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Arithmetic Transfinite Induction and Recursive Well-Orderings

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A uniform, algebraic proof that every number-theoretic assertion provable in any of the intuitionistic theories T listed below has a well-founded recursive proof tree (demonstraby in T) is given. Thus every such assertion is provable by transfinite induction over some primitive recursive well-ordering. T can be higher order number theory, set theory, or its extensions equiconsistent with large cardinals. It is shown that there is a number-theoretic assertion $B(n)$ (independent of T) with a parameter n such that any primitive recursive linear ordering R on ω for which transfinite induction on R for $B(n)$ is provable in T is in fact a well-ordering.

0. INTRODUCTION

Elementary number theory extended with the schema of transfinite induction on all primitive recursive well-orderings proves all true number-theoretic assertions [11]. This is not true for such an extension of constructive elementary number theory. We give a concise, uniform argument that this extension proves any number-theoretic assertion provable in practically any constructive theory consistent with the recursiveness of all functions $f: N \rightarrow N$. Besides the extended constructive elementary number theory under consideration, these include, e.g., higher order number theory, set theory, and its extensions equiconsistent with the existence of large cardinals. The result is known in the case of higher order number theory by elaborate proof-theoretic arguments [12, 18] which do not extend to stronger theories.

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Our proof involves the constructive metatheory T with the additional assumption that every function $f: N \to N$ is recursive. In this metatheory, the extension of constructive elementary number theory described above coincides with the extension with the full ω-rule

$$\frac{A(0)\ A(1) \cdots A(\bar{n}) \cdots, \text{ all } n}{\forall n A(n)}.$$

We embed its Lindenbaum algebra into a complete Heyting algebra, preserving all infs and sups (that already exist). This completion is given by all sup-closed ideals. Then it suffices to look at the Heyting-valued model over this completion. Finally, we use a transfer lemma that eliminates the additional assumption on the metatheory. This maneuver of switching metatheories is similar to the one we used in [5], except that here it is accompanied with an algebraic construction of a completion of a Heyting algebra and with a forcing extension, rather than with the slash.

In Section 1 we state constructive set theory, and discuss several equivalent versions of the extension of elementary number theory with transfinite induction schemata, particularly in terms of well-founded recursive proof trees in the system with ω-rule.

The main construction is given in Section 2. We give the transfer lemma on recursive realizability in Section 3.

Section 4 contains an application of the main theorem which shows that if transfinite induction for a particular number-theoretic formula on a primitive recursive binary relation R is provable in any of the constructive theories T mentioned above, then R is in fact well founded. This is an extension of the first author's result for elementary number theory [4]. It is known to be false for classical theories [10].

We conclude with the remark on the extension of these results to theories stronger than set theory.

1. Description of Theories

We shall use a formulation of intuitionistic set theory ZFI based on two-sorted Heyting's predicate logic, with variables $n, m, k, \ldots$, over natural numbers and variables $x, y, z, u, v, w, \ldots$, over sets, the number constant 0, and primitive recursive function symbols. Equality will be used only between numerical terms. The relation symbol $\in$ will be used only as $t \in x$, where t is a numerical term, or a set variable. The axioms are as follows:

(1) $\neg(s(n) = 0),\ s(n) = s(m) \to n = m$

$n = n, \quad n = m \to m = n, \quad n = m \wedge m = k \to n = k,$

$n_1 = m_1 \wedge \cdots \wedge n_i = m_i \to F(n_1, \ldots, n_i) = F(m_1, \ldots, m_i),$

$n = m \to (n \in x \leftrightarrow m \in x).$

(2) Primitive recursive defining equations.

(3) Induction. $A(0) \wedge \forall n(A(n) \rightarrow A(s(n))) \rightarrow \forall n A(n)$.

(4) Infinity. $\exists x. \forall n. n \in x$.

(5) Pairing. $\exists x(u \in x \wedge v \in x)$.

(6) Union. $\exists x. \forall y. \forall v(y \in v \wedge v \in u \rightarrow y \in x)$.

(7) Separation. $\exists x[\forall n(n \in x \leftrightarrow A(n)) \wedge \forall y(y \in x \leftrightarrow y \in u \wedge B(y))]$ where x is not free in $A(n)$, $B(y)$.

(8) Foundation. $\forall x(\forall y \in x. A(y) \rightarrow A(x)) \rightarrow \forall x A(x)$, where x does not occur in $A(y)$.

(9) Power set. $\exists x. \forall y(\forall z \in y. z \in u \rightarrow y \in x)$.

(10) Collection. $\forall x \in u. \exists y. A(x, y) \rightarrow \exists v. \forall x \in u. \exists y \in v. A(x, y)$,
$\forall n \in u. \exists y. B(n, y) \rightarrow \exists w. \forall n \in u. \exists y \in w. B(n, y)$
where v, w are not free in $A(x, y)$, $B(n, y)$.

(11) Extensionality.

$$\forall n(n \in x \leftrightarrow n \in y) \wedge \forall z(z \in x \leftrightarrow z \in y) \rightarrow (A(x) \leftrightarrow A(y)).$$

Axioms (1)–(7), (9), (11) with all quantifiers in $A(n)$, $B(y)$ bounded are equivalent to *higher order arithmetic* (HAH), which is often formulated in terms of finite types over natural numbers. Its fragment (1)–(3), (7) in the form $\exists x \forall n(n \in x \leftrightarrow A(n))$ with x not free in $A(n)$, and with or without (11) in the restricted language which allows only numerical terms as elements is called *second order arithmetic* (HAS). Further restriction of the language obtained by barring set variables altogether, and retaining the appropriate fragment of (1)–(3) as axioms, gives *first order* (*Heyting*) *arithmetic* (HA). Formulae of HA are often called arithmetic formulae.

Our main theorem concerns the extension of HA with the transfinite induction schema over recursive well-founded relations. We wish to give several equivalent formulations of this theory.

A binary relation R on a set S is said to be *well founded* if the following holds:

$$\forall X[\forall a \in S(\forall b \in S(bRa \rightarrow b \in X) \rightarrow a \in X) \rightarrow \forall a \in S. a \in X].$$

We shall also be interested in a more precise information on whether such a formula is provable in HAS or a stronger system.

Let R be a well-founded primitive recursive binary relation on natural numbers (i.e., given by a binary primitive recursive function symbol). Consider the following schema in the language of HA:

$$\forall n(\forall m(mRn \rightarrow A(m)) \rightarrow A(n)) \rightarrow \forall n A(n), \qquad \mathrm{TI}(R, A),$$

where $A(n)$ is an arithmetic formula. The theory HA* is obtained from HA by adding the schemata $\mathrm{TI}(R, A)$ for all well-founded primitive recursive (binary) relations (on natural numbers). Equivalently (cf. below), one can allow all recursive well founded relations, given by their recursive indices. In fact, it suffices to let R be a primitive recursive well-ordering, i.e., a well-founded ordering linear on its field.

PA* (obtained from HA* by adding the Law of Excluded Middle) proves all (classically) true arithmetic sentences [11]. HA* is complete for Π^0_2 sentences (cf. below). However, the schema

$$\forall n \exists m\, A(n, m) \rightarrow \exists e \forall n\, A(n, \{e\}(n)) \qquad \text{(CT)}$$

is known to be both consistent with and independent of HA*, so the completeness of HA* is inconsistent, as observed by Kreisel. The consistency of HA* + CT was established by the method of recursive realizability, the extension of which we use in Section 3. Here we briefly sketch known equivalences of several formulations of HA*, in particular in terms of infinitary systems. We include them here to provide the background for our main theorem. They do not appear to be readily available in the literature.

Let HA^∞ be HA with full ω-rule, i.e., the least collection of sentences satisfying the inductive definition of provability given, e.g., in a Gentzen-style system as follows.

A sequent is an expression of the form $\Gamma \vdash A$, where A is an arithmetic sentence, and Γ is a finite set of arithmetic sentences. The axioms are the sequents $\Gamma \vdash A$, where either A is atomic and true, or A is atomic and some element of Γ is atomic and false.

HA^∞ has the following rules:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad\qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}, \quad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}, \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad\qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}$$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists m A(m)} \qquad\qquad \frac{\Gamma, A(\bar{n}) \vdash B \qquad \text{all } n}{\Gamma, \exists m A(m) \vdash B}$$

$$\frac{\Gamma \vdash A(\bar{n}), \qquad \text{all } n}{\Gamma \vdash \forall m A(m)} \qquad\qquad \frac{\Gamma \vdash \forall m A(m)}{\Gamma \vdash A(t)}$$

$$\frac{\Gamma, A \vdash B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B} \qquad \text{(cut)}.$$

From a classical viewpoint, HA^∞ is the same as true arithmetic. From an intuitionistic viewpoint, it is not clear what HA^∞ is. In fact, under CT in the metatheory, it is equivalent to HA*. To see this, we will now consider recursive well founded proof trees.

One gives an inductive definition (of their gödelnumbers) analogous to the one of HA^∞ above, except that in the infinitary rules one requires a recursive sequence of (gödelnumbers of trees that end with) the premises. This is analogous to Kleene's $\mathcal{O}$.

Alternatively, e is a gödelnumber of a well-founded proof tree if

(i) $\{n \mid \{e\}(n) \neq 0\}$ is the set of (codes of) finite sequences of natural numbers that are nodes of a tree w.r.t. the reverse extension of sequences, so that

$$\{e\}(u) = 0 \rightarrow \{e\}(u * \langle n \rangle) = 0,$$

$$\{e\}(u * \langle n \rangle) = 0 \rightarrow \{e\}(u * \langle n+1 \rangle) = 0,$$

where * denotes concatenation.

(ii) For every node u, $\{e\}(u)$ gives the code of one of the inference rules given above in the definition of HA^∞, as well as the codes of the premises and the conclusion.

(iii) The tree is well founded.

One readily shows by induction on the first definition that it is included in the second. In the other direction, given an index e that satisfies (i)–(iii), one uses (iii) to show that for each node u, $\{v \mid v \leqslant u\}$ is a well-founded recursive proof tree in the first definition.

This defines the theory HA^∞_{rec}. It is worth pausing to note that HA^∞_{rec} is complete for Π^0_2 sentences. Let us show that it is equivalent to HA* in any metatheory extending HAS.

For any well-founded (in the metatheory) recursive binary relation R on ω, $TI(R, A)$ has a well-founded recursive proof tree (a related question is discussed in [12, Sect. A.2.3]). Indeed, if for each kRp, e_k is a proof (tree that ends with a code) for

$$\text{Prog} \vdash \forall m(mR\bar{\mathbf{k}} \rightarrow A(m)), \tag{1}$$

where Prog is the assumption of $TI(R, A)$, then the proof e_p of

$$\text{Prog} \vdash \forall n(nR\bar{p} \rightarrow A(n)) \tag{2}$$

is constructed by the infinitary $\forall$-introduction from the premises

$$\text{Prog} \vdash (\bar{\mathbf{k}}R\bar{\mathbf{p}} \rightarrow A(\bar{k})) \qquad \text{all } k, \tag{3}$$

that are easily obtained if $\bar{\mathbf{k}}R\bar{\mathbf{p}}$ is false, and if it is true, (1) and the clearly provable

$$\text{Prog} \vdash \forall m(mR\bar{\mathbf{k}} \rightarrow A(m)) \rightarrow A(\bar{\mathbf{k}})$$

give

$$\text{Prog} \vdash A(\bar{\mathbf{k}})$$

and thus the relevant instance of (3). Because R is well founded, the recursion theorem gives a recursive sequence $\{e_k\}_k$ of proof trees for (1). By another application of the infinitary $\forall$-introduction, one gets a recursive proof tree for

$$\text{Prog} \vdash \forall n.\forall m(mRn \rightarrow A(m)),$$

and thus a recursive proof tree for $\text{TI}(R, A)$. One shows that it is well founded by induction on R (in the metatheory).

For the other direction, let e be an index of a well founded recursive proof tree. We construct a well-founded primitive recursive linear ordering as follows. Suppose

$$\{e\}(\langle n_0, ..., n_s \rangle) = m$$

is computed in exactly k steps. We consider all pairs $(\langle n_0, ..., n_s \rangle, k\rangle$ for which $m = 0$, i.e., $\langle n_0, ..., n_s \rangle$ is a node of the given tree. This primitive recursive set can be linearly ordered à la Brouwer–Kleene: let $\langle \langle n_0, ..., n_s \rangle, k \rangle \prec_e \langle \langle p_0, ..., p_r \rangle, q \rangle$ iff either $s > r$ and $\langle n_0, ..., n_s \rangle$ extends $\langle p_0, ..., p_r \rangle$, or $n_i < p_i$ for the first $i \leqslant r, s$ for which $n_i \neq p_i$. Note that the ordering is itself primitive recursive. It is well founded because the given tree is well founded [17, Sect. 14.1].

Cut-elimination for HA^∞ works in $\text{HA}^\infty_{\text{rec}}$: indeed, there is an index c so that for any well-founded recursive proof tree with index d, $e = \{c\}(d)$ is an index of a well-founded recursive proof tree with the same conclusion in which the cut rule does not occur. Index c is obtained by recursive transfinite induction [13, Chap. 16]. All formulae in e are of bounded complexity, so one shows by transfinite induction on $\prec_e$ that the conclusion at any node is true. Here we have to use the fact that e describes a correct derivation. This is a true Π^0_1 sentence $\forall n E(n)$, E primitive recursive, and therefore provable by TI on the primitive recursive well-ordering given by $m <_E n$ iff either $m < n$ and $\forall k \leqslant n.\ E(k)$, or $n < m$ and $\exists k \leqslant n. \neg E(k)$ [12, TN4]. Thus every sentence provable in $\text{HA}^\infty_{\text{rec}}$ is provable by TI on primitive recursive well-orderings.

2. Embedding HA* into a Heyting-Valued Model of Set Theory

Lemma 2.1. HAH *proves that any Heyting algebra can be embedded in a complete Heyting algebra so that all sups and infs that exist are preserved.*

Proof. Such a completion is folklore from a classical viewpoint. We give a construction here because we need a sharper result concerning provability in HAH, and because it does not appear to be directly available in the literature. Given a Heyting algebra H, let L consist of all ideals in H closed w.r.t. the sups that exist in H. Define finite infs in L by intersection, and let $\bigvee \{X_i \in L \mid i \in I\}$ be the smallest $Y \in L$ for which $X_i \subseteq Y$, for all $i \in I$. The embedding of H into L is given by $p \rightsquigarrow \bar{p} = \{q \in H \mid q \leqslant p\}$, for all $p \in H$. It readily preserves all infs in H. We show that it preserves all sups and implication. If $p = \bigvee \{p_i \mid i \in I\}$ in H, it suffices to show that $\bar{p} \subseteq Y$ for any $Y \in L$ such that $\bar{p}_i \subseteq Y$, for each $i \in I$. But $p \in Y$ follows from $p_i \in Y$, each $i \in I$, because $Y \in L$. The implication in L is given by

$$X \Rightarrow Y = \bigvee \{S \in L \mid X \cap S \subseteq Y\}.$$

One readily has $\overline{p \to q} \subseteq \bar{p} \Rightarrow \bar{q}$ because $x \wedge z \leqslant y$ iff $z \leqslant x \to y$ in any Heyting algebra. For the reverse inclusion it suffices to show that for each $S \in L$:

$$\bar{p} \cap S \subseteq \bar{q} \qquad \text{implies} \qquad S \subseteq \overline{p \to q}.$$

The antecedent gives $s \wedge p \leqslant q$ for any $s \in S$ because S is downward closed in H. Therefore $s \leqslant p \to q$ in H for any $s \in S$. ▮

Similar issues are discussed in [9, Chap. 2]. The construction is an exemple of a general forcing method given in [16, Chap. 1].

For the rest of this section, fix the metatheory $T + CT$, where T is HAS, HAH, or ZFI; and specify H as the Lindenbaum algebra of HA with ω-rule, and let L be its completion. We refer to [1, 7, 15] for the details on Ω-valued models of HAS, HAH, and ZFI for any complete Heyting algebra Ω in T. We concentrate on truth-values of arithmetic sentences in the L-valued model:

$$[\![t = s]\!] = H \qquad \text{if } t = s,$$

$$\{\bot\} \qquad \text{otherwise},$$

where t, s are closed numerical terms,

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!],$$

$$[\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!],$$

$$[\![A \to B]\!] = [\![A]\!] \Rightarrow [\![B]\!],$$

$$[\![\forall n.A(n)]\!] = \bigcap_{n\in\omega} [\![A(\bar{n})]\!],$$

$$[\![\exists n.A(n)]\!] = \bigvee_{n\in\omega} [\![A(\bar{n})]\!].$$

LEMMA 2.2. *($T+CT$) Let A be an arithmetic sentence, and let $[A]$ be its equivalence class in H, the Lindenbaum algebra of* HA$^\infty$. *Let L be the completion of H given in Lemma 2.1. Then, in the L-valued model:*

$$[\![A]\!] = \{q \in H \mid q \leqslant [A]\}.$$

Proof. By induction on the complexity of A. For atomic arithmetic sentences, the statement is true by definition of the L-valued model. In the inductive step, note that for propositional connectives $\circ$, $[A \circ B] = [A] \circ [B]$ in H. Then the statement follows from Lemma 2.1. For the quantifiers, note that the infinitary rules (given in Section 1) state that $[\forall n.A(n)] = \bigwedge_{n\in\omega} [A(\bar{n})]$, $[\exists n.A(n)] = \bigvee_{n\in\omega} [A(\bar{n})]$ in H. Again, apply Lemma 2.1. ∎

Soundness theorem for Heyting-valued models [1, 7, 15] then gives

LEMMA 2.3. *Let A be an arithmetic sentence provable in T. Then $T+CT$ proves that* HA$^\infty$ *proves A.*

Proof. $[\![A]\!]$ is the top element in L, i.e., $[\![A]\!] = H$. By Lemma 2.2, $[A]$ is the top equivalence class in H, i.e., HA$^\infty$ proves A (provably in $T+CT$) ∎

Remark. For Lemma 2.2, we need only that certain subcountable sups in H are preserved by the embedding into L. Lemma 2.1 can be thus reformulated accordingly. By using a standard coding procedure, one can have $T=$ HAS in Lemma 2.3.

3. RECURSIVE REALIZABILITY

We now eliminate CT from the metatheory by the recursive realizability interpretation of $T+CT$ into T. This interpretation was first given for HA by Kleene, and extended to *ZFI* in [2]. We present it here as a syntactical translation.

Fix an enumeration of set variables. Without loss of generality, work with even-indexed ones only (keeping odd-indexed ones for the translation). Let $x'_{2i} = x_{2i+1}$ for set variables, and $t' = t$ for numerical terms. Given a formula A, we define a formula $n \mathbf{r} A$ with one additional free number variable n as follows (π_0, π_1 are primitive recursive coordinates of a pairing function):

$$\begin{array}{lll}
n\,\mathbf{r}\,t_1 = t_2 & \text{is} & t_1 = t_2, \\
n\,\mathbf{r}\,u \in x & \text{is} & \langle n, u' \rangle \in x',\ u \text{ any term}, \\
n\,\mathbf{r}\,A \wedge B & \text{is} & \pi_0(n)\,\mathbf{r}A, \text{ and } \pi_1(n)\,\mathbf{r}\,A, \\
n\,\mathbf{r}\,A \vee B & \text{is} & \begin{bmatrix} \pi_0(n) = 0 \rightarrow \pi_1(n)\,\mathbf{r}\,A \\ \wedge \\ \pi_0(n) \neq 0 \rightarrow \pi_1(n)\,\mathbf{r}\,B \end{bmatrix}, \\
n\,\mathbf{r}\,A \rightarrow B & \text{is} & \forall k(k\,\mathbf{r}\,A \rightarrow \{n\}(k)\downarrow \text{and } \{n\}(k)\,\mathbf{r}\,B), \\
n\,\mathbf{r}\,\exists m A(m) & \text{is} & \pi_1(n)\,\mathbf{r}\,A(\pi_0(n)), \\
n\,\mathbf{r}\,\forall m A(m) & \text{is} & \forall m(\{n\}(m)\downarrow \text{and } \{n\}(m)\,\mathbf{r}\,A(\bar{m})), \\
n\,\mathbf{r}\,\exists x A(x) & \text{is} & \exists x'.n\,\mathbf{r}\,A(x), \\
n\,\mathbf{r}\,\forall x A(x) & \text{is} & \forall x'.n\,\mathbf{r}\,A(x).
\end{array}$$

Soundness theorem for recursive realizability [2] states that if $T-$Extensionality proves A, then for some numeral $\bar{n}$, T proves $\bar{n}\,\mathbf{r}\,A$. One can interpret T in $T-$Extensionality by a translation that preserves formulae of HAS, as in [3].

LEMMA 3.1. *Let* WFPT(n) *be the formula of* HAS *stating that n is an index of a well-founded recursive proof tree (say, in the second definition, cf. Section* 1*). Then T proves* $(m\,\mathbf{r}\,\text{WFPT}(n)) \rightarrow \text{WFPT}(n)$.

Proof. The only problem is the condition (iii) requiring that the tree is well founded. Assume R is the tree ordering, and

$$m\,\mathbf{r}\,[\forall n(\forall i(iRn \rightarrow i \in x) \rightarrow n \in x) \rightarrow \forall n.n \in x],$$

i.e.,

$$\forall x'.m\,\mathbf{r}\,[\forall n(\forall i(iRn \rightarrow i \in x) \rightarrow n \in x) \rightarrow \forall n.n \in x], \tag{4}$$

i.e., for each x' and each k, if

$$k\,\mathbf{r}\,[\forall n(\forall i(iRn \rightarrow i \in x) \rightarrow n \in x], \tag{5}$$

then $\{m\}(k)$ is defined, and

$$\{m\}(k)\,\mathbf{r}\,\forall n.n \in x. \tag{6}$$

As (4) holds for each x', it holds in particular for x' of the kind $\bar{y} = \{\pi(l_0, l_1) \mid l_1 \in y\}$, where π is a primitive recursive pairing function with primitive recursive coördinates π_0, π_1. Given any set y such that

$$\forall n(\forall i(iRn \rightarrow i \in y) \rightarrow n \in y),$$

we wish to show $\forall n.n \in y$. So, for each n, let

$$\forall i(iRn \to i \in y) \to n \in y, \tag{7}$$

and let k be an index so that

$$\{\{k\}(l)\}(j) = 0$$

for every l, j. We claim that (5) holds. Indeed, let

$$j \,\mathbf{r}\, \forall i(iRn \to i \in y),$$

i.e., for each i, $\{j\}(i) \,\mathbf{r}\, (iRn \to i \in y)$, i.e., for each p, if $p \,\mathbf{r}\, (iRn)$, then $\{\{j\}(i)\}(p) \,\mathbf{r}\, (i \in y)$. Because the tree is recursive, iRn is a $\sum_1^0$ formula, so the last implication means that iRn implies $i \in y$, for each i. Therefore, $n \in y$ by (7). Thus $0 \,\mathbf{r}\, (n \in y)$, so (5) holds. Because of (4), one now has (6), i.e.,

$$\forall n.(\langle \{\{n\}(k)\}(n), n \rangle \in \bar{y}),$$

i.e., $\forall n.n \in y$, as required. ∎

Now we can eliminate CT from the metatheory.

LEMMA 3.2. *Let A be an arithmetic sentence such that $T + CT$ proves that* HA^∞ *proves A. Then T proves that* $\mathrm{HA}^\infty_{\mathrm{rec}}$ *proves A.*

Proof. $T + CT$ proves $\mathrm{HA}^\infty = \mathrm{HA}^\infty_{\mathrm{rec}}$. Apply Lemma 3.1. ∎

This gives our main result:

THEOREM 3.1. *Let A be an arithmetic sentence provable in T. Then there is a numeral $\bar{n}$ so that T proves that $\bar{n}$ is an index of a recursive well-founded proof tree with A as its conclusion. In particular, T is conservative over* HA^*.

Proof. By Lemma 2.3, HA^∞ proves A, demonstrably in $T + CT$. By Lemma 3.2, T proves that A has a recursive well-founded proof tree. Apply the numerical existence property for T, as in [5]. ∎

4. PROVABLE TRANSFINITE INDUCTION

There are primitive recursive linear orderings R which are not well founded, yet $\mathrm{TI}(R, A)$ is provable in PA for any arithmetic A [10]. On the other hand, it was shown in [4] that there is an arithmetic formula $B(n)$ with only n free, such that if R is any primitive recursive binary relation on ω for which HA proves $\mathrm{TI}(R, B)$, then R is in fact well founded. We now use Theorem 3.1 to extend this result from HA to theories T discussed at the beginning of Section 1.

$B(n)$ is obtained as follows. By a recursion-theoretic infinite injury argument [14], there is an effective sequence $\{Q_n\}_n$ of very independent r.e. subsets of ω, i.e., an r.e. set $Q \subseteq \omega \times \omega$ such that for each m, $Q_m = \{k \mid (m, k) \in Q\}$ is not recursive in $\{(p, k) \mid k \in Q_p, p \neq m\}$. Let $B(n)$ be an arithmetic formula saying $\forall m(m \in Q_n \vee m \notin Q_n)$. The following result was proved in [4]:

LEMMA 4.1. *Let R be a primitive recursive binary relation on ω. Then R is well founded iff* $\mathrm{HA}^{\infty}_{\mathrm{rec}}$ *proves* $\mathrm{TI}(R, B)$.

THEOREM 4.1. *There is an arithmetic formula $B(n)$ with n free, such that any primitive recursive binary relation R on ω for which T proves* $\mathrm{TI}(R, B)$ *is in fact well founded.*

Proof. If $\mathrm{TI}(R, B)$ is provable in T, it is provable in $\mathrm{HA}^{\infty}_{\mathrm{rec}}$ by Theorem 3.1. Thus by Lemma 4.1, R is well founded. ▮

The Heyting-valued model given in Section 2 corresponds to a mild forcing extension in the sense of [8, Sect. 37.1]. Because we need to consider only subcountable sups, the definition of the *cHa L* is absolute w.r.t. inner models. By the methods in [6, Sect. 5], Theorems 3.1 and 4.1 extend to all six theories given in [6] that claim the existence of large sets, and are equiconsistent, respectively, with *ZF* plus the existence of inaccessible, Mahlo, measurable, supercompact, and huge cardinals, and Reinhardt's Axiom.

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