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A Criterion for the Exponential Stability of Linear Difference Equations

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Abstract—We give an affirmative answer to a question formulated by Aulbach and Van Minh by showing that the linear difference equation

$$x_{n+1} = A_n x_n, \qquad \text{for } n \in \mathbb{N}$$

in a Banach space B is exponentially stable if and only if for every $f = \{f_n\}_{n=1}^{\infty} \in l_p(\mathbb{N}, B)$, where $1 < p < \infty$, the solution of the Cauchy problem

$$x_{n+1} = A_n x_n + f_n, \qquad \text{for } n \in \mathbb{N}, \quad x_1 = 0$$

is bounded on $\mathbb{N}$. © 2004 Elsevier Ltd. All rights reserved.

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Let B be a real or complex Banach space. Consider the linear difference equation

$$x_{n+1} = A_n x_n, \qquad \text{for } n \in \mathbb{N}, \tag{1}$$

where $x_n \in B$ and the operators A_n belong to $\mathcal{L}(B)$, the space of bounded linear operators acting in B. Let $|\cdot|$ denote the norm on B and the induced operator norm on $\mathcal{L}(B)$. Throughout the paper, we shall assume that the coefficients in (1) are uniformly bounded,

$$\sup_{n \in \mathbb{N}} |A_n| < \infty. \tag{2}$$

With equation (1), we can associate the nonhomogeneous equation

$$x_{n+1} = A_n x_n + f_n, \qquad \text{for } n \in \mathbb{N}, \tag{3}$$

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where $f = \{f_n\}_{n=1}^{\infty}$ belongs to one of the sequence spaces

$$l_p = \left\{ f\colon \mathbb{N} \to B \;\middle|\; \sum_{n=1}^{\infty} |f_n|^p < \infty \right\}, \qquad \text{for } 1 \le p < \infty,$$

$$l_\infty = \left\{ f\colon \mathbb{N} \to B \;\middle|\; \sup_{n\in\mathbb{N}} |f_n| < \infty \right\}.$$

(We use the notation $f(n) = f_n$ for $n \in \mathbb{N}$.) The spaces l_p, $1 \le p \le \infty$, equipped with the standard norms

$$\|f\|_p = \left(\sum_{n=1}^{\infty} |f_n|^p \right)^{1/p}, \qquad \text{for } 1 \le p < \infty,$$

$$\|f\|_\infty = \sup_{n\in\mathbb{N}} |f_n|,$$

are Banach spaces.

In [1], Aulbach and Van Minh formulated the following problem (see [1, p. 262]). Given $1 < p < \infty$, is it true that equation (1) is exponentially stable if and only if for every $f \in l_p$, the solution of the Cauchy problem

$$x_{n+1} = A_n x_n + f_n, \qquad \text{for } n \in \mathbb{N}, \quad x_1 = 0 \tag{4}$$

is bounded.

The following theorem gives an affirmative answer to the above question.

THEOREM 1. *Suppose (2) holds and let $1 < p < \infty$. Then, equation (1) is exponentially stable if and only if for every $f \in l_p$, the solution of the Cauchy problem (4) is bounded on $\mathbb{N}$.*

Some remarks are appropriate at this point.

REMARK 1. The conclusion of the theorem is also true for $p = \infty$ (see Theorem 2 below), but it is no longer valid when $p = 1$. As shown in [1, Theorem 6], the boundedness of the solution of the Cauchy problem (4) for every $f \in l_1$ is equivalent to the uniform stability of (1).

REMARK 2. Suppose that (1) is exponentially stable and let $f \in l_p$ for some p, $1 \le p \le \infty$. Then, all solutions of the nonhomogeneous equation (3) are bounded, not just the particular solution of the Cauchy problem (4). This is a simple consequence of the variation-of-constants formula and the inclusion $l_p \subset l_\infty$ for $1 \le p \le \infty$. With a little more effort, using the discrete version of Young's convolution theorem, it can be shown that all solutions of (3) belong even to l_p. Note also that this result is true without assuming the uniform boundedness of the coefficients in (1).

REMARK 3. A similar characterization of the exponential stability for linear ordinary differential equation was given by Kučer (see [2, Theorem 1]). For further related results and discussions in the continuous case, see the monographs by Daleckiĭ and Kreĭn [3, Chapter III] and Coppel [4, Chapter V].

The proof of Theorem 1 will be based on the following characterization of the exponential stability of (1) due to Aulbach and Van Minh [1].

THEOREM 2. *(See [1, Corollary 5].) Suppose (2) holds and let $1 \le p \le \infty$. Then, equation (1) is exponentially stable if and only if for every $f \in l_p$, the solution of the Cauchy problem (4) belongs to l_p.*

PROOF OF THEOREM 1.

NECESSITY. Suppose that (1) is exponentially stable. If $f \in l_p$, then by Theorem 2 the solution $x(f) = \{x_n(f)\}_{n=1}^{\infty}$ of the Cauchy problem (4) belongs to l_p. Since $l_p \subset l_\infty$, $x(f)$ is bounded on $\mathbb{N}$.

SUFFICIENCY. Suppose that for every $f \in l_p$, where $p \in (1,\infty)$ is fixed, the solution $x(f)$ of (4) is bounded on $\mathbb{N}$. We must show that (1) is exponentially stable.

It follows by easy induction on n that the solution $x(f)$ of (4) has the form

$$x_n(f) = \sum_{i=2}^{n} X(n,i) f_{i-1}, \qquad n \in \mathbb{N}, \tag{5}$$

where $X(n,i)$ is defined by the operator product

$$X(n,i) = A_{n-1}A_{n-2}\dots A_i, \qquad \text{for } n > i$$

and

$$X(n,n) = I, \qquad I \text{ being the identity on } B.$$

(As usual, an empty sum is defined to be zero.) It follows from (5) and Hölder's inequality that for every fixed $n \in \mathbb{N}$, the linear operator $x_n(\cdot)\colon l_p \to B$ is bounded. Further, by our assumption,

$$\sup_{n\in\mathbb{N}} |x_n(f)| < \infty, \qquad \text{for every } f \in l_p.$$

By the uniform boundedness principle,

$$K = \sup_{n\in\mathbb{N}} \|x_n(\cdot)\| < \infty,$$

where $\|\cdot\|$ denotes the operator norm. From this,

$$|x_n(f)| \le K\|f\|_p, \qquad \text{for every } n \in \mathbb{N} \text{ and } f \in l_p. \tag{6}$$

We shall show that

$$|X(n,k)| \le C(n-k)^{1/p} \exp\left(-\gamma(n-k)^{1/q}\right), \qquad \text{for } n > k \ge 2, \tag{7}$$

where q is the conjugate exponent of p $(1/p + 1/q = 1)$ and C, $\gamma > 0$ are constants (independent of n and k) which will be specified later.

Let $n > k \ge 2$ be fixed. If $|X(n,k)| = 0$, then (7) trivially holds. So from now on suppose that $|X(n,k)| > 0$. Since

$$|X(n,k)| = |X(n,i)X(i,k)| \le |X(n,i)|\,|X(i,k)|, \qquad \text{for } k < i < n,$$

it follows that

$$|X(i,k)| > 0, \qquad \text{for } k < i < n.$$

Therefore, we can define a sequence $g = \{g_i\}_{i=1}^{\infty} \in l_p$ by

$$g_i = \begin{cases} 0, & \text{for } 1 \le i < k, \\ \dfrac{X(i+1,k)y}{|X(i+1,k)|}, & \text{for } k \le i \le n-1, \\ 0, & \text{for } i \ge n, \end{cases}$$

where y is an arbitrary, but fixed element of B. The definition of g and the representation formula (5) yields for $n \in \mathbb{N}$,

$$x_n(g) = \sum_{i=k+1}^{n} X(n,i)\frac{X(i,k)y}{|X(i,k)|} = \left(\sum_{i=k+1}^{n} \frac{1}{|X(i,k)|}\right) X(n,k)y.$$

Consequently,

$$|X(n,k)y| \sum_{i=k+1}^{n} \frac{1}{|X(i,k)|} = |x_n(g)| \le K\|g\|_p \le K|y|(n-k)^{1/p},$$

the last but one inequality being a consequence of (6). Since $y \in B$ was arbitrary, the last inequality implies

$$|X(n,k)| \sum_{i=k+1}^{n} \frac{1}{|X(i,k)|} \le K(n-k)^{1/p}.$$

As noted before, $|X(j,k)| > 0$ for all j between k and n. Therefore, we can repeat the same procedure with n replaced by j and we conclude that

$$\alpha_j |X(j,k)| \le K(j-k)^{1/p}, \qquad \text{for } k < j \le n, \tag{8}$$

where

$$\alpha_j = \sum_{i=k+1}^{j} \frac{1}{|X(i,k)|}.$$

Without loss of generality, we may (and do) assume that the constant K in (8) is greater than one. From (8), we find for $k < j \le n$,

$$\alpha_j - \alpha_{j-1} = \frac{1}{|X(j,k)|} \ge \frac{\alpha_j}{K(j-k)^{1/p}},$$

$$\alpha_j \ge \left(1 - \frac{1}{K(j-k)^{1/p}}\right)^{-1} \alpha_{j-1}.$$

Hence,

$$\alpha_n \ge \prod_{m=2}^{n-k} \left(1 - \frac{1}{Km^{1/p}}\right)^{-1} \alpha_{k+1}.$$

(An empty product is defined to be one.) Taking into account that $\alpha_{k+1} = |A_k|^{-1}$ and using the last inequality in (8) (with $j = n$), we obtain

$$|X(n,k)| \le L(n-k)^{1/p} \prod_{m=2}^{n-k} \left(1 - \frac{1}{Km^{1/p}}\right), \tag{9}$$

where $L = K \sup_{k \in \mathbb{N}} |A_k|$. The last product can be written as

$$\prod_{m=2}^{n-k} \left(1 - \frac{1}{Km^{1/p}}\right) = \exp\left(\sum_{m=2}^{n-k} \ln\left(1 - \frac{1}{Km^{1/p}}\right)\right) \le \exp\left(-\frac{1}{K} \sum_{m=2}^{n-k} \frac{1}{m^{1/p}}\right),$$

where we have used the inequality $\ln(1-x) < -x$ for $x \in (0,1)$. For the last sum, we have the estimate

$$\sum_{m=2}^{n-k} \frac{1}{m^{1/p}} \ge \int_2^{n-k} \frac{1}{x^{1/p}}\, dx = q\left[(n-k)^{1/q} - 2^{1/q}\right],$$

where q is the conjugate exponent of p. Thus,

$$\prod_{m=2}^{n-k} \left(1 - \frac{1}{Km^{1/p}}\right) \le M \exp\left(-\gamma(n-k)^{1/q}\right),$$

where $\gamma = q/K$ and $M = \exp(q2^{1/q}/K)$. Using the last inequality in (9), we conclude that (7) holds with $C = LM$.

Using inequality (7), we can easily complete the proof. Let $f \in l_\infty$ and consider the solution $x(f)$ of (4). By virtue of (5) and (7), we have for $n \in \mathbb{N}$,

$$\begin{aligned} |x_n(f)| &\le \|f\|_\infty \sum_{k=2}^{n} |X(n,k)| = \|f\|_\infty \left(1 + \sum_{k=2}^{n-1} |X(n,k)|\right) \\ &\le \|f\|_\infty \left(1 + C\sum_{k=2}^{n-1} (n-k)^{1/p} \exp\left(-\gamma(n-k)^{1/q}\right)\right) \\ &\le \|f\|_\infty \left(1 + C\sum_{m=1}^{\infty} m^{1/p} \exp\left(-\gamma m^{1/q}\right)\right), \end{aligned}$$

where the convergence of the last infinite series follows from Cauchy's integral criterion. The last estimate shows that for every $f \in l_\infty$, the solution $x(f)$ of (4) belongs to l_∞. By the application of Theorem 2 ($p = \infty$), we conclude that equation (1) is exponentially stable. ∎

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