Long time behaviour for generalized complex Ginzburg–Landau equation

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Received 12 December 2002
Available online 11 September 2006
Submitted by C. Rogers

Abstract

In this paper, the two-dimensional generalized complex Ginzburg–Landau equation (CGL)

$$u_t = \rho u - \Delta \varphi(u) - (1 + i\gamma)\Delta u - \nu \Delta^2 u - (1 + i\mu)|u|^{2\sigma} u + \alpha \lambda_1 \cdot \nabla (|u|^2 u) + \beta (\lambda_2 \cdot \nabla)|u|^2$$

is studied. The existence of global attractor for this equation with periodic boundary condition is established and upper bounds of Hausdorff and fractal dimensions of attractor are obtained.

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Keywords: Generalized complex Ginzburg–Landau equation; Global attractor; Fractal dimension

1. Introduction

The description of spatial pattern formation or chaotic dynamics in continuum systems, in particular fluid dynamical system, is a challenging task in theoretical physics and applied mathematics. Due to the complexity of the corresponding nonlinear evolution equations, simpler model equations for which the mathematical issues can be solved with greater success, have been derived. The complex Ginzburg–Landau equation is one of these equations. It models the evolution of the amplitude of perturbations to steady-state solutions at the onset of instability. It

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0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.07.095
is a particularly interesting model because it is a dissipative version of the nonlinear Schrödinger equation—a Hamiltonian equation which can possess solutions that form localized singularities in finite time.

The complex Ginzburg–Landau equation (CGL) is of the form:

$$u_t = \rho u + (1 + i \nu) u_{xx} - (1 + i \mu) |u|^{2\sigma} u.$$  

Doering et al. [1], Ghidaglia and Héron [2], studied the finite dimension of global attractor and related dynamical issues for the one or two spatial dimensional GLE with cubic nonlinearity ($\sigma = 1$). Bartuccelli, Constantin, Doering, Gibbon and Gisselfält [3] investigated the “soft” and “hard” turbulent behavior for this equation. Doering, Gibbon, Levermore [4] presented the existence and uniqueness of global weak and strong solutions for this equation in all spatial dimensions and for all degree of nonlinearity $\sigma > 0$. Levermore and Oliver studied the well-posedness and regularity questions for this equation [5]. D. Li studied Cauchy problem for generalized complex Ginzburg–Landau equation [6].

In this paper, we consider two-dimensional generalized complex Ginzburg–Landau equation

$$u_t = \rho u - \Delta \varphi(u) - (1 + i \gamma) \Delta u - \nu \Delta^2 u - (1 + i \mu) |u|^{2\sigma} u 
+ \alpha \lambda_1 \cdot \nabla(|u|^2 u) + \beta (\lambda_2 \cdot \nabla)|u|^2 \ \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+,$$  

with the following usual initial-value and boundary condition:

$$u(x, 0) = u_0(x) \ \text{in} \ \mathbb{R}^2, $$  

and

$$u \ \text{is} \ \Omega\text{-periodic},$$  

where $u$ is a unknown complex-value function, $\Delta$ is a Laplacian, $\Omega = (-L, L) \times (-L, L)$, $\rho > 0$, $\gamma$, $\mu$, $\nu$ are real parameters.

The main result of this paper is to establish the existence of global attractor for Eqs. (1.1)–(1.3) under appropriate assumption on degree of nonlinearity $\sigma$, and to estimate the Hausdorff and fractal dimensions of this attractor.

In the following, we denote that $H^k = H^k(\Omega)$ be the usual $\Omega$-periodic Sobolev space with the norm $\|u\|_{H^k(\Omega)} = (\sum_{|x| \leq k} \|D^x u\|_{L^2(\Omega)}^2)^{1/2}$ and $\|\cdot\|_p$ with usual inner product $(\cdot, \cdot)$, $\|\cdot\|_{L^p(\Omega)}$ denotes the norm of $L^p(\Omega)$ for $1 \leq p \leq \infty$ ($\|\cdot\|_2 = \|\cdot\|$).

2. Existence and uniqueness of the solution

First, we discuss local existence of the problem (1.1)–(1.3).

We need the Agmon’s inequality

$$\|u\|_{L^\infty(\Omega)} \leq c \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2} \ \forall u \in H^2(\Omega),$$

and Gagliardo–Nirenberg inequality:

$$\|\nabla^j u\|_{p} \leq c \|\nabla^m u\|_{r}^a \|u\|_{q}^{1-a}$$

where

$$\frac{1}{p} = \frac{j}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + \frac{1-a}{q}$$

with $1 \leq q, r \leq \infty$, $0 \leq j < m$ and $\frac{j}{m} \leq a < 1$. 

We define a linear operator in \( H = L^2 \), \( Au = \nu \Delta u \) with definition domain \( D(A) = H^4 \cap H^2_0 \). By Lumer–Phillips theorem the linear operator \( A \) is the infinitesimal generator of a continuous semigroup of contraction \( s(t) = \exp At \) for \( t > 0 \) (see [7,8]).

Let
\[
N(u) = \rho u - (1 + i\gamma)\Delta u - \Delta \varphi(u) - (1 + i\mu)|u|^{2\sigma}u + \alpha \lambda_1 \cdot \nabla(|u|^2u) + \beta(\lambda_2 \cdot \nabla)|u|.
\]

Then (1.1) can be rewritten as
\[
 u_t + Au = N(u),
 u(0) = u_0.
\]

In order to obtain the existence of local solution of the problem (1.1)–(1.3) for every \( u_0 \in H^2(\Omega) \), we need the following lemma.

**Lemma 2.1.** The \( N(u) \) maps \( H^2(\Omega) \) into \( L^2 \) and satisfies for \( u, v \in H^2 \),
\[
\begin{align*}
\|N(u)\| & \leq C\|u\|_{H^2}, \\
\|N(u) - N(v)\| & \leq C(\|u\|_{H^2}, \|v\|_{H^2})\|u - v\|_{H^2}.
\end{align*}
\]

By using G–N inequality, the proof of the lemma is not difficult, we omit it here. Therefore we have

**Lemma 2.2.** For every \( u_0 \in H^2(\Omega) \) there exists a unique solution \( u \) of the problem (1.1), (1.2) on a finite time interval \( t \in [0, T_{\text{max}}) \) such that
\[
 u \in C^1((0, T_{\text{max}}) : H^2(\Omega)) \cap C([0, T_{\text{max}}) : H^2(\Omega))
\]
with the property that
\[
 T_{\text{max}} = \infty \quad \text{or if} \quad T_{\text{max}} < \infty \quad \text{then} \quad \lim_{t \to T_{\text{max}}} \|u(t)\|_{H^2} \to \infty.
\]

**Proof.** The result follows from Theorems 3.3.3, 3.3.4, 3.5.2 in [7].

In order to show the global existence of the solution for all \( t > 0 \), we need to establish some a priori estimates on \( u(t) \) in phase space \( L^2 \) and \( H^1, H^2 \).

**Lemma 2.3.** Assume that \( u_0 \in L^2(\Omega) \), and \( \varphi'(u) \leq 0 \), then for the solution \( u(t) \) of (1.1)–(1.3), we have
\[
\|u\|^2 \leq K_1(R, T_1), \quad t \in [0, T_1], \tag{2.1}
\]
where \( K_1(R, T_1) \) depends on data and \( R, T_1, \) and \( T_1 \) depends on the data and \( R \) when \( \|u_0\| \leq R \).

**Proof.** Taking inner product of (1.1) in \( H \) with \( u \) and then taking the real part of the resulting identity, it follows
\[
\frac{1}{2} \frac{d}{dt}\|u\|^2 = \rho \int |u|^2 + \|\nabla u\|^2 - v\Delta u|^2 - (\Delta \varphi(u), u) - \int |u|^{2\sigma + 2} + Re(u, \alpha \lambda_1 \cdot \nabla(|u|^2u) + \beta(\lambda_2 \cdot \nabla u)|u|^2). \tag{2.2}
\]

Obviously,
\[
Re(u, \alpha \lambda_1 \cdot \nabla(|u|^2u) + \beta(\lambda_2 \cdot \nabla u)|u|^2) = 0.
\]
We also have
\[ (\Delta \phi(u), u) = - (\nabla \phi(u), \nabla u) = - (\phi'(u) \nabla u, \nabla u) = -(\phi'(u), \|\nabla u\|^2) \geq 0 \]
and
\[ \|\nabla u\|^2 = -(\Delta u, u) \leq \|\Delta u\| \|u\| \leq \frac{\nu}{2} \|\Delta u\|^2 + \frac{1}{2\nu} \|u\|^2. \]
Thus synthesizing the above inequality we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{\nu}{2} \|\Delta u\|^2 + \int |u|^{2\sigma+2} \leq \left( \frac{1}{2\nu} + \rho \right) \|u\|^2.
\end{equation}
(2.3)
Since
\[ |u|^2 = |u|^2 \cdot 1 \leq \frac{1}{2k} |u|^{2\sigma+2} + c \]
for any positive constant $K$, integrating it we find that
\[ \|u\|^2 \leq \frac{1}{2k} \int |u|^{2\sigma+2} + c. \]
Therefore, we deduce
\begin{equation}
\frac{d}{dt} \|u\|^2 + \nu \|\Delta u\|^2 \leq \left(-4k + \frac{1+2\nu \rho}{\nu}\right) \|u\|^2 + 2c.
\end{equation}
(2.4)
Applying Gronwall’s inequality we conclude (2.1). □

Here and after we denote by $C$, $c$ any constants depending on the data $\sigma$, $\rho$, $\nu$, $\mu$, $\alpha$, $\beta$, $\lambda_1$, $\lambda_2$.

**Lemma 2.4.** Assume that the conditions of Lemma 2.3 hold, $\sigma < 2$ and
\[ |\phi'(u)| \leq C|u|^p, \quad 0 \leq p < 2, \]
then for the solution of the problem (1.1)–(1.3) we have
\[ \|\nabla u\|^2 \leq K_2(R, T_2), \quad t \in [0, T_2], \]
where $K_2(R, T_2)$ depends on data and $R, T_2$, and $T_2$ depends on the data and $R$ when $\|u_0\|_{H^1} \leq R$.

**Proof.** Multiplying (1.1) by $\Delta u$ then integrating on $\Omega$ and taking the real part of the resulting identity, we find that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 = \rho \|\nabla u\|^2 + \|\Delta u\|^2 - \nu \|\nabla \Delta u\|^2 + (\Delta \phi(u), \Delta u)
+ Re(1 + i \mu) \int |u|^{2\sigma} u \Delta \bar{u} + Re(\Delta u, \alpha \lambda_1 \cdot \nabla (|u|^2 u))
+ Re(\Delta u, \beta (\lambda_2 \cdot \nabla u) |u|^2). \tag{2.5}
\end{equation}
First, we have
\[ (\Delta \phi(u), \Delta u) = - (\nabla \phi(u), \nabla \Delta u) = - (\phi'(u) \nabla u, \nabla \Delta u) \]
\[ \leq c \|u\|_p^p \|\nabla u\|_4 \|\nabla \Delta u\|. \tag{2.6} \]
By G–N inequality
\[
\|u\|_r \leq c \|\nabla u\|_r^{\frac{1}{2}} \|u\|_r^{\frac{3}{2}}, \\
\|\nabla u\|_r \leq c \|\nabla \Delta u\|_r^{\frac{1}{2}} \|u\|_r^{\frac{3}{2}},
\]
(2.7)
we get
\[
(\Delta \varphi(u), \Delta u) \leq c \|\nabla \Delta u\|_r^{\frac{p+4}{p}} \|u\|_r^{\frac{2p+2}{p}} \quad \text{since } 0 \leq p < 2
\]
\[
\leq \frac{\nu}{10} \|\nabla \Delta u\|^2 + c \quad \text{by Lemma 2.3},
\]
(2.8)
and
\[
\|\nabla u\| \leq c \|\Delta \nabla u\|^{\frac{2}{3}} \|u\|^{\frac{1}{3}}.
\]
For the fifth term of right side in (2.5) we have
\[
\Re(1 + i\mu) \int |u|^{2\sigma} u \Delta \bar{u} \leq \sqrt{1 + \mu^2} \int |u|^{2\sigma+1} |\Delta u|
\]
\[
\leq \sqrt{1 + \mu^2} \|u\|^{2\sigma+1} \|\Delta u\|
\leq \sqrt{1 + \mu^2} \|\nabla \Delta u\|^{\frac{2+2\sigma}{3}} \|u\|^{\frac{4\sigma+4}{3}} \quad \text{since } \sigma < 2
\]
\[
\leq \frac{\nu}{10} \|\nabla \Delta u\|^2 + c(\mu, \nu) \|u\|^{\frac{2(4\sigma+4)}{2-2\sigma}},
\]
(2.9)
for the sixth term of right side in (2.5) we have
\[
\left| \Re(\Delta u, \beta(\lambda_2 \cdot \nabla (|u|^2 u))) \right| \leq |\beta\lambda_2| \int |\Delta u||\nabla u||u|^2
\]
\[
\leq |\beta\lambda_2| \|\Delta u\| \|\nabla u\| \|u\|^2
\leq |\beta\lambda_2| \|\nabla \Delta u\| \|\nabla u\| \|u\|^2
\leq \frac{\nu}{10} \|\nabla \Delta u\|^2 + c(\nu) |\beta\lambda_2| \|u\|^{14}.
\]
(2.10)
Similarly, due to
\[
\nabla(|u|^2 u) = |u|^2 \nabla u + \nabla |u|^2 u = 2|u|^2 \nabla u + u^2 \nabla \bar{u},
\]
we infer that
\[
\left| \Re(\Delta u, \alpha\lambda_1 \cdot \nabla (|u|^2 u)) \right| \leq \frac{\nu}{10} \|\nabla \Delta u\|^2 + c(\nu) |\alpha\lambda_1| \|u\|^{14}.
\]
(2.11)
By Lemma 2.3 and (2.7) with \( r = 2 \), we have
\[
\|\Delta u\|^2 = (\nabla \Delta u, \nabla u) \leq \|\nabla \Delta u\| \|\nabla u\|
\leq c \|\nabla \Delta u\| \|u\|^2 \leq \frac{\nu}{10} \|\nabla \Delta u\|^2 + c \|u\|^2.
\]
(2.12)
By (2.5)–(2.12) and Lemma 2.3, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \frac{\nu}{2} \|\nabla \Delta u\|^2 \leq \rho \|\nabla u\|^2 + C.
\]
(2.13)
The proof of the lemma follows from Gronwall’s inequality. \( \Box \)
Lemma 2.5. Assume that the conditions of Lemma 2.4 hold, and
\[ |\varphi''(u)| \leq c|u|^{p-1} \]
where \( p \) is as in Lemma 2.4. Then for \( u_0 \in H^2_{\text{per}}(\Omega) \), the solution of problem (1.1)–(1.3) satisfies
\[ \|\Delta u\|^2 \leq K_3(R, T_3), \quad t \in [0, T_3], \]
where \( K_3(R, T_3) \) depends on data and \( R, T_3 \), and \( T_3 \) depends on the data and \( R \) when \( \|u_0\|_{H^2} \leq R \).

Proof. Taking the inner product of (1.1) with \( \Delta^2 u \), then taking real part of the resulting identity, we find that
\[ \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 = \rho \|\Delta u\|^2 + \|\nabla \Delta u\|^2 - \nu \|\Delta^2 u\|^2 + (\Delta^2 u, \Delta^2 u) \]
\[ - \text{Re}(1 + i\mu) \int |u|^{2\sigma} u \Delta^2 u \]
\[ + (\Delta^2 u, \alpha \lambda_1 \cdot \nabla (|u|^2 u) + \beta (\lambda_2 \cdot \nabla u)|u|^2). \quad (2.14) \]

By G–N inequality, we have
\[ \|u\|_r \leq c\|\Delta^2 u\|^{\frac{r-2}{r}} \|u\|^{\frac{3r+2}{4r}}. \]

We estimate terms of right side in (2.14) now.

Noting that
\[ \|u\|_{4\sigma + 2} \leq \|\Delta^2 u\|^{\frac{\sigma}{4\sigma + 2}} \|u\|^{\frac{3\sigma + 2}{4\sigma + 2}}, \]
we deduce
\[ \text{Re}(1 + i\mu) \int |u|^{2\sigma} u \Delta^2 u \leq \sqrt{1 + \mu^2 \|\Delta^2 u\|^{\frac{\sigma}{2} + 1} \|u\|^{\frac{3\sigma + 2}{4\sigma + 2}}} \]
\[ \leq \sqrt{1 + \mu^2} \|\Delta^2 u\|^{\frac{\sigma}{2} + 1} \|u\|^{\frac{3\sigma + 2}{4\sigma + 2}} \quad \text{since } \sigma < 2 \]
\[ \leq \frac{\nu}{8} \|\Delta^2 u\|^2 + c(\mu)\|u\|^{\frac{2(3\sigma + 2)}{3 - \sigma}}. \quad (2.15) \]

By G–N inequality, we deduce
\[ \|\nabla u\|_{L^4} \leq \|\Delta^2 u\|^{\frac{3}{8}} \|u\|^{\frac{5}{8}}, \]
\[ \|u\|_{L^8} \leq \|\Delta^2 u\|^{\frac{3}{16}} \|u\|^{\frac{13}{16}}. \]

It is true that
\[ (\Delta^2 u, \alpha \lambda_1 \cdot \nabla (|u|^2 u)) \leq |\alpha \lambda_1| \left| \int \Delta^2 u \nabla (|u|^2 u) \right| \]
\[ \leq 3|\alpha \lambda_1| \int |\Delta^2 u||u|^2 |\nabla u| \]
\[ \leq 3|\alpha \lambda_1| \|\Delta^2 u\| \|u\|_4 \|\nabla u\|_4 \]
\[ \leq 3|\alpha \lambda_1| \|\Delta^2 u\|^{\frac{3}{2}} \|u\|^{\frac{9}{2}} \]
\[ \leq \frac{1}{16} \|\Delta^2 u\|^2 + 3c|\alpha \lambda_1| \|u\|^8. \quad (2.16) \]
Similar to (2.16), we have
\[
\left| (\Delta^2 u, \beta(\lambda_2 \cdot \nabla u)|u|^2) \right| \leq \frac{\nu}{8} \| \Delta^2 u \|^2 + c|\beta \lambda_2\| \|u\|^8. \tag{2.17}
\]
We also have
\[
-(\Delta \varphi(u), \Delta^2 u) \leq \| \Delta \varphi(u) \| \| \Delta^2 u \|
\leq \| \varphi''(u)(\nabla u)^2 + \varphi'(u) \Delta u \| \| \Delta^2 u \|
\leq c\|u\|^{p-1}_{4(p-1)} \| \nabla u \|_8^2 \| \Delta^2 u \| + c\|u\|^p_{4p} \| \Delta u \|_4 \| \Delta^2 u \|.
\]
By G–N inequality, we deduce
\[
\| \Delta u \|_4 \leq c\| \Delta^2 u \|^\frac{1}{2} \| \nabla u \|^\frac{1}{2},
\| \nabla u \|_8 \leq c\| \Delta^2 u \|^\frac{1}{2} \| \nabla u \|^\frac{1}{2},
\|u\|^{p-1}_{4(p-1)} \leq c\| \Delta^2 u \|^{\frac{2p-3}{8}} \|u\|^6^{\frac{p-5}{8}},
\|u\|^p_{4p} \leq c\| \Delta^2 u \|^{\frac{2p-1}{8}} \|u\|^6^{\frac{p+1}{8}}.
\]
Thus
\[
-(\Delta \varphi(u), \Delta^2 u) \leq c\|u\|^{p-1}_{4(p-1)} \| \nabla u \|_8^2 \| \Delta^2 u \| + c\|u\|^p_{4p} \| \Delta u \|_4 \| \Delta^2 u \|
\leq c\| \Delta^2 u \|^{1+\frac{2p-3}{8}} \|u\|^6^{\frac{p-5}{8}} + c\| \Delta^2 u \|^{1+\frac{2p-3}{8}} \| \nabla u \|^2 \|u\|^{\frac{1}{2}} \|u\|^{6^{\frac{p+1}{8}}}
\leq \frac{\nu}{8} \| \Delta^2 u \|^2 + c, \quad \text{since } p < 2. \tag{2.18}
\]
Combining (2.14)–(2.18), we get
\[
\frac{d}{dt} \| \Delta u \|^2 + \nu \| \Delta^2 u \|^2 \leq -2\rho \| \Delta u \|^2 + \| \nabla \Delta u \|^2 + C.
\]
Noting that
\[
\| \nabla \Delta u \|^2 = -(\Delta^2 u, \Delta u) \leq \| \Delta^2 u \| \| \Delta u \| \leq \frac{\nu}{2} \| \Delta^2 u \| + \frac{1}{2\nu} \| \Delta u \|^2,
\]
we find
\[
\frac{d}{dt} \| \Delta u \|^2 + \frac{\nu}{2} \| \Delta^2 u \|^2 \leq \left( \frac{1}{2\nu} + 2\rho \right) \| \Delta u \|^2 + c. \tag{2.19}
\]
Applying Gronwall inequality, we conclude the proof of the lemma. \(\square\)

From Lemmas 2.3–2.5, we obtain the existence of global solution of the problem (1.1)–(1.3).

**Theorem 2.6** (Global Existence). Assume that \(\sigma < 2\) and
\[
\varphi'(u) \leq 0, \quad |\varphi^{(k)}(u)| \leq C|u|^{p+1-k}, \quad k = 1, 2, \ 0 \leq p \leq 2,
\]
then there exists a unique global solution \(u(t)\) for problem (1.1)–(1.3) such that
\[
u \in C^1([0, \infty) : H^2(\Omega)) \cap C([0, \infty) : H^2(\Omega)).
\]
3. Existence of the global attractor

By Theorem 2.6, we know that there exists a dynamical system $S(t)$ $(t \geq 0)$ which maps $H_{\text{per}}^2(\Omega)$ to $H_{\text{per}}^2(\Omega)$ such that $S(t)u_0 = u(t)$, the solution of problem (1.1)–(1.3). It is clear that \{S(t)\}_{t \geq 0} enjoy the property of semigroup.

In this section, we construct the global attractor for the problem (1.1)–(1.3). We first prove the existence of absorbing sets. For the purpose, we need uniform a priori estimates in time.

Repeating the procedure of Lemma 2.3, we can derive

$$\frac{d}{dt} \|u\|^2 + \nu \|\Delta u\|^2 \leq -\left( k - \frac{1-2\nu\rho}{\nu} \right) \|u\|^2 + 2c. \tag{3.1}$$

Choosing $k$ large enough such that $k' = k - \frac{1+2\nu\rho}{\nu} \geq 0$, applying Gronwall’s inequality we conclude

$$\|u\|^2 \leq \|u_0\|^2 e^{-k't} + \frac{2c}{k'} \left( 1 - e^{-2k't} \right).$$

Thus

$$\|u\|^2 \leq \frac{4c}{k'} = \rho_1^2, \quad t \geq t_1, \tag{3.2}$$

where $t_1 = \frac{1}{k'} \ln \frac{k'\|u_0\|^2}{2c}$.

Repeating the procedure of Lemma 2.4, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \frac{\nu}{2} \|\nabla \Delta u\|^2 \leq \rho \|\nabla u\|^2 + C. \tag{3.3}$$

By G–N inequality and Young’s inequality we easily obtain

$$2\rho \|\nabla u\|^2 \leq \frac{\nu}{2} \|\nabla \Delta u\|^2 + C(\nu) \|u\|^2.$$

Thus, we give from (3.3),

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 \leq -\rho \|\nabla u\|^2 + C.$$

By Gronwall’s inequality, we obtain

$$\|\nabla u\|^2 \leq \|\nabla u_0\|^2 e^{-2\rho t} + \frac{C}{\rho} (1 - e^{-2\rho t})$$

and that

$$\|\nabla u\|^2 \leq \frac{2C}{\rho} = \rho_2^2, \quad \forall t > t_2, \tag{3.4}$$

where $t_2 = \frac{1}{2\rho} \ln \frac{\rho \|\nabla u_0\|^2}{2C}$.

Repeating the proof of Lemma 2.5, we also have

$$\frac{d}{dt} \|\Delta u\|^2 + \nu \|\Delta^2 u\|^2 \leq \left( \frac{1}{2\nu} + 2\rho \right) \|\Delta u\|^2 + c. \tag{3.5}$$

Due to
\[2 \left( \frac{1}{2v} + 2 \rho \right) \| \Delta u \|^2 = -2 \left( \frac{1}{2v} + 2 \rho \right) (\nabla \Delta u, \nabla u) \]
\[\leq c \| \nabla \Delta u \| \leq c \| \Delta^2 u \| \frac{1}{2} \| u \|^{\frac{3}{2}} \]
\[\leq \frac{V}{2} \| \Delta^2 u \|^2 + c(v) \| u \|^2 \]

we deduce that
\[
\frac{d}{dt} \| \Delta u \|^2 \leq - \left( \frac{1}{2v} + 2 \rho \right) \| \Delta u \|^2 + c. \tag{3.6}
\]

Applying Gronwall inequality, we obtain
\[
\| \Delta u \|^2 \leq \frac{4vc}{1 + 4v \rho} = \rho_3^2 \quad \forall t > t_3, \tag{3.7}
\]

where \( t_3 = \frac{2v}{1 + 4v \rho} \ln \left( \frac{1 + 4v \rho}{2vc} \| \Delta u_0 \|^2 \right) \).

(3.2), (3.4) with (3.7) together imply that the ball
\[
B = \{ u(x, t) \mid \| u(x, t) \|^2_{H^2(\Omega)} \leq \rho_1^2 + \rho_3^2 \}
\]
is an absorbing set in \( H^2(\Omega) \).

In the following, we show that
\[
S(t) (t \geq 0) : H^2_{\text{per}}(\Omega) \to H^2_{\text{per}}(\Omega)
\]
is compact for large \( t \).

**Lemma 3.1.** Assume that the conditions of Theorem 2.6 hold. The solution of problem (1.1), (1.2) satisfies
\[
\| \nabla \Delta u \|^2 \leq \rho_4^2, \quad \text{for } t \geq t_4, \tag{3.8}
\]

where \( \rho_4 \) depends on data, and \( t_4 \) depends on data and \( R \) when \( \| u_0 \|_{H^2} < R \).

**Proof.** Taking the inner product of (1.1) with \( \Delta^3 u \) in \( L^2(\Omega) \), and then taking real part of the resulting identity, we find that
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \Delta u \|^2 = \rho \| \nabla \Delta u \|^2 + \| \Delta^2 u \|^2 - \nu \| \nabla \Delta^2 u \|^2
\]
\[ + (\Delta \varphi(u), \Delta^3 u) - \text{Re}(1 + i \mu) \int |u|^{2\sigma} u \Delta^3 \bar{u}
\]
\[ + (\Delta^3 u, \alpha \lambda_1 \cdot \nabla (|u|^{2\sigma} u) + \beta (\lambda_2 \cdot \nabla u)|u|^2). \tag{3.9}
\]

We estimate terms of right side in (3.9) now,
\[
\text{Re}(1 + i \mu) \int |u|^{2\sigma} u \Delta^3 \bar{u} \leq \sqrt{1 + \mu^2} \int |\nabla (|u|^{2\sigma} u)| \| \nabla \Delta^2 u \|
\]
\[\leq (\sigma + 1) \sqrt{1 + \mu^2} \int |u|^{2\sigma} |\nabla u| \| \nabla \Delta^2 u \|
\]
\[\leq (\sigma + 1) \sqrt{1 + \mu^2} \| u \|^{2\sigma}_{\infty} \| \nabla u \| \| \nabla \Delta^2 u \|
\]
\[\leq \frac{\nu}{10} \| \nabla \Delta^2 u \|^2 + C(v, \sigma, \mu, \rho_2, \rho_3). \tag{3.10}
\]

Using
\[
\| \nabla u \|^4 \leq \| \nabla \Delta^2 u \|^3 \| u \|^2 \tag{3.11}
\]
we have

\[
(\Delta^3u, \alpha_1 \cdot \nabla(|u|^2u)) \leq |\alpha_1| \int |\nabla \Delta^2u| |\Delta(|u|^2u)|
\]

\[
\leq |\alpha_1| \int |\nabla \Delta^2u| (3|\Delta u||u|^2 + 4|\nabla u|^2|u|)
\]

\[
\leq 3|\alpha_1| \int |\nabla \Delta^2u| |\Delta u||u|^2 + 4|\alpha_1| \int |\nabla \Delta^2u| |\nabla u|^2|u|
\]

\[
\leq 3|\alpha_1| ||u||^2_{\infty} ||\nabla \Delta^2u|| ||\Delta u|| + 4|\alpha_1| ||u||_{\infty} ||\nabla \Delta^2u|| ||\nabla u||^2
\]

\[
\leq 3|\alpha_1| ||u||^2_{\infty} ||\nabla \Delta^2u|| ||\Delta u|| + 4|\alpha_1| ||u||_{\infty} ||\nabla \Delta^2u||^{\frac{n+1}{\alpha}} ||u||^{\frac{\alpha}{n}}
\]

\[
\leq \frac{\nu}{10} ||\nabla \Delta^2u||^2 + C_1(\alpha, \lambda_1, \nu, \rho_1, \rho_2, \rho_3).
\] (3.12)

Similar to (3.12), we have

\[
|(\Delta^2u, \beta (\lambda_2 \cdot \nabla u)|u|^2)| \leq \frac{\nu}{10} ||\nabla \Delta^2u||^2 + C_2(\beta, \lambda_2, \nu, \rho_1, \rho_2, \rho_3).
\] (3.13)

We also have

\[
(\Delta \varphi(u), \Delta^3u) = -|\nabla \Delta \varphi(u), \nabla \Delta^2u|
\]

\[
\leq ||\varphi'''(u)||_{\infty} ||\nabla u||^2_{\infty} ||\nabla \Delta^2u|| + 3||\varphi''(u)||_{\infty} ||\Delta u||_4 ||\nabla u||_4 ||\nabla \Delta^2u||
\]

\[
+ ||\varphi'(u)||_{\infty} ||\nabla \Delta u|| ||\nabla \Delta^2u||.
\] (3.14)

Noting that

\[
||\nabla u||_6 \leq ||\nabla \Delta^2u||^{\frac{1}{5}} ||\nabla u||^{\frac{4}{5}},
\]

\[
||\Delta u||_4 \leq ||\nabla \Delta^2u||^{\frac{1}{4}} ||u||^{\frac{3}{4}},
\]

\[
||\nabla \Delta u|| \leq ||\nabla \Delta^2u||^{\frac{3}{4}} ||u||^{\frac{1}{4}},
\] (3.15)

substituting (3.11), (3.15) into (3.14) we obtain

\[
(\Delta \varphi(u), \Delta^3u) \leq c ||\nabla \Delta^2u||^{1+\frac{1}{2}} ||u||^{\frac{3}{2}} + 3c ||\nabla \Delta^2u||^{1+\frac{8}{m}} ||u||^{\frac{12}{m}} + c ||\nabla \Delta^2u||^{1+\frac{1}{2}} ||u||^{\frac{3}{2}}
\]

\[
\leq \frac{\nu}{10} ||\nabla \Delta^2u||^2 + c.
\] (3.16)

Moreover, we deduce that

\[
||\nabla \Delta^2u||^2 = -(\nabla \Delta^2u, \nabla \Delta u) \leq ||\nabla \Delta^2u|| ||\nabla \Delta u||
\]

\[
\leq \frac{\nu}{10} ||\nabla \Delta^2u||^2 + \frac{5}{2\nu} ||\nabla \Delta u||^2.
\] (3.17)

Inserting (3.10), (3.12), (3.13), (3.16) and (3.17) into (3.9) we deduce

\[
\frac{1}{2} \frac{d}{dt} ||\nabla \Delta u||^2 + \frac{\nu}{2} ||\nabla \Delta^2u||^2 \leq \left( \frac{5}{2\nu} + \rho \right) ||\nabla \Delta u||^2 + c.
\] (3.18)

By G–N inequality and Young’s inequality again we infer from (3.18) that

\[
\frac{d}{dt} ||\nabla \Delta u||^2 \leq \left( \frac{5}{2\nu} + \rho \right) ||\nabla \Delta u||^2 + c.
\] (3.19)
Applying uniform Gronwall inequality we obtain
\[ \| \nabla \Delta u \|^2 \leq \frac{4vc}{5 + 2v_\rho} = \rho_4^2, \quad \forall t > t_4. \]

The proof of the lemma is completed. \(\square\)

Lemma 3.1 implies that dynamical system \(S(t)(t \geq 0)\) is uniformly compact for large \(t\) in \(H^2(\Omega)\) and then by [9] we obtain the following theorem.

**Theorem 3.2.** Assume that the conditions of Theorem 2.6 hold. Then the \(w\)-limit set of \(B\),
\[ A = w(B) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)B \]
is maximal, compact, connected global attractor for \(S(t)\) on \(H^2(\Omega)\), where the closure is taken in \(H^2(\Omega)\).

**4. Hausdorff dimension and the fractal dimension of the attractor**

To establish the estimates of the upper bounds of the Hausdorff dimension and the fractal dimension of the attractor \(A\), we consider the first variation equation of the problem (1.1)–(1.3)
\[ v_t = F'(u(t))v \quad \text{(4.1)} \]
with initial value condition
\[ v(0) = v_0 \in H \quad \text{(4.2)} \]
and periodic boundary condition
\[ v \text{ is } \Omega\text{-periodic} \quad \text{(4.3)} \]
where
\[ F'(u(t))v = \Delta(\varphi'(u)v) - \rho v - (1 + i\gamma)\Delta v - v\Delta^2 v + (1 + i\mu)(1 + \sigma)|u|^{2\sigma}v + (1 + i\mu)\sigma|u|^{2\sigma-2}u^2v^* + 2\alpha(\lambda_1 \cdot \nabla(|u|^2v)) + \alpha(\lambda_1 \cdot \nabla(u^2v^*)) + \beta(\lambda_2 \cdot \nabla v)|u|^2 + \beta(\lambda_2 \cdot \nabla u)(uv^* + uv^*). \quad \text{(4.4)} \]

By the standard methods we can show that \(\forall v_0 \in H\), the initial boundary value problem (1)–(3) possesses a unique \(v(t)\) such that
\[ v(t) \in L^2(0, T; H^2) \cap L^\infty(0, T; H), \quad \forall T > 0. \]

We can prove that semigroup \(S(t)\) is uniform differentiable on \(A\), namely

**Lemma 4.1.** For \(w_0, u_0 \in A\), and \(\|w_0\|_{H^2} \leq R, \|u_0\|_{H^2} \leq R, 0 \leq t \leq T\), then there exists a constant \(K\), such that
\[ \frac{\|S(t)w_0 - S(t)u_0 - G(t)(w_0 - u_0)\|}{\|w_0 - u_0\|} \leq K\|w_0 - u_0\|. \quad \forall 0 \leq t \leq T, \]
where \(K\) depends on the data \((\sigma, \rho, v, \mu, \Omega), T, \) and \(R\) when \(\|A\|_{H^2} \leq R\).
Theorem 4.2. Assume $V_1(t), \ldots, V_m(t)$ are the solutions of (4.1)–(4.3) corresponding $V_{01}, \ldots, V_{0m}, V_k(t) = S(t)V_{0k}$ ($k = 1, \ldots, m$), then

$$\left| V_1(t) \Lambda \cdots \Lambda V_m(t) \right|_{A^n H} = \left| V_{01} \Lambda \cdots \Lambda V_{0m} \right|_{A^n H} \exp \int_0^t \text{Re Tr} \ F'(u(\tau)) \circ Q_m(\tau) \, d\tau$$

$$\leq \left| V_{01} \Lambda \cdots \Lambda V_{0m} \right|_{A^n H} \exp \left( -C_1 m^{n+1/2} + C_2 |\Omega|^{-\frac{3}{n}} + C_3 |\Omega| \right)$$

(4.5)

where $Q_m(\tau) = Q_m(\tau, u_0, V_{01}, \ldots, V_{0m})$ is the orthonormal projection in $H$ onto the space spanned by $V_1(\tau), \ldots, V_m(\tau)$ and $\Lambda$ represents the exterior product, $Tr$ is the trace of the operator.

Proof. At a given time $\tau$, let $\varphi_j(\tau), j \in \mathbb{N}$, is an orthonormal basis of $H$ such that $Q_m(\tau)H = \text{span}\{V_1(\tau), \ldots, V_{0m}(\tau)\}$.

Since $V_j(\tau) \in H^2(\Omega)$ for a.e. $\tau$, $\varphi_1(\tau), \ldots, \varphi_m(\tau)$ belong to $H^2(\Omega)$ for a.e. $\tau$. Further, since $H = L^2(\Omega)$, the $\varphi_j(\tau)$ considered as vector of $L^2(\Omega)$ are orthogonal too.

And thus, we have

$$\text{Re Tr} \ F'(u(\tau)) \circ Q_m(\tau) = \sum_{j=1}^m \text{Re} \left( F'(u(\tau)) \circ Q_m(\tau) \varphi_j(\tau), \varphi_j(\tau) \right)$$

$$= \sum_{j=1}^m \text{Re} \left( F'(u(\tau)) \varphi_j(\tau), \varphi_j(\tau) \right).$$

(4.6)

Omitting temporarily the variable $\tau$, we see that

$$\text{Re} \left( F'(u)\varphi_j(\tau), \varphi_j(\tau) \right)$$

$$= \text{Re} \left( \Delta(\varphi'(u)) \varphi_j(\tau) - \rho \varphi_j(\tau) - (1 + i\gamma) \Delta \varphi_j(\tau) + \nu \Delta^2 \varphi_j(\tau) \right)$$

$$- (1 + i\mu)(1 + \sigma)|u|^{2\sigma} \varphi_j(\tau) - (1 + i\mu)\sigma |u|^{2\sigma - 2} u^2 \varphi_j^*(\tau)$$

$$+ 2\alpha(\lambda_1 \cdot \nabla(|u|^2 \varphi_j(\tau))) + \alpha(\lambda_1 \cdot \nabla(u^2 \varphi_j(\tau)^*))$$

$$+ \beta(\lambda_2 \cdot \nabla \varphi_j(\tau))|u|^2 + \beta(\lambda_2 \cdot \nabla u)(\varphi_j(\tau)u^* + u\varphi_j(\tau)^*), \varphi_j(\tau))$$

$$= \rho \| \varphi_j(\tau) \|^2 - \| \nabla \varphi_j(\tau) \|^2 - \text{Re} \left( \Delta(\varphi'(u)\varphi_j(\tau)), \varphi_j(\tau) \right) - \text{Re} \nu(\Delta^2 \varphi_j(\tau), \varphi_j(\tau))$$

$$- \text{Re}(1 + \sigma) \int |u|^{2\sigma} |\varphi_j(\tau)|^2 - \text{Re}(1 + i\mu)\sigma \int |u|^{2\sigma - 2} u^2 (\varphi_j^*(\tau))^2$$

$$+ \text{Re}(2\alpha(\lambda_1 \cdot \nabla(|u|^2 \varphi_j(\tau)))) + \alpha(\lambda_1 \cdot \nabla(u^2 \varphi_j(\tau)^*))$$

$$+ \beta(\lambda_2 \cdot \nabla \varphi_j(\tau))|u|^2 + \beta(\lambda_2 \cdot \nabla u)(\varphi_j(\tau)u^* + u\varphi_j(\tau)^*), \varphi_j(\tau)).$$

(4.7)

We now majorize every term in (4.7) as follows

$$- \text{Re} \left( \Delta(\varphi'(u)\varphi_j(\tau)), \varphi_j(\tau) \right) = \text{Re} \left( \varphi'(u)\varphi_j(\tau), \Delta \varphi_j(\tau) \right)$$

$$\leq \| \varphi'(u) \|_{\infty} \| \varphi_j(\tau) \| \| \Delta \varphi_j(\tau) \|$$

$$\leq \frac{\nu}{10} \| \Delta \varphi_j(\tau) \|^2 + \| \varphi'(u) \|_{\infty}^2 \| \varphi_j(\tau) \|^2,$$

(4.8)
\[-\text{Re } \nu (\Delta^2 \varphi_j (\tau), \varphi_j (\tau)) \leq -\nu \| \Delta \varphi_j (\tau) \|^2, \quad (4.9)\]
\[-\text{Re } (1 + i \mu)(1 + \sigma) \int |u|^{2\sigma} |\varphi_j (\tau)|^2 \leq (1 + \sigma) \sqrt{1 + \mu^2} \| u \|_{\infty}^{2\sigma} \| \varphi_j (\tau) \|^2 \leq c_1 \| \varphi_j (\tau) \|^2, \quad (4.10)\]
\[-\text{Re } (1 + i \mu) \sigma \int |u|^{2\sigma - 2} (\varphi_j^*(\tau))^2 \leq \sigma \sqrt{1 + \mu^2} \| u \|_{\infty}^{2\sigma} \| \varphi_j (\tau) \|^2 \leq c_2 \| \varphi_j (\tau) \|^2, \quad (4.11)\]
\[2 \text{Re } (\alpha (\lambda_1 \cdot \nabla |u|^2 \varphi_j (\tau)), \varphi_j (\tau)) \leq 2 \alpha \int (\lambda_1 \cdot \nabla \varphi_j (\tau)) |u|^2 \varphi_j (\tau)^* \leq 2 |\alpha \lambda_1| \| u \|_{\infty}^2 \| \nabla \varphi_j (\tau) \| \| \varphi_j (\tau) \| \leq \frac{\nu}{10} \| \Delta \varphi_j (\tau) \|^2 + c_3 \| \varphi_j (\tau) \|^2. \quad (4.12)\]

Similarly
\[\text{Re } (\alpha (\lambda_1 \cdot \nabla |u|^2 \varphi_j (\tau)^*)) + \beta (\lambda_2 \cdot \nabla \varphi_j (\tau)) |u|^2 + \beta (\lambda_2 \cdot \nabla u)(\varphi_j (\tau)u^* + u \varphi_j (\tau)^*), \varphi_j (\tau)) \leq \frac{3 \nu}{10} \| \Delta \varphi_j (\tau) \|^2 + c_3 \| \varphi_j (\tau) \|^2. \quad (4.13)\]

Therefore, we have from (4.6)–(4.13)
\[\text{Re } \text{Tr } F'(u(\tau)) \circ Q_m(\tau) \leq -\frac{\nu}{2} \sum_{j=1}^{m} \| \Delta \varphi_j (\tau) \|^2 + (\rho + C + \| \varphi'(u) \|_{\infty}^2) \sum_{j=1}^{m} \| \varphi_j (\tau) \|^2. \quad (4.14)\]

Let
\[\eta = \sum_{j=1}^{m} |\varphi_j (\tau)|^2. \]

Since the family \( \varphi_j \ (j = 1, \ldots, m) \) is orthonormal in \( H \), we see that
\[\sum_{j=1}^{m} \| \varphi_j (\tau) \|^2 = \int_{\Omega} \eta \, dx = m. \quad (4.15)\]

It follows from the Sobolev–Leib–Thirring inequality that
\[\int_{\Omega} \eta^{1 + \frac{\alpha}{n}} \, dx \leq \frac{C_0}{|\Omega|^{\frac{\alpha}{2}}} \int_{\Omega} \eta \, dx + C_0 \sum_{j=1}^{m} \| \Delta \varphi_j (\tau) \|^2 \quad (4.16)\]

where \( C_0 \) depends only on the sharp of \( \Omega \).

By the Hölder inequality we infer that
\[\int_{\Omega} \eta \, dx \leq |\Omega|^{\frac{\alpha}{n + \beta}} \left( \int_{\Omega} \eta^{1 + \frac{\alpha}{n}} \, dx \right)^{\frac{n}{n + \beta}}, \]
and therefore
\[ |\Omega|^{-\frac{4}{n}} \left( \int_{\Omega} \eta \, dx \right)^{\frac{n+4}{n}} \leq \int_{\Omega} \eta^{1+\frac{4}{n}} \, dx. \] (4.17)

By (4.16) and (4.17), it follows that
\[ -\sum_{j=1}^{m} \| \Delta \varphi_j (\tau) \|^2 \leq -\frac{1}{C_0} \int_{\Omega} \eta^{1+\frac{4}{n}} \, dx + \frac{1}{|\Omega|^{\frac{4}{n}}} \int_{\Omega} \eta \, dx \leq -\frac{1}{C_0} |\Omega|^{-\frac{4}{n}} m^{\frac{n+4}{n}} + |\Omega|^{-\frac{4}{n}} m. \] (4.18)

Summing up (4.14) and (4.18), we claim that
\[ \text{Re} \text{Tr} F' (u(\tau)) \circ Q_m (\tau) \leq -\frac{1}{2} |\Omega|^{-\frac{4}{n}} m^{\frac{n+4}{n}} + \frac{1}{2} |\Omega|^{-\frac{4}{n}} m + (\rho + C + \| \varphi'(u) \|_\infty)^2 m \]
\[ \leq -C_1 m^{\frac{n+4}{n}} + C_2 |\Omega|^{-\frac{4}{n}} + C_3 |\Omega| \] (4.19)

where we have used Young’s inequality and $C_1, C_2, C_3$ depend on the data, $n, \nu$, and the shape of $\Omega$, but are independent of the volume of $\Omega$.

And thus we obtain the proof of Theorem 4.2. □

For $i = 1, \ldots, m$, and $V_{0i} \in H^2(\Omega) \times H^1_0(\Omega)$ we define
\[ q_m(t) = \sup_{u_0 \in A} \sup_{\|v_0\| \leq 1} \left( \frac{1}{t} \int_0^t \text{Re} \text{Tr} F' (u(\tau)) \circ Q_m (\tau) \, d\tau \right), \]
\[ q_m = \lim_{t \to \infty} \sup q_m(t). \]

It follows from (4.19) that
\[ q_m \leq -C_1 m^{\frac{n+4}{n}} + C_2 |\Omega|^{-\frac{4}{n}} + C_3 |\Omega| \leq 0. \]

This shows that if $m$ is defined by
\[ m - 1 \leq \left( \frac{C_2 |\Omega|^{-\frac{4}{n}}}{C_1} + \frac{C_3 |\Omega|}{C_1} \right)^{\frac{n}{n+4}} \leq m \]
then $q_m < 0$ and then we obtain Theorem 4.3.

**Theorem 4.3.** Let $A$ be the compact attractor of the semigroup operator $S(t)$ defined by the initial value problem (1.1)–(1.3), then the Hausdorff dimension of $A$ is less than or equal to $m$, and its fractal dimension is less than or equal to $2m$.

**References**
