper bound, then DMi increases the minimal satisfactory level $\hat{\delta}_i$. Conversely, if the ratio Δ_i^ℓ is below its lower bound, then DMi decreases the minimal satisfactory level $\hat{\delta}_i$.
(3) If the ratio $\hat{\delta}_{i+1}/\hat{\delta}_i$ of the minimal satisfactory levels also is not in the valid interval of Δ_i^ℓ, then the minimal satisfactory level $\hat{\delta}_i$ is updated in a way similar to the updating Step 2.

Let $\hat{\delta}_i'$ denote the updated minimal satisfactory level. DMt solves the following maximization problem with the updated minimal satisfactory level $\hat{\delta}_i'$.

$$\begin{aligned}
\underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\
\text{subject to} \quad & A_1 x_1 + \cdots + A_t x_t \leq \mathbf{b}, \\
& \mu_i(z_i(\mathbf{x})) \geq \lambda, \quad i = 1, \ldots, q-1, \\
& \mu_i(z_i(\mathbf{x})) \geq \hat{\delta}_i', \quad i = q, \ldots, t-1, \\
& \mu_t(z_t(\mathbf{x})) \geq \lambda, \\
& \lambda \in [0, 1], \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned} \tag{20}$$

The above-mentioned algorithm is summarized as follows.

Algorithm of the Interactive Fuzzy Programming for Solving Multilevel Linear Programming Problems

STEP 1. For all $i = 1, \dots, t-1$, DMi elicits the membership function $\mu_i(z_i)$ of the fuzzy goal of DMi, and specifies the minimal satisfactory level $\hat{\delta}_i^1$ and the lower and the upper bounds of the ratio of satisfactory degrees Δ_i^1.

STEP 2. DMt elicits the membership function $\mu_t(z_t)$ of the fuzzy goal of DMt.

STEP 3. DMt solves the auxiliary problem (18). For all $i = t, t-1, \dots, 2$, DMi proposes a solution $(z_1^\ell, \dots, z_t^\ell)$ to problem (18), λ^ℓ, $\mu_i(z_i^\ell)$, and Δ_i^ℓ to DM$(i-1)$ successively.

STEP 4. If the solution proposed to DMs at all the levels except for the t^{th} level satisfies the termination conditions, they conclude the solution as a satisfactory one and the algorithm stops.

STEP 5. If the DMs from at the $(q+1)^{\text{th}}$ level to at the $(t-1)^{\text{th}}$ level, i.e., DM$(q+1)$, DM$(q+2)$, $\dots$, and DM$(t-1)$, satisfy the proposed solution but DMq does not satisfy it, DMi, $i = q, \dots, t-1$ update the minimal satisfactory levels $\hat{\delta}_i$, $i = q, \dots, t-1$ in accordance with the procedure of updating minimal satisfactory level.

STEP 6. DMt solves problem (20). For all $i = t, t-1, \dots, 2$, DMi proposes an optimal solution to problem (20) to DM$(i-1)$ successively. Return to Step 4. If there does not exist any feasible solution to problem (20), go to Step 7.

STEP 7. For all $i = q, \dots, t-1$, DMi update the minimal satisfactory levels $\hat{\delta}_i$ by decreasing the values $\hat{\delta}_i$, and return to Step 6.

4. NUMERICAL EXAMPLES

In this section, we provide illustrative numerical examples for two-level and three-level linear programming problems to demonstrate the feasibility of the proposed method.

EXAMPLE 1. Consider the following two-level linear programming problem:

$$\begin{aligned} \underset{\text{Level 1}}{\text{minimize}} \quad & z_1 = -\mathbf{c}_1\mathbf{x}_1 - \mathbf{c}_2\mathbf{x}_2, \\ \underset{\text{Level 2}}{\text{minimize}} \quad & z_2 = -\mathbf{c}_3\mathbf{x}_1 + \mathbf{c}_2\mathbf{x}_2, \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 \le \mathbf{b}, \\ & x_j \ge 0, \quad j = 1, 2, \dots, 20, \end{aligned} \tag{21}$$

where $\mathbf{x}_1 = (x_1, \dots, x_{10})^{\mathrm{T}}$, $\mathbf{x}_2 = (x_{11}, \dots, x_{20})^{\mathrm{T}}$; each entry of 25×10 coefficient matrices A_1 and A_2 is a random value in the interval $[-50, 50]$; each entry of the right-hand side constant column vector $\mathbf{b}$ is a sum of entries of the corresponding row vector of A_1 and A_2 multiplied by 0.6. Coefficients are shown in Table 1.

Suppose that the initial minimal satisfactory level as $\delta = 1.0$, and the lower and the upper bounds of Δ as $[0.6, 1.0]$. The membership functions (5) of the fuzzy goals are assessed by using values (6) and (7). The individual minima and the corresponding optimal solutions are shown in Table 2.

Then problem (11) for this numerical example can be formulated as

$$\begin{aligned} \underset{\mathbf{x}, \lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & \mathbf{x} \in X, \\ & \frac{(z_1(\mathbf{x}) + 384.325)}{(-783.988 + 384.325)} \ge \lambda, \\ & \frac{(z_2(\mathbf{x}) - 84.398)}{(-127.097 - 84.298)} \ge \lambda, \end{aligned} \tag{22}$$

Table 1. Coefficients for Example 1.

c_1	46	15	15	39	44	12	27	22	42	12	c_2	34	21	19	23	15	42	29	12	10	40	b
c_3	23	22	33	38	29	34	15	1	13	1												
A_1	25	50	29	16	−21	9	9	−47	−18	18	A_2	−1	35	13	−42	−13	21	17	22	18	41	82
	44	16	−27	−20	1	47	−21	31	8	−30		−24	−39	−25	−18	49	−12	−47	45	−43	17	−28
	−20	−15	21	19	13	−35	28	−38	−28	42		−6	−7	−45	45	−46	8	−34	−32	33	33	−38
	6	−34	−25	−23	9	36	−10	−4	5	−41		43	25	47	10	32	49	41	−39	3	23	91
	−38	−21	27	−15	42	2	−17	−7	−37	20		46	31	37	−36	−30	−47	18	26	22	36	35
	−38	−48	−37	12	−9	42	−2	−13	−5	41		20	−40	−25	45	50	−3	39	−24	33	19	34
	42	−21	−13	32	28	13	−35	49	11	−21		−18	−11	−17	−29	49	5	−28	−32	15	27	27
	30	18	22	3	−3	−2	−10	37	32	−15		26	−28	49	3	−41	−14	−15	−31	−17	29	43
	−13	−32	−25	−9	0	−32	−26	−48	−39	−27		8	15	−5	43	43	−10	−15	15	46	4	−64
	1	−24	43	31	−19	−7	−30	−4	−8	16		−31	35	−3	38	−18	−5	−17	14	37	42	54
	31	1	−33	−33	47	5	−37	−15	−8	40		−13	−49	−1	−13	48	44	−45	−6	28	−5	−8
	32	−32	10	40	50	−35	28	−4	33	−24		−46	34	47	39	−19	48	10	1	41	−12	144
	9	13	15	−38	44	5	7	34	29	17		2	−7	34	−17	−34	31	−30	11	45	−43	76
	1	29	29	5	−38	24	−39	−49	−47	8		23	4	−1	42	−25	−15	−22	−30	32	−43	−67
	12	−32	35	−35	29	−46	5	−21	−29	44		−31	−30	−18	34	−9	−4	−31	−43	−49	25	−116
	−1	20	−13	−48	24	24	5	10	47	18		41	−41	−49	−26	−23	36	46	−1	17	−12	44
	−26	22	18	7	−24	40	30	−18	31	−37		45	−15	14	−11	−39	−13	47	36	−9	26	74
	11	−46	8	−16	−42	−23	−17	−20	−24	−36		−16	48	48	14	−42	−25	8	−3	39	−42	−105
	44	−39	−31	−33	−18	−14	27	16	−23	−41		8	−12	50	26	−48	4	41	−33	−44	−49	−101
	−47	−15	14	7	−15	−3	34	−42	47	14		36	49	8	−24	−13	34	47	−49	50	−38	56
	−36	13	−5	42	24	−48	−18	−16	−48	44		−45	−17	−2	45	−33	26	14	−44	−17	−16	−82
	11	−17	−45	−1	2	−21	21	−35	41	16		−38	39	−11	−5	−28	36	−20	−30	−45	−45	−104
	3	−31	39	−46	−2	25	−39	−26	−47	12		−17	−28	−21	−32	−25	34	8	−19	−3	−22	−142
	−41	−29	7	28	−21	14	−2	40	−44	45		−42	38	−49	43	49	25	−4	−21	23	27	51
	−44	22	−24	25	−1	15	26	−5	−28	−8		13	28	1	−7	−34	30	−48	−7	25	−13	−20

Table 2. The solutions preferred by the DMs at both levels.

$z_1^{\min}$	−783.988									
$\mathbf{x}_1$	3.358	1.406	0.071	1.555	0.648	0	3.026	1.463	0.856	2.895
$\mathbf{x}_2$	4.636	2.794	0.196	2.386	2.003	0	0.035	2.309	0	0
$z_2^{\min}$	−127.097									
$\mathbf{x}_1$	0.582	0.881	1.431	2.020	0.720	0	1.653	0.959	0.929	1.488
$\mathbf{x}_2$	0.888	0.533	0.620	0	1.250	0	0.116	1.214	0	0

Table 3. The first iteration.

λ^1	0.704									
$\mathbf{x}_1^1$	1.141	2.715	1.382	1.794	2.334	0	2.145	0.930	0.834	1.442
$\mathbf{x}_2^1$	3.110	1.315	0.336	1.945	2.865	0	0.305	1.006	0	0
z_1^1	−665.638		$\mu_1(z_1^S)$				0.703874			
z_2^1	−64.468		$\mu_2(z_2^S)$				0.703874			
Δ^1	1.000000									

where X denotes the feasible area of problem (21). Data of the first iteration including an optimal solution to problem (22) are shown in Table 3.

Condition (1) of termination of the interactive process is not satisfied because the satisfactory degree $\mu_1^1 = 0.703874$ of DM1 does not exceed his/her minimal satisfactory level $\hat{\delta} = 1.0$. Consequently, suppose that DM1 changes the minimal satisfactory level from $\hat{\delta} = 1.0$ to $\hat{\delta}' = 0.75$. Then a problem corresponding to problem (13) is formulated as

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \mu_2(z_2(\mathbf{x})), \\ \text{subject to} \quad & \mathbf{x} \in X, \\ & \frac{(z_1(\mathbf{x}) + 384.325)}{(-783.988 + 384.325)} \geq 0.75. \end{aligned} \tag{23}$$

Data of the second iteration including an optimal solution to problem (23) are shown in Table 4.

Table 4. A satisfactory solution.

$\mathbf{x}_1^2$	1.471	2.524	1.166	1.784	2.064	0	2.221	0.997	0.877	1.695
$\mathbf{x}_2^2$	3.338	1.555	0.310	2.022	2.712	0	0.254	1.234	0	0
z_1^2	−684.072		$\mu_1(z_1^S)$				0.750000			
z_2^2	−41.473		$\mu_2(z_2^S)$				0.595151			
Δ^2	0.793534									

At the second iteration, the satisfactory degree $\mu_1^2 = 0.75$ of DM1 becomes equal to his/her minimal satisfactory level $\hat{\delta}' = 0.75$ and the ratio $\Delta^2 = 0.793534$ of satisfactory degrees is in the valid interval $[0.6, 1.0]$ of the ratio. Therefore, this solution satisfies the conditions of termination of the interactive process and becomes a satisfactory solution for both DMs.

EXAMPLE 2. As an example for a three-level linear programming problem, consider the following problem:

$$\begin{aligned} \underset{\text{Level 1}}{\text{minimize}} \quad & z_1 = \mathbf{c}_1\mathbf{x}_1 + \mathbf{c}_2\mathbf{x}_2 + \mathbf{c}_3\mathbf{x}_3, \\ \underset{\text{Level 2}}{\text{minimize}} \quad & z_2 = \mathbf{c}_3\mathbf{x}_1 + \mathbf{c}_4\mathbf{x}_2 + \mathbf{c}_6\mathbf{x}_3, \\ \underset{\text{Level 3}}{\text{minimize}} \quad & z_3 = \mathbf{c}_7\mathbf{x}_1 + \mathbf{c}_8\mathbf{x}_2 + \mathbf{c}_9\mathbf{x}_3, \\ \text{subject to} \quad & A_1\mathbf{x}_1 + A_2\mathbf{x}_2 + A_3\mathbf{x}_3 \leq \mathbf{b}, \\ & x_j \geq 0, \ j = 1, 2, \ldots, 15, \end{aligned} \tag{24}$$

Table 5. Coefficients for Example 2.

c_1	−45	−5	−45	−14	−9	c_2	−26	−47	−19	−27	−27	c_3	−33	−7	−28	−21	−12	
c_4	−50	−10	−30	−19	−2	c_5	−30	−30	−9	−10	−38	c_6	−17	−13	−48	−25	−2	b
c_7	−45	−17	−28	−49	−24	c_8	−4	−5	−49	−38	−32	c_9	−2	−5	−33	−4	−9	
A_1	46	−29	−48	31	21	A_2	−47	−37	37	32	19	A_3	21	2	−11	−35	−2	0
	0	39	12	−14	29		−42	−26	42	31	−19		−25	6	4	2	5	26
	38	−27	5	−31	14		−38	−29	−5	−47	49		−45	−45	5	10	−40	−111
	−26	16	44	6	19		17	27	32	17	−6		−27	1	18	−6	15	88
	−48	13	2	−33	19		22	−35	27	−35	−35		−26	−16	37	47	−2	−37
	9	−6	12	−17	−32		−8	24	−24	45	−31		16	−9	−19	17	44	12
	−9	2	−16	8	32		−6	−25	−25	−8	4		23	41	30	36	−11	45
	24	30	42	−26	16		19	−18	−18	9	−34		−46	30	3	−1	−45	−8
	−26	−8	0	41	−42		−19	13	−42	49	−27		4	−2	−12	24	−33	−47
	−29	16	−16	−4	18		45	−8	21	6	47		43	46	26	22	−5	136
	−7	1	−3	38	18		−43	−15	31	−34	23		−35	−34	20	−15	−26	−48
	−12	−4	47	0	−4		−18	−19	28	47	−36		−45	20	40	3	−15	19
	−12	−46	11	−47	−47		19	30	50	12	−24		13	20	−43	−8	20	−31
	5	−2	37	38	0		12	−34	34	28	−40		−18	33	39	14	2	88
	49	41	3	12	−48		15	12	32	31	−28		−25	−23	−6	−25	−15	14
	−17	−6	34	21	11		5	−28	−46	−15	9		12	49	4	−17	−47	−18

where $\mathbf{x}_1 = (x_1, \ldots, x_5)^T$, $\mathbf{x}_2 = (x_6, \ldots, x_{10})^T$, $\mathbf{x}_3 = (x_{11}, \ldots, x_{15})^T$; each entry of five-dimensional row constant vectors c_i, $i = 1, \ldots, 9$ and each entry of 16×5 coefficient matrices A_1, A_2, and A_3 are random values in the interval $[-50, 50]$; each entry of the right-hand side constant column vector $\mathbf{b}$ is a sum of entries of the corresponding row vector of A_1, A_2, and A_3 multiplied by 0.6. Coefficients are shown in Table 5.

Suppose that DM1 and DM2 determine the initial minimal satisfactory levels as $\delta_1 = \delta_2 = 1.0$, and the lower and the upper bounds of Δ_1 and Δ_2 as $[0.6, 1.0]$. The membership functions (5) of the fuzzy goals are assessed by using values (6) and $z_i^m = \max(z_i(\mathbf{x}^{1o}), \ldots, z_i(\mathbf{x}^{i-1,o}), z_i(\mathbf{x}^{i+1,o}), \ldots, z_i(\mathbf{x}^{ko}))$ instead of (7). The individual minima and the corresponding optimal solutions are shown in Table 6.

Table 6. The solutions preferred by the DMs at all the levels.

$z_1^{\min}$	−530.680591				
$\mathbf{x}_1$	2.722648	0	0	2.223594	2.565147
$\mathbf{x}_2$	1.642039	4.417235	0	0	1.051477
$\mathbf{x}_3$	1.289657	0	0	1.556818	0
$z_2^{\min}$	−466.089944				
$\mathbf{x}_1$	2.722648	0	0	2.223594	2.565147
$\mathbf{x}_2$	1.642039	4.417235	0	0	1.051477
$\mathbf{x}_3$	1.289657	0	0	1.556818	0
$z_3^{\min}$	−374.496510				
$\mathbf{x}_1$	2.151865	0	1.058096	2.857073	1.365113
$\mathbf{x}_2$	2.004901	1.582176	0	0	1.287546
$\mathbf{x}_3$	0.854287	0	0	1.507248	1.156342

Then problem (18) for this numerical example can be formulated as

$$\begin{aligned}
&\underset{\mathbf{x},\lambda}{\text{maximize}} \quad \lambda, \\
&\text{subject to} \quad \mathbf{x} \in X, \\
&\qquad \frac{(z_1(\mathbf{x}) + 431.70)}{(-530.68 + 431.70)} \geq \lambda, \\
&\qquad \frac{(z_2(\mathbf{x}) + 407.41)}{(-466.09 + 407.41)} \geq \lambda, \\
&\qquad \frac{(z_3(\mathbf{x}) + 364.14)}{(-374.49 + 364.14)} \geq \lambda,
\end{aligned} \tag{25}$$

where X denotes the feasible area of problem (24). Data of the first iteration including an optimal solution to problem (25) are shown in Table 7.

Table 7. The first iteration of the three-level problem.

λ^1	0.719718				
$\mathbf{x}_1^1$	2.560726	0	0.972062	2.312286	2.110154
$\mathbf{x}_2^1$	1.691225	3.141050	0	0	1.032234
$\mathbf{x}_3^1$	1.629141	0	0	1.284492	0.144631
z_1^1	−512.281943		$\mu_1(z_1^1)$	0.814106	
z_2^1	−449.642029		$\mu_2(z_2^1)$	0.719718	
z_3^1	−371.595715		$\mu_3(z_3^1)$	0.719718	
Δ_1^1	0.884058				
Δ_2^1	1.000000				

Condition (1) of termination of the interactive process is not satisfied because the satisfactory degree $\mu_2^1 = 0.719718$ of DM2 does not exceed his/her minimal satisfactory level $\hat{\delta}_2 = 1.0$. Consequently, suppose that DM2 changes the minimal satisfactory level from $\hat{\delta}_2 = 1.0$ to $\hat{\delta}_2' = 0.75$. Then a problem corresponding to problem (20) is formulated as

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & \mathbf{x} \in X, \\ & \frac{(z_1(\mathbf{x}) + 431.70)}{(-530.68 + 431.70)} \geq \lambda, \\ & \frac{(z_2(\mathbf{x}) + 407.41)}{(-466.09 + 407.41)} \geq 0.75, \\ & \frac{(z_3(\mathbf{x}) + 364.14)}{(-374.49 + 364.14)} \geq \lambda. \end{aligned} \tag{26}$$

Data of the second iteration including an optimal solution to problem (26) are shown in Table 8.

Table 8. The second iteration.

λ^1	0.707925				
$\mathbf{x}_1^2$	2.577929	0	0.968443	2.289364	2.141502
$\mathbf{x}_2^2$	1.678027	3.206640	0	0	1.021492
$\mathbf{x}_3^2$	1.661743	0	0	1.275119	0.102063
z_1^2	−515.672186		$\mu_1(z_1^2)$	0.848360	
z_2^2	−451.419102		$\mu_2(z_2^2)$	0.750000	
z_3^2	−371.473663		$\mu_3(z_3^2)$	0.707925	
Δ_1^2	0.884058				
Δ_2^2	0.943899				

Condition (1) of termination of the interactive process is not satisfied because the satisfactory degree $\mu_1^2 = 0.848360$ of DM1 does not exceed his/her minimal satisfactory level $\hat{\delta}_1 = 1.0$. Consequently, suppose that DM1 changes the minimal satisfactory level from $\hat{\delta}_1 = 1.0$ to $\hat{\delta}_1' = 0.9$. Then a problem corresponding to problem (20) is formulated as

$$\begin{aligned} \underset{\mathbf{x},\lambda}{\text{maximize}} \quad & \lambda, \\ \text{subject to} \quad & \mathbf{x} \in X, \\ & \frac{(z_1(\mathbf{x}) + 431.70)}{(-530.68 + 431.70)} \geq 0.90, \\ & \frac{(z_2(\mathbf{x}) + 407.41)}{(-466.09 + 407.41)} \geq 0.75, \\ & \frac{(z_3(\mathbf{x}) + 364.14)}{(-374.49 + 364.14)} \geq \lambda. \end{aligned} \tag{27}$$

Data of the second iteration including an optimal solution to problem (27) are shown in Table 9.

At the third iteration, the satisfactory degree $\mu_1^3 = 0.9$ of DM1 becomes equal to his/her minimal satisfactory level $\hat{\delta}_1' = 0.9$ and the satisfactory degree $\mu_2^3 = 0.795653$ of DM2 becomes larger than his/her minimal satisfactory level $\hat{\delta}_2' = 0.75$. The ratios $\Delta_1^3 = 0.884058$ and $\Delta_2^3 = 0.867396$ of satisfactory degrees are in the valid interval $[0.6, 1.0]$ of the ratios Δ_1 and Δ_2. Therefore, this solution satisfies the conditions of termination of the interactive process and then becomes a satisfactory solution for all the DMs.

5. CONCLUSIONS

In this paper, we have proposed interactive fuzzy programming for multilevel linear programming problems. In our interactive method, after determining the fuzzy goals of the decision

Table 9. A satisfactory solution to the three-level problem.

λ^1	0.690146				
$\mathbf{x}_1^2$	2.603864	0	0.962985	2.254808	2.188760
$\mathbf{x}_2^2$	1.658131	3.305521	0	0	1.005297
$\mathbf{x}_3^2$	1.710893	0	0	1.260989	0.037889
z_1^2	−520.783182		$\mu_1(z_1^2)$	0.900000	
z_2^2	−454.098147		$\mu_2(z_2^2)$	0.795653	
z_3^2	−371.289662		$\mu_3(z_3^2)$	0.690146	
Δ_1^2	0.884058				
Δ_2^2	0.867396				

makers at all levels, a satisfactory solution is derived efficiently by updating the satisfactory degrees of decision makers at the upper level with considerations of overall satisfactory balance among all levels. Illustrative numerical examples for two-level and three-level linear programming problems have been provided to demonstrate the feasibility of the proposed method.

APPENDIX
THE PROBLEM ARISING WHEN FUZZY GOALS FOR DECISION VARIABLES ARE INTRODUCED

In the methods by Lai [9] and Shih, Lai and Lee [10], fuzzy goals not only for objective functions but also for decision variables of DM1 are introduced. We will show an example producing an undesirable solution in such a case.

Suppose that DM1 defines membership functions of fuzzy goals for decision variables in the following by using a solution x^* yielding the individual minimum $z_1^{\min} = \min_{\mathbf{x}\in X} z_1(\mathbf{x})$ and a solution x_{L-} yielding the individual maximum $z_1^{\max} = \max_{\mathbf{x}\in X} z_1(\mathbf{x})$. For all $i = 1,\dots,10$,

$$\mu_{1i}^c(x_i) = \begin{cases} 0, & x_i > x_i^* + p_i, \\ \dfrac{-x_i + (x_i^* + p_i)}{p_i}, & x_i^* \le x_i \le x^* + p_i, \\ \dfrac{x_i - (x_i^* - p_i)}{p_i}, & x_i^* - p_i \le x_i \le x^*, \\ 0, & x_i < x_i^* - p_i, \end{cases}$$

where $p_i = |x_i^* - x_{iL-}|$.

Aggregating the membership functions of fuzzy goals for the objective function and the decision variables, Lai *et al.* defined a satisfactory degree of DM1 as

$$\mu_1 = \min(\mu_1(z_1(\mathbf{x})), \mu_{11}^c(\mathbf{x}_1), \dots, \mu_{1\,10}^c(\mathbf{x}_1)).$$

A solution to problem (22) substituted the above satisfactory degree μ_1 for that defined as (5) is shown in Table 10.

Table 10. A solution to the problem with fuzzy goals for decision variables.

λ^1	0.000000				
$\mathbf{x}_1^1$	1.148982	1.700706	0.383300	0.754668	0.991874
	0.000000	1.567263	0.951902	0.567960	0.889271
$\mathbf{x}_2^1$	1.287428	0.932108	0.394978	1.130434	1.487149
	0.118403	0.552626	0.264450	0.531488	0.132675
μ_{x_1}	0.342160	0.000000	0.383683	0.485395	0.468897
	0.000000	0.000000	0.327894	0.654584	0.000000
z_1^1	−408.924218		$\mu_1(z_1^1)$	0.061550	
z_2^1	−12.713343		$\mu_2(z_2^1)$	0.459167	
Δ^1	7.460065				

As seen in Table 10, the satisfactory degree of DMs at both levels is $\lambda^1 = 0.0$ because several satisfactory degrees of fuzzy goals for decision variables become zero. DMs cannot help updating their membership functions of fuzzy goals to improve their satisfactory degrees. It seems that the possibility that the overall satisfactory degree becomes zero gets larger if the number of fuzzy goals for decision variables increases.

Furthermore, we can point out another disadvantage from the following example. For the sake of simplicity, suppose that there is only one fuzzy goal for a decision variable. Thus, we have

$$\mu_1 = \min(\mu_1(z_1(\mathbf{x})), \mu_{11}^c(\mathbf{x}_1)).$$

At the ℓ^{th} iteration, suppose that

$$\mu_1 = 0.7, \quad \mu_1(z_1) = 0.88, \quad \mu_{11}^c(\mathbf{x}_1) = 0.8,$$
$$\mu_2 = 0.62, \quad \mu_2(z_2) = 0.62,$$

and this solution does not satisfy the condition of the bounds of Δ^ℓ. At the $(\ell+1)^{\text{th}}$ iteration, suppose that

$$\mu_1 = 0.8, \quad \mu_1(z_1) = 0.82, \quad \mu_{11}^c(\mathbf{x}_1) = 0.8,$$
$$\mu_2 = 0.58, \quad \mu_2(z_2) = 0.58,$$

and the interactive process terminates as this solution satisfies all the conditions.

It is seen that DM1 increases his/her satisfactory degree while DM2 decreases that of him/her because the pair (μ_1, μ_2) changes from $(0.7, 0.62)$ at the ℓ^{th} iteration to $(0.8, 0.58)$ at the $(\ell+1)^{\text{th}}$ iteration.

In contrast, for the pair $(\mu_1(z_1), \mu_2(z_2))$ of satisfactory degrees for the objective functions, both DM1 and DM2 decrease their satisfactory degrees because the pair $(\mu_1(z_1), \mu_2(z_2))$ changes from $(0.88, 0.62)$ at the ℓ^{th} iteration to $(0.82, 0.58)$ at the $(\ell+1)^{\text{th}}$ iteration. It should be noted that when fuzzy goals for decision variables are introduced, the iterative procedure may terminate, having an undesirable (dominated) solutions from the viewpoint of the satisfaction of the fuzzy goals for the objective functions.

REFERENCES

1. W.F. Bialas and M.H. Karwan, Two-level linear programming, *Management Science* **30**, 1004–1020, (1983).
2. M. Simaarn and J.B. Cruz, On the Stackelberg strategy in nonzero-sum games, *Journal of Optimization Theory and Applications* **11**, 533–555, (1973).
3. J.F. Bard and J.E. Falk, An explicit solution to the multi-level programming problem, *Computers and Operations Research* **9**, 77–100, (1983).
4. J.F. Bard, An efficient point algorithm for a linear two-stage optimization problem, *Operations Research* **38**, 556–560, (1983).
5. G. Ünlü, A linear bilevel programming algorithm based on bicriteria programming, *Computers and Operations Research* **14**, 173–179, (1973).
6. D.J. White and G. Anandalingam, A penalty function approach for solving bi-level linear programs, *Journal of Global Optimization* **3**, 397–419, (1973).
7. J.F. Bard, An investigation of the linear three-level programming problem, *IEEE Transactions on Systems, Man, and Cybernetics* **SMC-14**, 711–717, (1983).
8. U.-P. Wen and W.F. Bialas, The hybrid algorithm for solving the three-level linear programming problem, *Computers and Operations Research* **13**, 367–377, (1973).
9. Y.J. Lai, Hierarchical optimization: A satisfactory solution, *Fuzzy Sets and Systems* **77**, 321–335, (1983).
10. H.S. Shih, Y.J. Lai and E.S. Lee, Fuzzy approach for multi-level programming problems, *Computers and Operations Research* **23**, 73–91, (1983).
11. H.-J. Zimmermann, Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems* **1**, 45–55, (1978).
12. U.-P. Wen and S.-T. Hsu, Efficient solutions for the linear bilevel programming problem, *European Journal of Operational Research* **62**, 354–362, (1973).