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Iterated local reflection versus iterated consistency

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Abstract

For "natural enough" systems of ordinal notation we show that α times iterated local reflection schema over a sufficiently strong arithmetic T proves the same Π^0_1-sentences as ω^α times iterated consistency. A corollary is that the two hierarchies catch up modulo relative interpretability exactly at ε-numbers. We also derive the following more general "mixed" formulas estimating the consistency strength of iterated local reflection: for all ordinals $\alpha \geqslant 1$ and all β,

$$(T^\alpha)_\beta \equiv_{\Pi^0_1} T_{\omega^\alpha \cdot (1+\beta)}, \qquad (T_\beta)^\alpha \equiv_{\Pi^0_1} T_{\beta+\omega^\alpha}.$$

Here T^α stands for α times iterated local reflection over T, T_β stands for β times iterated consistency, and $\equiv_{\Pi^0_1}$ denotes (provable in T) mutual Π^0_1-conservativity.

In an appendix to this paper we develop our notion of "natural enough" system of ordinal notation and show that such systems do exist for every recursive ordinal.

1. Introduction

Since the fundamental works of Turing [17] and Feferman [6] transfinite recursive hierarchies of axiomatic theories have been playing a significant role in proof-theoretic studies, mainly as a kind of tool for measuring relative strength of theories. Historically the first and, probably, the most important example of such a hierarchy is the so-called *transfinite recursive progression based on iteration of consistency* defined (roughly) according to the following clauses:

(T1) $T_0 = T$, T being a given "initial" theory;

(T2) $T_{\alpha+1} = T_\alpha + \mathrm{Con}(T_\alpha)$;

(T3) $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$, for α a limit ordinal.

Here and below $\mathrm{Con}(U)$ denotes the standard arithmetical sentence expressing the consistency of a theory U.

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By Gödel's Theorem, whenever the initial theory T is sound,[1] the theories T_α form a strictly increasing transfinite sequence of sound extensions of T. This sequence can be used to associate an ordinal $\mathrm{ord}_T(U)$ with any theory U extending T as follows:

$$\mathrm{ord}_T(U) := \text{ least } \alpha \text{ such that } U \nvdash \mathrm{Con}(T_\alpha).$$

This definition is really only meaningful for those theories U which can, in a sense, be well approximated by the sequence T_α. For such theories (fortunately, these include the most natural extensions of T) one can usually show that $T_{\mathrm{ord}_T(U)}$ exhausts all arithmetical Π_1^0-consequences of U, that is,

$$U \equiv_{\Pi_1^0} T_{\mathrm{ord}_T(U)}. \tag{1}$$

(This equivalence can itself be considered as a definition of the property of well-approximation above.) Possible verification of the equivalence (1) within T immediately implies

$$T \vdash \mathrm{Con}(U) \leftrightarrow \mathrm{Con}(T_{\mathrm{ord}_T(U)}),$$

and thus, $\mathrm{ord}_T(U)$ can be thought of as an ordinal measuring the consistency strength of the theory U with respect to T.

A well-known difficulty in the way of this program roots in the fact that the clauses (T1)–(T3) do not uniquely define the sequence of theories T_α, that is, the theory T_α depends on the formal representation of the ordinal α within arithmetic rather than on the ordinal itself.

For the analysis of this problem Feferman [6] considered families of theories of the form $(T_c)_{c\in\mathcal{O}}$ satisfying (T1)–(T3) along every path within $\mathcal{O}$, where $\mathcal{O}$ is Kleene's universal system of ordinal notation. Using an idea of Turing, he showed that every true Π_1^0-sentence is provable in T_c for a suitable ordinal notation $c \in \mathcal{O}$ with $|c| = \omega + 1$. It follows that there are two ordinal notations $a, b \in \mathcal{O}$ with $|a| = |b| = \omega + 1$ such that T_a proves $\mathrm{Con}(T_b)$, and this observation seems to break down the program of associating ordinals to theories as described above, at least in the general case.

A possibility remains that for *natural* (mathematically meaningful) theories U, one can exhaust all Π_1^0-consequences of U using only specific *natural* ordinal notations, and a careful choice of such notations should yield proper ordinal bounds. This idea has been developed in the work of Schmerl [12] who showed among other things, e.g., that for natural ordinal notations,

$$\mathrm{PA} \equiv_{\Pi_1^0} \mathrm{PRA}_{\varepsilon_0}.$$

Thus, $\mathrm{ord}_{\mathrm{PRA}}(\mathrm{PA}) = \varepsilon_0$, which coincides with the ordinal associated with PA through other proof-theoretic methods. (In this formula PRA could be replaced by any finite subtheory of PA.)

[1] That is, if all theorems of T hold in the standard model of arithmetic.

The quoted result is a corollary of a more general theorem relating different restricted versions of iterated *uniform* reflection principles over PRA modulo $\equiv_{\Pi_1^0}$ and $\equiv_{\Pi_n^0}$ for $n > 1$. Recall (cf. [8, 14]) that the (full) uniform reflection principle for a theory T is the schema

$$\mathrm{RFN}(T) \rightleftharpoons \{\forall x\,(\mathrm{Pr}_T(\ulcorner A(\dot{x})\urcorner) \to A(x)) \mid A(x) \in \mathrm{Fm}_T\},$$

whereas the *local reflection principle* is defined as

$$\mathrm{Rfn}(T) \rightleftharpoons \{\mathrm{Pr}_T(\ulcorner A \urcorner) \to A \mid A \in \mathrm{St}_T\}.$$

Here Fm_T and St_T denote the sets of all formulas and sentences in the language of T, and $\mathrm{Pr}_T(\cdot)$ is the standard provability predicate for T.

Recursive hierarchies of theories based on iteration of the uniform and local reflection principles are defined in analogy with (T1)–(T3). The clause (T2) should then be replaced, respectively, by

$(\mathrm{T2}_{\mathrm{RFN}})\, T_{\alpha+1} = T_\alpha + \mathrm{RFN}(T_\alpha)$ and

$(\mathrm{T2}_{\mathrm{Rfn}})\, T_{\alpha+1} = T_\alpha + \mathrm{Rfn}(T_\alpha)$.

From the results of Schmerl it follows that, for natural systems of ordinal notation, $1 + \alpha$ times iterated uniform reflection principle over PRA proves the same Π_1^0-sentences as ε_α times iterated consistency. The corresponding question for iterated local reflection principles, however, remained open. In the following, the hierarchy of theories based on iteration of local reflection principles will be denoted T^α, and we shall keep the notation T_α for the hierarchy of iterated consistency assertions. We mention a few basic results on local reflection principles relevant for our work.

Löb [9] showed that an instance $\mathrm{Pr}_T(\ulcorner A \urcorner) \to A$ of the local reflection schema is provable in the theory T if and only if so is the sentence A (Löb's Theorem).

Feferman [6] considered transfinite recursive progressions $(T^c)_{c \in \mathcal{O}}$ based on iteration of local reflection. He showed that, for every ordinal notation $c \in \mathcal{O}$, the theory T^c is contained in the set of consequences of all true Π_1^0 arithmetical sentences over T. Therefore, the progression based on iteration of local reflection is "ultimately" of the same strength as the progression based on iteration of consistency:

$$\bigcup_{c \in \mathcal{O}} T^c \equiv \bigcup_{c \in \mathcal{O}} T_c \equiv T + \text{all true } \Pi_1^0\text{-sentences}.$$

On the other hand, Kreisel and Lévy [8] proved that $T^1 = T + \mathrm{Rfn}(T)$ cannot be majorized by any recursively enumerable set of true Π_1^0-sentences over T, and therefore $T^1 \nsubseteq T_c$ for all $c \in \mathcal{O}$.

Artemov [2] showed that, although obviously $T_\omega \subseteq T^1$, for no $c \in \mathcal{O}$ with $|c| > \omega$ do we have $T_c \subseteq T^1$. This result relies on a beautiful lemma, coming from provability logic and proved by Boolos [5] and independently by Artemov [1], stating that in order to derive $n < \omega$ times iterated consistency for any theory T no less than n instances of the local reflection schema for T are needed.

Goryachev [7] brought essentially the same idea to a particularly nice form by showing that the theories T^1 and T_ω are, in fact, mutually interpretable, and thus, prove the same Π_1^0-sentences (whatever the initial theory T, both T^1 and T_ω are always reflexive theories, cf. [3]).

In this paper we extend Goryachev's theorem to higher ordinals in the spirit of Schmerl's results, thus establishing precise relationship between the hierarchies of iterated local reflection principles and iterated consistency assertions for natural ordinal notation systems. We show that, for ordinals $\alpha \geqslant 1$, α times iterated local reflection schema over any sufficiently strong arithmetic T proves the same Π_1^0-sentences as ω^α times iterated consistency. (Notice that this ordinal does not essentially depend on the choice of the initial theory T, whereas for the case of iterated uniform reflection it does: we have $\mathrm{PRA} + \mathrm{RFN}(\mathrm{PRA}) \equiv_{\Pi_1^0} \mathrm{PRA}_{\varepsilon_0}$, but $\mathrm{PA} + \mathrm{RFN}(\mathrm{PA}) \equiv_{\Pi_1^0} \mathrm{PRA}_{\varepsilon_1}$.) We also derive the following more general "mixed" formulas:

$$(T^\alpha)_\beta \equiv_{\Pi_1^0} T_{\omega^\alpha \cdot (1+\beta)}, \qquad (T_\beta)^\alpha \equiv_{\Pi_1^0} T_{\beta + \omega^\alpha}. \tag{2}$$

In an appendix to this paper we isolate the properties of natural systems of ordinal notation needed for the above formulas to hold. This allows us to show that such systems exist for every constructive ordinal. Rather than saying much about the well-known problem of the choice of natural ordinal notations for large ordinals, this result merely shows that relationships such as (2) are general enough to hold even for those notation systems which, perhaps, would not serve as "natural" for some other proof-theoretic investigations. Therefore, a comprehensible formulation of the amount of natural properties of well-orders used here seems to be of some independent interest.

2. Preliminaries

2.1. Theories

All theories in this paper are assumed to be first order and to contain primitive recursive arithmetic (PRA) (cf. [14]). We also assume that each theory T comes together with a primitive recursive (p.r.) formula $\mathrm{Ax}_T(x)$ numerating the set of Gödel numbers of mathematical axioms of T, from which a p.r. formula $\mathrm{Prf}_T(y, x)$ expressing the predicate "y is (the G.n. of) a proof in T of the formula (with the G.n.) x" is constructed in the standard way. Let $\mathrm{Pr}_T(x)$ abbreviate $\exists y\, \mathrm{Prf}_T(y, x)$ and $\mathrm{Con}(T) := \neg \mathrm{Pr}_T(\ulcorner 0 = 1 \urcorner)$.

Parametric families of theories are numerated by p.r. formulas $\mathrm{Ax}(x)$ containing some free variables other than x. In particular, the formula $\mathrm{Ax}_{T \restriction n}(x) := (\mathrm{Ax}_T(x) \wedge x \leqslant n)$ numerates the canonical family $(T \restriction n)_{n \in N}$ of finite subtheories of a theory T.

Two theories U and V are *equivalent* iff they have the same set of theorems. In this case we also write $U \equiv V$. When used in a formalized context this notation is meant to abbreviate the formula $\forall x\,(\mathrm{Pr}_U(x) \leftrightarrow \mathrm{Pr}_V(x))$. Similarly, $U \subseteq V$ denotes $\forall x\,(\mathrm{Pr}_U(x) \to \mathrm{Pr}_V(x))$, $U \subseteq_{\Pi_1^0} V$ denotes $\forall x \in \Pi_1^0(\mathrm{Pr}_U(x) \to \mathrm{Pr}_V(x))$, and $U \equiv_{\Pi_1^0} V$ means $\forall x \in \Pi_1^0(\mathrm{Pr}_U(x) \leftrightarrow \mathrm{Pr}_V(x))$, where Π_1^0 stands for (a p.r. definition of) the set of Gödel numbers of arithmetical Π_1^0-sentences.

Further, we write $U \rhd V$ for $\forall n\; U \vdash \mathrm{Con}(T \restriction n)$. The following lemma is both well-known and easy to verify.

Lemma 2.1. *The following properties are provable in PRA for any theories U and V:*

1. $V \subseteq U \to V \subseteq_{\Pi_1^0} U$,
2. $U \rhd V \to V \subseteq_{\Pi_1^0} U$,
3. $V \subseteq_{\Pi_1^0} U \to (\mathrm{Con}(U) \to \mathrm{Con}(V))$.

If a theory V is reflexive, that is, if $V \rhd V$, we also have $V \subseteq_{\Pi_1^0} U \to U \rhd V$, and the two relations $V \subseteq_{\Pi_1^0} U$ and $U \rhd V$ become equivalent. It is also well known that for reflexive theories the relations $\subseteq_{\Pi_1^0}$ and $\rhd$ are equivalent to that of relative interpretability (cf. e.g. [18].)

2.2. *Recursive progressions*

There are at least two ways to formalize the definition (T1)–(T3) of transfinite progressions of theories based on iteration of consistency. One relies on the concept of a (primitive) recursive *well-ordering relation*, as e.g. in [12], and the other one relies on the concept of *constructive system of ordinal notation*, as in [6]. The difference between the two approaches is largely technical, but, at least for our present purposes, the former seems to be more convenient than the latter. The reason is that, under rather weak natural assumptions, transfinite recursive progressions of theories dealt with in this paper actually do not depend on the choice of fundamental sequences for ordinals. Therefore, our work will be greatly simplified, if these are avoided from the very beginning.

A *primitive recursive well-ordering* $(D, <)$ is a relative interpretation of the first-order theory of linear orderings in PRA with domain D, such that the predicates $x \in D$ and $x < y$ are (interpreted as) p.r. formulas and the relation $<$ well-orders the set D in the standard model of arithmetic. (The statement that $(D, <)$ is a relative interpretation of the theory of linear orderings in PRA essentially means that PRA proves that the relation $<$ linearly orders the set D.)

Suppose we are given a sound "initial" theory T and a p.r. well-ordering $(D, <)$. A p.r. formula $\mathrm{Ax}_T(z; x)$ is called a *smooth numeration of a progression based on iteration of consistency along* $(D, <)$ iff PRA proves

$$\forall z, x\,(\mathrm{Ax}_T(z;x) \leftrightarrow \mathrm{Ax}_T(x) \vee (z \in D \wedge \exists u \in D(u < z \wedge x = \ulcorner \mathrm{Con}(T_u) \urcorner))). \tag{3}$$

Here T_u denotes the theory numerated by $\mathrm{Ax}_T(u; x)$. For the sake of readability we shall also write $\mathrm{Pr}_T(z; x)$ for $\mathrm{Pr}_{T_z}(x)$.

Clearly, the definition (3) has the form of a fixed point equation; therefore, smooth numerations can be constructed for any given p.r. well-ordering. It only has to be noted that the existential quantifier in (3) can actually be bounded by x, assuming the Gödel numbering we use is standard, so the solution of the fixed point equation has to be equivalent to a p.r. formula. Then one can show, by metamathematical transfinite induction, that $(T_u)_{u \in D}$ is a strictly increasing sequence of sound theories satisfying (T1)–(T3). Moreover, the fact that the relation $<$ linearly orders the set D *provably in* PRA guarantees that PRA proves the formal analogs of (T1)–(T3):

(V1) $\text{“}u = 0\text{”} \vee u \notin D \rightarrow \forall x\, (\mathrm{Ax}_T(u;x) \leftrightarrow \mathrm{Ax}_T(x))$,
(V2) $\text{“}u = v + 1\text{”} \rightarrow \forall x\, (\mathrm{Ax}_T(u;x) \leftrightarrow (\mathrm{Ax}_T(v;x) \vee x = \ulcorner \mathrm{Con}(T_{\dot{v}}) \urcorner))$,
(V3) $\mathrm{LIM}(u) \rightarrow \forall x\, (\mathrm{Ax}_T(u;x) \leftrightarrow \exists z \in D(z < u \wedge \mathrm{Ax}_T(z;x)))$.

Here the expression “$u = 0$” abbreviates the formula

$$u \in D \wedge \forall z \in D(u < z \vee u = z),$$

“$u = v + 1$” means

$$u, v \in D \wedge v < u \wedge \forall z \in D(z < u \rightarrow z < v \vee z = v),$$

and $\mathrm{LIM}(u)$ denotes

$$u \in D \vee \neg \text{“}u = 0\text{”} \wedge \forall z \in D(z < u \rightarrow \exists v \in D(v < u \wedge z < v)).$$

From now on we shall adopt the following notational convention. Greek variables α, β, γ, etc., will always be assumed to range over ordinals, that is, over the domain D. Mathematical symbols like 0, 1, $<$, $+$, etc., will refer to the operations and predicates on ordinals. In the rare occasions when the ordinary arithmetical operations on natural numbers are used, they will be typed in boldface characters.

Formulas $\mathrm{Ax}_T(z;x)$ satisfying (the analogs of) (V1)–(V3) are called *verifiable numerations* for progressions based on iteration of consistency in [6]. Thus, smooth numerations are verifiable, but the converse is, generally, not true. We can only say that verifiable numerations are smooth in presence of transfinite induction for $(D, <)$, which is usually only the case for rather small ordinals.

Conditions like verifiability or smoothness can be thought of as coherence conditions on the simultaneous choice of numerations of theories $(T_\alpha)_{\alpha \in D}$ of a recursive progression. Whereas verifiability seems to be the weakest reasonable assumption of this sort, smoothness implies some additional natural properties of progressions. E.g., smooth numerations are *provably monotone* in the sense that they satisfy the following property provably in PRA:

(V4) $\forall \alpha, \beta(\alpha < \beta \rightarrow \forall x\, (\mathrm{Pr}_T(\alpha;x) \rightarrow \mathrm{Pr}_T(\beta;x)))$.

This property follows immediately from (3) and provable transitivity of $<$.

For smooth numerations we also have the following useful property.

Lemma 2.2. If $Ax_T(z;x)$ *is a smooth numeration of a recursive progression based on iteration of consistency, then* PRA *proves*

$$\forall\alpha,\beta\,(\alpha<\beta\rightarrow\forall x\,(\mathrm{Pr}_T(\beta;x)\leftrightarrow\exists\gamma<\beta\,\mathrm{Pr}_T(\alpha;\ulcorner\mathrm{Con}(T_{\dot\gamma})\urcorner\dot\rightarrow x))).$$

Proof. We give an informal argument that can be readily formalized in PRA.

By provable monotonicity (V4) $T_\alpha\subseteq T_\beta$, and by the definition of smoothness T_β is axiomatized over T_α by all sentences of the form $\mathrm{Con}(T_\gamma)$ with $\gamma<\beta$. The implication ($\leftarrow$) follows immediately.

To show ($\rightarrow$) consider an arbitrary derivation y of a formula x in T_β. From y one can primitively recursively reconstruct the finite set of all axioms of the form $\mathrm{Con}(T_\gamma)$ used in this derivation. Using provable linearity of $<$ pick the axiom corresponding to the largest ordinal out of this set. By (V4) this axiom will be the strongest one, so the other axioms can be replaced in y by their respective derivations from this axiom. Since the proof of (V4) is uniform in α and β, the total length of such derivations can be estimated by a p.r. function of y. This shows that such a proof transformation can be carried out inside PRA. It only remains to apply the formalized deduction theorem. □

Concerning the definitions of verifiability and smoothness a natural question arises: can one impose any additional natural requirements on the choice of numerations of recursive progressions that smooth numerations possibly lack? There is an easy, but nonetheless rather surprising answer to this question. No, smooth numerations are, in a very strong sense, *optimal*, because of the following uniqueness property.

Lemma 2.3 (Uniqueness). *Any two smooth numerations* $\mathrm{Ax}_T(z;x)$ *and* $\mathrm{Ax}'_T(z;x)$ *along one and the same p.r. well-ordering* $(D,<)$ *and satisfying the same initial conditions define equivalent progressions of theories based on iteration of consistency, i.e., provably in* PRA,

$$\forall\alpha\ T_\alpha\equiv T'_\alpha.$$

The uniqueness property is a robust background for our further treatment of recursive progressions of theories, and, in particular, it shows that these progressions, when smoothly defined, do not depend on the choice of fundamental sequences for ordinal notations.

The proof of Lemma 2.3 employs a trick from the work of Schmerl [12], which will also be extensively used later in this paper.

Lemma 2.4 (Reflexive induction). *For any p.r. well-ordering* $(D,<)$, *any theory* T *is closed under the following* reflexive induction rule:

$$\forall\alpha\,(\mathrm{Pr}_T(\ulcorner\forall\beta<\dot\alpha A(\beta)\urcorner)\rightarrow A(\alpha))\vdash\forall\alpha\ A(\alpha).$$

Proof. Assuming $T \vdash \forall\alpha\,(\mathrm{Pr}_T(\ulcorner \forall\beta < \dot{\alpha} A(\beta) \urcorner) \to A(\alpha))$ we derive

$$\begin{aligned} T \vdash \mathrm{Pr}_T(\ulcorner \forall\alpha\ A(\alpha) \urcorner) &\to \forall\alpha\ \mathrm{Pr}_T(\ulcorner \forall\beta < \dot{\alpha} A(\beta) \urcorner) \\ &\to \forall\alpha\ A(\alpha). \end{aligned}$$

Löb's Theorem for T then yields $T \vdash \forall\alpha\ A(\alpha)$. □

We shall also use reflexive transfinite induction on two variables, in the form of *double* induction:

$$\forall\alpha\forall\beta(\mathrm{Pr}_T(\ulcorner \forall\gamma\forall\delta\,((\gamma < \dot{\alpha} \vee (\gamma = \dot{\alpha} \wedge \delta < \dot{\beta})) \to A(\gamma,\delta)) \urcorner) \to A(\alpha,\beta))$$
$$\vdash \forall\alpha\forall\beta\, A(\alpha,\beta).$$

This rule is clearly reducible to the previous one for a suitable p.r. well-ordering.

Proof of Lemma 2.3. We prove $T_\alpha \subseteq T'_\alpha$ by reflexive transfinite induction on α reasoning informally inside PRA. Suppose x is an axiom of T_α, then either x is an axiom of T, or it has the form $\mathrm{Con}(T_\beta)$ for some $\beta < \alpha$ (by (3)). In the first case we are done; in the second case by Induction Hypothesis[2] we have

$$\mathrm{PRA} \vdash \forall\gamma < \alpha\, T_\gamma \subseteq T'_\gamma,$$

hence

$$\mathrm{PRA} \vdash T_\beta \subseteq T'_\beta$$

and

$$\mathrm{PRA} \vdash \mathrm{Con}(T'_\beta) \to \mathrm{Con}(T_\beta)$$

by Lemma 2.1. It follows that

$$\begin{aligned} T'_\alpha &\vdash \mathrm{Con}(T'_\beta) \\ &\vdash \mathrm{Con}(T_\beta), \end{aligned}$$

and thus we have shown that every axiom of T_α is provable in T'_α. To conclude from this fact that all theorems of T_α are provable in T'_α normally one would use Σ^0_1-collection schema, which is not available in PRA. However, for this particular case we can overcome this difficulty as follows.

First of all conclude using Σ^0_1-collection that $T_\alpha \subseteq T'_\alpha$. Then observe that the statement $T_\alpha \subseteq T'_\alpha$ is equivalent in PRA to a Π^0_2-sentence, and therefore the whole premise of the reflexive induction rule we have just proved using Σ^0_1-collection is.

[2] In an argument by reflexive induction, by the Induction Hypothesis we shall always mean the formalized statement, that is, $\mathrm{Pr}_T(\ulcorner \forall\beta < \dot{\alpha} A(\beta) \urcorner)$.

Now, by a well-known result of Parsons (cf. [13]), Σ^0_1-collection schema is conservative over PRA for Π^0_2-sentences so we infer that the premise of the reflexive induction rule is provable in PRA, and we can apply this rule to get the result. $\square$

For p.r. well-orderings satisfying a minor additional requirement, namely that

$$\mathrm{PRA} \vdash \forall \alpha \exists \beta \text{ “}\beta = \alpha + 1\text{”},$$

a similar argument can be used to show that smooth progressions are the weakest of all those defined by verifiable, provably monotone numerations. This shows that smooth numerations occupy a distinguished place among all the others.[3]

Smooth numerations for recursive progressions based on iteration of local reflection are defined in analogy with (3). A p.r. formula $\mathrm{Ax}_T(z;x)$ is called a *smooth numeration of a progression based on iteration of local reflection along* $(D, <)$ iff PRA proves

$$\forall z, x\,(\mathrm{Ax}_T(z;x) \leftrightarrow \mathrm{Ax}_T(x) \vee (z \in D \wedge \exists u \in D(u < z \wedge \exists v \in \mathrm{St}_T\, x = (\ulcorner \mathrm{Pr}_T(\dot{u};\dot{v}) \dot{\to} v)))).$$

Theories numerated by such $\mathrm{Ax}_T(z;x)$, for $z \in D$, will be denoted T^z. The analogs of verifiability conditions, provable monotonicity property, and the uniqueness lemma hold for smooth progressions based on iteration of local reflection, too, with similar proofs, so we shall not repeat them again.

2.3. Nice well-orderings

For the relationships such as (2) to hold the p.r. well-orderings under consideration must satisfy some additional “natural” requirements. For one thing, it is natural to require that the ordinal functions $+$, $\cdot$, and ω^x involved in these formulas are represented by p.r. terms, and that some basic properties of these operations are provable in PRA. These can be formulated in the following way.

Fix an arbitrary ε-number λ and consider λ as a first-order structure with individual constants 0, 1, ω; unary relations SUC, LIM defining the sets of successor and limit ordinals $<\lambda$, respectively; binary relations $<$, $=$; and the standard ordinal functions $+$, $\cdot$, and ω^x.

In the appendix we give a rather long list of axioms of a first-order theory NWO (for “nice well-orderings”) in the above language, which summarizes the basic properties of this structure that we need. For the first reading of this paper the reader is encouraged not to look there at all and to believe that all properties he/she can think of, but for transfinite induction, are present. For his/her convenience at latter stages, inside the formal proofs in the next two sections we added references to the axioms

[3] It is also worth noticing that, under some further natural requirements on the p.r. well-orderings in question, verifiable numerations become provably monotone. So, for natural well-orderings smooth numerations are the weakest of all verifiable ones.

or theorems of NWO really used. Thus, e.g., reference A9b points to the theorem of NWO numbered 9b in the appendix.

Keeping this information in mind, we give the following definition of a nice p.r. well-ordering.

Definition 1. A *nice well-ordering* is a relative interpretation of the theory NWO in PRA such that

- Its domain D and all atomic predicates $<$, SUC, LIM, functions $+$, $\cdot$, ω^x, and constants 0, 1, ω are defined by primitive recursive arithmetical formulas (terms).[4]
- The (interpreted) relation $<$ well-orders the domain D in the standard model of arithmetic.
- Natural numbers can be identified with ordinals $<\omega$, that is, for the p.r. function $(\cdot)^*$ given within PRA by the following schema:

  $$\mathbf{0}^* = 0; \qquad (n+1)^* = n^* + 1,$$

  we have

  $$\mathrm{PRA} \vdash \forall\alpha\, (\alpha < \omega \rightarrow \exists n\, \alpha = n^*). \tag{4}$$

  The latter property of nice well-orderings is a fairly strong and useful requirement, for it implies (and is essentially equivalent to) the following lemma.

Lemma 2.5. *For nice well-orderings, primitive recursive induction schema for ordinals* $<\omega$ *is available in* PRA, *that is,*

$$\mathrm{PRA} \vdash \forall\alpha < \omega (\forall\beta < \alpha\, A(\beta) \rightarrow A(\alpha)) \rightarrow \forall\alpha < \omega\, A(\alpha)$$

for every p.r. formula $A(\alpha)$.

Proof. Consider the p.r. formula $A'(n) := A(n^*)$ and prove $\forall n\ A'(n)$ by the ordinary p.r. induction on n. Then use condition (4). As an intermediate step one should establish within PRA that

$$\forall m, n (m < n \leftrightarrow m^* < n^*)$$

by straightforward p.r. induction using A10 and the definition of $(\cdot)^*$. $\square$

2.4. Composition properties

The uniqueness lemma for smooth recursive progressions allows us to use consistently notation like $(T_\alpha)_\beta$ or $(T^\alpha)_\beta$ for the composition of progressions of theories defined along the same p.r. well-ordering. For nice well-orderings we can verify the following "obvious" relationships that will later be used without notice.

[4] It is convenient here to think of the constants as of 0-ary function symbols. So, their interpretations must be closed p.r. terms (i.e., essentially, numerals).

Lemma 2.6. *For progressions of theories defined along nice well-orderings the following equivalences are provable in* PRA:

1. $\forall\alpha\forall\beta\,(T_\alpha)_\beta \equiv T_{\alpha+\beta}$,
2. $\forall\alpha\forall\beta\,(T^\alpha)^\beta \equiv T^{\alpha+\beta}$.

Proof. We prove only the first statement. The argument goes by reflexive transfinite induction on β in PRA and using Σ_1^0-collection as in the proof of Lemma 2.3.

For the inclusion ($\subseteq$) we have: any axiom of $(T_\alpha)_\beta$ is either an axiom of T_α, and in this case we are done, because by A4a $\alpha \leqslant \alpha + \beta$. Or it has the form $\mathrm{Con}(T_\alpha)_\delta$, for some $\delta < \beta$, and then by Induction Hypothesis

$$\mathrm{PRA} \vdash \mathrm{Con}(T_{\alpha+\delta}) \to \mathrm{Con}(T_\alpha)_\delta.$$

So, we conclude

$$\begin{aligned} T_{\alpha+\beta} &\vdash \mathrm{Con}(T_{\alpha+\delta}) \quad \text{(by A4c and the definition of smoothness)} \\ &\vdash \mathrm{Con}(T_\alpha)_\delta. \end{aligned}$$

For the converse inclusion we reason as follows: an axiom of $T_{\alpha+\beta}$ is either an axiom of T, in which case we are done, since $T \subseteq T_\alpha \subseteq (T_\alpha)_\beta$. Or it has the form $\mathrm{Con}(T_\gamma)$ for some $\gamma < \alpha + \beta$. By A9a either $\gamma < \alpha$ or $\exists\delta < \beta(\gamma = \alpha + \delta)$. In the first case we are done by provable monotonicity, and in the second case the Induction Hypothesis yields

$$\begin{aligned} \mathrm{PRA} \vdash \mathrm{Con}(T_\alpha)_\delta &\to \mathrm{Con}(T_{\alpha+\delta}) \\ &\to \mathrm{Con}(T_\gamma). \end{aligned}$$

So, by the definition of smoothness

$$(T_\alpha)_\beta \vdash \mathrm{Con}(T_\gamma). \qquad \square$$

3. The lower bound

In this section we shall prove the inclusion

$$T_{\omega^\alpha\cdot(1+\beta)} \subseteq (T^\alpha)_\beta,$$

which provides a lower bound to the consistency strength of iterated local reflection principles. From now on we assume a nice p.r. well-ordering fixed and consider only smoothly numerated recursive progressions. We shall need the following two auxiliary lemmas.

Lemma 3.1. *For any theory T,* PRA *proves* $\forall\alpha < \omega\, T^1 \vdash \mathrm{Con}(T_\alpha)$.

Lemma 3.2. *For any theory T,* PRA *proves* $\forall\gamma\,\forall\alpha < \omega\, T^1 \vdash \mathrm{Con}(T_\gamma) \to \mathrm{Con}(T_{\gamma+\alpha})$.

Proof. Lemma 3.1 obviously follows from Lemma 3.2. Also notice that, informally, both of the claims are rather straightforward. The proof of Lemma 3.2 relies on the fact that, for nice well-orderings, natural numbers can be identified with ordinals $<\omega$ by the mapping $(\cdot)^*$. So, for a fixed γ, we define the following arithmetical formula:

$$I(x) := (\mathrm{Con}(T_\gamma) \to \mathrm{Con}(T_{\gamma+x^*}))$$

and show within PRA that

$$\forall n\ T^1 \vdash I(n). \tag{5}$$

The argument goes by induction on n. Obviously, $T^1 \vdash I(0)$, and if $T^1 \vdash I(n)$ then $T^1 \vdash I(n+1)$, because

$$\begin{aligned} T^1 \vdash \mathrm{Con}(T_\gamma) &\to \mathrm{Con}(T_{\gamma+n^*}) && \text{(by IH)} \\ &\to \mathrm{Con}(T + \mathrm{Con}(T_{\gamma+n^*})) && \text{(by an instance of local reflection)} \\ &\to \mathrm{Con}(T_{\gamma+n^*+1}) && \text{(by Lemma 2.2)} \\ &\to \mathrm{Con}(T_{\gamma+(n+1)^*}) && \text{(by the definition of } *). \qquad \square \end{aligned}$$

It remains us to notice that the length of the derivation of $I(n)$ in T^1 can be estimated by a primitive recursive function of n; therefore, the above induction on n is available in PRA.

Now we recall that, for nice well-orderings, PRA proves $\forall\alpha < \omega\, \exists n\, \alpha = n^*$, and so, $\forall\alpha < \omega\, \exists n\ T^1 \vdash \alpha = n^*$, by Σ^0_1-completeness. Together with (5) this yields the result.

Lemma 3.3. PRA *proves* $\forall\alpha \geqslant 1\ \forall\beta\ \forall\gamma\ (T^\gamma)_{\omega^\alpha\cdot(1+\beta)} \subseteq (T^{\gamma+\alpha})_\beta$.

Proof. We prove the statement $\forall\gamma\, (T^\gamma)_{\omega^\alpha\cdot(1+\beta)} \subseteq (T^{\gamma+\alpha})_\beta$ arguing within PRA by double reflexive transfinite induction on $\langle\alpha, \beta\rangle$. As in the proof of Lemma 2.3 we may assume that Σ^0_1-collection schema is available.

It suffices to show that any axiom of $(T^\gamma)_{\omega^\alpha\cdot(1+\beta)}$ is a theorem of $(T^{\gamma+\alpha})_\beta$. By the definition of smoothness, an axiom of $(T^\gamma)_{\omega^\alpha\cdot(1+\beta)}$ is either an axiom of T^γ, in which case our claim is trivial, or has the form $\mathrm{Con}(T^\gamma)_\delta$ for some $\delta < \omega^\alpha\cdot(1+\beta)$. We distinguish several cases.

Case 1: $\alpha = 1$.

Case 1.1: $\beta = 0$. Then $\omega^\alpha\cdot(1+\beta) = \omega^1\cdot(1+0) = \omega\cdot 1 = \omega$ by A3b, A5d and A7b. So we have $\delta < \omega$ and Lemma 3.1 yields: $(T^{\gamma+1})_0 \equiv (T^\gamma)^1 \vdash \mathrm{Con}(T^\gamma)_\delta$.

Case 1.2: $\mathrm{SUC}(\beta)$, that is, β is a successor ordinal. By A11 there is a β' such that $\beta = \beta' + 1$. Then $\omega\cdot(1+\beta) = \omega\cdot(1+\beta') + \omega$ and hence, by A9b, $\delta \leqslant \omega\cdot(1+\beta') + \nu$ for some $\nu < \omega$. By provable monotonicity we have

$$T \vdash \mathrm{Con}(T^\gamma)_{\omega\cdot(1+\beta')+\nu} \to \mathrm{Con}(T^\gamma)_\delta. \tag{6}$$

On the other hand,

$$\begin{aligned}(T^{\gamma+1})_\beta &\vdash \mathrm{Con}(T^{\gamma+1})_{\beta'} \\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega\cdot(1+\beta')} && \text{(by IH)} \\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega\cdot(1+\beta')+\nu} && \text{(by Lemma 3.2)} \\ &\vdash \mathrm{Con}(T^{\gamma})_{\delta}, && \text{(by (6)).}\end{aligned}$$

Case 1.3: LIM(β), that is, β is a limit ordinal. By A15a there is a $\beta' < \beta$ such that $\delta \leqslant \omega\cdot(1+\beta')$. Induction Hypothesis along with the provable monotonicity property yields

$$\begin{aligned}T \vdash \mathrm{Con}(T^{\gamma+1})_{\beta'} &\to \mathrm{Con}(T^{\gamma})_{\omega\cdot(1+\beta')} \\ &\to \mathrm{Con}(T^{\gamma})_{\delta}.\end{aligned}$$

However, for all $\beta' < \beta$ by the definition of smoothness we have

$$(T^{\gamma+1})_\beta \vdash \mathrm{Con}(T^{\gamma+1})_{\beta'},$$

whence

$$(T^{\gamma+1})_\beta \vdash \mathrm{Con}(T^{\gamma})_{\delta}.$$

Case 2: $\alpha = \alpha' + 1$ is a successor ordinal, $\alpha' \geqslant 1$.

Case 2.1: $\beta = 0$. By Lemma 3.1 we have $T^{\gamma+\alpha'+1} \vdash \mathrm{Con}(T^{\gamma+\alpha'})_\nu$, for all $\nu < \omega$. It follows that $T^{\gamma+\alpha} \vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\cdot(1+\nu)}$, by Induction Hypothesis. Since $\delta < \omega^{\alpha} = \omega^{\alpha'+1} = \omega^{\alpha'}\cdot\omega$, we conclude that, for some $\nu < \omega$, $\delta \leqslant \omega^{\alpha'}\cdot\nu \leqslant \omega^{\alpha'}\cdot(1+\nu)$ (by A6a, A6b, A13), and hence $T^{\gamma+\alpha} \vdash \mathrm{Con}(T^{\gamma})_\delta$ by provable monotonicity.

Case 2.2: $\beta = \beta' + 1$ is a successor ordinal. Then $\omega^{\alpha}\cdot(1+\beta) = \omega^{\alpha}\cdot(1+\beta') + \omega^{\alpha}$. Since $\delta < \omega^{\alpha}\cdot(1+\beta)$, there is a $\nu < \omega$ such that $\delta \leqslant \omega^{\alpha}\cdot(1+\beta') + \omega^{\alpha'}\cdot\nu$ (by A9b and A6a). By Lemma 3.2 we have

$$T^{\gamma+\alpha'+1} \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')} \to \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\nu}. \tag{7}$$

On the other hand,

$$\begin{aligned}(T^{\gamma+\alpha})_\beta &\vdash \mathrm{Con}(T^{\gamma+\alpha})_{\beta'} \\ &\vdash \mathrm{Con}(T^{(\gamma+\alpha')+1})_{\beta'} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')} && \text{(by IH with } \tilde{\alpha} = 1, \tilde{\beta} = \beta', \tilde{\gamma} = \gamma + \alpha') \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\nu} && \text{(by (7))} \\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\omega\cdot(1+\beta')+\omega^{\alpha'}\cdot\nu} && \text{(by IH with } \tilde{\alpha} = \alpha') \\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta')+\omega^{\alpha'}\cdot\nu} \\ &\vdash \mathrm{Con}(T^{\gamma})_{\delta}, && \text{(by provable monotonicity).}\end{aligned}$$

Case 2.3: β is a limit ordinal. Then there is a $\beta' < \beta$ such that $\delta \leqslant \omega^\alpha \cdot (1 + \beta')$, by A15a. So we obtain

$$\begin{aligned}(T^{\gamma+\alpha})_\beta &\vdash \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^\alpha\cdot(1+\beta')} \quad \text{(by IH)}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\delta}, \quad \text{(by monotonicity).}\end{aligned}$$

Case 3: α is a limit ordinal.

Case 3.1: $\beta = 0$. We have that $\delta < \omega^{\alpha'}$ for a suitable $\alpha' < \alpha$, by A8b. Then, clearly,

$$\begin{aligned}(T^{\gamma+\alpha})_0 \equiv T^{\gamma+\alpha} &\vdash \mathrm{Con}(T^{\gamma+\alpha'})\\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}} \quad \text{(by IH)}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\delta}\end{aligned}$$

Case 3.2: $\beta = \beta' + 1$ is a successor ordinal.

Then $\delta \leqslant \omega^\alpha \cdot (1 + \beta') + \omega^{\alpha'}$, for some $\alpha' < \alpha$ (by A9b and A8b). Let λ be such that $\alpha' + \lambda = \alpha$ (A4b); clearly $1 \leqslant \lambda \leqslant \alpha$ (A3b, A4c, A4d). By Lemma 3.2 we have

$$T^{\gamma+\alpha} \supseteq T^{\gamma+\alpha'+1} \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^\lambda\cdot(1+\beta')} \to \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^\lambda\cdot(1+\beta')+1}. \tag{8}$$

On the other hand,

$$\begin{aligned}(T^{\gamma+\alpha})_\beta &\vdash \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}\\ &\vdash \mathrm{Con}(T^{(\gamma+\alpha')+\lambda})_{\beta'}\\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^\lambda\cdot(1+\beta')} \quad \text{(by IH with } \tilde{\alpha} = \lambda,\ \tilde{\beta} = \beta', \tilde{\gamma} = \gamma + \alpha')\\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^\lambda\cdot(1+\beta')+1} \quad \text{(by (8))}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\cdot\omega^\lambda\cdot(1+\beta')+\omega^{\alpha'}} \quad \text{(by IH with } \tilde{\alpha} = \alpha')\\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta')+\omega^{\alpha'}}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\delta} \quad \text{(by provable monotonicity).}\end{aligned}$$

Case 3.3: β is a limit ordinal. This case is fully similar to Case 2.3.

The nine cases just considered exhaust all possibilities by the axioms A2c, A2d defining the predicates SUC and LIM. This observation completes our proof of Lemma 3.3. □

4. The upper bound

In this section we shall prove the inclusion

$$(T^\alpha)_\beta \subseteq_{\Pi_1^0} T_{\omega^\alpha\cdot(1+\beta)},$$

which provides an upper bound to the consistency strength of iterated local reflection principles. The following two auxiliary lemmas are crucial for our proof.

Lemma 4.1. *For any theory T*, PRA *proves* $T_\omega \rhd T^1$.

Proof. This fact is just a formalization of Goryachev's theorem [7]. For reader's convenience we give an easy informal proof and then explain why it works within PRA.

The underlying idea comes from provability logic in the form of the following lemma essentially due to Boolos [5] and Artemov [1]. Let H_n denote the following propositional modal formula:

$$H_n := \bigwedge_{i=1}^{n} (\Box p_i \to p_i),$$

and let $\Box^n \bot$ abbreviate $\underbrace{\Box\Box \dots \Box}_{n \text{ times}}\bot$, where $\bot$ is the constant "falsum".

Lemma 4.2. *The following formula is a theorem of the provability logic* GL *for every n*:

$$\neg \Box^{n+1} \bot \to \neg \Box \neg H_n. \tag{9}$$

Proof. Consider an arbitrary finite irreflexive tree-like Kripke model for GL. If the formula $\neg \Box^{n+1} \bot$ is forced at the root of this model, then there is a chain of at least $n + 1$ nodes above it. However, any conjunct of the form $\Box p_i \to p_i$ can be false at no more than one node of this chain. Therefore, by Pigeon Hole Principle, there is a node above the root of this model that forces H_n. $\Box$

Proceeding with the proof of Goryachev's theorem recall that according to the arithmetical interpretation of provability logic the modality $\Box$ is translated as the provability predicate in T. Therefore, under this interpretation, $\neg \Box^{n+1} \bot$ is equivalent to the statement $\mathrm{Con}(T_n)$, whereas $\neg \Box \neg H_n$ asserts the consistency of T together with (arbitrary) n instances of local reflection. As T_ω contains $\mathrm{Con}(T_n)$ and arithmetical interpretations of all theorems of GL are provable in PRA, the result follows.

To formalize the previous argument in PRA, first of all, notice that in Lemma 4.2 the proof in GL of the formula (9) can be found as a p.r. function of n. This follows, essentially, from the fact that the decision procedure for GL is primitive recursive (and verifiable in PRA). Further, observe that a proof in PRA of an arithmetical interpretation of (9) is obtained from that in GL, roughly, by substituting everywhere arithmetical sentences (or their Gödel numbers) for propositional variables, so that the result is p.r. in the size of these sentences, and the fact that it is a PRA-proof can be verified in PRA (by induction on the length of the GL-derivation). Finally, given an n we can primitively recursively find a substitution, say f_n, of arithmetical sentences to propositional variables $p_1, \dots, p_n$ such that (verifiably in PRA)

$$\forall n\ \mathrm{PRA} \vdash f_n(\neg \Box \neg H_n) \to \mathrm{Con}(T^1 \restriction n).$$

This follows from the understanding that, under the standard Gödel numbering, no more than n instances of local reflection may have Gödel numbers smaller than n.

Combining these three things together we conclude that, for each n, the proof of $\mathrm{Con}(T^1 \restriction n)$ within PRA from the arithmetical interpretation of $\neg\Box^{n+1}\bot$ is found (verifiably) primitively recursively in n. (We denote this interpretation by $f(\neg\Box^{n+1}\bot)$ dropping the subscript n at f to stress the fact that the result does not depend on a particular substitution of arithmetical sentences for propositional variables.) What remains to be seen for the proof of Lemma 4.1 is essentially contained in the second part of the following lemma.

Lemma 4.3. PRA *proves*

1. $\forall n\ \mathrm{PRA} \vdash f(\neg\Box^{n+1}\bot) \leftrightarrow \mathrm{Con}(T_{n^*})$,
2. $\forall n\ T_\omega \vdash f(\neg\Box^{n+1}\bot)$.

Proof. Part 1 is proved by straightforward p.r. induction on n within PRA using Lemma 2.2 at the induction step. Part 2 follows from Part 1 and the following property of nice well-orderings:

$$\mathrm{PRA} \vdash \forall n\ (n^* < \omega).$$

This property can easily be established by p.r. induction on n using A14d, and this observation completes our proof of Lemmas 4.3 and 4.1. $\Box$

Lemma 4.4. *For any theory* T, PRA *proves* $\forall \alpha\ T_{\alpha+\omega} \rhd (T^1 + \mathrm{Con}(T_\alpha))$.

Proof. We argue informally within PRA. Since the statement to be proved has Π^0_2 form, w.l.o.g. we may assume that Σ^0_1-collection principle is available. By Lemmas 4.1 and 2.6 we have

$$T_{\alpha+\omega} \rhd (T_\alpha)^1. \tag{10}$$

On the other hand, by provable monotonicity

$$(T^1 + \mathrm{Con}(T_\alpha)) \subseteq (T_\alpha)^1,$$

and thus, for all n there is an m such that

$$(T^1 \restriction n + \mathrm{Con}(T_\alpha)) \subseteq (T_\alpha)^1 \restriction m. \tag{11}$$

Since the theory $T^1 \restriction n + \mathrm{Con}(T_\alpha)$ has finitely many axioms, Σ^0_1-collection implies that statement (11) is equivalent to a Σ^0_1-formula. Therefore, by Σ^0_1-completeness principle, (11) must be provable in PRA together with Σ^0_1-collection, ergo in PRA itself. (Here we use the fact that Parsons' theorem is actually formalizable in PRA, which can be readily seen from its proof given in [13].) So, we conclude that

$$\mathrm{PRA} \vdash \mathrm{Con}((T_\alpha)^1 \restriction m) \rightarrow \mathrm{Con}(T^1 \restriction n + \mathrm{Con}(T_\alpha)).$$

By (10) we have

$$\forall m \quad T_{\alpha+\omega} \vdash \mathrm{Con}((T_\alpha)^1 \restriction m).$$

It follows that

$$\forall n \quad T_{\alpha+\omega} \vdash \mathrm{Con}(T^1 \restriction n + \mathrm{Con}(T_\alpha)). \quad \square$$

Remark. Notice that Lemmas 4.1 and 4.4 can be stated in a strengthened form. E.g., for Lemma 4.1 we also have

$$T_\omega \vdash \mathrm{Con}(T + T^1 \restriction n),$$

for every n, although, in general, T need not be a *finite* subtheory of the theory T^1. The possibility of such a strengthening follows immediately from the given proof of Goryachev's theorem, and we shall use it in the proof of our main technical lemma below.

Lemma 4.5. PRA *proves* $\forall\alpha \geqslant 1\, \forall\beta\, \forall\gamma\ (T^\gamma)_{\omega^\alpha\cdot(1+\beta)} \rhd (T^{\gamma+\alpha})_\beta$.

Proof. We argue by double reflexive transfinite induction on $\langle \alpha, \beta \rangle$ within PRA assuming Σ^0_1-collection, as we did before. We consider the following cases.

Case 1: $\alpha = 1$.

Case 1.1: $\beta = 0$. We have to show that $(T^\gamma)_{\omega^1\cdot(1+0)} \equiv (T^\gamma)_\omega \rhd T^{\gamma+1}$. But this is, essentially, the claim of Lemma 4.1.

Case 1.2: $\beta = \beta' + 1$ is a successor ordinal. First of all, notice that, for all n,

$$\begin{aligned}(T^{\gamma+1})_\beta \restriction n &\equiv ((T^{\gamma+1})_{\beta'} + \mathrm{Con}(T^{\gamma+1})_{\beta'}) \restriction n \\ &\subseteq (T^{\gamma+1})_{\beta'} \restriction n + \mathrm{Con}(T^{\gamma+1})_{\beta'} \\ &\subseteq T + (T^{\gamma+1}) \restriction n + \mathrm{Con}(T^{\gamma+1})_{\beta'},\end{aligned}$$

because by provable monotonicity $T \vdash \mathrm{Con}(T^{\gamma+1})_{\beta'} \to \mathrm{Con}(T^{\gamma+1})_\delta$ for all $\delta < \beta'$.

Induction Hypothesis yields

$$T \vdash (T^\gamma)_{\omega\cdot(1+\beta')} \rhd (T^{\gamma+1})_{\beta'},$$

whence

$$T \vdash \mathrm{Con}(T^\gamma)_{\omega\cdot(1+\beta')} \to \mathrm{Con}(T^{\gamma+1})_{\beta'}$$

by Lemma 2.1, and we conclude

$$(T^{\gamma+1})_\beta \restriction n \subseteq T^\gamma + (T^{\gamma+1}) \restriction n + \mathrm{Con}(T^\gamma)_{\omega\cdot(1+\beta')}. \tag{12}$$

As in the proof of Lemma 4.4, by Σ^0_1-collection and Σ^0_1-completeness principles, and using the formalization of Parsons' theorem in PRA we obtain

$$T \vdash \mathrm{Con}(T^\gamma + (T^{\gamma+1}) \restriction n + \mathrm{Con}(T^\gamma)_{\omega\cdot(1+\beta')}) \to \mathrm{Con}((T^{\gamma+1})_\beta \restriction n). \tag{13}$$

Applying (the strengthened form of) Lemma 4.4 to the theory T^{γ} we get

$$\forall n \quad (T^{\gamma})_{\omega\cdot(1+\beta')+\omega} \vdash \mathrm{Con}(T^{\gamma} + (T^{\gamma+1})\restriction n + \mathrm{Con}(T^{\gamma})_{\omega\cdot(1+\beta')}),$$

and together with (13) this yields the result.

Case 1.3: β is a limit ordinal. For every n there is a $\beta' < \beta$ such that $(T^{\gamma+1})_{\beta}\restriction n \subseteq (T^{\gamma+1})_{\beta'}$. Take a β' bigger than any $\delta < \beta$ such that $\ulcorner \mathrm{Con}(T^{\gamma+1})_{\delta} \urcorner \leqslant n$. (Since there is an a priori upper bound ($=n$) on the size of the code of any such δ, the finite set of all δ's exists even in absence of Σ^0_1-collection.) So, by Σ^0_1-completeness principle, as in the previous case, we conclude

$$T \vdash \mathrm{Con}(T^{\gamma+1})_{\beta'} \to \mathrm{Con}((T^{\gamma+1})_{\beta}\restriction n). \tag{14}$$

On the other hand, we have

$$\begin{aligned} (T^{\gamma})_{\omega\cdot(1+\beta)} &\vdash \mathrm{Con}(T^{\gamma})_{\omega\cdot(1+\beta')} && \text{(by A15b)} \\ &\vdash \mathrm{Con}(T^{\gamma+1})_{\beta'} && \text{(by IH and Lemma 2.1)} \\ &\vdash \mathrm{Con}((T^{\gamma+1})_{\beta}\restriction n) && \text{(by (14))}. \end{aligned}$$

Case 2: $\alpha = \alpha' + 1$ is a successor ordinal, $\alpha' \geqslant 1$.

Case 2.1: $\beta = 0$. We have to show

$$\forall n \quad (T^{\gamma})_{\omega^{\alpha'+1}} \vdash \mathrm{Con}(T^{\gamma+\alpha'+1}\restriction n).$$

Notice that by Lemma 4.1 for every n there is a $\delta < \omega$ such that

$$(T^{\gamma+\alpha'})_{\delta} \vdash \mathrm{Con}(T^{\gamma+\alpha'+1}\restriction n).$$

By Σ^0_1-completeness (applied twice) we obtain

$$\begin{aligned} T \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\delta} &\to \mathrm{Con}(T + \mathrm{Con}(T^{\gamma+\alpha'+1}\restriction n)) \\ &\to \mathrm{Con}(T^{\gamma+\alpha'+1}\restriction n). \end{aligned} \tag{15}$$

By A15b, A14a, and algebraic properties of ω^{x}, for all $\delta < \omega$,

$$\omega^{\alpha'}\cdot(1+\delta) < \omega^{\alpha'}\cdot\omega = \omega^{\alpha'+1} = \omega^{\alpha}.$$

It follows that

$$\begin{aligned} (T^{\gamma})_{\omega^{\alpha}} &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\cdot(1+\delta)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\delta} && \text{(by IH and Lemma 2.1)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha}\restriction n) && \text{(by (15))}. \end{aligned}$$

Case 2.2: $\beta = \beta' +$ is a successor ordinal. First of all, similarly to Case 1.2 we have that, for all n,

$$(T^{\gamma+\alpha})_{\beta}\restriction n \subseteq T + (T^{\gamma+\alpha})\restriction n + \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}$$

and

$$T \vdash \mathrm{Con}(T + (T^{\gamma+\alpha})\restriction n + \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}) \to \mathrm{Con}((T^{\gamma+\alpha})_{\beta}\restriction n). \tag{16}$$

On the other hand, by Lemma 4.4,

$$\forall n \quad (T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\omega} \vdash \mathrm{Con}(T^{\gamma+\alpha'} + (T^{\gamma+\alpha}) \restriction n + \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')}),$$

and Induction Hypothesis with $\tilde{\alpha} = 1$, $\tilde{\beta} = \beta'$, $\tilde{\gamma} = \gamma + \alpha'$ together with Lemma 2.1 yields

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')} \to \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}.$$

So, by (16), A9b and properties of smooth numerations, for all n there is a $\delta < \omega$ such that

$$(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\delta} \vdash \mathrm{Con}((T^{\gamma+\alpha})_{\beta} \restriction n).$$

By Σ^0_1-completeness (applied twice) it follows that

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\delta} \to \mathrm{Con}((T^{\gamma+\alpha})_{\beta} \restriction n). \tag{17}$$

So we obtain

$$\begin{aligned} (T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta)} \equiv (T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta')+\omega^{\alpha'}\cdot\omega} &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\cdot(\omega\cdot(1+\beta'))+\omega^{\alpha'}\cdot\delta} && \text{(by A6b, A4c)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega\cdot(1+\beta')+\delta} && \text{(by IH)} \\ &\vdash \mathrm{Con}((T^{\gamma+\alpha})_{\beta} \restriction n) && \text{(by (17))}. \end{aligned}$$

Case 2.3: β is a limit ordinal. As in Case 1.3, for every n we find a $\beta' < \beta$ such that $(T^{\gamma+\alpha})_{\beta} \restriction n \subseteq (T^{\gamma+\alpha})_{\beta'}$ and

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha})_{\beta'} \to \mathrm{Con}((T^{\gamma+\alpha})_{\beta} \restriction n). \tag{18}$$

On the other hand, we have

$$\begin{aligned} (T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta)} &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta')} && \text{(by A15b)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha})_{\beta'} && \text{(by IH and Lemma 2.1)} \\ &\vdash \mathrm{Con}((T^{\gamma+\alpha})_{\beta} \restriction n) && \text{(by (18))}. \end{aligned}$$

Case 3: α is a limit ordinal.

Case 3.1: $\beta = 0$. As in Case 1.3, for every n there is an ordinal $\alpha' < \alpha$ such that $(T^{\gamma+\alpha}) \restriction n \subseteq T^{\gamma+\alpha'}$ and

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'}) \to \mathrm{Con}(T^{\gamma+\alpha} \restriction n). \tag{19}$$

So, we obtain

$$\begin{aligned} (T^{\gamma})_{\omega^{\alpha}} &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}} && \text{(by A8a)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'}) && \text{(by IH and Lemma 2.1)} \\ &\vdash \mathrm{Con}(T^{\gamma+\alpha} \restriction n) && \text{(by (19))}. \end{aligned}$$

Case 3.2: $\beta = \beta' + 1$ is a successor ordinal. As in Case 2.2, for every n we find an $\alpha' < \alpha$ such that $(T^{\gamma+\alpha})\restriction n \subseteq T^{\gamma+\alpha'} + \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}$ and

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'} + \mathrm{Con}(T^{\gamma+\alpha})_{\beta'}) \to \mathrm{Con}((T^{\gamma+\alpha})_{\beta}\restriction n). \tag{20}$$

Let λ be such that $\alpha' + \lambda = \alpha$, then clearly $1 \leqslant \lambda \leqslant \alpha$. By Induction Hypothesis with $\tilde{\alpha} = \lambda$, $\beta = \beta'$, $\tilde{\gamma} = \gamma + \alpha'$ and Lemma 2.1 we have

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^{\lambda}\cdot(1+\beta')} \to \mathrm{Con}(T^{\gamma+\alpha})_{\beta'},$$

and so, by (20)

$$\begin{aligned}(T^{\gamma+\alpha'})_{\omega^{\lambda}\cdot(1+\beta')+1} &\vdash \mathrm{Con}(T^{\gamma+\alpha'} + \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^{\lambda}\cdot(1+\beta')})\\ &\vdash \mathrm{Con}((T^{\gamma+\alpha})_{\beta}\restriction n).\end{aligned}$$

By Σ_1^0-completeness (applied twice) it follows that

$$T \vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^{\lambda}\cdot(1+\beta')+1} \to \mathrm{Con}((T^{\gamma+\alpha})_{\beta}\restriction n). \tag{21}$$

Now we consequtively derive

$$\begin{aligned}(T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta'+1)} &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha}\cdot(1+\beta')+\omega^{\alpha'}} && \text{(by A8a, A4c)}\\ &\vdash \mathrm{Con}(T^{\gamma})_{\omega^{\alpha'}\cdot(\omega^{\lambda}\cdot(1+\beta')+1)} &&\\ &\vdash \mathrm{Con}(T^{\gamma+\alpha'})_{\omega^{\lambda}\cdot(1+\beta')+1} && (\text{by IH with } \tilde{\alpha} = \alpha')\\ &\vdash \mathrm{Con}((T^{\gamma+\alpha})_{\beta}\restriction n), && \text{(by (21))}\end{aligned}$$

Case 3.3: β is a limit ordinal. This case is fully similar to Case 2.3, and this completes our proof of Lemma 4.5. □

Now we are ready to prove our main result.

Theorem 1. *For nice well-orderings, smoothly defined recursive progressions of theories based on iteration of local reflection principles and on iteration of consistency assertions, respectively, provably in* PRA *satisfy the following relationships: for all* $\alpha \geqslant 1$ *and all* β,
1. $(T^{\alpha})_{\beta} \equiv_{\Pi_1^0} T_{\omega^{\alpha}\cdot(1+\beta)}$;
2. $(T_{\beta})^{\alpha} \equiv_{\Pi_1^0} T_{\beta+\omega^{\alpha}}$.

Proof. Statement 1 follows from Lemmas 3.3 and 4.5. Statement 2 follows from 1 and Lemma 2.6. □

Corollary 1. *Under the assumptions of the previous theorem, for* $\alpha \geqslant 1$ *we have*
1. $T^{\alpha} \equiv_{\Pi_1^0} T_{\omega^{\alpha}}$;
2. $T^{\alpha} \equiv_{\Pi_1^0} T_{\alpha}$ *if and only if* α *is an* ε*-number.*

Appendix

The theory NWO is formulated in a first-order language with equality containing individual constants 0, 1, ω; unary predicates SUC, LIM; binary predicate $<$; and functions $+, \cdot$ and ω^x. NWO has the following mathematical axioms:

1. $(\lambda, <)$ is a linear ordering:
   (a) $x < y \wedge y < z \to x < z$,
   (b) $\neg x < x$,
   (c) $x < y \vee y < x \vee x = y$.
2. Axioms defining 0, 1, ω, SUC, LIM in terms of $<$:
   (a) $0 \leqslant x$ ($x \leqslant y$ abbreviates $x < y \vee x = y$),
   (b) $0 < 1 \wedge \forall y\ (y < 1 \to y = 0)$,
   (c) $\mathrm{SUC}(x) \leftrightarrow \exists z < x \forall y\ (y < x \to y \leqslant z)$,
   (d) $\mathrm{LIM}(x) \leftrightarrow x \neq 0 \wedge \neg \mathrm{SUC}(x)$,
   (e) $\mathrm{LIM}(\omega) \wedge \forall x < \omega \neg \mathrm{LIM}(x)$.
3. $(\lambda, +, 0)$ is an associative monoid:
   (a) $x + (y + z) = (x + y) + z$,
   (b) $x + 0 = 0 + x = x$.
4. Properties relating $+$ and $<$:
   (a) $x \leqslant x + y$,
   (b) $x \leqslant y \to \exists u(y = x + u)$,
   (c) $x < y \to u + x < u + y$,
   (d) $x \leqslant y \to x + u \leqslant y + u$.
5. Algebraic properties of $\cdot$.
   (a) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
   (b) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$,
   (c) $x \cdot 0 = 0 \cdot x = 0$,
   (d) $x \cdot 1 = 1 \cdot x = x$.
6. Properties relating $\cdot$ and $<$:
   (a) $\mathrm{LIM}(y) \wedge z < x \cdot y \to \exists u < y(z < x \cdot u)$,
   (b) $u < v \wedge x \neq 0 \to x \cdot u < x \cdot v$.
7. Algebraic properties of ω^x:
   (a) $\omega^0 = 1$,
   (b) $\omega^1 = \omega$,
   (c) $\omega^{x+y} = \omega^x \cdot \omega^y$.
8. Properties relating $<$ and ω^x:
   (a) $x < y \to \omega^x < \omega^y$,
   (b) $\mathrm{LIM}(y) \wedge z < \omega^y \to \exists u < y(z < \omega^u)$.

Next we list some theorems of NWO used in the proof of our main result. To simplify the references we enumerate them on a par with the axioms of NWO.

9. (a) $z < x + y \to z < x \vee \exists u(u < y \wedge z = x + u)$.
   The proof is as follows: By A1 and A4b $\neg z < x$ implies $x \leqslant z$ and $\exists u(z = x + u)$. By A4c we have $z = x + u \wedge z < x + y \to u < y$, therefore $z < x + y \wedge \neg z < x \to \exists u(u < y \wedge z = x + u)$. □
   (b) $z < x + y \wedge y \neq 0 \to \exists u < y(z \leqslant x + u)$ (by A9a).

10. $x + 1$ is the successor of x:
    (a) $x < x + 1$ (because $0 < 1 \to x = x + 0 < x + 1$ by A2b, A3b, A4c),
    (b) $u < x + 1 \to u \leqslant x$ (by A9a, A2b and A3b).
11. $\mathrm{SUC}(x) \leftrightarrow \exists z(x = z + 1)$ (by A10 and A2c).
12. (a) $\mathrm{SUC}(y) \to \mathrm{SUC}(x + y)$ (by A11 and A3a),
    (b) $\mathrm{LIM}(y) \to \mathrm{LIM}(x + y)$ (by A9a, A2d and A4c).
13. $x \leqslant 1 + x$ (by A4d, A3b).
14. (a) $1 + \omega = \omega$.
    The proof is as follows: By A13 we have $\omega \leqslant 1 + \omega$. If $\omega < 1 + \omega$ then $\exists u < \omega(\omega = 1 + u)$ (by A9a and A2b). Then A2e and A12a imply $\mathrm{SUC}(u)$ (since $u < \omega$) and $\mathrm{SUC}(1 + u)$, ergo $\mathrm{SUC}(\omega)$, a contradiction. $\square$
    (b) $\omega \leqslant x \to 1 + x = x$ (by A4b, A14a and A3a),
    (c) $x < \omega \to 1 + x < \omega$ (by A4c and A14a),
    (d) $x < \omega \to x + 1 < \omega$ (by A2e and A10).
15. (a) $z < x \cdot (1 + y) \wedge \mathrm{LIM}(y) \to \exists u < y(z < x \cdot (1 + u))$.
    The proof is as follows: $z < x \cdot (1 + y) = x + x \cdot y$ implies $\exists v < x \cdot y(z \leqslant x + v)$ by A9b. The result follows by A6a, A4c and algebraic properties of. $\square$
    (b) $x < y \to \omega^u \cdot (1 + x) < \omega^u \cdot (1 + y)$.
    The proof is as follows: Suppose $x < y$, then A8a implies $\forall u\ \omega^u \neq 0$, whence $1 + x < 1 + y$ by A4c and $\omega^u \cdot (1 + x) < \omega^u \cdot (1 + y)$ by A6b.

Recall (cf. Section 2) that by a *nice well-ordering* we mean a relative interpretation of the theory NWO in PRA such that

- Its domain D and all atomic predicates, functions, and constants are defined by primitive recursive arithmetical formulas (terms).
- The (interpreted) relation $<$ well-orders the domain D in the standard model of arithmetic.
- Natural numbers can be identified with ordinals $< \omega$.

Clearly, the interpretation of any theorem of NWO is provable in PRA. On the other hand, it is also interesting to notice that PRA proves a lot more about nice well-orderings than NWO itself. This can be seen from the fact that PRA proves induction up to ω for arbitrary p.r. predicates (Lemma 2.5), including those in the language of NWO, whereas NWO does not.

Our final goal is the following theorem.

Theorem A.1. *For every recursive ε-number λ, there is a nice well-ordering having order type λ in the standard model of arithmetic.*

Proof. This theorem is a typical representative of those results in proof theory which can be characterized as "being essentially well-known, but not necessarily well-documented". The closest documented results of similar character that we know about are contained in the work of Sommer [15, 16], who proves, among many other things, that the part of NWO not involving ordinal multiplication and exponentiation functions can be interpreted in PRA in this way. Moreover, he deals with much more restrictive kind of interpretations, namely, with those in a very weak arithmetical theory $I\Delta_0$, and such that all atomic relations are interpreted by Δ_0 formulas. However, for our present purposes we do not need these results in such a strengthened form.

The method that can be used for a proof of Theorem A.1 is standard. So, instead of going into the technical details of an honest proof, we shall just indicate the main ideas. The construction of the required interpretation goes in two steps.

Let $\lambda = \varepsilon_\alpha$. First of all, we construct a p.r. well-ordering $(E, <_E)$ of type α, that is, an interpretation in PRA of the group of axioms A1 only. This can be done either by referring to the quoted theorem from [15], or, alternatively, one can use a standard theorem (cf. e.g. [11]) stating that every recursive well-ordering can be embedded as a p.r. subset into the set of rationals $\mathbf{Q}$. Then it only remains to notice that the usual ordering relation on $\mathbf{Q}$ is primitive recursive and provably in PRA linear.

At the second step we stipulate that the elements of the ordering $(E, <_E)$ code ε-numbers $<\lambda$, and we use Cantor normal forms of the terms build-up from the elements of E and 0 by the functions $+$ and ω^x to code the ordinals occurring between the ε-numbers. Cantor's normal form theorem then shows that this construction gives a unique notation to every ordinal $<\lambda$.

Formally, one can define the set NF of normal forms and the ordering relation $<$ by simultaneous primitive recursion, e.g., in analogy with the definition given in [10, p. 86]. (However, additional clauses corresponding to ε-numbers will be present.) Provable linearity of the ordering $(E, <_E)$ then guarantees that the ordering $<$ thus defined on NF will be provably linear, too.

Relations SUC, LIM, and functions $+$ and ω^x are easily and primitively recursively explained in terms of Cantor normal forms. The definition of $\cdot$ is somewhat more complicated, but can be carried out primitively recursively using the formula

$$(\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}) \cdot \omega^\gamma = \begin{cases} \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k} & \text{if } \gamma = 0, \\ \omega^{\alpha_1 + \gamma} & \text{if } \gamma > 0 \end{cases}$$

and the distributivity law. (Here we assume $\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$ to be in Cantor normal form, that is, $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$.)

Most of the axioms A1–A8 are then easy to verify, although some of them, most notably the associativity of multiplication, require some patience, because there are so many cases to consider.

To finish with our sketch of the proof of Theorem 2 we mention that the requirement that natural numbers can be identified with ordinals $<\omega$ in our construction is obviously satisfied, because finite ordinals are coded as something like strings of 0's, and the fact that any such string has a certain natural number as its length is clearly verifiable in PRA.

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