Asymptotic Eigenvalues of Sturm-Liouville Systems*

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1. INTRODUCTION

The first few terms in the asymptotic expansion

\[ \lambda_n^{1/2} = n + a_0 + a_1 n^{-1} + \cdots \]

for the eigenvalues \( \lambda_n \) of smooth Sturm-Liouville systems have been computed in ([1]-[3]). In this paper, we shall extend these results by developing recurrence formulas (Corollaries 1 and 2) with which all higher order terms in an infinite asymptotic expansion can be determined. These formulas appear to be new.

Once the asymptotic expansion for the eigenvalues has been determined, the corresponding expansions for the eigenfunctions can be obtained by using Horn's method ([4, Chap. 3]). Although the author made these calculations, they will not be given in this paper since there does not appear to be a compact recurrence formula, analogous to that given in Corollaries 1 and 2, for obtaining all the terms in the expansion.

The asymptotic expansions for the eigenvalues are compared elsewhere [5] with the numerical accuracy of methods employing high speed computing machines.

We recall ([6], pp. 265-267) that any smooth Sturm-Liouville system can be written

\[ u'' + [\lambda - p(x)] u = 0, \quad 0 \leq x \leq \pi. \tag{1} \]

subject to suitable endpoint conditions such as

\[ u'(0) - \alpha u(0) = u'(\pi) + \beta u(\tau) = 0, \tag{2} \]

or,

\[ u(0) = u(\pi) = 0. \tag{3} \]

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If $p$ is piecewise continuous on $[0, \pi]$, then the above system has a sequence of distinct real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{n \to \infty} \lambda_n = +\infty. \quad (4)$$

In the sequel, we shall assume that $p$ is continuously differentiable on $[0, \pi]$.

2. MODIFIED PRüFER SUBSTITUTION

To study the asymptotic behavior of eigenvalues for large $n$, we shall introduce the function $\phi(x; \lambda)$, the modifier Prüfer phase, which is defined for any given solution $u(x; \lambda)$ of (1) by the equation

$$\tan \phi = [\lambda - p(x)]^{-1/2} u^{-1} u'.$$

The differential equation for $\phi$ corresponding to (1) is

$$\phi' = - [\lambda - p(x)]^{1/2} p'(x) [4(\lambda - p(x))]^{-1} \sin 2\phi. \quad (6)$$

**Theorem 1.** Let $\phi_0$ and $\lambda$ be given with $\lambda \geq p_0 > p(x)$ for all $x \in [0, \pi]$. Then there exists a unique solution $\phi(x; \lambda)$ of (5) such that $\phi(0; \lambda) = \phi_0$. Moreover, if the sequence $\{\phi_n(n) : n = 1, 2, \ldots\}$ is defined recursively by

$$\phi_1(x; \lambda) = \phi_0 - \lambda^{1/2} x + \frac{1}{2} \lambda^{-1} P_1(x)$$

$$\phi_{n+1}(x; \lambda) = \phi_0 - \sum_{\ell=0}^{n+1} H(x; \ell) \lambda^{\ell+1/2} + \sum_{\ell=0}^{n-1} K(x; \ell; n) \lambda^{\ell-1}, \quad (7)$$

where

$$H(x; \ell) = \binom{1}{\ell} (1 - 1)^\ell P_\ell(x),$$

$$K(x; \ell; n) = \binom{-1}{\ell} (1 - 1)^\ell \int_0^x P'(t)^\ell \sin [2\phi_n] \ dt \quad (7')$$

$$P_\ell(x) = \int_0^x p(t)^\ell dt, \quad \binom{a}{n} = \frac{a(a-1) \cdots (a-n+1)}{n!}, \quad (7'')$$

then

$$|\phi(x, \lambda) - \phi_n(x; \lambda)| \leq O(\lambda^{-n}) \quad (8)$$

for all $x \in [0, \pi]$. 
PROOF. For existence and uniqueness, see [7, Chap. 5]. We shall prove (8) by induction on n. First, observe that

\[
[\lambda - p(x)]^{1/2} = \sum_{\ell=0}^{\infty} \left(\frac{2}{\ell}\right)(-1)^{\ell} p(x)^{\ell} \lambda^{-\ell-1/2}
\]

(9)

and

\[
[\lambda - p(x)]^{-1} = \sum_{\ell=0}^{\infty} \left(-\frac{1}{\ell}\right)(-1)^{\ell} p(x)^{\ell} \lambda^{-\ell-1},
\]

(10)

the series being uniformly convergent for \(x \in [0, \pi]\). This implies

\[
|\phi'_1 + [\lambda - p]^{1/2} - p'[4(\lambda - p)^{-1}] \sin 2\phi| = O(\lambda^{-1}).
\]

(11)

Therefore, from a classical theorem [1, p. 105], it follows that

\[
|\phi_1(x; \lambda) - \phi(x; \lambda)| = O(\lambda^{-1}), \quad x \in [0, \pi].
\]

(12)

This proves (8) for \(n = 1\). Assume (8) is true for \(n = k\). Then

\[
\sin [2\phi(x; \lambda)] = \sin [2\phi_k(x; \lambda)] + O(\lambda^{-k}).
\]

(13)

Also, from (7)

\[
\phi_{k+1} = -\sum_{\ell=0}^{k+1} \left(\frac{3}{\ell}\right)(-p)^{\ell} \lambda^{-\ell+1/2} + \frac{1}{2} \sum_{\ell=0}^{k-1} \left(-\frac{1}{\ell}\right)\lambda^{-\ell-1} p'(-p)^{\ell} \sin [2\phi_k]
\]

\[
= -[\lambda - p]^{1/2} + \frac{1}{2} p'[\lambda - p]^{-1} \sin 2\phi
\]

\[
+ \frac{1}{2} p'[\lambda - p]^{-1} [\sin 2\phi_k - \sin 2\phi] + O(\lambda^{-k-1})
\]

\[
= \phi' + O(\lambda^{-k-1}).
\]

So, again appealing to [6, p. 105], we have

\[
|\phi_{k+1}(x; \lambda) - \phi(x; \lambda)| = O(\lambda^{-k-1}), \quad x \in [0, \pi].
\]

This proves (8) for \(n = k + 1\).

3. Asymptotic Expansions

In this section we shall use Theorem 1 to obtain expansions for the eigenvalues.
LEMMA 2. Let \( \lambda_n \) be the \( n \)th eigenvalue of (1) and (3). Let \( \phi(x; \lambda_n) \) be the modified phase of the associated eigenfunction. Then

\[
\phi(0; \lambda_n) = -\frac{\pi}{2}, \quad \phi(\pi; \lambda_n) = -\left(n + \frac{1}{2}\right)\pi.
\]

Now, let \( \lambda_n \) be the \( n \)th eigenvalue of (1)-(2) and let \( \phi(x; \lambda_n) \) be the phase of the associated eigenfunction. Then

\[
\phi(0; \lambda_n) = + \omega \lambda_n^{1/2} - \left[ \frac{\alpha^3}{3} + \frac{\alpha \beta}{2} \right] \lambda_n^{-3/2} + O(\lambda_n^{-4})
\]

\[
\phi(\pi; \lambda_n) - \phi(0; \lambda_n) = -n\pi - [\beta + \alpha] \lambda_n^{-1/2}
\]

\[
\quad + \left[ \frac{\beta^3 - \alpha^3}{3} + \frac{\beta \beta^2}{2} \right] \lambda_n^{-3/2} + O(\lambda_n^{-4}).
\]

PROOF. See [6, Chap. 10].

COROLLARY 1. Let \( \lambda_n \) be the \( n \)th eigenvalue of (1) and (3). Put \( \mu_n = \lambda_n^{1/2} \). Then for any positive integer \( m \),

\[
\mu_n = n + \sum_{\ell=1}^{m+1} H(\pi; \ell) \mu_n^{-2\ell+1} + \sum_{\ell=0}^{m-1} K(\pi; \ell; m) \mu_n^{-2\ell-2} + O(\mu_n^{-2m-2}),
\]

where \( H(\pi; \ell) \) and \( K(\pi; \ell; m) \) are given by (7').

COROLLARY 2. Let \( \lambda_n \) be the \( n \)th eigenvalue of (1)-(2). Put \( \mu_n = \lambda_n^{1/2} \). Then for any positive integer \( m \),

\[
\mu_n = n + \mu_n^{-1}Q_1 + \mu_n^{-3}Q_2 + \sum_{\ell=1}^{m+1} H(\pi; \ell) \mu_n^{-2\ell+1}
\]

\[
\quad + \sum_{\ell=0}^{m-1} K(\pi; \ell; m) \mu_n^{-2\ell-2} + O(\mu_n^{-2m-2}),
\]

where \( H_\ell \) and \( K_{m\ell} \) are given by (7')

\[
Q_1 = \frac{1}{\pi} \left[ \alpha + \beta + \frac{1}{2} P_1(\pi) \right]
\]

\[
Q_2 = \frac{1}{\pi} \left[ \frac{1}{8} P_2(\pi) - \frac{1}{3} \beta^3 - \frac{1}{3} \alpha^3 - \frac{1}{2} \beta^2(\pi) - \frac{1}{2} \alpha \beta \right].
\]

From (4), (16), and (17) it follows that the eigenvalues of (1)-(2) as well as (1) and (3) have an infinite asymptotic expansion in powers of \( n^{-1} \). Moreover, (16) and (17) can be used to compute as many terms as desired.
COROLLARY 3. Let $\lambda_n$ be the $n$th eigenvalue of (1) and (3). Then

$$\lambda_n^{1/2} = n + (2\pi n)^{-1} P_1(\pi) + (4\pi n^3)^{-1} \left[ \left( \frac{1}{2} \right) P_2(\pi) - \left( \frac{1}{\pi} \right) P_1(\pi)^2 + \left( \frac{1}{2} \right) P'(\pi) - \left( \frac{1}{2} \right) P'(0) \right] + O(n^{-4}).$$  \hspace{1cm} (21)

COROLLARY 4. Let $\lambda_n$ be the $n$th eigenvalue of (1)-(2). Then

$$\lambda_n^{1/2} = n + \frac{1}{n} Q_1 + \frac{1}{n^3} \left[ Q_2 - Q_1^2 - \frac{1}{8} P'(\pi) + \frac{1}{8} P'(0) \right] + O(n^{-4}),$$  \hspace{1cm} (22)

where $Q_1$ and $Q_2$ are given by (18) and (19).

4. Examples

In this section we shall consider several numerical examples of interest.

Example 1. The Mathieu equation can be written

$$u'' + \left[ \lambda - s \cos^2 x \right] u = 0, \hspace{1cm} 0 < x < \pi. \hspace{1cm} (23)$$

This equation has a basis of even and odd periodic eigenfunctions, the even eigenfunctions being determined by (2) with $\alpha = \beta = 0$ while the odd eigenfunctions are determined by (4). Using Corollaries 3 and 4, we have

$$\lambda_n^{1/2} = n + \frac{s}{4\pi} - \frac{s^2}{64\pi^3} + O(n^{-4}),$$  \hspace{1cm} (24)

or

$$\lambda_n = n^2 + \frac{s}{2} + \frac{s^2}{32\pi^2} + O(n^{-3})$$  \hspace{1cm} (25)

for both the eigenvalues associated with the even periodic eigenfunctions and the eigenvalues associated with odd periodic eigenfunctions. Formula (25) gives very accurate approximations for all but the first few eigenvalues; for example, letting $\nu_n$ denote the exact eigenvalue, we have for $s = 2$

$$\nu_4 - \lambda_4 = .00050, \hspace{1cm} \nu_{12} - \lambda_{12} = .000006,$$  \hspace{1cm} (26)

the values of $\nu_n$ being obtained from [8].

Example 2. Ince’s equation [7, p. 145] is

$$v'' + 4\beta_2^{1/2} \sin 2xv' + \left[ \lambda + 2\beta_1 + 4\beta_2^{1/2} \right] \cos 2x \right] v = 0.$$  \hspace{1cm} (27)
Letting
\[ w = - \theta^{1/2} \cos 2\pi v, \]
we obtain the three term Hill equation
\[ w'' + [\lambda + \theta_1 \cos 2x + \theta_2 \cos 4x] w = 0. \] (28)
As with the Mathieu Eq. (28) has a basis of even and odd periodic eigenfunctions. Corollaries 3 and 4 yield
\[ \lambda_n^{1/2} = n + \frac{1}{4n^3} [\theta_1^2 + \theta_2^2] + O(n^{-4}) \] (29)
\[ \lambda_n = n^2 + \frac{1}{2n^2} [\theta_1^2 + \theta_2^2] + O(n^{-3}) \] (29')
for both the eigenvalues associated with even periodic eigenfunctions and the eigenvalues associated with odd periodic eigenfunctions.

**EXAMPLE 3.** A boundary value problem involving the parabolic cylinder equation is
\[ u'' + [\lambda - \frac{1}{4} x^2] u = 0 \quad 0 < x < \pi \]
\[ u(0) = u(\pi) = 0. \] (30)
The nth eigenfunction of (30) is
\[ u_n(x) = x \exp \left( - \frac{1}{4} x^2 \right) M\left( \frac{3}{4}, \frac{3}{2}, \frac{1}{2} \pi \right), \] (31)
where \( M(a, b, x) \) is the Kummer function [9, p. 337] and the associated eigenvalue \( \nu_n \) is the nth root of
\[ M\left( \frac{3}{4} - \frac{1}{2} \lambda, \frac{3}{2}, \frac{1}{2} \pi \right) = 0. \] (32)
From Corollary 3 we have for the asymptotic approximation \( \lambda_n \) to \( \nu_n \)
\[ \lambda_n^{1/2} = n + \frac{\pi^2}{24n} + \frac{1}{16n^3} \left[ 1 - \frac{\pi^4}{270} \right] + O(n^{-4}). \] (33)

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**REFERENCES**


