Mixed matrices and binomial ideals

Klaus G. Fischer, Jay Shapiro*

Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA

Received 18 April 1995; revised 28 August 1995
Communicated by C.A. Weibel

Abstract

Given an integral vector \( u \in \mathbb{Z}^n \), one may associate with it the binomial \( f_u = X^{u^+} - X^{u^-} \) in \( \mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_n] \) where \( u^+ \) and \( u^- \) are the positive and negative supports of \( u \), respectively. We say that \( u \) is mixed if \( u^+, u^- \neq 0 \) and a matrix \( M \) is mixed if all its rows are mixed. We investigate relationships between the matrix \( M \) whose rows are \( u_1, \ldots, u_r \) and the ideal \( I = \langle f_{u_1}, \ldots, f_{u_r} \rangle \).

For example, if \( M \) contains no square mixed submatrix of any size, then the vectors \( u_1, \ldots, u_r \) are linearly independent and \( f_{u_1}, \ldots, f_{u_r} \) form a regular sequence in \( \mathbb{Z}[X] \). This allows us to decide if a semigroup ring is a complete intersection. When applied to numerical semigroups, the results give an alternate proof of a theorem by Delorme which characterizes numerical semigroups that are complete intersections.

1991 Math. Subj. Class.: Primary 13C40; secondary 14M10

1. Introduction

In this paper we consider a purely combinatorial property of a matrix – the existence of mixed submatrices – and apply it to obtain commutative algebraic properties of certain binomial ideals in the integral polynomial ring and, in particular, semigroup rings. The setting is as follows.

Let \( \mathbb{Z}[X] \) be the polynomial ring in the variables \( X_1, \ldots, X_n \) over the integers \( \mathbb{Z} \). For any integral vector \( u \in \mathbb{Z}^n \), we write \( u = u^+ - u^- \) where \( u^+ \) and \( u^- \) are, respectively, the positive and negative parts of the vector \( u \). That is, if \( [u]_i \) is the \( i \)th coordinate of \( u \), then we set the \( i \)th coordinate \( [u^+]_i, [u^-]_i \) of \( u^+ \) equal to \( \max\{0, [u]_i\} \). Similarly, \( [u^-]_i = \max\{0, -[u]_i\} \). To each such \( u \) we associate the binomial \( f_u = X^{u^+} - X^{u^-} \in \mathbb{Z}[X] \). To

* Corresponding author. E-mail: jshapiro@gmu.edu.
each set of integral vectors $u_1, \ldots, u_r$, we associate three binomial ideals within $\mathbb{Z}[X]$ as follows.

1. $I = \langle f_{u_1}, \ldots, f_{u_r} \rangle$,
2. $I^* = \langle f_u^* : u \in \text{span}_{\mathbb{Z}}(u_1, \ldots, u_r) \rangle$,
3. $\tilde{I} = \langle f_u : u \in \text{span}_{\mathbb{Q}}(u_1, \ldots, u_r) \cap \mathbb{Z}^n \rangle$.

Although it is clear that $I \subset I^* \subset \tilde{I}$, in general these ideals are not the same without some assumptions on the $r \times n$ matrix $M$ whose rows consist of the coefficients of the vectors $u_1, \ldots, u_r$. For example, if we define the content of $M$ to be the gcd of all $r \times r$ minors of $M$, then it follows that $I^* = \tilde{I}$ if and only if $M$ has content 1 (see Section 2). In this case, the rows of $M$ are necessarily independent over $\mathbb{Q}$ and so to say that $M$ has content 1 means that the vectors $u_1, \ldots, u_r$ span the set $\text{span}_{\mathbb{Q}}(u_1, \ldots, u_r) \cap \mathbb{Z}^n$ over $\mathbb{Z}$.

We will examine and compare these ideals using only the sign pattern of $M$.

The ideal $\tilde{I}$ is of particular significance because of the following consideration. Let $S$ be a subsemigroup of $\mathbb{Z}^d$ generated by the elements $s_1, \ldots, s_n$ (i.e., the subsemigroup of non-negative integral combinations of the $s_i$). Let $W$ be the rational vector space generated by the set

$$\left\{ u \in \mathbb{Z}^n : \sum_{i=1}^n [u] s_i = 0 \right\}.$$ 

We will call $W$ the relation space of $S$. If $u_1, \ldots, u_r$ is a set of integral vectors which form a basis for the vector space $W$ over the rationals $\mathbb{Q}$, we will call this set an integral basis of $W$. If $\tilde{I}$ is the ideal defined above with respect to an integral basis of $W$, then the semigroup ring $\mathbb{Z}[S]$ may be realized as $\mathbb{Z}[t^*: s \in S]$, which is naturally a homomorphic image of $\mathbb{Z}[X]$ with kernel $\tilde{I}$ [6, Proposition 1.11]. Furthermore, by [5], the prime ideal $I$ has height $\dim_{\mathbb{Q}} W$.

We will usually assume that our semigroups have the property that they contain no invertible elements. This means that there is no positive integral combination of the $s_i$'s which equals zero. In this case, the coefficient matrix of any basis of the space of relations of $S$ has the property that every row contains both a positive and negative entry. This leads to our first definition which will be used throughout the paper.

**Definition 1.1.** A matrix $M$ will be called mixed if every row of $M$ contains both a positive and negative entry.

In Section 2 we show that the height of the ideal $I$ is determined by the size of mixed submatrices of $M$. As a corollary, we show that the height of $I$ equals $r$, i.e., $f_{u_1}, \ldots, f_{u_r}$ is a regular sequence in $\mathbb{Z}[X]$, if and only if $M$ contains no mixed submatrices of size $s \times t$ where $s > t$. These results depend on a theorem by Eisenbud and Sturmfels [4], which states that when $u_1, \ldots, u_r$ are linearly independent, then $f_{u_1}, \ldots, f_{u_r}$ is a regular sequence in the Laurent polynomial ring $\mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. 


If we examine the situation where \( M \) does not contain even a square mixed submatrix, then the ring implications are even sharper.

**Definition 1.2.** A matrix \( M \) is called *dominating* if it does not contain a square mixed submatrix.

**Example 1.3.**

\[
\begin{pmatrix}
3 & 2 & -1 & 0 \\
-1 & 1 & 2 & 3 \\
2 & 1 & -1 & 1
\end{pmatrix}
\]

Not dominating

\[
\begin{pmatrix}
3 & 2 & -1 & 0 \\
1 & 1 & 2 & 3 \\
0 & 0 & -1 & 1
\end{pmatrix}
\]

Dominating

\[
\begin{pmatrix}
0 & 2 & -1 & 0 \\
-1 & 1 & 0 & 0 \\
2 & 1 & 1 & -1
\end{pmatrix}
\]

Mixed dominating

If we assume that \( M \) is dominating and has content 1, then \( I = I^* = \tilde{I} \) (Theorem 2.9). Hence, if \( M \) is the coefficient matrix of an integral basis for the space of relations of a semigroup \( S \), it follows that \( \tilde{I} \) is generated by \( ht(I) \) elements and therefore \( Z[S] \) is a complete intersection. We will simply say \( S \) is a complete intersection in this case. These results depend on the linear algebraic properties of a dominating matrix which are explored in Proposition 2.6 and which also makes clear the use of the term “dominating”. Corollary 2.8 shows that if \( M \) is a mixed dominating matrix with real entries, then the rows of \( M \) are linearly independent. Since the definition of a dominant matrix depends only on the sign of the entries and not on their magnitude, the linear independence of the rows means that mixed dominating matrices are a subclass of L-matrices [1]. Specifically, if \( \tilde{M} \) is any matrix with the same sign pattern as \( M \), i.e., \( [\tilde{M}]_{ij} = e_{ij}[M]_{ij} \) for positive numbers \( e_{ij} \), then the rows of \( \tilde{M} \) are also linearly independent. We prove in Proposition 3.1 that \( r \times (r + 1) \) mixed dominating matrices are a special class of L-matrices and precisely the same as what are called S-matrices ([1, 2] and [7]).

For an \( r \times (r + 1) \) mixed dominating matrix \( M \), we prove the existence of a “decomposition” of \( M \) into smaller mixed dominating matrices (Theorem 3.4). This result, along with those in Section 2, are applied to numerical semigroups \( S \) and allow us to give a straightforward proof of a theorem by Delorme which characterizes when \( S' \) is a complete intersection. We conclude by showing that if \( M \) has a very simple mixed dominating form – principally dominating – then the semigroup \( S \) satisfies a condition formulated by Herzog in [6].

### 2. Dominating matrices

Let \( u_1, u_2, \ldots, u_r \) be a linearly independent set of integral \( n \)-tuples. We want to examine the height of the ideal \( I = (f_{u_1}, \ldots, f_{u_r}) \) in the polynomial ring \( Z[X] = Z[X_1, \ldots, X_n] \). Eisenbud and Sturmfels [4, Theorem 2.1] showed that these binomials
form a regular sequence in the Laurent polynomial ring. This condition is clearly not sufficient in the ordinary polynomial ring as the following example shows.

**Example 2.1.** Consider the following three vectors:

\[
\begin{align*}
\mathbf{u}_1 &= (1, -2, 0, -1, -2), \\
\mathbf{u}_2 &= (1, -3, -1, 0, -1), \\
\mathbf{u}_3 &= (1, -4, -1, -2, 0).
\end{align*}
\]

The ideal \(\langle f_{u_1}, f_{u_2}, f_{u_3} \rangle\) is contained in \(\langle X_1, X_2 \rangle\) which is a prime ideal of height two. Therefore, the elements \(f_{u_1}, f_{u_2}, f_{u_3}\) cannot possibly form a regular sequence.

From the result of Eisenbud and Sturmfels [4, Theorem 2.1] one can deduce that if all the variables \(X_1, \ldots, X_n\) are invertible modulo \(I\), then the binomials will form a regular sequence in the ordinary polynomial ring. Thus, when we consider a sequence \(f_{u_1}, \ldots, f_{u_r}\) for an arbitrary set of linearly independent vectors \(u_1, \ldots, u_r\) we can “delete” the variables that become units mod \(I\). Observe that if a row \(u\) of \(M\) is not mixed, then for all \(j\) such that \([u]_j \neq 0\), the variable \(X_j\) is a unit mod \(I\). More generally, even when \(u\) is mixed, suppose that \(X_j\) is a unit mod \(I\) whenever \([u]_j < 0\), then \(X_k\) is unit mod \(I\) whenever \([u]_k > 0\). This leads us to our next construction. For a given \(r \times (r + j)\) matrix \(M\) we will derive an \(r \times k\) submatrix \(H\) such that every row of \(H\) is either mixed or the zero vector. To build such an \(H\) we delete columns of \(M\) as follows. Pick a row \(u\) of \(M\) that is not mixed and not the zero vector. For each \(j\) with \([u]_j \neq 0\), delete the \(j\)th column of \(M\). Repeat this procedure on the new matrix and keep repeating as long as there exists a non-zero row that is not mixed. Eventually, this stops with a submatrix \(H\) having the desired properties. Note that no rows have been deleted and the columns that have been deleted from \(M\) correspond to variables that become invertible mod \(I\). We will call \(H\) the *derived* submatrix of \(M\).

**Example 2.2.** Let \(M\) be the matrix given by

\[
\begin{pmatrix}
2 & 3 & 0 & 0 & 0 & 0 \\
0 & 3 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & -2 & -3 \\
0 & 0 & 0 & 2 & -3 \\
\end{pmatrix}
\]

Here, the derived submatrix \(H\) of \(M\) is obtained by deleting the first three columns of \(M\). Hence,

\[
H = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
3 & -2 & -3 \\
0 & 2 & -3 \\
\end{pmatrix}
\]
Theorem 2.3. Let $u_1, \ldots, u_r$ be a set of linearly independent vectors in $\mathbb{Z}^n$ and as usual let $I = \langle f_{u_1}, \ldots, f_{u_r} \rangle$. Let $M$ be the matrix whose $i$th row is $u_i$ and let $H$ be the derived submatrix of $M$. If

$$k = \max\{0, s - t \mid H \text{ contains a mixed } s \times t \text{ submatrix}\},$$

then $ht(I) = r - k$.

**Proof.** First we will show that $r - k$ is an upper bound for the height of $I$. If $k = 0$, then clearly $r$ is an upper bound on the height of an ideal generated by $r$ elements. Hence, we can assume that $k > 0$. Let $N$ be a mixed $s \times t$ submatrix of $H$, such that $k = s - t$ and assume that $N$ is of maximal size with this property. By rearranging rows and columns of $M$ we can assume that $H$ consists of the first $l$ columns of $M$ (and all the rows), while $N$ consists of the first $s$ rows and $t$ columns of $H$. In other words,

$$M = \left( H \mid A \right), \quad \text{where } H = \begin{pmatrix} N & B \\ C & D \end{pmatrix}. $$

We also claim that $C$ is the zero matrix. For suppose that some entry of $C$, say $[u]_{jm} \neq 0$, where $j > s$ and $m < t$. Since every row of $H$ is either identically zero or mixed, some other entry of $u_j$, say $[u]_{jp}$, where $p \leq l$, must also be nonzero and have the opposite sign of $[u]_{jm}$. We could then add the $j$th row and $p$th column of $H$ to $N$ and we would still have a mixed matrix. But this would contradict the maximality of $N$.

Let $J = \langle f_{u_1}, \ldots, f_{u_r} \rangle$ and let $Q$ be the ideal $\langle X_1, X_2, \ldots, X_t, J^\perp \rangle$, where $J^\perp = \langle J; (x_q) \rangle$. Since $J$ is a prime ideal and since the variables $X_i$, $i = 1, 2, \ldots, t$ do not appear in the polynomials $f_{u_q}$, $q > s$, $Q$ is a prime ideal. Furthermore, we know that $J$ has height $r - s$; thus $Q$ has height $r - s + t = r - k$, giving the upper bound.

For the converse, let $Q$ be a prime ideal over $I$. If $Q$ does not contain any variables, then the image of $Q$ in the Laurent polynomial ring is a prime ideal containing $f_{u_1}, \ldots, f_{u_r}$, which by [4, Theorem 2.1] is a regular sequence. Hence, it follows that $Q$ must have height at least $r$. Now suppose that $Q$ contains some variables. The derived submatrix $H$ consists of the first $l$ columns of $M$ and any column not in $H$ corresponds to a variable that is a unit mod $I$ and so is a unit mod $Q$. Hence, we can assume that $Q$ contains the variables $X_1, \ldots, X_t$ where $t \leq l$. Without loss of generality, we can also assume that the first $s$ rows of $H$ are the only rows of $H$ with support in the first $t$ entries. Thus, we can write

$$H = \begin{pmatrix} N & B \\ O & D \end{pmatrix},$$

where $N$ is an $s \times t$ submatrix and $O$ is the zero matrix. We also claim that $N$ must be a mixed matrix. For let $u_j$, $1 \leq j \leq s$ be a row of $M$. Since $f_{u_j}$ is a binomial contained in $Q$ and since $Q$ contains a variable appearing in one of the monomial terms of $f_{u_j}$,
(\(u_j\) has nonempty support in the first \(t\) entries), it follows from the fact that \(Q\) is prime that \(Q\) must contain a variable occurring in the other monomial term of \(f_u\). This variable must be one of \(X_1, \ldots, X_t\), which proves that \(N\) is mixed. Therefore, by our assumption about the maximality of \(k\), we have \(s - t \leq k\). Now consider the image of \(Q\) in the ring \(\mathbb{Z}[X_{t+1}, \ldots, X_r]\) under the obvious map that sends \(X_i\) to zero for \(i = 1, 2, \ldots, t\). This ideal, which we will denote by \(Q'\), is a prime ideal over \(J = \langle f_{u_1}, \ldots, f_u \rangle\) (note that this notation is consistent since none of these \(r - s\) binomials contain any of the variables \(X_1, \ldots, X_t\)). Now \(Q'\) does not contain any variables so again, by [4, Theorem 2.1], \(Q'\) has height at least \(r - s\). Therefore, \(Q\) has height at least \(r - s + t \geq r - k\).

We now establish when the elements \(f_{u_1}, \ldots, f_u\) form a regular sequence.

**Corollary 2.4.** Let \(u_1, \ldots, u_r\) and \(M\) be as above and let \(H\) be the derived submatrix of \(M\). Then \(f_{u_1}, \ldots, f_u\) is a regular sequence if and only if \(H\) does not contain an \(s \times t\) mixed submatrix where \(s > t\).

**Proof.** If the sequence of elements is a regular sequence, then in the above theorem we must have \(k = 0\). Conversely, if \(H\) does not contain any \(s \times t\) mixed submatrices with \(s > t\), then the ideal generated by the sequence \(f_{u_1}, \ldots, f_u\) must have height \(r\). However, in a Cohen–Macaulay ring such as \(\mathbb{Z}[X]\), any set that generates an ideal of height equal to the number of elements in the set must be a regular sequence [8, Theorem 29].

We note that if \(u_1, u_2, u_3, u_4\) are the rows of the matrix \(M\) in Example 2.1, then by the corollary \(f_{u_1}, f_{u_2}, f_{u_3}, f_{u_4}\) is a regular sequence in the ordinary polynomial ring.

**Lemma 2.5.** Suppose that \(M\) is an \(n \times n\) mixed matrix which contains no square mixed submatrix of smaller size. Then up to a permutation of the rows and columns of \(M\) and multiplication of the rows by \(\pm 1\), \(M\) has the sign pattern

\[
\begin{pmatrix}
+ & + & - \\
- & + & - \\
- & - & + \\
- & + & +
\end{pmatrix}
\]

where the symbols + and − indicate positive and negative terms, respectively, and all other entries are zero.

**Proof.** First observe that the elementary row and column operations of the hypothesis do not change the basic assumptions on the matrix \(M\). We can assume that \(n > 2\) for otherwise the result is trivial. We will create a (possibly) new \(n \times n\) matrix from \(M\) which we call \(N\). From each row of \(M\) select one term that is positive and one term
that is negative and put that in the corresponding position of $N$. In other words, for each $i, i = 1, \ldots, n$, pick $1 \leq j, k \leq n$, such that $[M]_{i,j} > 0$ and $[M]_{i,k} < 0$. Then let $[M]_{i,j} = [N]_{i,j}$ and $[M]_{i,k} = [N]_{i,k}$. All other terms in $N$ are zero. By construction, $N$ is mixed while every smaller minor is unmixed since $M$ has this property.

We will first show that $N$ can be put into the above form using the indicated operations. Without loss of generality, we may assume that $[N]_{1,1} > 0$. There must be another nonzero entry in the first column of $N$, for if not, deleting the first column and first row of $N$ gives a mixed minor of $N$ of size $(n - 1) \times (n - 1)$. Multiplying by $-1$ and interchanging rows if necessary, we may assume that $[N]_{2,1} < 0$. Now the second row of $N$ must contain a positive entry which we can assume is $[N]_{2,2}$. As before, the second column must contain another nonzero entry. If this entry is $[N]_{1,2}$, then it must be negative, since $[N]_{1,1}$ is positive. But then the first two rows and columns of $N$ give a square mixed minor of $N$ of size $2 \times 2$. Hence, we may assume that $[N]_{3,2} < 0$. Continuing in this fashion we have $[N]_{1,3} > 0$, and if $n > 3$ we have $[N]_{1,3} < 0$, etc. Finally, we have $[N]_{n,n} > 0$ and the $n$th column of $N$ must contain another nonzero entry. However, the only row of $N$ that does not already contain two nonzero terms is the first. Hence, $[N]_{1,n}$ is nonzero and it is necessarily negative, proving that $N$ fits the required form.

We now claim that $M = N$. For assume that $M$ has been presented in such a way so that $N$ has the form of the lemma. If $[M]_{i,j} > 0$ where $i \neq j$, then deleting column $i$ and row $(i + 1)$ or row $1$ if $i = n$, produces a mixed minor of size $(n - 1) \times (n - 1)$. If $[M]_{i,j} < 0$, where $j \neq i - 1$, then deleting column $i - 1$ and row $i - 1$ or row $n$ if $i = 1$, produces a mixed minor of size $(n - 1) \times (n - 1)$. In either case we reach a contradiction. Hence $M = N$, proving the lemma. □

**Proposition 2.6.** Let $M$ be an $r \times n$ matrix, where $r < n$. Denote the rows of $M$ by $u_i, i \in [r]$. The following are equivalent:

1. $M$ is dominating.
2. For any subset $[k] \subset [r]$ and nonzero integers $\varepsilon_i, i \in [k]$ where $|\varepsilon_i| = 1$, there exists $j \in [k]$ such that $(\sum \varepsilon_i u_i)^+ \geq (\varepsilon_j u_j)^+$.
3. For any subset $[k] \subset [r]$ and nonzero numbers $a_i, i \in [k]$, there exists $j \in [k]$ such that $(\sum a_i u_i)^+ \geq (a_j u_j)^+$.

**Proof.** We first show that (1) implies (3). If $M$ is dominating, removing rows from $M$ or multiplying these rows by nonzero integers does not alter this fact. If $v_i = a_i u_i$, we will show that if $(\sum_{i=1}^m v_i)^+ \not\geq v_j^+$ for any $j, 1 \leq j \leq r$, then we arrive at a contradiction. If $(\sum v_i)^+ \not\geq (v_1)^+$, then it must be that there is some $t$ so that $[v_t]_t > 0$ and $[\sum v_i]_t < [v_1]_t$. Hence, there must exist some $m$ so that $[v_m]_t < 0$. Without loss of generality, assume that $t = 1$ and $m = 2$ ($m$ cannot be 1). Since $(\sum v_i)^+ \not\geq v_2^+$, some coordinate of $v_2^+$ (other than the first) must be positive. Hence, assume that $[v_2]_2 > 0$ and $[\sum v_i]_2 < [v_1]_2$. Therefore, there exists a $t$ such that $[v_t]_2 < 0$. If $t = 1$, we obtain a $2 \times 2$ mixed matrix and hence we can assume that $t = 3$. Continuing in this fashion, we obtain a sequence $[v_1], \ldots, [v_k]_k > 0$ and $[v_{i+1}]_t < 0, i = 1, \ldots, k - 1$. However,
since \((\sum v_t)^+ \not\geq v^+_k\), it follows that for some \(t < r\), \([v_k]_t < 0\). But then the square submatrix \((M|_{ij})\), \(t \leq i, j \leq k\) is mixed, contradicting our assumption about \(M\).

Since (3) implies (2) we need only show that (2) implies (1). But suppose that \(M\) has a square mixed submatrix \(N\) and assume that \(N\) is of minimal size. By multiplication of the rows by \(\pm 1\) and by rearranging rows and columns, we may assume that \(N\) has the sign pattern of Lemma 2.5. It is then clear that if \(v_1, \ldots, v_k\) are the rows of \(M\) corresponding to the rows in \(N\), \((\sum v_t)^+ \not\geq v_j\) for any \(j\). □

We say that the vector \(w\) dominates the vector \(v\), if for each \(i = 1, \ldots, n\) \([w]_i \geq [v]_i\).

We note that (3) above implies that if \(M\) is a mixed matrix and if \(w = \sum a_iu_i\), then for some \(j\), \(w^+\) dominates \((a_ju_j)^+\) and hence dominates \(u^+_j\) or \(u^-_j\) if the \(a_i\)'s are integers.

**Corollary 2.7.** If \(M\) is a mixed dominating matrix, then any linear combination of the rows of \(M\) is a mixed vector.

**Proof.** Suppose that there is a linear combination of the rows of \(M\) that is not a mixed vector. Then we can find a linear combination \(w\) such that \([w]_i \leq 0\) for all \(i\), i.e., \(w^+\) is the zero vector. But since \(M\) is mixed it would be impossible for \(w^+\) to dominate any \(u^+_j\) or \(u^-_j\), a contradiction. □

The following is now immediate.

**Corollary 2.8.** Let \(M\) be an \(r \times n\) mixed dominating matrix. Then the rows of \(M\) are linearly independent.

**Theorem 3.4** of [5] says that \(f_w \in (f_{u_1}, \ldots, f_{u_r})\) if and only if there exists a sequence of (not necessarily distinct) elements \(v_i \in \{\pm u_1, \ldots, \pm u_r\}, i = 1, \ldots, s\) so that \(w = v_1 + \cdots + v_s\), \(w^+ \geq v^+_i\) and for every \(i, 1 \leq i \leq s - 1\), \(w^+ - (v_1 + \cdots + v_i) \geq v^+_i\).

It is the above quoted theorem that forces \(I^* = \hat{I}\) precisely when the coefficient matrix \(M\) of the vectors \(u_1, \ldots, u_r\) has content 1. For \(M\) has content 1 if and only if \(\text{span}_Z\{u_1, \ldots, u_r\} = \mathbb{Z}^n \cap \text{span}_Q\{u_1, \ldots, u_r\}\). But if \(f_w \in \hat{I} = I^*\) with \(u \in \mathbb{Z}^n \cap \text{span}_Q\{u_1, \ldots, u_r\}\), then by this theorem, \(u\) is a linear combination of elements in \(\text{span}_Z\{u_1, \ldots, u_r\}\) and hence in \(\text{span}_Z\{u_1, \ldots, u_r\}\) and so \(M\) has content 1. The reverse implication is clear.

**Theorem 2.9.** Let \(M\) be a dominating \(r \times n\) matrix whose rows are linearly independent. As usual we denote the \(i\)th row of \(M\) by \(u_i\). Then the ideal generated by the elements \(f_{u_1}, \ldots, f_{u_r}\) is equal to \(\{f_w : w \in \text{span}_Z\{u_1, \ldots, u_r\}\}\) (i.e., \(I = I^*\)). Conversely, if the rows of \(M\) are linearly independent and if \(M\) is mixed, then \(I = I^*\) implies that \(M\) is dominating.

**Proof.** Assume that \(M\) is dominating and let \(w \in \text{span}_Z\{u_1, \ldots, u_r\}\). Write \(w = \sum a_iv_i\) where the integers \(a_i \geq 0\) and \(v_i = \pm u_i\). If \(\sum a_i = 1\), then \(w = v_j\) for some \(j\) and
since $f_{-u} = -f_u$ it follows that $f_w \in \langle f_{u_1}, \ldots, f_{u_r} \rangle$. If $\sum a_i = n$, then by (3) of the previous result $(\sum a_i v_i) = (a_i v_i) \geq v_j^+$ for some $j$. Since $w^+ \geq v_j^+$, it follows that $(w - v_i)^+ = w^+ - v_j - \delta$ where $[\delta]_i = \min\{[w^+]_i, [v_j]_i\}$ for all $i$. Hence,

$$f_w = X^{w^+ - v_j^+} \cdot f_{v_j} + X^\delta f_{w - v_j}$$

and by induction on $n$, $f_{w - v_j}$ is in $\langle f_{u_1}, \ldots, f_{u_r} \rangle$ and so is, therefore, $f_w$.

For the converse, assume that $M$ is mixed. We will show that $M$ cannot have a square mixed submatrix and hence $M$ is dominating. For if $N$ is a $t \times t$ mixed submatrix we can assume that it is of maximal size and consists of the first $t$ rows and $t$ columns of $M$. As in Theorem 2.3, because $M$ is mixed, the maximality of $N$ forces $[M]_{k,i} = 0$ for $i = 1, \ldots, t$, whenever $k > t$.

We can always find a nonzero linear combination $w = \sum_{i=1}^t a_i u_i$ of the rows of $N$ such that $[w^+]_i = 0$ for $1 \leq i \leq t$. For if $\det(N) = 0$, then there is a linear combination $w$ so that $[w]_i = 0$, $1 \leq i \leq t$ and $w \neq 0$ since these rows in $M$ are linearly independent. On the other hand, if $\det(N) \neq 0$, there is a linear combination $w$ such that $[w]_i < 0$, $1 \leq i \leq t$. We will use the theorem quoted above to arrive at a contradiction. Since $f_w \in \langle f_{u_1}, \ldots, f_{u_r} \rangle$, $w^+$ dominates some $v_1 \in \{\pm u_1, \ldots, \pm u_r\}$. But since $[w^+] = 0$, $1 \leq i \leq t$, it follows that $v_1 \in \{\pm u_{t+1}, \ldots, \pm u_r\}$ and by the assumptions on this latter set, $[w^+ - v_1]_i = 0$, $1 \leq i \leq t$. But $w^+ - v_1$ must dominate some $v_2 \in \{\pm u_1, \ldots, \pm u_r\}$ and again this implies that $v_2 \in \{\pm u_{t+1}, \ldots, \pm u_r\}$. Therefore $[w^+ - v_1 - v_2]_i = 0$ for $1 \leq i \leq t$. Continuing with this construction, it follows that $w = v_1 + \cdots + v_x$ where $v_i \in \{\pm u_{t+1}, \ldots, \pm u_r\}$. But since $w$ is also a nontrivial linear combination of the first $t$ rows of $M$, the linear independence of the rows of $M$ give a contradiction.

Corollary 2.10. Let $S$ be a finitely generated subsemigroup of $\mathbb{Z}^r$ that contains no invertible elements. Then $S$ is a complete intersection if and only if there exists an integral basis of the relation space of $S$ whose coefficient matrix $M$ is dominating with content 1.

Proof. Assume that $S$ is a complete intersection so that $\hat{I}$ is generated by $r = \dim Q W$ elements where $W$ is the space of relations of $S$. By [6, Proposition 1.11] we may assume that the $r$ generators are of the form $f_{u_1}, \ldots, f_{u_r}$, where the $u_i$'s are integral vectors in $W$, i.e., $I = \langle f_{u_1}, \ldots, f_{u_r} \rangle = I$. Furthermore, if $u \in \text{span}_Q \{u_1, \ldots, u_r\} \cap \mathbb{Z}^n$, then $f_u \in I = I$ forces $u \in \text{span}_\mathbb{Z} \{u_1, \ldots, u_r\}$ (see the discussion after Corollary 2.8). It follows that $u_1, \ldots, u_r$ is a basis for $W$ and if $M$ is the coefficient matrix of the $u_i$'s, then $M$ has content 1. Because $S$ has no invertible elements, $M$ is mixed. Therefore, since $I = I^*$, it follows from the theorem that $M$ is dominating.

Conversely, suppose that there exists vectors $u_1, \ldots, u_r$ as given in the hypothesis. It follows from the theorem that $I = \langle f_{u_1}, \ldots, f_{u_r} \rangle = I^*$. Additionally, since $M$ has content 1, we have $I^* = \hat{I}$. Since $M$ is a mixed matrix, by Corollary 2.4, $f_{u_1}, \ldots, f_{u_r}$ is a regular sequence and so $r = \text{ht}(\hat{I}) = \dim Q W$. \qed
3. Dominating matrices of size $r \times (r \times 1)$

We will apply some of our results to numerical semigroups. Specifically, we describe a decomposition that a mixed dominating $r \times (r + 1)$ matrix must possess and use this to easily recover a theorem of Delorme [3] which characterizes when a numerical semigroup is a complete intersection. Although not needed in the sequel, we first show that a mixed dominating $r \times (r + 1)$ matrix is what Klee in [7] and Brualdi and Shader in [2] call an S-matrix. An $r \times (r + 1)$ matrix is an S-matrix if every matrix $\tilde{M}$ with the same sign pattern as $M$, has a right null space that is generated by a vector all of whose entries are positive.

**Proposition 3.1.** An $r \times (r + 1)$ matrix $M$ is an S-matrix, if and only if it is mixed dominating.

**Proof.** If $M$ is mixed dominating, then the same is true for any matrix $\tilde{M}$ with the same sign pattern as $M$. It follows from Corollary 2.8 that $\tilde{M}$ has full rank and by Corollary 2.7 that every linear combination of the rows of $\tilde{M}$ must be mixed. Hence, if $u$ is the nonzero vector which generates the right nullspace of $\tilde{M}$, then $u$ cannot have a zero coordinate and cannot be mixed.

Conversely, if $M$ is an S-matrix, then $M$ is of full rank. If $M$ contains a mixed square submatrix, then choose one of largest size $(k \times k)$ and denote it by $N$. Since the rows of $N$ are mixed, it follows that for some matrix $\hat{N}$, with the same sign pattern as $N$, the columns of $\hat{N}$ are linearly dependent and therefore $\det(\hat{N}) = 0$. As we saw in Theorem 2.3, since $N$ was chosen to be mixed and of the largest size, we may assume that

$$\tilde{M} = \begin{pmatrix} \hat{N} & A \\ O & B \end{pmatrix},$$

where $O$ denotes the $(r - k) \times k$ zero matrix and $A$ and $B$ are matrices. By inspection, the determinant $\tilde{M}_{r+1}$ of the $r \times r$ submatrix obtained by deleting the $(r + 1)$st column of $\tilde{M}$ is zero. By assumption, the right null space of $\tilde{M}$ is generated by a vector $u$ each of whose coordinates are not zero. But this cannot be since $0 = \tilde{M}_{r+1}$ is a nonzero multiple of $\{u\}_{r+1}$. \(\square\)

**Definition 3.2.** Let $M$ be an $r \times n$ matrix. We will say that the $i$th row $u_i$ of $M$ isolates on the $j$th column if $u_i$ is mixed and there exists $j$ such that $[u_i]_j \neq 0$ and for all $k \neq j$, $[u_i]_k [u_i]_k \leq 0$.

Observe that a row can isolate on at most two columns and if it does, it has exactly two nonzero entries of opposite sign. The following result is known for S-matrices. We give a proof using the definition of mixed dominating matrices.

**Lemma 3.3.** Let $M$ be a mixed dominating $r \times (r + 1)$ matrix. Then there is a row of $M$ that contains exactly two nonzero entries of opposite sign.
Proof. Since $M$ is a dominating matrix, the square submatrix obtained by deleting the $j$th column must have a row, say the $i$th, which is not mixed. Therefore, the $i$th row of $M$ must isolate on the $j$th column. In particular, every column of $M$ is isolated by some row of $M$. Since there are more columns than rows, some row must isolate on two columns. Clearly, this row contains exactly two nonzero terms. □

Let $M$ be a mixed dominating $r \times (r + 1)$ matrix, and suppose that the $i$th row of $M$ contains exactly two nonzero terms, say in the $j$th and $k$th term. Denote the $j$th and $k$th column of $M$ by $v_j$ and $v_k$, respectively. Then for $l \neq i$, $[v_j][v_k][l] \geq 0$, for otherwise $M$ would contain a $2 \times 2$ mixed submatrix. Let $M^*$ be the $(r - 1) \times r$ matrix obtained from $M$ by deleting the $i$th row and replacing columns $v_j$ and $v_k$ with $v_j + v_k$. This new column will have a zero in the $i$th position if and only if $[v_j][l] = [v_k][l] = 0$. Following Brualdi and Shader we call $M^*$ the conformal contraction of $M$ along columns $j$ and $k$. Using the equivalence of S-matrices to mixed dominating $r \times (r + 1)$ matrices, Brualdi and Shader have shown that $M^*$ is a mixed dominating matrix. This can be seen directly using the definition of mixed dominating matrix in the following argument. For if $N^*$ is a square mixed $t \times t$ submatrix of $M^*$, then one of the columns of $N^*$ must contain $t$ coordinates of the “contracted” column $v_j + v_k$ since otherwise $N^*$ would have existed in $M$. But then one may form a $(t + 1) \times (t + 1)$ matrix $N$ by replacing the “contracted” column of $N^*$ with two columns corresponding to the $t$ coordinates of columns $v_j$ and $v_k$ and then addending the corresponding entries of the $i$th row of $M$. This square matrix $N$ is a submatrix of $M$ and row $i$ is mixed by the way it was selected. Since for $l \neq i$ $[v_j][l][v_k][l] \geq 0$, it follows that every row of $N$ is mixed, contradicting that $M$ is dominating.

In the following we denote the content of a matrix $M$ by $\text{cont}(M)$. We also denote by $M_{(i)}$, the determinant of the $r \times r$ submatrix obtained by deleting the $i$th column of $M$.

**Theorem 3.4.** Let $M$ be an $r \times (r + 1)$ matrix. Then $M$ is mixed dominating if and only if there exists a rearrangement of the rows and columns of $M$ such that

$$M = \begin{pmatrix} A & O \\ O & B \\ a & b \end{pmatrix},$$

where $A$ and $B$ are mixed dominating matrices of sizes $t \times (t + 1)$ and $s \times (s + 1)$, respectively, with $t \geq 0$, $s > 0$ and $s + t + 1 = r$. Additionally, $a$ and $b$ are $1 \times (t + 1)$ and $1 \times (s + 1)$ nonzero, nonmixed matrices, respectively, of opposite sign. Furthermore, $\text{cont}(M) = 1$ if and only if $\text{cont}(A) = \text{cont}(B) = 1$ and the integers $\det(A)$ and $\det(B)$ are relatively prime where $\bar{A}$ and $\bar{B}$ are the $(t + 1) \times (t + 1)$ and $(s + 1) \times (s + 1)$ matrices obtained by adjoining $a$ to $A$ and $b$ to $B$, respectively.

Proof. Note that we allow $A$ to be a $0 \times 1$ matrix in which case the first column of $M$ consists of zeroes except for the last entry and $\bar{A}$ is the $1 \times 1$ matrix $(a)$. We make the convention in this case that $\text{cont}(A) = 1$. 

If $M$ can be put into the form of (1), then clearly $M$ is mixed. It is also dominating. For suppose that $N$ is a square mixed submatrix of $M$. Then the set of indices of the rows of $N$ is not a subset of the set $\{1, \ldots, t, r\}$. If it were, then by deleting the $r$th row and all columns beyond the $(t + 1)$st from $N$ one obtains a mixed submatrix of $A$ that has at least as many rows as columns. This contradicts the fact that $A$ is dominating. Similarly, the set of indices of the rows of $N$ is not a subset of $\{t + 1, \ldots, r\}$.

If $A'$ is the submatrix of $A$ consisting of all entries that are in $N$, then $A'$ is nonempty and it is mixed since $N$ is mixed and $[M]_{ij} = 0$ for $1 \leq i \leq t$ and $j > t + 1$. Hence, $A'$ must have more columns than rows. The same can be said for $B'$, the analogous submatrix of $B$. However, the number of columns of $N$ is equal to the sum of the number of columns in $A'$ and $B'$, while the number of rows is at most the sum of the number of rows in $A'$ and $B'$ plus one. Hence, $N$ has more columns than rows, contradicting the fact that $N$ is square.

We will prove the converse by induction on $r$. If $M$ is a $2 \times 3$ mixed dominating matrix, then by inspection one checks that $M$ can be put into the form

$$
\begin{pmatrix}
0 & c_1 & c_2 \\
-d_1 & -d_2 & -d_3
\end{pmatrix}
$$

where $c_1, c_2 > 0$, $d_1 > 0$ and $d_j$, $j = 2, 3$ is nonnegative with at least one of them positive. Hence, the result is true for $r = 2$. Now assume that $M$ is mixed dominating and $r > 2$. By Lemma 3.3, some row of $M$ consists of exactly two nonzero entries, say in columns $j$ and $k$. Let $M^*$ be the conformal contraction of $M$ along columns $j$ and $k$. Since $M^*$ is mixed dominating, by induction it can be put into the form of (1) by rearranging rows and columns. Since the contracted column of $M^*$ has a zero in the $i$th row if and only if the $i$th row entries of columns $j$ and $k$ of $M$ are both zero, it follows that $M$ can also be put into the form of (1).

To show the second part of the theorem, assume that $\text{cont}(A) = \text{cont}(B) = 1$ and $\det(A)$ and $\det(B)$ are relatively prime. For $i = 1, 2, \ldots, t + 1$, $M_{(i)} = A_{(i)} \det(B)$ and for $j = t + 2, t + 3, \ldots, r + 1$, $M_{(j)} = B_{(j-t-1)} \det(A)$. Since $\text{cont}(M) = \gcd(M_{(i)} : i = 1, 2, \ldots, r + 1)$, it follows that $\text{cont}(M) = 1$.

Now suppose that $\text{cont}(M) = 1$. It follows from the description of $M_{(i)}$ above, that $\det(A)$ and $\det(B)$ must be relatively prime. Moreover, if $d = \text{cont}(A)$, then $d$ divides $M_{(i)}$ for $i = 1, 2, \ldots, t + 1$, since it divides $A_{(i)}$. From the equation

$$
\det(A) = \sum_{i=1}^{t+1} (-1)^{i+1}[a]_i A_{(i)}
$$

we see that $d$ divides $\det(A)$ and hence $d$ divides $M_{(j)}$ for $j = t + 2, t + 3, \ldots, r + 1$. Therefore $d$ divides $\text{cont}(M)$, proving that $d = 1$. An analogous argument shows that $\text{cont}(B) = 1$. □

Recall that a semigroup $S$ is called numerical if $S$ is a subset of the natural numbers. If $S$ is generated by $r + 1$ positive integers $\{n_1, n_2, \ldots, n_{r+1}\}$, then the dimension of the relation space $W$ is $r$. It also follows that the coefficient matrix $M$ of an integral basis
of \( W \) is a mixed \( r \times (r+1) \) matrix. Since the vector \((M_{(1)}, -M_{(2)}, \ldots, (-1)^{r+1}M_{(r+1)})\) is orthogonal to the row space of \( M \) as is the vector \((n_1, n_2, \ldots, n_{r+1})\), it follows that these two vectors are rational multiples of each other. In fact, we may present \( M \) so that they are positive rational multiples of each other. Therefore, if \( M \) has content 1, and if \( \gcd(n_1, n_2, \ldots, n_{r+1}) = 1 \), then \((-1)^{i+1}M_{(i)} = n_i \) for \( i = 1, \ldots, r+1 \). More generally, if \( \gcd(n_1, n_2, \ldots, n_{r+1}) = d \), then \((-1)^{i+1}M_{(i)} = n_i/d \).

In the next result we show how the decomposition of \( M \) in the previous theorem is essentially what is needed to prove a result by Delorme [3, Proposition 9]. This proposition characterizes numerical semigroups which are complete intersections by partitioning the generating set so that each subset generates a complete intersection. If \( G \) is a set of positive integers we denote by \( \gcd(G) \) the greatest common divisor of \( G \) and by \( \langle G \rangle \) the semigroup generated by \( G \).

\[ \text{Theorem 3.5.} \quad \text{Let } G = \{n_1, n_2, \ldots, n_{r+1}\} \text{ be a subset of } \mathbb{N}. \text{ Then the semigroup } \langle G \rangle \text{ is a complete intersection if and only if } G \text{ may be partitioned into two subsets } G_1 \text{ and } G_2 \text{ so that the semigroups } \langle G_1 \rangle \text{ and } \langle G_2 \rangle \text{ are complete intersections and } \gcd(G_1)\gcd(G_2)/\gcd(G) \in \langle G_1 \rangle \cap \langle G_2 \rangle. \]

\[ \text{Proof.} \quad \text{Suppose that } \langle G \rangle \text{ is a complete intersection. Let } M \text{ be a mixed dominating matrix with } \text{cont}(M) = 1 \text{ which is the coefficient matrix of an integral basis of the relation space of } \langle G \rangle. \text{ Assume that } M \text{ is so presented that } (-1)^{i+1}M_{(i)} = n_i/d, \text{ where } d = \gcd(n_1, \ldots, n_{r+1}) \text{ and is written in the form of Theorem 3.4. Using the notation from that theorem, let } G_1 \text{ consist of the first } t+1 \text{ elements of } G \text{ and } G_2 \text{ the rest. Since the rows of } A \text{ and } B \text{ are linearly independent they form the coefficient matrix of an integral basis of the relation space of } \langle G_1 \rangle \text{ and } \langle G_2 \rangle, \text{ respectively. Since } A \text{ and } B \text{ are mixed dominating of content 1}, \langle G_1 \rangle \text{ and } \langle G_2 \rangle \text{ are complete intersections.}

\text{But } n_i/d = (-1)^{i+1}A_{(i)}\det(B) \text{ for } i = 1, 2, \ldots, t+1 \text{ and hence it follows that } \gcd(G_1) = \det(B)d \text{ and similarly, } \gcd(G_2) = \det(A)d. \text{ The last row of } M \text{ says that}

\[ a_1n_1 + \cdots + a_{t+1}n_{t+1} = b_1n_{t+2} + \cdots + b_{r+1}n_{r+1}, \]

and therefore this integer is in both \( \langle G_1 \rangle \) and \( \langle G_2 \rangle \). Replacing \( n_i \) with \((-1)^{i+1}A_{(i)}\det(B)d \), for \( i = 1, \ldots, t+1 \), one has that \( \gcd(G_1)\gcd(G_2)/d \in \langle G_1 \rangle \cap \langle G_2 \rangle \).

Conversely, assume that \( G_1 \) and \( G_2 \) are partitions of \( G \) with the described properties. Let \( A \) and \( B \) be coefficient matrices for an integral bases of the space of relations of \( G_1 \) and \( G_2 \), respectively, where \( A \) and \( B \) are mixed dominating with content 1. Since \( \gcd(G) = d \), it follows that \( d \) is the greatest common divisor of \( \gcd(G_1) \) and \( \gcd(G_2) \). By assumption

\[ \gcd(G_1)\gcd(G_2)/d = a_1n_1 + \cdots + a_{t+1}n_{t+1} = b_1n_{t+2} + \cdots + b_{r+1}n_{r+1}, \]

where \( a_i, b_j \geq 0 \). If \( a = (a_1, a_2, \ldots, a_{t+1}) \) and \( b = -(b_1, b_2, \ldots, b_{r+1}) \), then the matrix

\[ M = \begin{pmatrix} A & O \\ O & B \\ a & b \end{pmatrix} \]
is mixed dominating by Theorem 3.4 and therefore of full rank. Hence, it is a coefficient
matrix of an integral basis of the relation space of \( \langle G \rangle \). To conclude the proof we
need only show by Corollary 2.10 that \( \text{cont}(M) = 1 \) and for this it suffices to prove
by Theorem 3.4 that \( \det(A) \) and \( \det(B) \) are relatively prime. But for \( i = 1, \ldots, t + 1, \)
\( n_i = (-1)^{i+1} \gcd(G_i) A_{(i)} \) and hence
\[
\gcd(G_1) \gcd(G_2) / d = a_1 n_1 + \cdots + a_{t+1} n_{t+1} = \gcd(G_1) \det(A).
\]
It follows that \( \gcd(G_2) / d = \det(A) \) and a similar argument shows that \( \gcd(G_2) / d = \det(B) \) and therefore, \( \det(A) \) and \( \det(B) \) are relatively prime. □

Note that the proof of Theorem 3.4 shows that all \( 2 \times 3 \) mixed dominating matrices
can be put into the form
\[
M = \begin{pmatrix}
0 & c_1 & -c_2 \\
d_1 & -d_2 & -d_3
\end{pmatrix},
\]
where \( c_1, c_2 > 0 \) and \( d_1 > 0 \) and \( d_2, d_3 \) are nonnegative and \( d_2 + d_3 > 0 \). According to
the above theorem, \( \langle n_1, n_2, n_3 \rangle \) is a complete intersection if and only if
\[
\gcd(n_1) \gcd(n_2, n_3) / \gcd(n_1, n_2, n_3) \in \langle n_1 \rangle \cap \langle n_2, n_3 \rangle
\]
or, equivalently \( \text{lcm}(n_1, \gcd(n_2, n_3)) \in \langle n_2, n_3 \rangle \). The matrix \( M \) decomposes into matrices
\( A \) and \( B \) where \( A \) is empty and \( B = (c_1, -c_2) \). This points out a specific case when
a numerical semigroup is a complete intersection and also forms the inductive start of
the “if” portion of Corollary 3.8.

**Definition 3.6.** An \( r \times n \) mixed matrix \( M \) will be called *principally dominating*
if, after interchanging rows and columns and multiplying rows by \(-1\), \( M \) has the sign pattern
\[
\begin{pmatrix}
0 & + & * & * \\
\vdots & + & * & * \\
+ & * & * & *
\end{pmatrix}
\]
where \( [M]_{ij} = 0 \) for \( i + j \leq r, [M]_{ij} > 0 \) for \( i + j = r + 1 \) and \( [M]_{ij} \leq 0 \) for \( i + j > r + 1 \).
Furthermore, for each row at least one of the entries \( * \) is negative.

Clearly, such a matrix is mixed dominating and if \( \text{cont}(M) = 1 \) it represents the
coefficient matrix of an integral basis for a semigroup which is a complete intersection.
If \( M \) is of size \( r \times (r + 1) \), there is an analogous criteria for numerical semigroups that
is due to Herzog [6] which describes the simplest form for a numerical semigroup to
be a complete intersection.

**Definition 3.7.** We say that the numerical semigroup \( S \) satisfies condition (H) if the
generating set \( \{n_1, \ldots, n_{r+1}\} \) of \( S \), after a suitable reordering, satisfies \( \text{lcm}(n_i, \gcd(n_{i+1}, \ldots, n_{r+1})) \in \langle n_i+1, \ldots, n_{r+1} \rangle \), for \( i = 1, 2, \ldots, r \).
Herzog showed that for a numerical semigroup \( S \), condition (H) implies that the semigroup is a complete intersection. Furthermore, if \( S \) is three generated he showed, as does the argument prior to Definition 3.6, that condition (H) is equivalent to \( S \) being a complete intersection. We show in general that condition (H) is equivalent to principally dominating.

**Corollary 3.8.** Let \( S = \langle n_1, \ldots, n_{r-1} \rangle \). The semigroup \( S \) satisfies condition (H) if and only if there exists an integral basis for the relation space of \( S \) whose coefficient matrix \( M \) is principally dominating with content 1.

**Proof.** The key to this result are the facts that

\[
\text{lcm}(n_i, \text{gcd}(n_{i+1}, \ldots, n_{r-1})) = n_i \text{gcd}(n_{i+1}, \ldots, n_{r-1})/d_i,
\]

where \( d_i = \text{gcd}(n_i, n_{i+1}, \ldots, n_{r-1}) \) and that \( \text{lcm}(n_i, \text{gcd}(n_{i+1}, \ldots, n_{r-1})) \) is always in \( \langle n_i \rangle \).

Suppose that \( S \) satisfies condition (H) with respect to the generating set \( \{n_1, n_2, \ldots, n_{r+1}\} \). If \( G_1 = \{n_1\} \) and \( G_2 = \{n_2, \ldots, n_{r+1}\} \), then clearly \( \langle G_1 \rangle \) is a complete intersection. By induction there is a coefficient matrix \( B \) of a basis of the space of relations of \( \langle G_2 \rangle \) that is principally dominating with content 1. Therefore, \( \langle G_2 \rangle \) is a complete intersection. Furthermore, it is clear by our initial observation that we may write \( \text{gcd}(G_1) \text{gcd}(G_2)/d_1 = a_1 n_1 = b_2 n_2 + \cdots + b_{r+1} n_{r+1} \in \langle G_1 \rangle \cap \langle G_2 \rangle \). Hence, the matrix

\[
M = \begin{pmatrix} O \\ a \\ b \end{pmatrix} B
\]

where \( a = (a_1) \) and \( b = (b_2, \ldots, b_{r+1}) \) is a coefficient matrix of a basis of the relation space of \( S \) having the desired form.

Conversely, let \( M \) be a coefficient matrix of a basis of the space of relations of \( S \) such that \( M \) that is principally dominating and has content 1. We can assume that the matrix is already in the form given in Definition 3.6 and that after a suitable reordering, \( n_i/d = (-1)^t M(i) \). Since \( M \) is in the form of Theorem 3.4 with \( t = 0 \), the corresponding partition of \( G \) in Theorem 3.5 is \( G_1 = \{n_1\} \) and \( G_2 = \{n_2, \ldots, n_{r+1}\} \). It follows that \( \text{lcm}(n_1, \text{gcd}(n_2, \ldots, n_{r+1})) = n_1 \text{gcd}(n_2, \ldots, n_{r+1})/d_1 \) is in \( \langle G_1 \rangle \cap \langle G_2 \rangle \). This is the first step of condition (H). Repeating the argument on \( G_2 \) shows that \( S \) satisfies condition (H). \( \square \)

The following example shows that condition (H) is a special case of a complete intersection or, equivalently, principally dominating is a special case of mixed dominating.

**Example 3.9.** Consider the numerical semigroup \( S = \langle 33, 44, 20, 30 \rangle \). One checks that this semigroup does not satisfy condition (H) by observing that no matter how the elements are ordered, \( \text{lcm}(n_1, \text{gcd}(n_2, n_3, n_4)) \) is never in \( \langle n_2, n_3, n_4 \rangle \). Thus, \( S \) cannot be represented by a principally dominating matrix of content 1. However, \( S \) is a complete
intersection because the following dominating content 1 matrix is the coefficient matrix of an integral basis of the space of relations of $S$:

$$
\begin{pmatrix}
4 & -3 & 0 & 0 \\
0 & 0 & 3 & -2 \\
2 & 1 & -1 & -3
\end{pmatrix}.
$$

Here, $A = (4, -3)$ and $B = (3, -2)$ are mixed dominating, content 1 matrices. Therefore, $\langle 33, 44 \rangle$ and $\langle 20, 30 \rangle$ form complete intersections with $11 \cdot 10/1 \in \langle 33, 44 \rangle \cap \langle 20, 30 \rangle$.

References


