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A rate of convergence for asymptotic contractions

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Abstract

In [P. Gerhardy, A quantitative version of Kirk's fixed point theorem for asymptotic contractions, J. Math. Anal. Appl. 316 (2006) 339–345], P. Gerhardy gives a quantitative version of Kirk's fixed point theorem for asymptotic contractions. This involves modifying the definition of an asymptotic contraction, subsuming the old definition under the new one, and giving a bound, expressed in the relevant moduli and a bound on the Picard iteration sequence, on how far one must go in the iteration sequence to at least once get close to the fixed point. However, since the convergence to the fixed point needs not be monotone, this theorem does not provide a full rate of convergence. We here give an explicit rate of convergence for the iteration sequence, expressed in the relevant moduli and a bound on the sequence. We furthermore give a characterization of asymptotic contractions on bounded, complete metric spaces, showing that they are exactly the mappings for which every Picard iteration sequence converges to the same point with a rate of convergence which is uniform in the starting point.

Keywords: Metric fixed point theory; Asymptotic contractions; Proof mining

1. Introduction

The notion of asymptotic contractions was introduced by W.A. Kirk in [8]. There also a fixed point theorem is proved, stating that given a complete metric space (X, d) and a continuous asymptotic contraction $f : X \to X$, if for some $x \in X$ the Picard iteration sequence $(f^n(x))$ is bounded, then for every starting point $x \in X$ the iteration sequence $(f^n(x))$ converges to the

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unique fixed point of f. (Note that, as remarked in e.g. [1,7], in the statement of the theorem in [8] the assumption that the mapping must be continuous was inadvertently left out.) Kirk's proof of this theorem uses ultrapowers. An elementary proof was given by I.D. Arandelović in [1]. We will first for reference give the definition of an asymptotic contraction in the sense of Kirk.

Definition 1.1. [8] A function $f : X \to X$ on a metric space (X, d) is called an *asymptotic contraction in the sense of Kirk* with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ if ϕ, ϕ_n are continuous, $\phi(s) < s$ for all $s > 0$ and for all $x, y \in X$,

$$d\big(f^n(x), f^n(y)\big) \leqslant \phi_n\big(d(x, y)\big),$$

and moreover $\phi_n \to \phi$ uniformly on the range of d.

In [7], J. Jachymski and I. Jóźwik prove a similar theorem, relaxing the requirements on ϕ, ϕ_n, but in addition assuming f to be uniformly continuous. They also give a criterion which makes the assumption that one iteration sequence is bounded superfluous. Also Y.-Z. Chen [4] develops a version of the theorem proved by Kirk. For details, see the remarks preceding Proposition 3.10. However, none of the above treatments gives explicit numerical information concerning the convergence to the fixed point. A quantitative version of Kirk's theorem on asymptotic contractions is given by P. Gerhardy in [6]. Here Gerhardy makes use of techniques from the program of proof mining, as developed by U. Kohlenbach (see e.g. [9,10]). In [6], an alternative definition of asymptotic contractions is given, according to which a function is an asymptotic contraction if it has certain moduli expressing its asymptotic contractivity. This covers the usual definition. An explicit bound is then presented, expressed by these moduli and the bound on the iteration sequence, on how far one must go in the iteration sequence to at least once get within a specified distance of the fixed point. The proof is completely elementary. This theorem does not, however, give a rate of convergence to the fixed point in the general case. The convergence needs not be monotone, and so for $m > n$ it is not the case that $f^m(x)$ needs to be close to the fixed point if $f^n(x)$ is. For an example of such a function, see Example 2 in [7]. In contrast to this, the results in [6] do give a rate of convergence when the convergence to the fixed point is monotone, and this is the case for a very large class of functions, including the nonexpansive ones.

We here give for the general case an explicit rate of convergence to the unique fixed point for sequences $(f^n(x))$. The assumptions are in general the same as in [6]. We will, however, consider a slightly more general definition of asymptotic contractions. This will be of importance in relation to a further result. We show also that the rate of convergence only depends on the starting point and the function through a bound on the iteration sequence and the moduli mentioned above.

2. Preliminaries

We give for reference the alternative definition of an asymptotic contraction from [6]. We will in the following make reference to certain results from [6], including some lemmas, some propositions and a theorem. These will be readily identifiable, but we will not repeat them here.

Definition 2.1. [6] A function $f : X \to X$ on a metric space (X, d) is called an *asymptotic contraction in the sense of Gerhardy* if for each $b > 0$ there exist moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ and the following hold:

(1) There exists a sequence of functions $\phi_n^b : (0, \infty) \to (0, \infty)$ such that for all $x, y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$,

$$b \geqslant d(x, y) \geqslant \varepsilon \quad \text{gives} \quad d\big(f^n(x), f^n(y)\big) \leqslant \phi_n^b(\varepsilon) d(x, y).$$

(2) For each $0 < l \leqslant b$ the function $\beta_l^b := \beta^b(l, \cdot)$ is a modulus of uniform convergence for $(\phi_n^b)_{n \in \mathbb{N}}$ on $[l, b]$, i.e.

$$\forall \varepsilon > 0\ \forall s \in [l, b]\ \forall m, n \geqslant \beta_l^b(\varepsilon)\big(\big|\phi_m^b(s) - \phi_n^b(s)\big| \leqslant \varepsilon\big).$$

(3) Define $\phi^b := \lim_{n \to \infty} \phi_n^b$. Then for each $0 < \varepsilon \leqslant b$ we have

$$\phi^b(s) + \eta^b(\varepsilon) \leqslant 1$$

for each $s \in [\varepsilon, b]$.

We modify this definition as follows.

Definition 2.2. A function $f : X \to X$ on a metric space (X, d) is called a (*generalized*) *asymptotic contraction* if for each $b > 0$ there exist moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ and the following hold:

(1) There exists a sequence of functions $\phi_n^b : (0, \infty) \to (0, \infty)$ such that for each $0 < l \leqslant b$ the function $\beta_l^b := \beta^b(l, \cdot)$ is a modulus of uniform convergence for $(\phi_n^b)_{n \in \mathbb{N}}$ on $[l, b]$, i.e.

$$\forall \varepsilon > 0\ \forall s \in [l, b]\ \forall m, n \geqslant \beta_l^b(\varepsilon)\big(\big|\phi_m^b(s) - \phi_n^b(s)\big| \leqslant \varepsilon\big).$$

Furthermore, if $\varepsilon < \varepsilon'$ then $\beta_l^b(\varepsilon) \geqslant \beta_l^b(\varepsilon')$.

(2) For all $x, y \in X$, for all $b \geqslant \varepsilon > 0$ and for all $n \in \mathbb{N}$ such that $\beta_\varepsilon^b(1) \leqslant n$, we have:

$$b \geqslant d(x, y) \geqslant \varepsilon \quad \text{gives} \quad d\big(f^n(x), f^n(y)\big) \leqslant \phi_n^b(\varepsilon) d(x, y).$$

(3) Define $\phi^b := \lim_{n \to \infty} \phi_n^b$. Then for each $0 < \varepsilon \leqslant b$ we have

$$\phi^b(s) + \eta^b(\varepsilon) \leqslant 1$$

for each $s \in [\varepsilon, b]$.

If f is an asymptotic contraction in the sense of Gerhardy, then it is also an asymptotic contraction in our sense. However, one might have to modify the moduli β_l^b. As pointed out in Remark 8 in [6], we can equivalently give the moduli η^b, β^b as functions $\eta^b : \mathbb{N} \to \mathbb{N}$ and $\beta^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, with real numbers approximated from below by rationals 2^{-n}. This is the case for the moduli in Definition 2.2 as well as the moduli in Definition 2.1. Then β^b in the sense of Definition 2.2 is effectively computable in β^b in the sense of Definition 2.1. We will for short call the mappings defined in Definition 2.2 just asymptotic contractions.

Unless otherwise specified we will throughout let (X, d) be a complete metric space, and given $x_0 \in X$ and $f : X \to X$ we will let (x_n) be the sequence defined by $x_{n+1} := f(x_n)$. When there is no risk of ambiguity we will drop the superscripts from η^b and β^b. We first note that in particular the following results from [6] go through also for our generalized definition of asymptotic contractions: Proposition 4, Lemmas 10, 11, 13–15, and Theorem 16. This can be verified by inspection of the proofs, and by in so doing noting that $\eta(\varepsilon)/2 < 1$. We mention in passing that in Lemma 11 in [6] one must modify the conclusion by writing $d(x_m, f^N(x_m)) \leqslant \delta$

instead of $d(x_m, f^N(x_m)) < \delta$, whether one considers asymptotic contractions in the sense of Definition 2.1 or in the sense of Definition 2.2. This has no further implications, since the lemma is only used in the proofs of Lemmas 13 and 14, and there the modified conclusion of Lemma 11 works just as well. We will from now on assume Lemma 11 thus modified. (One could of course instead have slightly modified the functional M appearing in the lemma.)

We note also that the results in [6] which subsume Definition 1.1 under Definition 2.1 tacitly assume the equivalence of Definition 1.1 with a version of the definition where the sequence of moduli ϕ_n is required to converge to ϕ uniformly on $[0, \infty)$. It is indeed straightforward to see that given a mapping f on a nonempty metric space (X, d) satisfying Definition 1.1 with moduli ϕ_n, ϕ, one may modify the moduli as follows to get uniform convergence on $[0, \infty)$. Denote by $\overline{\mathrm{ran}(d)}$ the closure of the range of d. For $x \in [0, \infty)$ define $a(x) := \sup\{y \in \overline{\mathrm{ran}(d)}\colon x > y\}$ and $b(x) := \inf\{y \in \overline{\mathrm{ran}(d)}\colon x < y\}$ when possible. Define $\phi'_n : [0, \infty) \to [0, \infty)$ by

$$\phi'_n(x) := \begin{cases} \phi_n(x) & \text{if } x \in \overline{\mathrm{ran}(d)}, \\ \phi_n(a(x)) + \left(\frac{x-a(x)}{b(x)-a(x)}\right)(\phi_n(b(x)) - \phi_n(a(x))) & \text{if } x \notin \overline{\mathrm{ran}(d)} \text{ and } b(x) \text{ exists}, \\ \phi_n(a(x)) & \text{if } x \notin \overline{\mathrm{ran}(d)} \text{ and} \\ & b(x) \text{ does not exist.} \end{cases}$$

Define likewise ϕ' from ϕ. Since ϕ_n, ϕ are continuous, and $\phi_n \to \phi$ uniformly on $\mathrm{ran}(d)$, it follows that ϕ'_n, ϕ' are continuous and that $\phi'_n \to \phi'$ uniformly on $[0, \infty)$.

Our first result is an improvement on the bound in Theorem 16 in [6]. The following theorem is identical to Theorem 16 in [6], except that it involves asymptotic contractions in our sense, and that $\eta(\varepsilon) \cdot \varepsilon/4$ is replaced by ε in the definition of M_ε. So the 'modulus of uniqueness' from Lemma 10 in [6] no longer plays any part in the bound. This will in most cases, depending on η, constitute a significant numerical improvement. The following theorem, as well as Theorem 16 in [6], does not provide a rate of convergence, but rather what in [3] is called a rate of proximity.

Theorem 2.3. *Let (X, d) be a complete metric space, let $f : X \to X$ be a continuous asymptotic contraction and let $b > 0$ and η, β be given. If for some $x_0 \in X$ the sequence (x_n) is bounded by b then f has a unique fixed point z, (x_n) converges to z and for every $\varepsilon > 0$ such that $b \geqslant \varepsilon$ there exists $m \leqslant M_\varepsilon$ such that*

$$d(x_m, z) \leqslant \varepsilon,$$

where

$$M_\varepsilon(\eta, \beta, b) := k\left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta(\varepsilon)}{2}\right)} \right\rceil,$$

with $k := \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$.

Proof. Suppose (x_n) is bounded by b. Let $b \geqslant \varepsilon > 0$. Let

$$M_\varepsilon := k\left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta(\varepsilon)}{2}\right)} \right\rceil,$$

where $k := \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$. By Theorem 16 in [6] we have that (x_n) converges to the unique fixed point z of f. Let $l \in \mathbb{N}$ be arbitrary and let N be such that $d(x_n, z) < 2^{-l}$ for all $n \geqslant N$. By Lemma 11 in [6] there exists $m \leqslant M_\varepsilon$ such that

$$d\left(x_m, f^N(x_m)\right) \leqslant \varepsilon.$$

Note that M_ε does not depend on N. Since $f^N(x_m) = x_{m+N}$ and $m + N \geqslant N$, we have $d(f^N(x_m), z) < 2^{-l}$. Therefore

$$d(z, x_m) \leqslant d\bigl(x_m, f^N(x_m)\bigr) + d\bigl(f^N(x_m), z\bigr) < \varepsilon + 2^{-l}.$$

Since there are only finitely many $m \leqslant M_\varepsilon$ there must exist $m_1 \leqslant M_\varepsilon$ such that

$$d(z, x_{m_1}) < \varepsilon + 2^{-l}$$

holds for infinitely many l. Hence

$$d(z, x_{m_1}) \leqslant \varepsilon.$$

All the rest follows from Theorem 16 in [6]. □

3. Main results

Our main result is an explicit rate of convergence for asymptotic contractions. We begin with the continuous case.

Theorem 3.1. *Let (X, d) be a complete metric space, let $b > 0$ be given, and let $f : X \to X$ be a continuous asymptotic contraction with moduli η and β. If for some $x_0 \in X$ the sequence (x_n) is bounded by b, then (x_n) has the following rate of convergence. Let z be the unique fixed point. Let $b \geqslant \varepsilon > 0$ and let $n \in \mathbb{N}$ satisfy*

$$n \geqslant \max\bigl\{k \cdot \bigl(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\bigr), (k-1) \cdot \bigl(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\bigr) + M_\gamma + 1\bigr\},$$

where

$$k := \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\bigl(1 - \frac{\eta(\gamma)}{2}\bigr)} \right\rceil, \qquad M_\gamma := K_\gamma \cdot \left\lceil \frac{\lg(\gamma) - \lg(b)}{\lg\bigl(1 - \frac{\eta(\gamma)}{2}\bigr)} \right\rceil, \qquad K_\gamma := \beta_\gamma\left(\frac{\eta(\gamma)}{2}\right),$$

and $\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}$, $\gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}$. Then

$$d(x_n, z) \leqslant \varepsilon.$$

Proof. Let $b \geqslant \varepsilon > 0$. Let $\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}$ and $\gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}$. Let $x_0 \in X$ be such that (x_n) is bounded by b. For $a > 0$ let

$$B_a := \bigl\{x \in X\colon\ d(x, z) \leqslant a\bigr\}.$$

By Theorem 2.3 there exists $m' \leqslant M_\gamma$ such that $x_{m'} \in B_\gamma$. Suppose there exists $m > m'$ such that $x_m \notin B_\varepsilon$. Then let

$$m := \min\{n\colon\ n > m' \text{ and } x_n \notin B_\varepsilon\}.$$

Then for $x_n \in B_\gamma$ we get $d(x_n, x_m) > \frac{\varepsilon}{2}$ since

$$d(x_n, x_m) \geqslant d(x_m, z) - d(x_n, z) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Assume $m - n \geqslant \beta_{(\frac{\varepsilon}{2})}(\delta)$. Note that $\delta < 1$. Then for all $k \geqslant m - n$ we have $|\phi_k^b(\frac{\varepsilon}{2}) - \phi_{m-n}^b(\frac{\varepsilon}{2})| \leqslant \delta$, and hence $|\phi^b(\frac{\varepsilon}{2}) - \phi_{m-n}^b(\frac{\varepsilon}{2})| \leqslant \delta$. The definition of an asymptotic contraction gives

$$\phi^b\left(\frac{\varepsilon}{2}\right) + \eta\left(\frac{\varepsilon}{2}\right) \leqslant 1,$$

and so

$$\phi^b\left(\frac{\varepsilon}{2}\right) \leqslant 1 - \eta\left(\frac{\varepsilon}{2}\right),$$

and

$$\phi^b_{m-n}\left(\frac{\varepsilon}{2}\right) \leqslant 1 - \eta\left(\frac{\varepsilon}{2}\right) + \left|\phi^b\left(\frac{\varepsilon}{2}\right) - \phi^b_{m-n}\left(\frac{\varepsilon}{2}\right)\right|.$$

We therefore have

$$\phi^b_{m-n}\left(\frac{\varepsilon}{2}\right) \leqslant 1 - 2\delta + \delta = 1 - \delta.$$

Since we by definition have

$$d\big(f^{m-n}(x_n), f^{m-n}(x_m)\big) = d(x_m, x_{2m-n}) \leqslant \phi^b_{m-n}\left(\frac{\varepsilon}{2}\right) \cdot d(x_n, x_m),$$

we get

$$d(x_m, x_{2m-n}) \leqslant (1-\delta) \cdot d(x_n, x_m).$$

So in this case

$$d(x_{2m-n}, x_n) \geqslant d(x_n, x_m) - d(x_m, x_{2m-n})$$

gives

$$d(x_{2m-n}, x_n) \geqslant d(x_n, x_m) - (1-\delta) \cdot d(x_n, x_m) = \delta \cdot d(x_n, x_m) > \frac{\delta\varepsilon}{2}.$$

If $x_{2m-n} \in B_\gamma$ then we would have

$$d(x_{2m-n}, x_n) \leqslant d(x_{2m-n}, z) + d(z, x_n) \leqslant 2\gamma \leqslant \frac{\delta\varepsilon}{2}.$$

So $x_{2m-n} \notin B_\gamma$. Let

$$m'' := \min\{n\colon\ n > m' \text{ and } x_n \notin B_\gamma\}.$$

If

$$m'' - m' = M' + \beta_{(\frac{\varepsilon}{2})}(\delta)$$

for some $M' \geqslant 0$, then since $m \geqslant m''$ we have

$$m - m', m - (m'+1), \ldots, m - (m'+M') \geqslant \beta_{(\frac{\varepsilon}{2})}(\delta),$$

and $x_{m'}, x_{m'+1}, \ldots, x_{m'+M'} \in B_\gamma$. By the above argument this gives that respectively $x_{2m-m'}$, $x_{2m-m'-1}, \ldots, x_{2m-m'-M'+1}$ and $x_{2m-m'-M'}$ are not in B_γ. By arranging the indices in increasing order, we have

$$x_{2m-m'-M'}, x_{2m-m'-M'+1}, \ldots, x_{2m-m'} \notin B_\gamma.$$

By taking $x_{2m-m'-M'}$ as the starting point of a b-bounded Picard iteration sequence defined by $x_{n+1} := f(x_n)$, we get by Theorem 2.3 that there exists $m''' \leqslant M_\gamma$ such that $x_{2m-m'-M'+m'''} \in B_\gamma$. So $M' < M_\gamma$. (And so in this case $0 < M_\gamma$.) In total, if there exists $m > m'$ such that $x_m \notin B_\varepsilon$, then we get that for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$ we have

$$\gamma < d(x_n, z).$$

Since (x_n) converges to z, we have

$$d(x_n, z) \leqslant b.$$

So in this case by Proposition 4 in [6], for $n \in \mathbb{N}$ such that

$$n \geqslant 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$$

we have

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b.$$

Likewise, by then treating $x_{2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1}$ as the starting point y_0 of a Picard iteration sequence (y_n) bounded by b with the property that

$$d(y_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for all $n \geqslant 0$, either there exists no $n \in \mathbb{N}$ with

$$n > 3M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$$

such that $x_n \notin B_\varepsilon$, or else for $n \in \mathbb{N}$ such that

$$n \geqslant 2 \cdot \left(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right)$$

we have

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b.$$

We get that for $n \in \mathbb{N}$ such that

$$n \geqslant \max\left\{k \cdot \left(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right), (k-1) \cdot \left(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right) + M_\gamma + 1\right\},$$

where $k \geqslant 1$, have

$$x_n \in B_\varepsilon \quad \text{or} \quad x_n \in B_{(1-\frac{\eta(\gamma)}{2})^k \cdot b}.$$

By letting

$$k := \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta(\gamma)}{2}\right)} \right\rceil$$

we get for $n \in \mathbb{N}$ such that

$$n \geqslant \max\left\{k \cdot \left(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right), (k-1) \cdot \left(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right) + M_\gamma + 1\right\},$$

that

$$x_n \in B_\varepsilon. \qquad \square$$

We note that completeness and continuity in the above theorem was only needed to show the existence of a fixed point z. If a fixed point exists, then by Proposition 4 in [6] every Picard iteration sequence is bounded, irrespective of completeness and continuity, and hence by Lemma 15 in [6] it is Cauchy. By Lemma 10 in [6] it converges to the fixed point z, and by inspection we see that the proof of Theorem 3.1 goes through. Then Theorem 3.1 gives a rate of convergence for a b-bounded Picard iteration sequence. Hence we have the following theorem.

Theorem 3.2. *Let (X, d) be a metric space, let $b > 0$ be given, and let $f : X \to X$ be an asymptotic contraction with moduli η and β. Assume that f has a fixed point z. Then every Picard iteration sequence is bounded, and if for $x_0 \in X$ the sequence (x_n) is bounded by b then (x_n) converges to z with the rate of convergence specified in Theorem* 3.1.

Proof. Follows by the above remarks. □

If in the metric space (X, d) some iteration sequence $(f^n(x))$ is bounded, where f is an asymptotic contraction with moduli η^b, β^b for $b > 0$, then by Proposition 4 and Lemmas 15 and 10 in [6] all iteration sequences are Cauchy even if none of them converges, and if some $z \in X$ is the limit of one sequence, then z is the limit of all the iteration sequences. Namely, by Lemma 15 $(f^n(x))$ is Cauchy, and if we let $n \in \mathbb{N}$ be such that $m \geqslant n$ gives $d(f^n(x), f^m(x)) < 1$, then taking $f^n(x)$ as x in Proposition 4 gives that any $(f^n(y))$ is bounded. Then $(f^n(y))$ is Cauchy by Lemma 15 and $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$ by Lemma 10. If $(f^n(x))$ does not converge then we consider the completion $\overline{X}$ of X, in which the limit z exists. We can then extend f to be defined on $X \cup \{z\}$ by letting $f(z) = z$. It is then easy to see that f is an asymptotic contraction with moduli $\eta_1^b : (0, b] \to (0, 1)$ and $\beta_1^b : (0, b] \times (0, \infty) \to \mathbb{N}$ defined by for example $\eta_1^b(\varepsilon) := \eta^{2b}(\varepsilon/2)$, $\beta_1^b(l, \varepsilon) := \beta^{2b}(l/2, \varepsilon)$.

If the b-bounded iteration sequence (x_n) converges in X to z, and z is not a fixed point, then we have the following.

Theorem 3.3. *Let (X, d) be a metric space, and let $f : X \to X$ be an asymptotic contraction with moduli η^b and β^b for each $b > 0$. Let $x_0 \in X$ be such that the Picard iteration sequence (x_n) is bounded. Then all Picard iteration sequences are Cauchy. Assume that $z := \lim_{n\to\infty} x_n$ exists. Then for any $x_0 \in X$ the iteration sequence (x_n) converges to z, irrespective of whether z is a fixed point or not. If (x_n) is bounded by $b > 0$ then (x_n) converges to z with the rate of convergence specified in Theorem* 3.1.

Proof. Proposition 4 and Lemmas 15 and 10 in [6] still imply that all iteration sequences converge to z. The rate of proximity in Theorem 2.3 only depends on Lemma 11 in [6] and the fact that (x_n) converges to z, all of which is independent of whether z is a fixed point or not. However, in the proof of Theorem 3.1 we use that z is a fixed point when we use Proposition 4 in [6] to infer

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $n \geqslant 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$ from the fact that $\gamma < d(x_n, z) \leqslant b$ for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$. In this manner we in the proof repeatedly make use of the fact that z is a fixed point. When z is not a fixed point we can proceed as follows. Assuming that there exists $n > M_\gamma$ such that $x_n \notin B_\varepsilon$, we have $\gamma < d(x_n, z) \leqslant b$ for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$. Choose such $n \in \mathbb{N}$, and choose a real number $a > 0$. Choose then $K' \in \mathbb{N}$ such that for this n we have

$$d(x_k, z) < \min\{a, (d(x_n, z) - \gamma)\}$$

for all $k \geqslant K'$. We can find such K' since (x_n) converges to z. Then $\gamma < d(x_n, x_{K'}) \leqslant b$, so Proposition 4 in [6] gives

$$d(f^k(x_n), f^k(x_{K'})) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $k \geqslant K_\gamma$. Now the triangle inequality gives

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b + a$$

for $n \geqslant 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$. Since $a > 0$ was arbitrary we get

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $n \geqslant 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$. Then, following the proof of Theorem 3.1 we get that either there does not exist $n \in \mathbb{N}$ with $n > 3M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$ and $x_n \notin B_\varepsilon$, or else we have

$$\gamma < d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1 + (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta))$. Choose such $n \in \mathbb{N}$, and choose a real number $a > 0$. Then we can choose $K' \in \mathbb{N}$ as above and get

$$\gamma < d(x_n, x_{K'}) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b + a.$$

We can assume $(1 - \frac{\eta(\gamma)}{2}) \cdot b + a < b$, so

$$d\big(f^k(x_n), f^k(x_{K'})\big) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b + \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot a$$

for $k \geqslant K_\gamma$. And so

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b + \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot a + a$$

for $n \geqslant 2 \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1)$. Since this holds for all sufficiently small $a > 0$ we get for such n that

$$d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b.$$

We can now obviously employ the same strategy each time we have that for a given $k \in \mathbb{N}$ either there does not exist $n \in \mathbb{N}$ with $n > k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + M_\gamma$ and $x_n \notin B_\varepsilon$, or else we have

$$\gamma < d(x_n, z) \leqslant \left(1 - \frac{\eta(\gamma)}{2}\right)^k \cdot b$$

for some $n < k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta))$. We get that for $n \in \mathbb{N}$ such that

$$n \geqslant \max\big\{k \cdot \big(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\big), (k-1) \cdot \big(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\big) + M_\gamma + 1\big\},$$

where $k \geqslant 1$, we have

$$x_n \in B_\varepsilon \quad \text{or} \quad x_n \in B_{(1 - \frac{\eta(\gamma)}{2})^k \cdot b}.$$

Thus we have the same rate of convergence as in Theorem 3.1. □

We will use the following theorem to give a characterization of asymptotic contractions on bounded, complete metric spaces.

Theorem 3.4. *Let (X,d) be a metric space, and let $f: X \to X$. Let now $\Psi: \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{N}$ satisfy*

$$\forall \varepsilon \in \mathbb{R}_+^*\ \forall x, y \in X\ \forall b \geqslant d(x,y)\ \forall n \geqslant \Psi(\varepsilon, b)\big(d\big(f^n(x), f^n(y)\big) \leqslant \varepsilon\big).$$

Assume further that $\varepsilon < \varepsilon'$ implies $\Psi(\varepsilon, b) \geqslant \Psi(\varepsilon', b)$. Then f is an asymptotic contraction.

Proof. For $b > 0$ and $n \in \mathbb{N}$ define $\phi_n^b: (0,\infty) \to (0,\infty)$ by $\phi_n^b(\varepsilon) := 1/2$. Define further $\eta^b: (0,b] \to (0,1)$ by $\eta^b(\varepsilon) := 1/2$ and $\beta_l^b: (0,\infty) \to \mathbb{N}$ by $\beta_l^b(\varepsilon) := \Psi(l/2, b)$. These moduli satisfy Definition 2.2. □

Corollary 3.5. *Let (X,d) be a complete metric space, and let $f: X \to X$. Let for each $x_0 \in X$ the Picard iteration sequence converge to the point $z \in X$ with a rate of convergence which is uniform in the starting point. Then f is an asymptotic contraction.*

Proof. By assumption there exists $\Psi: \mathbb{R}_+^* \to \mathbb{N}$ such that

$$\forall \varepsilon \in \mathbb{R}_+^*\ \forall x, y \in X\ \forall n \geqslant \Psi(\varepsilon)\big(d\big(f^n(x), f^n(y)\big) \leqslant \varepsilon\big).$$

We can furthermore assume that $\varepsilon < \varepsilon'$ implies $\Psi(\varepsilon) \geqslant \Psi(\varepsilon')$. Thus Theorem 3.4 applies. □

The above corollary also follows from Proposition 3 in [7], which implies that f in this case is an asymptotic contraction in the sense of Kirk.

Corollary 3.6. *Let (X,d) be a nonempty, bounded, complete metric space, and let $f: X \to X$. Then f is an asymptotic contraction if and only if there exists $z \in X$ such that for each $x_0 \in X$ the Picard iteration sequence converges to z with a rate of convergence which is uniform in the starting point.*

Proof. That f is an asymptotic contraction if such a $z \in X$ exists follows from Corollary 3.5. The other implication follows from Theorem 3.3, since we assume that the space is bounded. □

Proposition 3.7. *Let (X,d) be a nonempty, bounded, complete metric space, and let $f: X \to X$. Then the following are equivalent*:

(1) *The function f is an asymptotic contraction.*
(2) *The function f is an asymptotic contraction in the sense of Gerhardy.*
(3) *The function f is an asymptotic contraction in the sense of Kirk.*
(4) *There exists $z \in X$ such that for each $x_0 \in X$ the Picard iteration sequence converges to z with a rate of convergence which is uniform in the starting point.*
(5) *There exists $\alpha: (0,\infty) \to \mathbb{N}$ such that*

$$\forall x, y \in X\ \forall \varepsilon > 0\ \forall n \geqslant \alpha(\varepsilon)\left(d(x,y) \geqslant \varepsilon \to d\big(f^n(x), f^n(y)\big) \leqslant \frac{1}{2} d(x,y)\right).$$

Proof. Assume first that f is an asymptotic contraction in the weakest sense, i.e., in the sense of Definition 2.2. Then by the previous corollary, (4) holds. Furthermore, by Theorem 3.3 and the proofs of Corollary 3.5 and Theorem 3.4, (5) holds. Now assume that (4) holds. Then $\operatorname{diam} f^n(X) \to 0$. Following the proof of Proposition 3 in [7], we define $\phi, \phi_n: [0,\infty) \to [0,\infty)$

by $\phi_n(t) := \operatorname{diam} f^n(X)$ and $\phi(t) := 0$. These moduli satisfy Definition 1.1, so f is an asymptotic contraction in the sense of Kirk. Thus (1)–(4) are equivalent. Now assume that (5) holds. For $b > 0$ and $n \in \mathbb{N}$ define $\phi_n^b : (0, \infty) \to (0, \infty)$ by $\phi_n^b(\varepsilon) := 1/2$. Define further $\eta^b : (0, b] \to (0, 1)$ by $\eta^b(\varepsilon) := 1/2$ and $\beta_l^b : (0, \infty) \to \mathbb{N}$ by $\beta_l^b(\varepsilon) := \alpha(l)$. These moduli satisfy Definition 2.2, so (5) is equivalent to (1). □

Theorem 3 in [7] gives a characterization of continuous asymptotic contractions in the sense of Kirk on *compact* metric spaces, showing among other things that they are exactly the continuous functions such that the core $Y := \bigcap_{n\in\mathbb{N}} f^n(X)$ is a singleton (assuming the space is nonempty). If we in Proposition 3.7 require that f be continuous, we get a generalization of this fact from the compact case to the case where the space is bounded and complete. Namely, we get by Theorem 3.1 that if a continuous f is an asymptotic contraction, then there exists a fixed point z, and $Y = \{z\}$. If on the other hand the core Y is a singleton $\{z\}$, then Proposition 3.7 implies that f is an asymptotic contraction (in all three senses considered).

The above proposition also has consequences for other kinds of contractive type mappings on bounded, complete metric spaces. In [11], B.E. Rhoades systematized 25 basic definitions of various contractive type mappings, and also considered several standard generalizations of these. The comparison between the 25 basic definitions was completed by P. Collaço and J. Carvalho e Silva in [5]. In [2] we treat so-called uniformly generalized p-contractive mappings, and we now get the following results regarding this.

Corollary 3.8. *Let (X, d) be a bounded, complete metric space, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. Then f is an asymptotic contraction.*

Proof. Let b be a bound on the space. Then for each $x_0 \in X$ we have that b is a bound on the Picard iteration sequence (x_n). We can assume X nonempty, for else the proof is trivial. Thus Theorem 3.1 in [2] (and the comments directly following it) assures the existence of a fixed point $z \in X$ and a rate of convergence for Picard iteration sequences (x_n) to z, and this rate is moreover uniform in the starting point x_0. Then by Proposition 3.7 we have that f is an asymptotic contraction. □

Corollary 3.9. *Let (X, d) be a compact metric space. Let $f : X \to X$ be continuous and such that it satisfies one of the conditions* (1)–(50) *from* [11]. *Then f is an asymptotic contraction.*

Proof. Since f satisfies one of the requirements (1)–(50) we know from [11] and [5] that there exists $k \in \mathbb{N}$ such that f^k satisfies (25). Then in the terminology of [2] f is generalized p-contractive. Since X is compact we know that f is uniformly continuous, and by Proposition 2.5 in [2] f is uniformly generalized p-contractive. Thus by the previous corollary it follows that f is an asymptotic contraction. □

In [4], Y.-Z. Chen proves Kirk's theorem on asymptotic contractions under conditions which are weaker than the ones in [8]. In particular, it is no longer assumed that f is continuous, and it is enough that ϕ and one particular $\phi_{n'}$ are upper semicontinuous (here ϕ, $\phi_{n'}$ are as in Kirk's definition). It is furthermore enough that $\lim_{n\to\infty} \phi_n = \phi$ uniformly on any bounded interval $[0, b]$. (A condition which allows one to drop the requirement that one iteration sequence is bounded is also specified.) It is, however, assumed that $\phi_{n'}(0) = 0$. We can adapt a part of the argument in [4] to the situation with asymptotic contractions in the sense of Gerhardy. In the following proposition we develop a criterion which allows us to infer the existence of a fixed point without the assumption of continuity. This will in a sense work like the condition $\phi_{n'}(0) = 0$ in [4].

(Note that the arguments in [6] which allow us to subsume Definition 1.1 under Definition 2.1 would work just as well if the ϕ, ϕ_n in Definition 1.1 were assumed to be upper semicontinuous instead of continuous, since upper semicontinuous functions $\phi, \phi_n : [0, \infty) \to [0, \infty)$ are bounded on bounded closed intervals $[s, b]$. So Definition 6 and Propositions 7 and 9 in [6] would remain unchanged. These arguments would also work if for ϕ, ϕ_n in Definition 1.1 we had $\lim_{n\to\infty} \phi_n = \phi$ uniformly only on bounded intervals $[0, b]$. Definition 6 in [6] would be unchanged, in Proposition 7 one would have to say that the sequence $(\tilde{\phi}_n)$ converges uniformly to $\tilde{\phi}$ on $[l, b]$ for all $b > l > 0$ instead of saying that it converges uniformly on $[l, \infty)$ for all $l > 0$, but the second part of Proposition 7 and also Proposition 9 would remain unchanged.)

Proposition 3.10. *Let (X, d) be a complete metric space, and let $f : X \to X$ be an asymptotic contraction in the sense of Gerhardy with moduli η^b and β^b for each $b > 0$. For each $b > 0$ let (ϕ_n^b) be a sequence of functions which satisfy Definition* 2.1. *Let $b' > 0$ and let $x_0 \in X$ be such that the sequence (x_n) is b'-bounded. Let $z := \lim_{n\to\infty} x_n$. Let $m \in \mathbb{N}$ be such that $\limsup_{t\to 0} \phi_m^{b'}(t) < \infty$. Then $f(z) = z$.*

Proof. We have for each $n \in \mathbb{N}$ that

$$d\big(f^{n+m}(x_0), f^m(z)\big) \leqslant \phi_m^{b'}\big(d\big(f^n(x_0), z\big)\big) \cdot d\big(f^n(x_0), z\big).$$

Since $\lim_{n\to\infty} d(f^n(x_0), z) = 0$ and $\limsup_{t\to 0} \phi_m^{b'}(t) < \infty$, we get

$$\lim_{n\to\infty} d\big(f^{n+m}(x_0), f^m(z)\big) = 0,$$

i.e. $\lim_{n\to\infty} f^{n+m}(x_0) = f^m(z)$. Thus $f^m(z) = z$. We know by Lemma 15 in [6] that $(f^n(z))$ is a Cauchy sequence, hence $f(z) = z$. □

We note that in the case covered by Proposition 3.10 each iteration sequence converges to z, and the rate of convergence from Theorem 3.1 applies. This follows from Theorem 3.2 or Theorem 3.3.

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