ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Advances in Mathematics 194 (2005) 225–245

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Bosonic and fermionic representations of Lie algebra central extensions

Michael Lau*

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

Received 31 July 2003; accepted 16 June 2004
Available online 24 August 2004

Abstract

Given any representation of an arbitrary Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0, we construct representations of a central extension of $\mathfrak{g}$ on bosonic and fermionic Fock space. The method gives an explicit formula for a (sometimes trivial) 2-cocycle in $H^2(\mathfrak{g};\mathbb{K})$. We illustrate these techniques with several concrete examples.

MSC: 17B10; 17B56; 17B65; 17B68

Keywords: Lie algebras; Fock space; Central extensions; Cohomology

1. Introduction

Clifford and Weyl algebras have natural representations on exterior and symmetric algebras, respectively. In the early 1980s, Frenkel, Kac, and Peterson ([7,12]) explicitly constructed the orthogonal affine Lie algebra $\widehat{\mathfrak{so}_N}$ using the quadratic elements of a Clifford algebra C. These elements were viewed as quadratic operators on a certain highest weight C-module, called fermionic Fock space. Feingold and Frenkel [5] later gave an analogous construction of the symplectic affine Lie algebra $\widehat{\mathfrak{sp}_{2N}}$ from quadratic elements of a Weyl algebra A acting on bosonic Fock space, a highest weight A-module. The natural inclusion of $\widehat{\mathfrak{gl}_N}$ into both $\widehat{\mathfrak{so}_{2N}}$ and $\widehat{\mathfrak{sp}_{2N}}$ meant that the affine Lie algebra

* Present address: Department of Mathematics, University of Ottawa, 585 King Edward Ave., Ottawa, Ontario K1N 6N5, Canada.
E-mail address: Michael.Lau@mathstat.uottawa.ca.

0001-8708/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2004.06.005

of type $A^{(1)}_{N-1}$ admits a uniform construction of both fermionic and bosonic modules. Notably, the bosonic construction (which has level -1) was the first known construction of a nonstandard irreducible highest weight representation[1] for an affine Lie algebra. Seligman [18] later modified the Feingold–Frenkel construction to produce a large class of irreducible integrable highest-weight modules for $\widehat{\mathfrak{so}_N}$ and $\widehat{\mathfrak{sp}_{2N}}$. The fermionic construction (for types A and D) is isomorphic to the vertex operator construction (in [8,14] for instance), giving a boson–fermion correspondence. A generalization of this boson–fermion correspondence appears in [15].

In 2002, Gao [10] used techniques similar to those of Feingold and Frenkel [5] to construct bosonic and fermionic representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$, where $\mathbb{C}_q$ is the quantum torus in two variables. This was accomplished by defining an interesting module for $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$, a central extension of $\mathfrak{gl}_N(\mathbb{C}_q)$.

The Feingold–Frenkel construction for $\widehat{\mathfrak{gl}_N(\mathbb{C})}$ and Gao's construction for $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$ are special cases of a more general phenomenon addressed in this paper. Both constructions define a Weyl or Clifford algebra $\mathfrak{A}$ from generators that we view as basis elements of modules for the Lie algebras. These modules, $\mathbb{C}^N \otimes \mathbb{C}[t, t^{-1}]$ and its dual module $(\mathbb{C}^N \otimes \mathbb{C}[t, t^{-1}])^*$, can be thought of as natural modules for the Lie algebras $\mathfrak{gl}_N(\mathbb{C}[t, t^{-1}])$ and $\mathfrak{gl}_N(\mathbb{C}_q)$. Selecting some of these generators to be "positive", Feingold–Frenkel and Gao define an associative subalgebra $\mathfrak{A}^+$ of $\mathfrak{A}$. Their Fock spaces are trivial $\mathfrak{A}^+$-modules induced to all of $\mathfrak{A}$.

Our construction replaces $\mathfrak{gl}_N(\mathbb{C}[t, t^{-1}])$ and $\mathfrak{gl}_N(\mathbb{C}_q)$ with *any* Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0, and replaces $\mathbb{C}^N \otimes \mathbb{C}[t, t^{-1}]$ with an *arbitrary* $\mathfrak{g}$-module W. We use Weyl- (resp., Clifford-) type relations to generate a unital associative algebra $\mathfrak{a}$. This algebra is constructed from a basis of $W \oplus W'$ where W' is the $\mathfrak{g}$-submodule of the dual space W^* generated by the restricted dual $\mathrm{Span}_{\mathbb{K}}\{w^*_\alpha \mid \alpha \in \mathcal{I}\}$, where $\{w_\alpha \mid \alpha \in \mathcal{I}\}$ is a basis for W. The generators for $\mathfrak{a}$ are partitioned into "positive" and "negative" elements, and the division is used to define a vacuum vector and an induced module for $\mathfrak{a}$, called bosonic (resp., fermionic) Fock space V. Some care is needed so that V remains a well-defined module under the action of quadratic operators f_x defined for each element x in $\mathfrak{g}$. We treat the operators f_x as elements of a completion of $\mathfrak{a}$. The fact that the completion is itself an associative algebra simplifies some of the most difficult computations of Feingold–Frenkel and Gao.

The assignment $x \mapsto f_x$ extends to a representation of a certain one-dimensional central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. For Feingold–Frenkel, $\widetilde{\mathfrak{g}}$ is simply the affine Lie algebra $\widehat{\mathfrak{gl}_N(\mathbb{C})}$ and for Gao, $\widetilde{\mathfrak{g}}$ is a homomorphic image of $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$. The representation he constructs for $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$ is the pullback of the Fock representation we construct for $\widetilde{\mathfrak{g}}$.

When this representation is faithful, it gives an embedding of $\widetilde{\mathfrak{g}}$ into a Lie algebra of "infinite matrices", analogous to that found by Kac and Peterson [12] for $\widehat{\mathfrak{gl}_N(\mathbb{C})}$. It also affords a simple formula for the 2-cocycle defining the central extension, and we use it to explicitly compute the 2-cocycle of the (universal) central extension of

[1] The *level* k of a representation of an affine Lie algebra is the constant by which the canonical central element acts. If k is anything other than a nonnegative integer, the representation is said to be *nonstandard*.

the Lie algebra of $N \times N$ matrices over the ring of differential operators of the form $\sum_{n=0}^{\infty} f_n(t)(\frac{d}{dt})^n$ (where $f_n(t) \in \mathbb{C}[t, t^{-1}]$ is 0 for $n \gg 0$).

2. Bosonic and fermionic realizations

Assume $\rho = 1$ or -1. If $\rho = -1$ (resp., $+1$), we call the resulting constructions *bosonic* (resp., *fermionic*). For elements a, b of any associative algebra A, let $\{a, b\}_\rho = ab + \rho ba$. Note that $\{a, b\}_\rho = \rho\{b, a\}_\rho$, and $[ab, c] = a\{b, c\}_\rho - \rho\{a, c\}_\rho b$ for $a, b, c \in A$, where $[a, b]$ is the usual commutator $ab - ba$.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ of characteristic 0, and let W be an arbitrary $\mathfrak{g}$-module with $\mathbb{K}$-basis $\mathfrak{B} = \{w_\alpha \mid \alpha \in \mathcal{I}\}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has a natural action on the dual module $W^* = \mathrm{Hom}_{\mathbb{K}}(W, \mathbb{K})$ coming from the $\mathfrak{g}$-action $(x \cdot \lambda)(w) = -\lambda(x \cdot w)$ for all $x \in \mathfrak{g}$, $\lambda \in W^*$, $w \in W$. Let $W' = \sum_{\alpha \in \mathcal{I}} \mathcal{U}(\mathfrak{g}) \cdot w_\alpha^*$ where the linear functionals $w_\alpha^* : W \to \mathbb{K}$ are defined by $w_\alpha^*(w_\beta) = \delta_{\alpha,\beta}\ \forall\, \alpha, \beta \in \mathcal{I}$. If W is infinite-dimensional, it is possible that $W' \supsetneq \mathrm{Span}_{\mathbb{K}}\{w_\alpha^* \mid \alpha \in \mathcal{I}\}$, so we fix a $\mathbb{K}$-basis $\mathfrak{B}' = \{w_\alpha^*, \lambda_\beta \mid \alpha \in \mathcal{I},\ \beta \in \mathcal{I}'\}$ for W'.

Definition 2.1. The choice of $\mathfrak{B}$, $\mathfrak{B}'$, and a subset $\mathcal{J} \subseteq \mathcal{I}$ is called a *realization* of $(\mathfrak{g}, W)$ if $\mathrm{card}\{\alpha \in \mathcal{J} \mid x \cdot w_\alpha \notin S_{\mathcal{J}}\} < \infty$ for each $x \in \mathfrak{g}$, where $S_{\mathcal{J}} = \mathrm{Span}_{\mathbb{K}}\{w_\beta \mid \beta \in \mathcal{J}\}$.

Realizations always exist—for example, any finite subset $\mathcal{J} \subseteq \mathcal{I}$ will trivially satisfy the "finiteness condition", $\mathrm{card}\{\alpha \in \mathcal{J} \mid x \cdot w_\alpha \notin S_{\mathcal{J}}\} < \infty$. The purpose of a realization is to define an associative algebra $\mathfrak{a}$, with "positive" and "negative" parts, $\mathfrak{a}^+$ and $\mathfrak{a}^-$, for later use in constructing bosonic (resp., fermionic) Fock space.

Given a realization of a Lie algebra $\mathfrak{g}$ and representation W, let $\mathfrak{a} = \mathfrak{a}(\mathfrak{g}, W, \mathcal{I}, \mathcal{I}', \rho)$ be the (unital) associative algebra generated by $\{w_\alpha, w_\alpha^*, \lambda_\beta \mid \alpha \in \mathcal{I}, \beta \in \mathcal{I}'\}$, modulo the relations

$$\text{(R1)} \quad \{v, w\}_\rho = \{\lambda, \eta\}_\rho = 0,$$

$$\text{(R2)} \quad \{\lambda, w\}_\rho = \lambda(w)$$

for all $v, w \in W$, $\lambda, \eta \in W'$. Let $\mathfrak{a}^+ \subseteq \mathfrak{a}$ denote the (unital) subalgebra generated by those elements $w_\alpha \in \mathfrak{B}$, $\lambda \in \mathfrak{B}'$ such that $\alpha \in \mathcal{J}$ and $\lambda(w_\beta) = 0$ for all $\beta \in \mathcal{J}$. Likewise, let $\mathfrak{a}^-$ be the (unital) subalgebra generated by $\{w_\alpha \in \mathfrak{B}, \lambda \in \mathfrak{B}' \mid \alpha \in \mathcal{I} \setminus \mathcal{J}$, and $\lambda(w_\beta) \neq 0$ for some $\beta \in \mathcal{J}\}$. We will sometimes use the *restricted algebra* $\mathfrak{a}_{\mathrm{res}}$, defined as the (unital) subalgebra generated by $\{w_\alpha, w_\alpha^* \mid \alpha \in \mathcal{I}\}$. Analogously, $\mathfrak{a}_{\mathrm{res}}^+$ and $\mathfrak{a}_{\mathrm{res}}^-$ are the (unital) subalgebras generated by $\{w_\alpha \in \mathfrak{B}, w_\beta^* \in \mathfrak{B}' \mid \alpha \in \mathcal{J},\ \beta \in \mathcal{I} \setminus \mathcal{J}\}$ and $\{w_\alpha \in \mathfrak{B}, w_\beta^* \in \mathfrak{B}' \mid \alpha \in \mathcal{I} \setminus \mathcal{J},\ \beta \in \mathcal{J}\}$, respectively. Note that in many interesting cases (see Examples 6.1–6.7 below), $\mathcal{I}' = \emptyset$, so $\mathfrak{a}_{\mathrm{res}} = \mathfrak{a}$ and $\mathfrak{a}_{\mathrm{res}}^{\pm} = \mathfrak{a}^{\pm}$.[2]

[2] An example where $\mathcal{I}' \neq \emptyset$ is where $\mathfrak{g} = \mathrm{Span}_{\mathbb{K}}\{\sum_{n=0}^{\infty} f_n(t)(\frac{d}{dt})^n \mid f_n(t) \in \mathbb{K}[t]\}$, viewed as a Lie algebra of infinite series of differential operators on $W = \mathbb{K}[t]$.

We will use the multiindex notation $w_{\underline{\alpha}} = w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_r}$, $w^*_{\underline{\alpha}} = w^*_{\alpha_1} w^*_{\alpha_2} \cdots w^*_{\alpha_r}$, and $\lambda_{\underline{\beta}} = \lambda_{\beta_1} \lambda_{\beta_2} \cdots \lambda_{\beta_s}$ where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_s)$. Fix a total ordering $\preceq$ on both $\mathcal{I}$ and $\mathcal{I}'$. We say that $w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}} \neq 0$ is a *standard monomial* if $\alpha_i \preceq \alpha_j$, $\beta_i \preceq \beta_j$, and $\gamma_i \preceq \gamma_j$ whenever $i < j$. By relations (R1) and (R2), the standard monomials span $\mathfrak{a}$. The *length* $\ell(\underline{\alpha})$ of a multiindex $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ is r, the number of its entries. If $\underline{\alpha} = \emptyset$, we say that $\ell(\underline{\alpha}) = 0$, and we let $w_{\underline{\alpha}} = \lambda_{\underline{\alpha}} = w^*_{\underline{\alpha}} = 1$.

Let $\widehat{\mathfrak{a}}$ be the $\mathbb{K}$-vector space of linear combinations of (possibly infinitely many) distinct standard monomials $w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}}$ of $\mathfrak{a}$. That is, the elements of $\widehat{\mathfrak{a}}$ are those that can be expressed as (possibly infinite) sums $\sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}} c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}}$ where $c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} \in \mathbb{K}$, and the sum runs over all finite multiindices $\underline{\alpha}, \underline{\gamma}$ with entries in $\mathcal{I}$ and $\underline{\beta}$ with entries in $\mathcal{I}'$ such that $w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}}$ is a standard monomial. Elements written in this way (as linear combinations of distinct standard monomials) are said to be in a *standard form*. For the remainder of the paper, we will restrict our attention to the subspace $\widehat{\mathfrak{a}}_0$ consisting of those elements $\sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}} c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}} \in \widehat{\mathfrak{a}}$ with the following property:

(P1) For each $\underline{\gamma}$, there are only finitely many pairs $(\underline{\alpha}, \underline{\beta})$ such that $c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} \neq 0$.

Lemma 2.2. *Term-by-term multiplication gives* $\widehat{\mathfrak{a}}_0$ *the structure of an associative algebra.*

Proof. It is enough to show that this multiplication is well-defined and the resulting products are in $\widehat{\mathfrak{a}}_0$. Associativity then follows immediately from the associativity of $\mathfrak{a}$. Fix a multiindex $\underline{\zeta} = (\zeta_1, \ldots, \zeta_t)$ of elements from $\mathcal{I}$. Let

$$T = \sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}} c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}}$$

and

$$T' = \sum_{\underline{\alpha'},\underline{\beta'},\underline{\gamma'}} d_{\underline{\alpha'}\underline{\beta'}\underline{\gamma'}} w_{\underline{\alpha'}} \lambda_{\underline{\beta'}} w^*_{\underline{\gamma'}}$$

be elements of $\widehat{\mathfrak{a}}_0$ written in a standard form.

We consider the products $P = P_{\underline{\alpha}\underline{\beta}\underline{\gamma}\underline{\alpha'}\underline{\beta'}\underline{\gamma'}} = c_{\underline{\alpha}\underline{\beta}\underline{\gamma}} w_{\underline{\alpha}} \lambda_{\underline{\beta}} w^*_{\underline{\gamma}} d_{\underline{\alpha'}\underline{\beta'}\underline{\gamma'}} w_{\underline{\alpha'}} \lambda_{\underline{\beta'}} w^*_{\underline{\gamma'}}$ which, when written in a standard form, contain a nonzero multiple of a standard monomial of the form $w_{\underline{\eta}} \lambda_{\underline{\mu}} w^*_{\underline{\zeta}}$. Note that $w^*_{\gamma_i} \in \{w^*_{\zeta_j} \mid 1 \leq j \leq t\}$ for all $w^*_{\gamma_i}$ occurring in the expression for P. Thus, there are only finitely many possible $w^*_{\underline{\gamma'}}$ that may occur. For each such $\underline{\gamma'}$, there are only finitely many $\underline{\alpha'} = (\alpha'_1, \ldots, \alpha'_r)$, $\underline{\beta'} = (\beta'_1, \ldots, \beta'_s)$ for which $d_{\underline{\alpha'}\underline{\beta'}\underline{\gamma'}} \neq 0$, since $T' \in \widehat{\mathfrak{a}}_0$. For each of these (finitely many) possible triples $(\underline{\alpha'}, \underline{\beta'}, \underline{\gamma'})$ occurring in the expression for P, we see that every $w^*_{\gamma_i} \in \{w^*_{\alpha'_j}, w^*_{\zeta_k} \mid 1 \leq j \leq r,\ 1 \leq k \leq t\}$. (Any $w^*_{\gamma_i} \notin \{w^*_{\alpha'_j} \mid 1 \leq j \leq r\}$ would commute with $w_{\underline{\alpha'}} \lambda_{\underline{\beta'}}$ and thus contribute a $w^*_{\gamma_i}$ term

to the expression $w^*_{\underline{\zeta}}$, which is impossible unless $w^*_{\gamma_i} \in \{w^*_{\zeta_k} \mid 1 \le k \le t\}$.) For each of these finitely many possible $\underline{\gamma}$, there are only finitely many $(\underline{\alpha}, \underline{\beta})$ such that $c_{\underline{\alpha\beta\gamma}} \neq 0$, so there are only finitely many $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\alpha'}, \underline{\beta'}, \underline{\gamma'})$ such that a standard form expression for P contains a term of the form $\overline{k w_{\underline{\eta}} \lambda_{\underline{\mu}} w^*_{\underline{\zeta}}}$, where k is a nonzero scalar. That is, the product TT' is well-defined and in $\widehat{\mathfrak{a}}_0$. □

For each $x \in \mathfrak{g}$ and $\alpha \in \mathcal{I}$, write $x \cdot w_\alpha = \sum_{\gamma \in \mathcal{I}} x^\alpha_\gamma w_\gamma$ with $x^\alpha_\gamma \in \mathbb{K}$. Note that $x^\alpha_\gamma = 0$ for all but finitely many γ. Define the normal ordering $: \sum c_{\alpha\beta} w_\alpha w^*_\beta := \sum c_{\alpha\beta} : w_\alpha w^*_\beta :$ where $c_{\alpha\beta} \in \mathbb{K}$ and

$$: w_\alpha w^*_\beta := \begin{cases} -\rho w^*_\beta w_\alpha & \text{if } \alpha = \beta \in \mathcal{J}, \\ w_\alpha w^*_\beta & \text{otherwise.} \end{cases} \tag{2.3}$$

Let

$$f_x = \sum_{\alpha \in \mathcal{I}} : (x.w_\alpha) w^*_\alpha : . \tag{2.4}$$

Note that $f_x = \sum_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma : w_\gamma w^*_\alpha :$ is a well-defined member of $\widehat{\mathfrak{a}}_0$ for each $x \in \mathfrak{g}$. Thus, we may conduct the multiplications of the following two lemmas within $\widehat{\mathfrak{a}}_0$.

Lemma 2.5. *For every $x \in \mathfrak{g}$, $\eta \in \mathcal{I}$, and $\lambda \in W'$,*

(i) $[f_x, w_\eta] = \sum_{\gamma \in \mathcal{I}} x^\eta_\gamma w_\gamma = x \cdot w_\eta$,
(ii) $[f_x, \lambda] = -\sum_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma \lambda(w_\gamma) w^*_\alpha = x \cdot \lambda$.

Proof. If $\gamma = \alpha \in \mathcal{J}$, then $: w_\gamma w^*_\alpha : = -\rho w^*_\alpha w_\gamma = w_\gamma w^*_\alpha + 1$. Otherwise, $: w_\gamma w^*_\alpha : = w_\gamma w^*_\alpha$. Therefore $\mathrm{ad} : w_\gamma w^*_\alpha : = \mathrm{ad}\ w_\gamma w^*_\alpha$, so

$$\begin{aligned} [f_x, w_\eta] &= \sum\nolimits_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma [: w_\gamma w^*_\alpha :, w_\eta] \\ &= \sum\nolimits_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma [w_\gamma w^*_\alpha, w_\eta] \\ &= \sum\nolimits_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma w_\gamma \{w^*_\alpha, w_\eta\}_\rho \\ &= \sum\nolimits_{\gamma \in \mathcal{I}} x^\eta_\gamma w_\gamma = x \cdot w_\eta. \end{aligned}$$

Similarly,

$$\begin{aligned} [f_x, \lambda] &= -\rho \sum\nolimits_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma \{w_\gamma, \lambda\}_\rho w^*_\alpha \\ &= -\sum\nolimits_{\alpha,\gamma \in \mathcal{I}} x^\alpha_\gamma \lambda(w_\gamma) w^*_\alpha \end{aligned}$$

and

$$\begin{aligned}(x\cdot\lambda)(w_\alpha) &= -\lambda(x\cdot w_\alpha)\\ &= -\sum\nolimits_{\gamma\in\mathcal{I}} x_\gamma^\alpha \lambda(w_\gamma).\end{aligned}$$

Thus, $x\cdot\lambda = -\sum_{\alpha,\gamma\in\mathcal{I}} x_\gamma^\alpha \lambda(w_\gamma) w_\alpha^* = [f_x,\lambda]$. □

Lemma 2.6. $[f_x, f_y] = f_{[x,y]} + \Omega_{x,y}^\rho$ *for some* $\Omega_{x,y}^\rho \in \widehat{\mathfrak{a}}_0$ *such that* $[\Omega_{x,y}^\rho, \mathfrak{a}] = 0$.

Proof. By the Jacobi identity (on $\widehat{\mathfrak{a}}_0$, viewed as a Lie algebra), if $u \in W$ or W', then $[[f_x, f_y], u] = [f_x, [f_y, u]] - [f_y, [f_x, u]] = x\cdot(y\cdot u) - y\cdot(x\cdot u) = [x, y]\cdot u = [f_{[x,y]}, u]$. Thus $[f_x, f_y] - f_{[x,y]}$ commutes with the generators of $\mathfrak{a}$, and hence centralizes $\mathfrak{a}$. □

3. Fock space

Define an augmentation map $\varepsilon: \mathfrak{a}^+ \to \mathbb{K}$ by the rule that

$$\varepsilon(w_{\underline{\alpha}}\lambda_{\underline{\beta}} w_{\underline{\gamma}}^*) = \begin{cases} 1 & \text{if } \underline{\alpha} = \underline{\beta} = \underline{\gamma} = \emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathbb{K}v_0$ be the one-dimensional left $\mathfrak{a}^+$-module given by

$$\mu v_0 = \varepsilon(\mu) v_0 \text{ for } \mu \in \mathfrak{a}^+.$$

Inducing to $\mathfrak{a}$ gives the *Fock space* $V = V(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}', \rho) = \mathfrak{a} \otimes_{\mathfrak{a}^+} \mathbb{K}v_0$.

Analogously, we define the left $\mathfrak{a}_{\text{res}}$-module $V_{\text{res}} = V(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \rho) = \mathfrak{a}_{\text{res}} \otimes_{\mathfrak{a}_{\text{res}}^+} \mathbb{K}v_0$, where $\mathfrak{a}_{\text{res}}^+$ acts on $\mathbb{K}v_0$ by restriction of the $\mathfrak{a}^+$-action. We write av_0 to denote $a \otimes v_0$ in either of V or V_{res}. Which module av_0 inhabits should be clear from context.

By the relations of $\mathfrak{a}$, each $v \in V$ (resp., V_{res}) can be written in the form av_0 for some $a \in \mathfrak{a}^-$ (resp., $\mathfrak{a}_{\text{res}}^-$). From the relations of $\mathfrak{a}$, we have the following useful formulas:

Lemma 3.1. *Let* $\alpha, \gamma \in \mathcal{I}$. *Then*

$$\text{(i) } w_\alpha w_\gamma = \begin{cases} w_\gamma w_\alpha & \text{if } \rho = -1,\\ -w_\gamma w_\alpha & \text{if } \rho = 1;\end{cases}$$

$$\text{(ii) } w_\alpha w_\gamma^* = \begin{cases} w_\gamma^* w_\alpha - \delta_{\alpha,\gamma} 1 & \text{if } \rho = -1,\\ -w_\gamma^* w_\alpha + \delta_{\alpha,\gamma} 1 & \text{if } \rho = 1;\end{cases}$$

$$\text{(iii)}\ \ w_\alpha^* w_\gamma = \begin{cases} w_\gamma w_\alpha^* + \delta_{\alpha,\gamma} 1 & \text{if } \rho = -1, \\ -w_\gamma w_\alpha^* + \delta_{\alpha,\gamma} 1 & \text{if } \rho = 1; \end{cases}$$

$$\text{(iv)}\ \ w_\alpha^* w_\gamma^* = \begin{cases} w_\gamma^* w_\alpha^* & \text{if } \rho = -1, \\ -w_\gamma^* w_\alpha^* & \text{if } \rho = 1. \end{cases} \qquad \square$$

Iterating Lemma 3.1 gives

Lemma 3.2. *Suppose $\alpha \in \mathcal{I}$, and let*

$$0 \neq w_{\underline{\gamma}} w_{\underline{\mu}}^* = w_{\gamma_1}^{m_1} w_{\gamma_2}^{m_2} \cdots w_{\gamma_r}^{m_r} w_{\mu_1}^{*n_1} w_{\mu_2}^{*n_2} \cdots w_{\mu_s}^{*n_s} \in \mathfrak{a}_{\mathrm{res}},$$

where each m_i, n_j is a positive integer,[3] *the γ_i are all pairwise distinct, and the μ_j are all pairwise distinct.*

Then

$$\text{(i)}\ \ w_\alpha w_{\underline{\gamma}} w_{\underline{\mu}}^* = (-\rho)^{\ell(\underline{\gamma})+\ell(\underline{\mu})} w_{\underline{\gamma}} w_{\underline{\mu}}^* w_\alpha - w_{\underline{\gamma}} \sum_{j=1}^{s} (-\rho)^{\ell(\underline{\gamma})+j+1} n_j \delta_{\alpha,\mu_j} w_{\mu_1}^{*n_1} \cdots w_{\mu_{j-1}}^{*n_{j-1}} w_{\mu_j}^{*n_j-1} w_{\mu_{j+1}}^{*n_{j+1}} \cdots w_{\mu_s}^{*n_s},$$

$$\text{(ii)}\ \ w_\alpha^* w_{\underline{\gamma}} w_{\underline{\mu}}^* = (-\rho)^{\ell(\underline{\gamma})+\ell(\underline{\mu})} w_{\underline{\gamma}} w_{\underline{\mu}}^* w_\alpha^* + \sum_{i=1}^{r} (-\rho)^i m_i \delta_{\alpha,\gamma_i} w_{\gamma_1}^{m_1} \cdots w_{\gamma_{i-1}}^{m_{i-1}} w_{\gamma_i}^{m_i-1} w_{\gamma_{i+1}}^{m_{i+1}} \cdots w_{\gamma_r}^{m_r} w_{\underline{\mu}}^*. \qquad \square$$

Thus if $w_{\underline{\gamma}} w_{\underline{\mu}}^* \in \mathfrak{a}^-$, the action of w_α (resp., w_β^*)$\in \mathfrak{a}^+$ on $w_{\underline{\gamma}} w_{\underline{\mu}}^* v_0$ is, up to a factor of ± 1, $\frac{\partial}{\partial w_\alpha^*}$ (resp., $\frac{\partial}{\partial w_\beta}$):

Lemma 3.3. *Let $w_\alpha, w_\beta^* \in \mathfrak{a}^+$, and suppose $w_{\underline{\gamma}} w_{\underline{\mu}}^*$ is as in Lemma* 3.2. *Then*

$$\text{(i)}\ \ w_\alpha w_{\underline{\gamma}} w_{\underline{\mu}}^* v_0 = -\sum_{j=1}^{s} (-\rho)^{\ell(\underline{\gamma})+j+1} n_j \delta_{\alpha,\mu_j} w_{\underline{\gamma}} w_{\mu_1}^{*n_1} \cdots w_{\mu_{j-1}}^{*n_{j-1}} w_{\mu_j}^{*n_j-1} w_{\mu_{j+1}}^{*n_{j+1}} \cdots w_{\mu_s}^{*n_s} v_0,$$

$$\text{(ii)}\ \ w_\alpha^* w_{\underline{\gamma}} w_{\underline{\mu}}^* v_0 = \sum_{i=1}^{r} (-\rho)^i m_i \delta_{\alpha,\gamma_i} w_{\gamma_1}^{m_1} \cdots w_{\gamma_{i-1}}^{m_{i-1}} w_{\gamma_i}^{m_i-1} w_{\gamma_{i+1}}^{m_{i+1}} \cdots w_{\gamma_r}^{m_r} w_{\underline{\mu}}^* v_0. \qquad \square$$

[3] Note that if $\rho = 1$, then each m_i and n_j is equal to 1.

Corollary 3.4. *Suppose* $\ell(\underline{\alpha}) + \ell(\underline{\beta}) \geq \ell(\underline{\gamma}) + \ell(\underline{\mu})$, $w_{\underline{\alpha}} w_{\underline{\beta}}^* \in \mathfrak{a}^*$, *and* $w_{\underline{\gamma}} w_{\underline{\mu}}^*$ *is as in Lemma* 3.2. *Then*

$$w_{\underline{\alpha}} w_{\underline{\beta}}^* w_{\underline{\gamma}} w_{\underline{\mu}}^* v_0 = k(\underline{\gamma}, \underline{\mu}) \delta_{\underline{\alpha},\underline{\mu}} \delta_{\underline{\beta},\underline{\gamma}} m_1! m_2! \cdots m_r! n_1! n_2! \cdots n_s! v_0,$$

where $k(\underline{\gamma}, \underline{\mu}) = \pm 1$. □

Proposition 3.5. V_{res} *is a simple* $\mathfrak{a}_{\mathrm{res}}$*-module with* $\mathbb{K}$*-basis*

$$\mathcal{B} = \{w_{\underline{\alpha}} w_{\underline{\beta}}^* v_0 \mid w_{\underline{\alpha}} w_{\underline{\beta}}^* \in \mathfrak{a}^- \text{ is a standard monomial}\}.$$

Proof. By the observations before Lemma 3.1, the elements of $\mathcal{B}$ span V_{res}. Let $v \in V_{\mathrm{res}}$. Write $v = \sum_{\underline{\alpha},\underline{\beta}} c_{\underline{\alpha}\underline{\beta}} w_{\underline{\alpha}} w_{\underline{\beta}}^* v_0 = 0$ where each $w_{\underline{\alpha}} w_{\underline{\beta}}^*$ is a standard monomial in $\mathfrak{a}^-$. Choose multiindices $\underline{\alpha}'$ and $\underline{\beta}'$ so that $w_{\underline{\alpha}'} w_{\underline{\beta}'} \in \mathfrak{a}^-$ and $\ell(\underline{\alpha}') + \ell(\underline{\beta}') = \max\{\ell(\underline{\alpha}) + \ell(\underline{\beta}) \mid c_{\underline{\alpha}\underline{\beta}} \neq 0\}$. Then by Corollary 3.4,

$$w_{\underline{\beta}'} w_{\underline{\alpha}'}^* v = k c_{\underline{\alpha}'\underline{\beta}'} v_0 \tag{3.6}$$

for some $k \in \mathbb{K} \setminus 0$. If $v \neq 0$, then $c_{\underline{\alpha}'\underline{\beta}'}$ can be taken to be nonzero. Hence every nonzero submodule V' contains v_0. But since v_0 generates V_{res}, V' must be V_{res}, so V_{res} is simple.

If $v = 0$, then (3.6) reads $0 = w_{\underline{\beta}'} w_{\underline{\alpha}'}^* v = k c_{\underline{\alpha}'\underline{\beta}'} v_0$. Hence $c_{\underline{\alpha}'\underline{\beta}'} = 0$, and by the maximality of $\ell(\underline{\alpha}') + \ell(\underline{\beta}')$, we see that every $c_{\underline{\alpha}\underline{\beta}}$ is zero. Therefore the elements of $\mathcal{B}$ are linearly independent and form a $\mathbb{K}$-basis for V_{res}. □

Proposition 3.7. *Let* $v \in V_{\mathrm{res}}$. *Then* $v \in \mathbb{K} v_0$ *if and only if*

$$((1 - \varepsilon)(\mathfrak{a}_{\mathrm{res}}^+))v = 0.$$

Proof. If $v \in \mathbb{K} v_0$, then $((1 - \varepsilon)(\mathfrak{a}_{\mathrm{res}}^+))v = 0$, by the definition of V_{res}. Conversely, assume that $((1-\varepsilon)(\mathfrak{a}_{\mathrm{res}}^+))v = 0$. Write $v = \sum_{\underline{\alpha},\underline{\beta}} c_{\underline{\alpha}\underline{\beta}} w_{\underline{\alpha}} w_{\underline{\beta}}^* v_0 = 0$ where each $c_{\underline{\alpha}\underline{\beta}} \in \mathbb{K}$, and $w_{\underline{\alpha}} w_{\underline{\beta}}^*$ is an element of $\mathfrak{a}_{\mathrm{res}}^-$ in standard form. Choose multiindices $\underline{\alpha}'$, $\underline{\beta}'$ such that $\ell(\underline{\alpha}') + \ell(\underline{\beta}') = \max\{\ell(\underline{\alpha}) + \ell(\underline{\beta}) \mid c_{\underline{\alpha}\underline{\beta}} \neq 0\}$ and $c_{\underline{\alpha}'\underline{\beta}'} \neq 0$. By Corollary 3.4, $w_{\underline{\beta}'} w_{\underline{\alpha}'}^* v = k c_{\underline{\alpha}'\underline{\beta}'} v_0$ for some $k \in \mathbb{K} \setminus 0$. But $w_{\underline{\alpha}'} w_{\underline{\beta}'}^* \in \mathfrak{a}_{\mathrm{res}}^-$, so by the definition of $\mathfrak{a}_{\mathrm{res}}^+$, $w_{\underline{\beta}'} w_{\underline{\alpha}'}^* \in \mathfrak{a}_{\mathrm{res}}^+$. If $\underline{\alpha}'$ and $\underline{\beta}'$ are not both equal to $\emptyset$, then $w_{\underline{\beta}'} w_{\underline{\alpha}'}^* v = ((1 - \varepsilon)(w_{\underline{\beta}'} w_{\underline{\alpha}'}^*))v = 0$, a contradiction. Hence $\underline{\alpha}' = \underline{\beta}' = \emptyset$, so we are done by the maximality of $\ell(\underline{\alpha}') + \ell(\underline{\beta}')$. □

Modules like V play an important role in statistical mechanics, where they represent the "space of states" for a given system. The cyclic vector v_0 is viewed as a vacuum, and the element

$$w_{\alpha_1}^{m_1} w_{\alpha_2}^{m_2} \cdots w_{\alpha_r}^{m_r} \lambda_{\beta_1}^{n_1} \cdots \lambda_{\beta_s}^{n_s} w_{\gamma_1}^{*q_1} \cdots w_{\gamma_t}^{*q_t} v_0$$

(with every m_i, n_j, q_k a nonnegative integer, and $w_{\alpha_i}, \lambda_{\beta_j}, w_{\gamma_k}^* \in \mathfrak{a}^-$) corresponds to a state with m_i particles in state w_{α_i}, n_j particles in state λ_{β_j}, and q_k particles in state $w_{\gamma_k}^*$. When $\rho = -1$ (resp., $+1$), V is called *bosonic* (resp., *fermionic*) *Fock space*, since the "particles" in V satisfy Bose–Einstein (resp., Fermi–Dirac) occupancy statistics.

Due to the normal ordering and the "finiteness condition" in our definition of a realization, the elements f_x defined in Section 2 have a well-defined left-multiplication action on V. This follows easily from Lemma 2.5 and the fact that all but finitely many monomials in any f_x act as 0 on the vacuum vector v_0. We can therefore interpret the elements f_x as operators on the Fock space V.

Moreover, for any $\alpha, \beta, \gamma, \eta \in \mathcal{I}$, the bracket $[: w_\alpha w_\beta^* :, : w_\gamma w_\eta^* :]$ is an element of $\mathrm{Span}_{\mathbb{K}}\{: w_\mu w_\nu^* : \ \mid \mu, \nu \in \mathcal{I}\} \oplus \mathbb{K}1$, so $[f_x, f_y]$ may be written in the form $\sum_{\mu,\nu\in\mathcal{I}} c_{\mu\nu} : w_\mu w_\nu^* :$ for some $c_{\mu\nu} \in \mathbb{K}$. Since f_x and f_y are given by a realization $(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}')$, the "finiteness condition" ensures that there are only finitely many $(\mu, \nu) \in (\mathcal{I} \setminus \mathcal{J}) \times \mathcal{J}$ such that $c_{\mu\nu} \neq 0$. Thus $[f_x, f_y]v_0 \in V_{\mathrm{res}}$. Now by Lemma 2.6, $\Omega_{x,y}^\rho v_0 \in V_{\mathrm{res}}$. But since $\Omega_{x,y}^\rho$ commutes with the elements of $\mathfrak{a}$, we see that

$$((1-\varepsilon)(\mathfrak{a}_{\mathrm{res}}^+))\Omega_{x,y}^\rho v_0 = \Omega_{x,y}^\rho((1-\varepsilon)(\mathfrak{a}_{\mathrm{res}}^+))v_0 = 0.$$

Hence by Proposition 3.7, $\Omega_{x,y}^\rho v_0 \in \mathbb{K}v_0$. Writing $\Omega_{x,y}^\rho v_0 = c_{x,y}^\rho v_0$ with $c_{x,y}^\rho \in \mathbb{K}$ gives

Corollary 3.8. $\mathrm{Span}_{\mathbb{K}}\{f_x \mid x \in \mathfrak{g}\} \oplus \mathbb{K}e$ *is a Lie algebra with bracket defined by* $[f_x, f_y] = f_{[x,y]} + c_{x,y}^\rho e$ *and* $[e, f_x] = 0$ *for all* $x, y \in \mathfrak{g}$.

4. Central extensions

A *central extension* of $\mathfrak{g}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ and an epimorphism $\phi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ with $\ker\phi$ contained in the center $Z(\widetilde{\mathfrak{g}})$ of $\widetilde{\mathfrak{g}}$. Given two central extensions $(\widehat{\mathfrak{g}}, \pi)$ and $(\widetilde{\mathfrak{g}}, \phi)$ of $\mathfrak{g}$, a *morphism* (from $(\widehat{\mathfrak{g}}, \pi)$ to $(\widetilde{\mathfrak{g}}, \phi)$) in the category of central extensions is a Lie algebra homomorphism $\mu : \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ such that $\phi\mu = \pi$. We say that $(\widehat{\mathfrak{g}}, \pi)$ and $(\widetilde{\mathfrak{g}}, \phi)$ are *isomorphic* if the morphism μ is a bijection. The central extension $(\widehat{\mathfrak{g}}, \pi)$ is *universal* if there is a unique morphism from it to every other central extension of $\mathfrak{g}$.

Isomorphism classes of one-dimensional central extensions are in bijective correspondence with cohomology classes in $H^2(\mathfrak{g}; \mathbb{K})$. In particular, each class $[c] \in H^2(\mathfrak{g}; \mathbb{K})$ determines a central extension $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ with the bracket $[\cdot, \cdot]^\sim$ in $\widetilde{\mathfrak{g}}$ given by

$[x, y]\tilde{} = [x, y] + c(x, y)\mathbf{c}$ for $x, y \in \mathfrak{g}$, where $\mathbf{c}$ is central and c is a representative of the class $[c]$. Conversely, if $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is $\mathbb{K}$-bilinear and $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ is a Lie algebra under the bracket $[\cdot, \cdot]\tilde{}$ defined above, then c is a representative of a cohomology class $[c]$ in $H^2(\mathfrak{g}; \mathbb{K})$. Direct computation from the chain complex defining $H^2(\mathfrak{g}; \mathbb{K})$ shows that $[c] \in H^2(\mathfrak{g}; \mathbb{K})$ if and only if c satisfies the *2-cocycle identities*

(i) $c(x, y) = -c(y, x)$ and
(ii) $c(x, [y, z]) + c(y, [z, x]) + c(z, [x, y]) = 0$.

Theorem 4.1. *For every $x, y \in \mathfrak{g}$, let $c(x, y) = \rho c^{\rho}_{x,y}$. Then $[c] \in H^2(\mathfrak{g}; \mathbb{K})$.*

Proof. Since $x \mapsto f_x$ is a linear map, the space $S = \mathrm{Span}_{\mathbb{K}}\{f_x \mid x \in \mathfrak{g}\}$ is a Lie algebra under the bracket $[f_x, f_y]_S = f_{[x,y]}$. Then by Corollary 3.8, the map $(f_x, f_y) \mapsto c^{\rho}_{x,y}$ is a 2-cocycle for S. Finally, $[c] \in H^2(\mathfrak{g}; \mathbb{K})$ since $x \mapsto f_x$ is a homomorphism from $\mathfrak{g}$ to S, and $c(x, y)$ is a constant multiple of $c^{\rho}_{x,y}$. □

Corollary 4.2. *Let $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ with $[x, y]\tilde{} = [x, y] + c(x, y)\mathbf{c}$ where $\mathbf{c}$ is central, and let $\pi : \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ be the canonical projection. Then $(\widetilde{\mathfrak{g}}, \pi)$ is a central extension of $\mathfrak{g}$.* □

Theorem 4.3. *The action $\mathbf{c}.v = \rho v$, $x.v = f_x v$ for $x \in \mathfrak{g}$, $v \in V$ gives a representation of $\widetilde{\mathfrak{g}}$ on V.*

Proof. The map $x \mapsto f_x$, $\mathbf{c} \mapsto \rho 1$ extends to a linear transformation on $\widetilde{\mathfrak{g}}$, so it suffices to check that

$$\begin{aligned}
[x, y]\tilde{} \cdot v &= ([x, y] + c(x, y)\mathbf{c}) \cdot v \\
&= f_{[x,y]}v + \rho c(x, y)v \\
&= ([f_x, f_y] - \Omega^{\rho}_{x,y})v + \rho c(x, y)v \\
&= [f_x, f_y]v \\
&= f_x f_y v - f_y f_x v \\
&= x \cdot (y \cdot v) - y \cdot (x \cdot v). \qquad \square
\end{aligned}$$

The relation $c^{\rho}_{x,y}v_0 = [f_x, f_y]v_0 - f_{[x,y]}v_0$ (Lemma 2.6 and the discussion after Proposition 3.7) gives a way to calculate $c(x, y)$ explicitly. In the notation of Section 2, $x \cdot w_\alpha = \sum_{\gamma \in \mathcal{I}} x^{\alpha}_{\gamma} w_\gamma$ for every $x \in \mathfrak{g}$ and $\alpha \in \mathcal{I}$, and

Theorem 4.4. $c(x, y) = \displaystyle\sum_{\alpha \in \mathcal{J}, \gamma \in \mathcal{I} \setminus \mathcal{J}} x^{\gamma}_{\alpha} y^{\alpha}_{\gamma} - \sum_{\alpha \in \mathcal{I} \setminus \mathcal{J}, \gamma \in \mathcal{J}} x^{\gamma}_{\alpha} y^{\alpha}_{\gamma}.$

Proof. Let $\kappa : V_{\mathrm{res}} \to \mathbb{K}$ be the linear map defined by

$$\kappa(w_{\underline{\alpha}} w^*_{\underline{\beta}} v_0) = \begin{cases} 1 & \text{if } \underline{\alpha} = \underline{\beta} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

for $w_{\underline{\alpha}} w^*_{\underline{\beta}} \in \mathfrak{a}^-$. Since $c^{\rho}_{x,y}$ is a scalar, $c^{\rho}_{x,y} v_0 = [f_x, f_y] v_0 - f_{[x,y]} v_0 = \kappa([f_x, f_y] v_0) v_0 - \kappa(f_{[x,y]} v_0) v_0$. For any $z \in \mathfrak{g}$, we have

$$\begin{aligned} \kappa(f_z v_0) &= \kappa\left(\sum_{\alpha,\gamma \in \mathcal{I}} z^{\alpha}_{\gamma} : w_\gamma w^*_\alpha : v_0 \right) \\ &= \kappa\left(\sum_{\alpha \in \mathcal{I}} z^{\alpha}_{\alpha} : w_\alpha w^*_\alpha : v_0 \right) = 0, \end{aligned}$$

by normal ordering since $w_\alpha \in \mathfrak{a}^+$ or $w^*_\alpha \in \mathfrak{a}^+$. Thus

$$\begin{aligned} c^{\rho}_{x,y} &= \kappa([f_x, f_y] v_0) \\ &= \kappa\left(\sum_{\alpha,\gamma \in \mathcal{I}} y^{\alpha}_{\gamma} [f_x, : w_\gamma w^*_\alpha :] v_0 \right) \\ &= \kappa\left(\sum_{\alpha,\gamma \in \mathcal{I}} y^{\alpha}_{\gamma} ([f_x, w_\gamma] w^*_\alpha + w_\gamma [f_x, w^*_\alpha]) v_0 \right) \\ &= \kappa\left(\sum_{\alpha,\gamma \in \mathcal{I}} x^{\gamma}_{\alpha} y^{\alpha}_{\gamma} (w_\alpha w^*_\alpha - w_\gamma w^*_\gamma) v_0 \right). \end{aligned}$$

Let $\chi : \mathcal{I} \to \{0, 1\}$ with

$$\chi(\eta) = \begin{cases} 1 & \text{if } \eta \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $w_\alpha \in \mathfrak{a}^+$ or $w^*_\alpha \in \mathfrak{a}^+$ (and likewise for w_γ and w^*_γ). Therefore

$$w_\alpha w^*_\alpha v_0 = \begin{cases} 0 & \text{if } w^*_\alpha \in \mathfrak{a}^+, \\ -\rho w^*_\alpha w_\alpha v_0 + \rho v_0 = \rho v_0 & \text{if } w^*_\alpha \notin \mathfrak{a}^+. \end{cases}$$

Hence $(w_\alpha w_\alpha^* - w_\gamma w_\gamma^*)v_0 = \rho(\chi(\alpha) - \chi(\gamma))v_0$, so

$$\begin{aligned} c(x, y) = \rho c_{x,y}^{\rho} &= \rho\kappa\left(\sum_{\alpha,\gamma\in\mathcal{I}} x_\alpha^\gamma y_\gamma^\alpha \rho(\chi(\alpha) - \chi(\gamma))v_0\right) \\ &= \sum_{\alpha,\gamma\in\mathcal{I}} x_\alpha^\gamma y_\gamma^\alpha(\chi(\alpha) - \chi(\gamma)) \\ &= \sum_{\alpha\in\mathcal{J},\gamma\in\mathcal{I}\setminus\mathcal{J}} x_\alpha^\gamma y_\gamma^\alpha - \sum_{\alpha\in\mathcal{I}\setminus\mathcal{J},\gamma\in\mathcal{J}} x_\alpha^\gamma y_\gamma^\alpha \end{aligned}$$

and both of these sums are finite by the finiteness condition on realizations. □

5. Embeddings into Lie algebras of infinite matrices

Date et al. [3] and Kac and Peterson [12] have introduced the Lie algebra $\overline{\mathfrak{gl}}_\infty$ consisting of infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with $a_{ij} \in \mathbb{K}$ and only a finite number of nonzero diagonals. The Lie algebra $\overline{\mathfrak{gl}}_\infty$ has a central extension $\mathfrak{gl}_\infty = \overline{\mathfrak{gl}}_\infty \oplus \mathbb{K}\mathbf{c}$ given by the 2-cocycle $\alpha(E_{ij}, E_{ji}) = -\alpha(E_{ji}, E_{ij}) = 1$ for $i \le 0,\ j > 0$, and $\alpha(E_{ij}, E_{mn}) = 0$ otherwise (cf. [13]).

By analogy, for any realization $(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}', \rho)$, we may view the space

$$\overline{\mathfrak{A}} := \left\{ \sum_{\alpha,\beta\in\mathcal{I}} c_{\alpha\beta} : w_\alpha w_\beta^* :\in \widehat{\mathfrak{a}}_0 \mid c_{\alpha\beta} \in \mathbb{K} \right\}$$

as a Lie algebra of (card $\mathcal{I}$) × (card $\mathcal{I}$) "matrices" $\sum_{\alpha,\beta\in\mathcal{I}} c_{\alpha\beta}E_{\alpha\beta}$, where $E_{\alpha\beta}$ is the matrix with 1 in the (α, β) position and 0 elsewhere. The Lie bracket in $\overline{\mathfrak{A}}$ is obtained by linearly extending the ordinary "matrix Lie bracket" $[E_{\alpha\beta}, E_{\gamma\eta}] = \delta_{\beta,\gamma}E_{\alpha\eta} - \delta_{\alpha,\eta}E_{\gamma\beta}$ to $\overline{\mathfrak{A}}$. Although the matrices in $\overline{\mathfrak{A}}$ may be of arbitrary dimension and need not have only finitely many nonzero "diagonals," this bracket is well-defined, since it is simply the result of restricting the Lie bracket on $\widehat{\mathfrak{a}}_0$ to $\overline{\mathfrak{A}} \oplus \mathbb{K}1 \subset \widehat{\mathfrak{a}}_0$ and then projecting onto $\overline{\mathfrak{A}}$.

It is clear that the space $\mathfrak{A} = \overline{\mathfrak{A}} \oplus \mathbb{K}1$, under the restriction of the Lie bracket of $\widehat{\mathfrak{a}}_0$, is a central extension of $\overline{\mathfrak{A}}$, and the resulting 2-cocycle is simply the "constant term" that occurs in a given bracket—i.e., $c(E_{\alpha\beta}, E_{\gamma\eta}) = \varepsilon([: w_\alpha w_\beta^* :, : w_\gamma w_\eta^* :]) = \delta_{\alpha,\eta}\delta_{\beta,\gamma}(\chi(\alpha) - \chi(\gamma))$, in the notation of Sections 3 and 4. The restriction of c to the Lie algebra of Corollary 4.2 is the 2-cocycle of Section 4. If $\mathfrak{g} = \overline{\mathfrak{gl}}_\infty$, W is the natural representation on (doubly infinite) column vectors (with canonical basis indexed by $\mathcal{I} = \mathbb{Z}$), $\mathcal{I}' = \emptyset$, and $\mathcal{J}$ is the nonpositive integers, then we get the 2-cocycle α and the Lie algebra $\mathfrak{gl}_\infty$ of [3] and [12].

In particular, if V is faithful, the identification of the operator $:w_\alpha w_\beta^*:$ with the matrix $E_{\alpha\beta}$ and $\mathbf{c}$ with $\rho 1$ gives an embedding of $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ into $\mathfrak{A}$. The following proposition gives an easy criterion for faithfulness.

Proposition 5.1. *The $\widetilde{\mathfrak{g}}$-module $V = V(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}', \rho)$ is faithful if and only if W is a faithful $\mathfrak{g}$-module.*

Proof. Suppose W is not faithful. Then there is a nonzero $x \in \mathfrak{g}$ such that $x \cdot w = 0$ for all $w \in W$. Thus $f_x = \sum_{\alpha\in\mathcal{I}} :(x \cdot w_\alpha)w_\alpha^*: = 0$, so for every $v \in V$, $x \cdot v = f_x v = 0$. Hence V is not faithful.

Conversely, suppose W is faithful and $k\mathbf{c} + x$ acts as zero on V for some $k \in \mathbb{K}$ and $x \in \mathfrak{g}$. Then for every $a \in \mathfrak{a}$,

$$\begin{aligned} 0 = (k\mathbf{c} + x) \cdot (av_0) &= (k\rho + f_x)av_0 \\ &= [k\rho + f_x, a]v_0 + a(k\rho + f_x)v_0 \\ &= [f_x, a]v_0 + a(k\mathbf{c} + x) \cdot v_0 \\ &= [f_x, a]v_0. \end{aligned}$$

Therefore by Lemma 2.5, $(x \cdot w_\alpha)v_0 = 0 = (x \cdot w_\alpha^*)v_0$ for all $\alpha \in \mathcal{I}$. By Proposition 3.5, it now follows that $xw_\alpha = \sum_{\gamma\in\mathcal{J}} x_\gamma^\alpha w_\gamma$. Then for any $\alpha, \beta \in \mathcal{I}$,

$$\begin{aligned} 0 = (k\mathbf{c} + x) \cdot (w_\alpha w_\beta^* v_0) &= [f_x, w_\alpha w_\beta^*]v_0 \\ &= [f_x, w_\alpha]w_\beta^* v_0 + w_\alpha[f_x, w_\beta^*]v_0 \\ &= (x \cdot w_\alpha)w_\beta^* v_0 + w_\alpha(x \cdot w_\beta^*)v_0 \\ &= (x \cdot w_\alpha)w_\beta^* v_0 \\ &= \sum_{\gamma\in\mathcal{J}} x_\gamma^\alpha w_\gamma w_\beta^* v_0 \\ &= -\rho \sum_{\gamma\in\mathcal{J}} x_\gamma^\alpha w_\beta^* w_\gamma v_0 + \sum_{\gamma\in\mathcal{J}} x_\gamma^\alpha \{w_\gamma, w_\beta^*\}_\rho v_0 \\ &= \rho x_\beta^\alpha \chi(\beta) v_0, \end{aligned}$$

in the notation of the proof of Theorem 4.4. Hence $x_\beta^\alpha = 0$ for all $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$. But $x \cdot w_\alpha = \sum_{\gamma\in\mathcal{J}} x_\gamma^\alpha w_\gamma$, so this gives $x \cdot w_\alpha = 0$ for all $\alpha \in \mathcal{I}$. Since W is faithful, $x = 0$, so $k\mathbf{c} = k\mathbf{c} + x$ acts as zero on V. Hence $k = 0$, and V is faithful. □

We can use the "identity matrix" $J := \sum_{\gamma\in\mathcal{I}} :w_\gamma w_\gamma^*:$ to decompose V into submodules:

Proposition 5.2. *The $\widetilde{\mathfrak{g}}$-module V has a $\mathbb{Z}$-grading $V = \bigoplus_{r\in\mathbb{Z}} V_r$, where V_r is the J-eigenspace with eigenvalue r. Each V_r is a $\widetilde{\mathfrak{g}}$-submodule of V. Explicitly, $V_r = \mathrm{Span}_{\mathbb{K}}\{w_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}v_0 \mid \ell(\underline{\alpha}) - \ell(\underline{\beta}) - \ell(\underline{\gamma}) = r\}$.*

Proof. For any

$$\lambda \in W',$$
$$[J,\lambda] = \sum_{\gamma\in\mathcal{I}} [:w_\gamma w^*_\gamma:,\lambda] = -\rho\sum_{\gamma\in\mathcal{I}}\{w_\gamma,\lambda\}_\rho w^*_\gamma = -\sum_{\gamma\in\mathcal{I}}\lambda(w_\gamma)w^*_\gamma$$

and for any

$$\alpha \in \mathcal{I},$$
$$-\sum_{\gamma\in\mathcal{I}}\lambda(w_\gamma)w^*_\gamma(w_\alpha) = -\lambda(w_\alpha),$$
$$\text{so } [J,\lambda] = -\lambda.$$

Likewise, for $w \in W$, $[J,w] = w$, and $Jv_0 = 0$. Thus

$$\begin{aligned} Jw_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}v_0 &= [J, w_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}]v_0 + w_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}Jv_0 \\ &= [J, w_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}]v_0 \\ &= (\ell(\underline{\alpha}) - \ell(\underline{\beta}) - \ell(\underline{\gamma}))w_{\underline{\alpha}}\lambda_{\underline{\beta}}w^*_{\underline{\gamma}}v_0. \end{aligned}$$

To see that V_r is a $\widetilde{\mathfrak{g}}$-submodule, it is enough to observe that J commutes with f_x and $\mathbf{c}$ for all $x \in \mathfrak{g}$. But this is trivial, since $[J,:w_\alpha w^*_\beta:] = [J,w_\alpha]w^*_\beta + w_\alpha[J,w^*_\beta] = w_\alpha w^*_\beta - w_\alpha w^*_\beta = 0$ for all $\alpha,\beta \in \mathcal{I}$. □

As we will see in Example 6.1, the modules V_r need not be irreducible, even if W is irreducible.

6. Examples

In this section, we show that our notion of bosonic and fermionic realizations includes representations on symmetric and skew-symmetric tensors, as well as generalizes the Fock space constructions given by Feingold and Frenkel [5] for types A_ℓ and $A^{(1)}_\ell$ and by Gao [10] for $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$, where $\mathbb{C}_q$ is the quantum torus in two variables. Also, Theorem 4.4 may be used to compute interesting central extensions. We illustrate this with the Virasoro Lie algebra and a central extension of $\mathfrak{gl}_N(A)$, where A is the ring of differential operators on the punctured plane $\mathbb{C}^\times$. We anticipate that the techniques of this paper may be used to produce nontrivial representations of toroidal and other interesting Lie algebras, and we plan to investigate these and further applications in a later work.

Example 6.1 (*Symmetric and skew-symmetric tensors*). Let $\mathfrak{f}\mathfrak{g}$ be any Lie algebra with a module W. Consider a realization $(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}')$ with $\mathcal{J} = \emptyset$. Then $\mathfrak{a}^-$ is generated by $\{w_\alpha \mid \alpha \in \mathcal{I}\}$ and $f_x v_0 = 0$ for every $x \in \mathfrak{g}$. The defining relations of $\mathfrak{a}$ make it clear that V is the module $S(W)$ of symmetric tensors in the tensor algebra $T(W)$ if $\rho = -1$, and the module of skew-symmetric tensors $\bigoplus_{r\geq 1} \bigwedge^r W$ if $\rho = 1$. The central extension $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ is obviously split, since the 2-cocycle c is trivial. Moreover, the submodules

$$V_r \cong \begin{cases} S^r(W), \text{ the } r\text{th symmetric power of } W & \text{if } \rho = -1 \\ \bigwedge^r W & \text{if } \rho = 1 \end{cases}$$

are seldom irreducible, even if W is irreducible.

For instance, take $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $W = S^m(\mathbb{C}^2)$, $\mathcal{J} = \emptyset$, and $\rho = -1$, where $\mathbb{C}^2$ is the natural module and $m > 1$. Then W is irreducible, but the Clebsch–Gordan rule (see [9], for instance) gives $V_2 = S^2(S^m(\mathbb{C}^2)) \cong \bigoplus_{0\leq n\leq \frac{m}{2}} S^{2m-4n}(\mathbb{C}^2)$, so V_2 is not irreducible. Similar arguments can be made if $\rho = 1$.

Example 6.2 (*Oscillator and spinor representations for* $\mathfrak{gl}_N(\mathbb{K})$). The natural representation $W = \mathbb{K}^N$ of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{K})$, with $\mathcal{I} = \{1, 2, \ldots, N\}$ and $\mathcal{J} = \emptyset$ gives the usual oscillator and spinor representations (as described in [5], for example) for $\rho = -1$ and $\rho = 1$, respectively. These are representations that come from the natural isomorphism $\mathfrak{gl}_N(\mathbb{K}) \cong W \otimes W^*$, and can be thought of as the model for all bosonic and fermionic representations described in this paper (as discussed in Section 5).

Example 6.3 (*Matrices over associative algebras*). Let A be an associative algebra over the field $\mathbb{K}$, and let M be an A-module. The Lie algebra $\mathfrak{gl}_N(A)$ of $N \times N$ matrices over A has a natural "left-multiplication" action on the space M^N of $N \times 1$ column vectors with entries in M. Specifically, let $x(a) = (x_{ij}a)_{1\leq i,j\leq N}$ where $x = (x_{ij})_{1\leq i,j\leq N} \in \mathfrak{gl}_N(\mathbb{K})$ and $a \in A$, and let

$$v(m) = \begin{pmatrix} v_1 m \\ v_2 m \\ \vdots \\ v_N m \end{pmatrix}, \text{ where } v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \in \mathbb{K}^N \text{ and } m \in M.$$

Then $x(a) \cdot v(m) = xv(a \cdot m)$. Realizations of $(\mathfrak{gl}_N(A), M^N)$ give representations for a variety of interesting Lie algebras, including Examples 6.4, 6.5, and 6.7 below.

Example 6.4 (*Affine Lie algebra* $A_{N-1}^{(1)}$). In the notation of Example 6.3, let $A = \mathbb{K}[t, t^{-1}]$, and let $M = \mathbb{K}[t, t^{-1}]$ be the (left) regular A-module. Then $W = M^N$ is a module for $\mathfrak{g} = \mathfrak{gl}_N(A)$. Let $\mathcal{I} = \{1, 2, \ldots, N\} \times \mathbb{Z}$ with $w_{(i,n)} = e_i(n)$ and $\mathcal{J} = \{1, 2, \ldots, N\} \times \mathbb{N}$, where $e_i(n) = e_i(t^n)$ and $\mathbb{N}$ denotes the nonnegative integers. Then $\mathcal{I}' = \emptyset$, and the realization $(\mathfrak{g}, W, \mathcal{J}, \mathcal{I}, \mathcal{I}')$ is (with the exception of a

small change to the normal order)[4] the Fock space construction given by Feingold and Frenkel for the affine Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$ in [5]. If $\mathcal{J} = \{1, 2, \ldots, N\} \times \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the positive integers, then we recover the representation given by Gao [10, Prop 2.25] for his "vertical Lie algebra" $\mathcal{L}_v \cong \widetilde{\mathfrak{g}}$.

Example 6.5 ($\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$). Let $\mathbb{C}_q = \mathbb{C}[x^{\pm}, y^{\pm}]_{n.c.}/(yx - qxy)$ be the quantum torus in two variables.[5] The associative algebra $\mathbb{C}_q$ admits a representation on $M = \mathbb{C}[t, t^{-1}]$ by $x \cdot t^{\ell} = t^{\ell+1}$ and $y \cdot t^{\ell} = q^{\ell} t^{\ell}$. Taking $w_{(i,n)}$, $\mathcal{I}$, $\mathcal{I}'$, and W as in Example 6.4, we let $\mathcal{J} = \{1, 2, \ldots, N\} \times \mathbb{Z}^+$. This realization gives Gao's representation $E_{ij}(q^m p^n) \mapsto f_{ij}(m, n) = \sum_{s \in \mathbb{Z}} q^{ns} : e_i(s + m) e_j(s)^* :$. In order to have the desired commutation relations between the operators $f_{ij}(m, n)$, Gao adjusts his basis of the central extension spanned by $f_{ij}(m, n)$ and 1 by replacing the $f_{ij}(m, n)$ with $F_{ij}(m, n) = f_{ij}(m, n) + k(i, j, m, n)1$ for some constants $k(i, j, m, n) \in \mathbb{C}$. Note that such a change of basis is included in the boundary map $\partial : \mathcal{C}^1 \to \mathcal{C}^2$ of the Chevalley–Eilenberg complex for $H^*(\mathfrak{gl}_N(\mathbb{C}_q); \mathbb{C})$, so does not alter the 2-cocycle or the central extension described above.[6]

Example 6.6 (*Virasoro Lie algebra*). Let $\mathfrak{g}$ be the Witt algebra $\mathrm{Span}_{\mathbb{K}}\{L_m \mid m \in \mathbb{Z}\}$ with commutation relations $[L_m, L_n] = (m - n)L_{m+n}$ for all $m, n \in \mathbb{Z}$. The Witt algebra has a natural (and faithful) representation as the derivations $L_m = -t^{m+1}\frac{d}{dt}$ on the space of Laurent polynomials, $W = \mathbb{K}[t, t^{-1}]$. Taking $\mathcal{I} = \mathbb{Z}$, $\mathcal{I}' = \emptyset$, and $\mathcal{J} = \mathbb{Z}^+$, we obtain the 2-cocycle

$$c(L_m, L_n) = \sum_{i \geq 0, j < 0} (L_m)_i^j (L_n)_j^i - \sum_{i < 0, j \geq 0} (L_m)_i^j (L_n)_j^i.$$

Clearly $L_m \cdot t^j = -jt^{m+j}$, so $(L_m)_i^j = -j\delta_{i,m+j}$ and

$$\begin{aligned} c(L_m, L_n) &= \sum_{i \geq 0, j < 0} ij\delta_{i,m+j}\delta_{j,n+i} - \sum_{i < 0, j \geq 0} ij\delta_{i,m+j}\delta_{j,n+i} \\ &= \sum_{i=0}^{m-1} \delta_{m,-n} i(i - m) - \sum_{i=m}^{-1} \delta_{m,-n} i(i - m) = \delta_{m,-n}\left(\frac{m - m^3}{6}\right). \end{aligned}$$

[4] The normal ordering used by Feingold and Frenkel is

$$: e_i(m) e_j(n)^* := \frac{1}{2} e_i(m) e_j(n)^* - \frac{\rho}{2} e_j(n)^* e_i(m) = \begin{cases} -e_j(n)^* e_i(m) - \frac{\rho}{2} & \text{if } i = j \text{ and } m = n \\ e_i(m) e_j(n)^* & \text{otherwise.} \end{cases}$$

Their change in normal order amounts only to replacing $E_{ii}(0)$ with $E_{ii}(0) - \frac{1}{2}\mathbf{c}$ in our representation. They (and Gao [10]) use the notation $a_j^*(-n)$ to denote what we call $e_j(n)^*$.

[5] Here $q \in \mathbb{C}^{\times}$, and $\mathbb{C}[x^{\pm}, y^{\pm}]_{n.c.}$ is the space of Laurent polynomials in two noncommuting variables x and y.

[6] Gao's central elements $c(n)$ all act as our central element c and his c_y acts as 0.

Adjusting our 2-cocycle c by a constant factor of $-\frac{1}{2}$ gives the usual normalization of the central extension $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}\mathbf{c}$, called the *Virasoro Lie algebra*. It is well known that this central extension is universal (cf. [1], [13], or [17], for instance).

Example 6.7 ($\widehat{\mathfrak{gl}_N(\mathcal{W}_{1+\infty})}$). Let $\mathbb{K}[p, q^{\pm 1}]_{n.c.}$ be the (unital) associative algebra of noncommuting polynomials in two variables p and q, localized on the multiplicative set $\{q^m | m \in \mathbb{N}\}$. Let $A = \mathbb{K}[p, q^{\pm 1}]_{n.c.}/([p, q] - 1)$. The algebra A is the first Weyl algebra A_1 localized on the multiplicative set $\{q^m \mid m \in \mathbb{N}\}$, and thus has a natural representation on $\mathbb{K}[t, t^{-1}]$ given by $p \cdot t^\ell = \ell t^{\ell-1}$, $q \cdot t^\ell = t^{\ell+1}$, and $q^{-1} \cdot t^\ell = t^{\ell-1}$. If $\mathbb{K}$ is algebraically closed, A is also the ring of differential operators on the "circle" $\{(x, y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}$ or the scheme $\mathbb{K} \setminus \{0\}$ (cf. [2,11]).[7] Viewed as a Lie algebra in the usual way, the algebra A arises as a limit of objects called W_n-algebras (see [19], for instance), and is usually denoted $\mathcal{W}_{1+\infty}$.

In the notation of Example 6.3, $W = (\mathbb{K}[t, t^{-1}])^N$ is a representation for the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(A)$, and we obtain a realization by taking $w_{(i,n)} = e_i(t^n) \in W$, $\mathcal{I} = \{1, 2, \ldots, N\} \times \mathbb{Z}$, $\mathcal{I}' = \emptyset$, and $\mathcal{J} = \{1, 2, \ldots, N\} \times \mathbb{N}$. Let $x(k, \ell) = x(q^k p^\ell)$ where $x \in \mathfrak{gl}_N(\mathbb{K})$, $k \in \mathbb{Z}$, and $\ell \in \mathbb{N}$. Since the module action is $E_{ij}(k, \ell) \cdot w_{(m,n)} = \delta_{j,m}(n)_\ell w_{(i,n+k-\ell)}$ where $(n)_\ell = n(n-1)\cdots(n-(\ell-1))$, we have $E_{ij}(k, \ell)^{(m,n)}_{(r,s)} = \delta_{i,r}\delta_{j,m}\delta_{n+k-\ell,s}(n)_\ell$. Thus the Fock space representation is

$$E_{ij}(k, \ell) \mapsto \sum_{n \in \mathbb{Z}} (n)_\ell : w_{(i,n+k-\ell)} w^*_{(j,n)} :,$$

and by Theorem 4.4, the resulting 2-cocycle is

$$\begin{aligned}
&c(E_{ij}(k, \ell), E_{mn}(r, s)) \\
&\quad = \sum_{(a,b) \in \mathcal{J}, (u,v) \in \mathcal{I} \setminus \mathcal{J}} E_{ij}(k, \ell)^{(u,v)}_{(a,b)} E_{mn}(r, s)^{(a,b)}_{(u,v)} \\
&\qquad\qquad - \sum_{(a,b) \in \mathcal{I} \setminus \mathcal{J}, (u,v) \in \mathcal{J}} E_{ij}(k, \ell)^{(u,v)}_{(a,b)} E_{mn}(r, s)^{(a,b)}_{(u,v)} \\
&\quad = \delta_{i,n}\delta_{j,m} \left(\sum_{b \geq 0, v < 0} \delta_{v+k-\ell,b}\delta_{b+r-s,v}(v)_\ell (b)_s - \sum_{b < 0, v \geq 0} \delta_{v+k-\ell,b}\delta_{b+r-s,v}(v)_\ell (b)_s \right) \\
&\quad = \delta_{k+r,\ell+s}\delta_{j,m}\delta_{i,n} \left(\sum_{b=0}^{s-r-1} (b + r - s)_\ell (b)_s - \sum_{b=s-r}^{-1} (b + r - s)_\ell (b)_s \right)
\end{aligned}$$

with the convention that empty sums are 0.

[7] Which is the punctured plane in the case $\mathbb{K} = \mathbb{C}$.

By the lemma and identities in the appendix, if $s \geq r$, then

$$\begin{aligned}\sum_{b=0}^{s-r-1} (b+r-s)_\ell (b)_s &= (-1)^\ell \ell! s! \binom{s-r+\ell}{-r-1} \\ &= (-1)^\ell \ell! s! \binom{s-r+\ell}{s+\ell+1} \\ &= (-1)^{s+1} \ell! s! \binom{r}{s+\ell+1}.\end{aligned}$$

Likewise, if $s < r$, then $r > 0$, so

$$\begin{aligned}\sum_{b=s-r}^{-1} (b+r-s)_\ell (b)_s &= (-1)^s \ell! s! \binom{r}{r-s-\ell-1} \\ &= (-1)^s \ell! s! \binom{r}{s+\ell+1}.\end{aligned}$$

Hence $c(E_{ij}(k,\ell), E_{mn}(r,s))$

$$\begin{aligned}&= \delta_{k+r,\ell+s}\delta_{j,m}\delta_{i,n}(-1)^{s+1}\ell! s! \tbinom{r}{s+\ell+1} \\ &= \operatorname{tr}(E_{ij}E_{mn})\delta_{k+r,\ell+s}(-1)^{s+1}\ell! s! \tbinom{r}{s+\ell+1},\end{aligned}$$

so by the linearity of the trace,

$$c(x(k,\ell), y(r,s)) = \operatorname{tr}(xy)\delta_{k+r,\ell+s}(-1)^{s+1}\ell! s! \binom{r}{\ell+s+1}$$

for any $x, y \in \mathfrak{gl}_N(\mathbb{K})$.

The 2-cocycle c was first computed by Kac and Peterson [12], and the corresponding central extension $\widehat{\mathfrak{gl}_N(A)}$ is universal ([4,15,16]). With a slight abuse of notation, we denote the central extension as $\widehat{\mathfrak{gl}_N}(\mathcal{W}_{1+\infty})$. By the 2-cocycle formula obtained above, $\widehat{\mathfrak{gl}_N}(\mathcal{W}_{1+\infty})$ contains full copies of the affine and Virasoro Lie algebras

$$\widehat{\mathfrak{gl}_N(\mathbb{K})} \cong \operatorname{Span}_{\mathbb{K}}\{x(k,0) \mid x \in \mathfrak{gl}_N(\mathbb{K}),\ k \in \mathbb{Z}\} \oplus \mathbb{K}\mathbf{c},$$

$$\operatorname{Vir} \cong \operatorname{Span}_{\mathbb{K}}\{I(k,1) \mid k \in \mathbb{Z}\} \oplus \mathbb{K}\mathbf{c},$$

where $I \in \mathfrak{gl}_N(\mathbb{K})$ is the identity matrix.

Acknowledgments

The author thanks Professor G. Benkart for her many helpful suggestions, and Professor B.N. Allison for his comments on a previous draft. This work was partially supported by N.S.A. Grant MDA 904-03-1-0068.

Appendix. A combinatorial lemma

In the statements below, we assume that all variables represent integers, and we follow the standard conventions that empty sums are 0, empty products are 1, and $\binom{a}{b} = \frac{(a)_b}{b!}$ where

$$(a)_b = \begin{cases} a(a-1)\cdots(a-(b-1)) & \text{if } b \geq 0, \\ 0 & b < 0. \end{cases}$$

We will use the following well-known identities without explicit mention:

$$\text{(I1)} \quad \binom{a}{b} = (-1)^b \binom{b-a-1}{b},$$

$$\text{(I2)} \quad \sum_{n=0}^{c} \binom{a}{n}\binom{b}{c-n} = \binom{a+b}{c},$$

$$\text{(I3)} \quad \binom{d}{b} = \binom{d}{d-b}$$

for any $a, b, c \in \mathbb{Z}$ and $d \in \mathbb{N}$ (See [6], for instance.)

Lemma A.1. *Let* $m, \ell, s \geq 0$. *Then*

$$\text{(i)} \quad \sum_{b=0}^{m-1} (b-m)_\ell (b)_s = (-1)^\ell \ell! s! \binom{m+\ell}{m-s-1},$$

$$\text{(ii)} \quad \sum_{b=-m}^{-1} (b+m)_\ell (b)_s = (-1)^s \ell! s! \binom{m+s}{m-\ell-1}.$$

Proof. Note that (ii) will follow from (i) by interchanging ℓ and s and making the change of variables $b \mapsto b+m$ in (i). Thus, we may restrict our attention to the first identity:

$$\sum_{b=0}^{m-1} (b-m)_\ell (b)_s = \sum_{b=0}^{m-1} (-1)^\ell (m-b+\ell-1)_\ell (b)_s$$

$$= \sum_{b=s}^{m-1} (-1)^{\ell} \ell! s! \binom{m-b+\ell-1}{\ell} \binom{b}{s}$$

$$= (-1)^{\ell} \ell! s! \sum_{b=s}^{m-1} \binom{m-b+\ell-1}{m-b-1} \binom{b}{b-s}$$

$$= (-1)^{\ell+m-s-1} \ell! s! \sum_{b=s}^{m-1} \binom{-\ell-1}{m-b-1} \binom{-s-1}{b-s}.$$

Changing the index of summation from b to $b-s$ now gives

$$\sum_{b=0}^{m-1} (b-m)_{\ell} (b)_s = (-1)^{\ell+m-s-1} \ell! s! \sum_{b=0}^{m-1-s} \binom{-\ell-1}{m-b-s-1} \binom{-s-1}{b}$$

$$= (-1)^{\ell+m-s-1} \ell! s! \binom{-\ell-s-2}{m-s-1}$$

$$= (-1)^{\ell} \ell! s! \binom{m+\ell}{m-s-1}. \qquad \square$$

References

[1] G. Benkart, R.V. Moody, Derivations, central extensions, and affine Lie algebras, Algebras Groups Geom. 3 (1986) 456–492.

[2] J.-E. Björk, Rings of Differential Operators, North-Holland Mathematics Library, vol. 21, North-Holland, Amsterdam, 1979.

[3] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations. III. Operator approach to the Kadomtsev–Petviashvili equation, J. Phys. Soc. Japan 50 (1981) 3806–3812.

[4] B.L. Feigin, The Lie algebras $\mathfrak{gl}$ and the cohomology of the Lie algebra of differential operators, Usp. Mat. Nauk. 35 (1988) 157–158.

[5] A. Feingold, I.B. Frenkel, Classical affine algebras, Adv. Math. 56 (1985) 117–172.

[6] W. Feller, An Introduction to Probability Theory and its Applications, third ed., Wiley, New York, 1968.

[7] I.B. Frenkel, Spinor representations of affine Lie algebras, Proc. Natl. Acad. Sci. USA 77 (1980) 6303–6306.

[8] I.B. Frenkel, Two constructions of affine Lie algebra representations and boson–fermion correspondence in quantum field theory, J. Funct. Anal. 44 (1981) 259–327.

[9] W. Fulton, J. Harris, Representation Theory: A First Course, Graduate Texts in Mathematics, vol. 129, Springer, New York, 1991.

[10] Y. Gao, Fermionic and bosonic representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$, Canad. Math. Bull. 45 (2002) 623–633.

[11] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer, New York, 1977.

[12] V.G. Kac, D. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Natl. Acad. Sci. USA 78 (1981) 3308–3312.

[13] V.G. Kac, A.K. Raina, Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras, Advanced Series in Mathematical Physics, vol. 2, World Scientific, Singapore, 1987.
[14] F. ten Kroode, J. van de Leur, Bosonic and fermionic realizations of the affine algebra $\widehat{\mathfrak{gl}_n}$, Comm. Math. Phys. 137 (1991) 67–107.
[15] M. Lau, Fock Representations and Central Extensions, Ph.D. dissertation, University of Wisconsin-Madison, 2004.
[16] W.L. Li, 2-cocycles on the algebra of differential operators, J. Algebra 122 (1989) 64–80.
[17] R.V. Moody, A. Pianzola, Lie Algebras with Triangular Decompositions, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1995.
[18] G.B. Seligman, in: Kac-Moody modules and generalized Clifford algebras, Lie Algebras, Madison 1987, Lecture Notes in Mathematics, vol. 1373, Springer, Berlin, 1989, pp. 124–143.
[19] A.B. Zamolodchikov, Infinite additional symmetries in 2-dimensional conformal field theory, Theoret. Math. Phys. 65 (1985) 1205–1213.