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Permutation statistics on the alternating group

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Abstract

Let $A_n \subseteq S_n$ denote the alternating and the symmetric groups on $1, \ldots, n$. MacMahon's theorem [P.A. MacMahon, Combinatory Analysis I–II, Cambridge Univ. Press, 1916], about the equi-distribution of the length and the major indices in S_n, has received far reaching refinements and generalizations, by Foata [Proc. Amer. Math. Soc. 19 (1968) 236], Carlitz [Trans. Amer. Math. Soc. 76 (1954) 332; Amer. Math. Monthly 82 (1975) 51], Foata-Schützenberger [Math. Nachr. 83 (1978) 143], Garsia–Gessel [Adv. Math. 31 (1979) 288] and followers. Our main goal is to find analogous statistics and identities for the alternating group A_n. A new statistics for S_n, *the delent number*, is introduced. This new statistics is involved with new S_n identities, refining some of the results in [D. Foata, M.P. Schützenberger, Math. Nachr. 83 (1978) 143; A.M. Garsia, I. Gessel, Adv. Math. 31 (1979) 288]. By a certain covering map $f : A_{n+1} \to S_n$, such S_n identities are 'lifted' to A_{n+1}, yielding the corresponding A_{n+1} equi-distribution identities.

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1. Introduction

1.1. General outline

One of the most active branches in enumerative combinatorics is the study of *permutation statistics*. Let S_n be the symmetric group on $1, \dots, n$. One is interested in the refined count of permutations according to (non-negative, integer valued) combinatorial parameters. For example, the number of inversions in a permutation—namely its *length*—is such a parameter. Another parameter is MacMahon's *major index*, which is defined via the *descent* set of a permutation—see below.

Two parameters that have the same generating function are said to be *equi-distributed*. Indeed, MacMahon [12] proved the remarkable fact that the inversions and the major-index statistics are equi-distributed on S_n. MacMahon's classical theorem [12] has received far reaching refinements and generalizations, including: multivariate refinements which imply equi-distribution on certain subsets of permutations (done by Carlitz [3,4], Foata–Schützenberger [6] and Garsia–Gessel [7]); analogues for other combinatorial objects, cf. [5,10,18]; generalizations to other classical Weyl groups, cf. [1,2,15].

Let $A_n \subseteq S_n$ denote the alternating group on $1, \dots, n$. Easy examples show that the above statistics fail to be equi-distributed when restricted to A_n. Our main goal is to find statistics on A_n which are natural analogues of the above S_n statistics and are equi-distributed on A_n, yielding analogous identities for their generating functions. This goal is achieved by proving further refinements of the above S_n-identities.

It is well known that the above statistics on S_n may be defined via the Coxeter generators $\{(i, i+1) \mid 1 \leqslant i \leqslant n-1\}$ of S_n. Mitsuhashi [13] pointed out at a certain set of generators of the alternating group A_n, which play a role similar to that of the above Coxeter generators of S_n, see Section 1.3. We use these generators to define statistics which are analogous to the above length and descent statistics.

The S_n-Coxeter generators allow one to introduce the classical canonical presentation of the elements of S_n, see Section 3.1. Similarly, the above Mitsuhashi's 'Coxeter' generators allow us to introduce the corresponding canonical presentation of the elements of A_{n+1}, see Section 3.3. We remark that usually, S_n is viewed as a double cover of A_n. However, the above canonical presentations enable us to introduce a covering map f from the alternating group A_{n+1} onto S_n, and thus A_{n+1} can be viewed as a covering of S_n.

A new statistic, the *delent number*, plays a crucial role in the paper, and allows us to 'lift' S_n identities to A_{n+1}. The delent number on S_n may be defined as follows: if the transposition $(1, 2)$ appears r times in the canonical presentation of $\sigma \in S_n$ then the delent number of σ, $\mathrm{del}_S(\sigma)$, is r. An analogous statistic is defined for A_{n+1}, see Definition 4.3. We give direct combinatorial characterizations of this statistic (see Propositions 1.7 and 1.8) and show that this statistic is involved in new S_n equi-distribution identities, refining some of the results of Foata–Schützenberger [6] and of Garsia–Gessel [7]. Identities involving the delent number are then 'lifted' by the covering map f, yielding A_{n+1} equi-distribution identities, see Theorems 6.1, 9.1 and Corollary 9.2.

In Appendix A we present different statistics on A_n, and a consequent different analogue of MacMahon's equi-distribution theorem. These statistics are compatible with the usual point of view of S_n as a double cover of A_n.

The above setting and results are connected with enumeration of other combinatorial objects, such as permutations avoiding patterns, leading to q-analogues of the classical S_n statistics and of the Bell and Stirling numbers. A detailed study of these q-analogues is given in [14] (a few of these results appear in Section 5.3).

The paper is organized as follows: The rest of this section surveys briefly the classical background and lists our main results. Background and notations are given in detail in Section 2, while the A-canonical presentation is analyzed in Section 3. In Section 4 we study the length statistics, and in Section 5 we discuss the relations between various S- and A-statistics, relations given by the map $f: A_{n+1} \to S_n$. In Section 6 we study the ordinary and the reverse major indices, together with the delent statistics. Additional properties of the delent numbers are given in Section 7. In Section 8 we prove some lemmas on shuffles—lemmas that are needed for the proof of the main theorem. The main theorem (Theorem 9.1) and its proof are given in Section 9. Finally, Appendix A presents other statistics.

1.2. Classical S_n-statistics

Recall that the Coxeter generators $S := \{(i, i+1) \mid 1 \leqslant i \leqslant n-1\}$ of S_n give rise to various combinatorial statistics, like the *length* statistic, etc. As we show later, most of these S_n statistics have A_n analogues, therefore we add "S-" and "A-" to the titles of the corresponding statistics.

- The *S-length*: For $\pi \in S_n$ let $\ell_S(\pi)$ be the standard length of π with respect to these Coxeter generators, see [9].
- The *S-descent*: Given a permutation π in the symmetric group S_n, the *S-descent set* of π is defined by

$$\mathrm{Des}_S(\pi) := \bigl\{i \mid \ell_S(\pi) > \ell_S(\pi s_i)\bigr\} = \bigl\{i \mid \pi(i) > \pi(i+1)\bigr\}.$$

- The *descent number* of π, $\mathrm{des}_S(\pi)$, is defined by $\mathrm{des}_S(\pi) := |\mathrm{Des}_S(\pi)|$.
- The *major index*, $\mathrm{maj}_S(\pi)$ is

$$\mathrm{maj}_S(\pi) := \sum_{i \in \mathrm{Des}_S(\pi)} i.$$

The corresponding *reverse major index* does depend on n, and is denoted

$$\mathrm{rmaj}_{S_n}(\pi) := \sum_{i \in \mathrm{Des}_S(\pi)} (n-i).$$

- The reverse major index $\mathrm{rmaj}_{S_n}(\pi)$ is implicit in [6].

These statistics are involved in many combinatorial identities. First, MacMahon proved the following equi-distribution of the length and the major indices [12]:

$$\sum_{\sigma\in S_n} q^{\ell_S(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{maj}_S(\sigma)}.$$

Foata [5] gave a bijective proof of MacMahon's theorem, then Foata and Schützenberger [6] applied this bijection to refine MacMahon's identity by analyzing bivariate distributions. Garsia and Gessel [7] extended the analysis to multivariate distributions. Extensions of MacMahon's identity to hyperoctahedral groups appear in [1].

Combining Theorems 1 and 2 of [6] one deduces the following identity:

Theorem 1.1. *For any subset* $D_1 \subseteq \{1,\dots,n-1\}$,

$$\sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})\subseteq D_1\}} q^{\mathrm{maj}_{S_n}(\pi)} = \sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})\subseteq D_1\}} q^{\mathrm{rmaj}_{S_n}(\pi)} = \sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})\subseteq D_1\}} q^{\ell_S(\pi)}.$$

A bivariate equi-distribution follows.

Corollary 1.2.

$$\sum_{\pi\in S_n} q_1^{\mathrm{maj}_{S_n}(\pi)} q_2^{\mathrm{des}_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{\mathrm{rmaj}_{S_n}(\pi)} q_2^{\mathrm{des}_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{\ell_S(\pi)} q_2^{\mathrm{des}_S(\pi^{-1})}.$$

As already mentioned, one of the main goals in this paper is to find analogous statistics and identities for the alternating group A_n. In the process, we first prove some further refinements of some of the above identities for S_n, refinements involving the new *delent* statistic, see Theorems 6.1.1 and 9.1.1.

1.3. Main results

Here is a summary of the main results of this paper.

1.3.1. A_n-statistics

Following Mitsuhashi [13], we let

$$a_i := s_1 s_{i+1} = (1,2)(i+1,i+2) \quad (1 \leqslant i \leqslant n-1).$$

Thus $a_i = a_i^{-1}$ if $i \neq 1$, while $a_1^2 = a_1^{-1}$. The set $A := \{a_i \mid 1 \leqslant i \leqslant n-1\}$ generates the alternating group on $n+1$ letters A_{n+1} (see, e.g., [13]). It is the above exceptional property of a_1 among the elements of A—which naturally leads to the 'delent' statistic (Definition 1.5 below), both for S_n and for A_{n+1}. This new statistic enables us to deduce new refinements of the MacMahon-type identities for S_n, and for each such an identity to derive the analogous identity for A_{n+1}.

The canonical presentation in S_n by the Coxeter generators is well known, and is discussed in Section 3, see Theorem 3.1. With the above generating set A of A_{n+1} we also have canonical presentations for the elements of A_{n+1}, as follows. For each $1 \leqslant j \leqslant n-1$, define

$$R_j^A = \left\{1, a_j, a_j a_{j-1}, \ldots, a_j \cdots a_2, a_j \cdots a_2 a_1, a_j \cdots a_2 a_1^{-1}\right\}, \tag{1}$$

where $R_1^A = \{1\}$.

Theorem 1.3 (see Theorem 3.4). *Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R_j^A$, $1 \leqslant j \leqslant n-1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique. Call that presentation $v = v_1 \cdots v_{n-1}$ the A-canonical presentation of v.*

The A-canonical presentation allows us to introduce the A-length of an element in A_{n+1}.

Definition 1.4. Let $v \in A_{n+1}$ with $v = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$ ($\epsilon_i = \pm 1$) its A-canonical presentation, then its A-length is $\ell_A(v) = r$.

A combinatorial interpretation of the A-length in terms of inversions is given below, see Proposition 4.5.

The A-descent statistic is defined using the above generating set A.

Definition 1.5. (1) The *alternating-descent* (i.e. the A-descent) set of $\sigma \in A_{n+1}$ is defined by

$$\mathrm{Des}_A(\sigma) := \left\{1 \leqslant i \leqslant n-1 \mid \ell_A(\sigma) \geqslant \ell_A(\sigma a_i)\right\},$$

and the *A-descent number* of $\sigma \in A_{n+1}$ is defined by

$$\mathrm{des}_A(\sigma) := \left|\mathrm{Des}_A(\sigma)\right|$$

(note that the strict relation $>$ in the definition of an S-descent in Section 1.2 is replaced in the A-analogue by $\geqslant$).

(2) Define the *alternating reverse major index* of $\sigma \in A_{n+1}$ as

$$\mathrm{rmaj}_{A_{n+1}}(\sigma) := \sum_{i \in \mathrm{Des}_A(\sigma)} (n-i).$$

1.3.2. The delent number

New statistics, for the alternating group, as well as for the symmetric group, are introduced.

Definition 1.6 (see *Definition* 4.3).

(1) Let $w \in S_n$. The S-delent number of w is the number of times that $s_1 = (1, 2)$ occurs in the S-canonical presentation of w, and is denoted by $\mathrm{del}_S(w)$.
(2) Let $v \in A_{n+1}$. The A-delent number of v is the number of times that $a_1^{\pm 1}$ occur in the A-canonical presentation of v, and is denoted by $\mathrm{del}_A(v)$.

A combinatorial interpretation of the delent numbers, del_S and del_A, is given in Section 7. Let $w \in S_n$, then j is a l.t.r.min (left-to-right minimum) of w if $w(i) > w(j)$ for all $1 \leqslant i < j$.

Proposition 1.7 (see Proposition 7.7). *For every permutation $w \in S_n$, let*

$$\mathrm{Del}_S(w) = \{1 < i \leqslant n \mid i \textit{ is a l.t.r.min}\};$$

then

$$\mathrm{del}_S(w) = \left|\mathrm{Del}_S(w)\right|.$$

Notice that in the above definition of $\mathrm{Del}_S(w)$, the first l.t.r.min (i.e. $i = 1$) does not count.

Similar to l.t.r.min, we define an *almost left to right minimum* (a.l.t.r.min) of $w \in A_{n+1}$ as follows:

- j is an a.l.t.r.min of w if $w(i) < w(j)$ for at most one j less than i. Define $\mathrm{Del}_A(w)$ as the set of the *almost left-to-right minima* of w. Then $\mathrm{del}_A(v) = |\mathrm{Del}_A(w)|$, i.e. is the number of a.l.t.r.min of w, see Proposition 7.7.

We also have

Proposition 1.8 (see Proposition 4.4). *Let $w \in A_{n+1}$, then*

$$\mathrm{del}_S(w) = \ell_S(w) - \ell_A(w).$$

1.3.3. Equi-distribution identities

The covering map $f : A_{n+1} \to S_n$, presented in Definition 5.1, allows us to translate S_n-identities, which involve the delent statistic, into corresponding A_{n+1}-identities. This strategy is used in the proofs of part (2) of the following theorems.

Part (1) of the following theorem is a new generalization of MacMahon's classical identity, and part (2) is its A-analogue.

Theorem 1.9 (see Theorem 6.1).

(1) $$\sum_{\sigma\in S_n} q^{\ell_S(\sigma)} t^{\mathrm{del}_S(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{rmaj}_{S_n}(\sigma)} t^{\mathrm{del}_S(\sigma)} = (1+qt)\bigl(1+q+q^2t\bigr)\cdots\bigl(1+q+\cdots+q^{n-1}t\bigr).$$

(2) $$\sum_{w\in A_{n+1}} q^{\ell_A(w)} t^{\mathrm{del}_A(w)} = \sum_{w\in A_{n+1}} q^{\mathrm{rmaj}_{A_{n+1}}(w)} t^{\mathrm{del}_A(w)} = (1+2qt)\bigl(1+q+2q^2t\bigr)\cdots\bigl(1+q+\cdots+q^{n-2}+2q^{n-1}t\bigr).$$

Recall the standard notation $[m]=\{1,\ldots,m\}$. The main theorem in this paper strengthens Theorem 1.1, and also gives its A-analogue. This is

Theorem 1.10 (see Theorem 9.1). *For every subsets $D_1\subseteq[n-1]$ and $D_2\subseteq[n]$,*

(1) $$\sum_{\left\{\pi\in S_n \,\middle|\, \substack{\mathrm{Des}_S(\pi^{-1})\subseteq D_1\\ \mathrm{Del}_S(\pi^{-1})\subseteq D_2}\right\}} q^{\mathrm{rmaj}_{S_n}(\pi)} = \sum_{\left\{\pi\in S_n \,\middle|\, \substack{D_S(\pi^{-1})\subseteq D_1\\ \mathrm{Del}_S(\pi^{-1})\subseteq D_2}\right\}} q^{\ell_S(\pi)},$$

(2) $$\sum_{\left\{\sigma\in A_{n+1} \,\middle|\, \substack{\mathrm{Des}_A(\sigma^{-1})\subseteq D_1\\ \mathrm{Del}_A(\sigma^{-1})\subseteq D_2}\right\}} q^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} = \sum_{\left\{\sigma\in A_{n+1} \,\middle|\, \substack{\mathrm{Des}_A(\sigma^{-1})\subseteq D_1\\ \mathrm{Del}_A(\sigma^{-1})\subseteq D_2}\right\}} q^{\ell_A(\sigma)}.$$

This shows that the delent set and the descent set play a similar role in these identities.

The A-analogue of Corollary 1.2 follows. It is obtained as a special case of Corollary 9.2(2) (by substituting $q_3=1$).

Corollary 1.11 (see Corollary 9.2).

$$\sum_{\sigma\in A_{n+1}} q_1^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} q_2^{\mathrm{des}_A(\sigma^{-1})} = \sum_{\sigma\in A_{n+1}} q_1^{\ell_A(\sigma)} q_2^{\mathrm{des}_A(\sigma^{-1})}.$$

Note that, while the S-identity holds for maj_{S_n} as well as for rmaj_{S_n}, it is not possible to replace $\mathrm{rmaj}_{A_{n+1}}$ by $\mathrm{maj}_{A_{n+1}}$ in the A-analogue.

2. Preliminaries

2.1. Notation

For an integer a, we let $[a]:=\{1,2,\ldots,a\}$ (where $[0]\ :=\emptyset$). Let $n_1,\ldots,n_r$ be non-negative integers such that $\sum_{i=1}^r n_i=n$. Recall that the *q-multinomial coefficient*

$\left[{n \atop n_1,\ldots,n_r}\right]_q$ is defined by

$$[0]!_q := 1, \qquad [n]!_q := [n-1]!_q \left(1 + q + \cdots + q^{n-1}\right) \quad (n \geqslant 1),$$

$$\begin{bmatrix} n \\ n_1 \ldots n_r \end{bmatrix}_q := \frac{[n]!_q}{[n_1]!_q \cdots [n_r]!_q}.$$

Represent $\sigma \in S_n$ by 'its second row' $\sigma = [\sigma(1), \ldots, \sigma(n)]$. We also use the cycle-notation; in particular, we define $s_i := (i, i+1)$, the transposition of i and $i+1$. Thus

$$\left[\ldots, \sigma(r), \sigma(r+1), \ldots\right] s_r = \left[\ldots, \sigma(r+1), \sigma(r), \ldots\right] \tag{2}$$

(i.e. only $\sigma(r), \sigma(r+1)$ switch places).

2.2. The Coxeter system of the symmetric group

The symmetric group on n letters, denoted by S_n, is generated by the set of adjacent transpositions $S := \{(i, i+1) \mid 1 \leqslant i < n\}$. The defining relations of S are the Moore–Coxeter relations:

$$(s_i s_{i+1})^3 = 1 \quad (1 \leqslant i < n); \qquad (s_i s_j)^2 = 1 \quad (|i-j| > 1); \qquad s_i^2 = 1 \quad (\forall i).$$

This set of generators is called the *Coxeter system* of S_n.

For $\pi \in S_n$ let $\ell_S(\pi)$ be the standard length of π with respect to S (i.e. the length of the canonical presentation of π, see Section 3). Let w be a word on the letters S. A *commuting move* on w switches the positions of successive letters $s_i s_j$ where $|i-j| > 1$. A *braid move* replaces $s_i s_{i+1} s_i$ by $s_{i+1} s_i s_{i+1}$ or vice versa. The following is a well-known fact, but we shall not use it in this paper.

Fact 2.1. *All irreducible expressions of $\pi \in S_n$ are of length $\ell_S(\pi)$. For every pair of irreducible words of $\pi \in S_n$, it is possible to move from one to another along commuting and braid moves.*

2.3. Permutation statistics

There are various statistics on the symmetric groups S_n, like the *descent* number and the *major* index. We introduce and study analogue statistics on the alternating groups A_n. To distinguish, we add 'sub-S' and 'sub-A' accordingly.

Given a permutation $\pi = [\pi(1), \ldots, \pi(n)]$ in the symmetric group S_n, we say that a pair (i, j), $1 \leqslant i < j \leqslant n$, is an *inversion* of π if $\pi(i) > \pi(j)$. The set of inversions of π is denoted by $\mathrm{Inv}_S(\pi)$ and its cardinality is denoted by $\mathrm{inv}_S(\pi)$. Also $1 \leqslant i < n$ is a *descent* of π if $\pi(i) > \pi(i+1)$. For the definitions of the descent set $\mathrm{Des}_S(\pi)$, the descent number $\mathrm{des}_S(\pi)$, the major index $\mathrm{maj}_S(\pi)$ and the reverse major index $\mathrm{rmaj}_{S_n}(\pi)$, see Section 1.2.

Note that i is a descent of π if and only if $\ell_S(\pi s_i) < \ell_S(\pi)$. Thus (as already mentioned in Section 1.2), the descent set, and consequently the other statistics, have an algebraic interpretation in terms of the Coxeter system. In particular, for every $\pi \in S_n$,

$$\mathrm{inv}_S(\pi) = \ell_S(\pi). \tag{3}$$

The following well-known identity is due to MacMahon [12]. See, e.g., [5] and [17, Corollaries 1.3.10 and 4.5.9].

Theorem 2.2.

$$\begin{aligned}\sum_{\pi\in S_n} q^{\mathrm{inv}_S(\pi)} &= \sum_{\pi\in S_n} q^{\mathrm{maj}_S(\pi)} = [n]!_q \\ &= (1+q)\big(1+q+q^2\big)\cdots\big(1+q+\cdots+q^{n-2}+q^{n-1}\big).\end{aligned}$$

The following theorem is a reformulation of [6, Theorem 1].

Theorem 2.3. *For every $B \subseteq [n-1]$,*

$$\sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})=B\}} q^{\mathrm{inv}_S(\pi)} = \sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})=B\}} q^{\mathrm{maj}_S(\pi)}.$$

2.3.1. Shuffles

Let $1 \leqslant i \leqslant n-1$, then $w \in S_n$ is an $\{i\}$-shuffle if it shuffles $\{1,\ldots,i\}$ with $\{i+1,\ldots,n\}$; in other words, if $1 \leqslant a < b \leqslant i$ then $w^{-1}(a) < w^{-1}(b)$, and similarly, if $i+1 \leqslant k < \ell \leqslant n$, then $w^{-1}(k) < w^{-1}(\ell)$.

Example. Let $n=4$ and $B=\{2\}$, then $\{1,2\}$ and $\{3,4\}$ are being shuffled, hence

$$[1,2,3,4],\quad [1,3,2,4],\quad [1,3,4,2],\quad [3,1,2,4],\quad [3,1,4,2],\quad [3,4,1,2]$$

are all the $\{2\}$-shuffles.

More generally, let $B=\{i_1,\ldots,i_k\} \subseteq [n-1]$, where $i_1 < \cdots < i_k$. Set $i_0 := 0$ and $i_{k+1} := n$. A *B-shuffle* is a permutation which shuffles $\{1,\ldots,i_1\}, \{i_1+1,\ldots,i_2\},\ldots$ Thus $\pi \in S_n$ is a B-shuffle if it satisfies: if $i_j \leqslant a < b \leqslant i_{j+1}$ for some $0 \leqslant j \leqslant k$, then $\pi = [\ldots,a,\ldots,b,\ldots]$ (i.e. a is left of b in π). Notice that in particular there can be no descent for π^{-1} at any a, $i_j < a < i_{j+1}$, hence $\mathrm{Des}_S(\pi^{-1}) \subseteq B$. The opposite is also clear, hence

Fact 2.4. *For every $B \subseteq [n-1]$,*

$$\big\{\pi \in S_n \mid \mathrm{Des}_S\big(\pi^{-1}\big) \subseteq B\big\} = \{\pi \in S_n \mid \pi \textit{ is a } B\textit{-shuffle}\}.$$

For a permutation $\pi \in S_n$, let

$$\mathrm{supp}(\pi) := \{1 \leqslant i \leqslant n \mid \pi(i) \neq i\}$$

be the *support* of π.

Let $k \in [n-1]$, and let π_1, π_2 be permutations in S_n, such that $\mathrm{supp}(\pi_1) \subseteq [k]$ and $\mathrm{supp}(\pi_2) \subseteq [k+1, n]$. A permutation $\sigma \in S_n$ is called a shuffle of π_1 and π_2 if $\sigma = \pi_1\pi_2 r$ for some $\{k\}$-shuffle r. Equivalently, σ is a shuffle of π_1 and π_2 if and only if the letters of $[k]$ appear in σ in the same order as they appear in π_1 and the letters of $[k+1, n]$ appear in σ in the same order as they appear in π_2. The following is a special case of [17, Proposition 1.3.17].

Fact 2.5. *Let $k \in [n]$, and let π_1, π_2 be permutations in S_n such that* $\mathrm{supp}(\pi_1) \subseteq [k]$ *and* $\mathrm{supp}(\pi_2) \subseteq [k+1, n]$. *Then*

$$\sum_{\mathrm{Des}(r^{-1}) \subseteq \{k\}} q^{\mathrm{inv}_S(\pi_1\pi_2 r) - \mathrm{inv}_S(\pi_1) - \mathrm{inv}_S(\pi_2)} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The following analogue is a special case of a well-known theorem of Garsia and Gessel. It should be noted that, while Garsia–Gessel's Theorem is stated in terms of sequences, our reformulation is in terms of permutations.

Theorem 2.6 [7, Theorem 3.1]. *Let $k \in [n-1]$, and let π_1, π_2 be permutations in S_n, such that* $\mathrm{supp}(\pi_1) \subseteq [k]$ *and* $\mathrm{supp}(\pi_2) \subseteq [k+1, n]$. *Let* $\nu_k := (1, k+1)(2, k+2)\cdots(n-k, n) \in S_n$. *Then*

$$\sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{k\}} q^{\mathrm{maj}_S(\pi_1\pi_2 r) - \mathrm{maj}_S(\pi_1) - \mathrm{maj}_S(\nu_k^{-1}\pi_2\nu_k)} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

In order to translate Theorem 2.6 into Garsia–Gessel's terminology, note that $\pi_1\pi_2 r$ are shuffles of π_1 and π_2 (as mentioned above); thus the sum runs over all shuffles of π_1 and π_2. Also, $\mathrm{maj}_S(\nu_k^{-1}\pi_2\nu_k)$ is the major index of π_2, when it is considered as a sequence on the letters $[k+1, n]$.

Remark 2.7. In general, it is possible to replace a statement involving maj by a corresponding statement involving rmaj, using the automorphism $\sigma \to \hat{\sigma}$ which reverses the order of the letters and replace each letter i by $n+1-i$:

$$\hat{\sigma} := \rho_n \sigma \rho_n \quad \text{where } \rho_n := (1, n)(2, n-1)\cdots\big(\lfloor n/2 \rfloor, \lfloor (n+3)/2 \rfloor\big).$$

Then

$$\mathrm{rmaj}_{S_n}(\sigma) = \mathrm{maj}_S(\hat{\sigma}), \qquad \mathrm{inv}_S(\sigma) = \mathrm{inv}_S(\hat{\sigma}), \tag{4}$$

and σ is an $\{i\}$-shuffle if and only if $\hat{\sigma}$ is an $\{n-i\}$-shuffle, i.e.

$$\mathrm{Des}_S(\sigma^{-1}) \subseteq \{i\} \quad \Longleftrightarrow \quad \mathrm{Des}_S(\hat{\sigma}^{-1}) \subseteq \{n-i\}. \tag{5}$$

Note that by (4) and [16, Claim 0.4], for every $\pi \in S_n$,

$$\mathrm{rmaj}_{S_n}(\pi) = \mathrm{charge}(\pi^{-1}),$$

where the charge is defined as in [11, p. 242].

3. The S- and A-canonical presentations

In this section we consider canonical presentations of elements in S_n and in A_n by the corresponding Coxeter generators. This presentation for S_n is well known, see, for example, [8, pp. 61–62]. The analogous presentation for A_n follows from the properties of the Mitsuhashi's Coxeter generators.

3.1. The S_n case

The S_n-canonical presentation is proved below, using the S-procedure, which is also applied later.

Recall that $s_i = (i, i+1)$, $1 \leqslant i < n$, are the Coxeter generators of S_n. For each $1 \leqslant j \leqslant n-1$, define

$$R_j^S = \{1, s_j, s_j s_{j-1}, \ldots, s_j s_{j-1} \cdots s_1\} \tag{6}$$

and note that $R_1^S, \ldots, R_{n-1}^S \subseteq S_n$.

Theorem 3.1. *Let $w \in S_n$, then there exist unique elements $w_j \in R_j^S$, $1 \leqslant j \leqslant n-1$, such that $w = w_1 \cdots w_{n-1}$. Thus, the presentation $w = w_1 \cdots w_{n-1}$ is unique.*

Definition 3.2. Call the above $w = w_1 \cdots w_{n-1}$ in Theorem 3.1 the S-canonical presentation of $w \in S_n$.

Proof of Theorem 3.1. If follows from the following S-procedure.

The S-procedure. The following is a simple procedure for calculating the S-canonical presentation of a given $w \in S_n$. It can also be used to prove Theorem 3.1, as well as various other facts. Let $\sigma \in S_n$, $\sigma(r) = n$, $\sigma = [\ldots, n, \ldots]$, then apply Eq. (2) to 'pull n to its place on the right': $\sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots, n]$. This gives $w_{n-1} = s_{n-1} \cdots s_{r+1} s_r$. Next, in

$$\sigma w_{n-1}^{-1} = \sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots, n-1, \ldots, n],$$

pull $n-1$ to its right place (second from right) by a similar product $s_t s_{t+1} \cdots s_{n-2}$. This yields $w_{n-2} = s_{n-2} \cdots s_t$. Continue! Finally, $\sigma = w_1 \cdots w_{n-1}$.

For example, let $\sigma = [2,5,4,1,3]$, then $w_{n-1} = w_4 = s_4s_3s_2$; $\sigma w_4^{-1} = [2,4,1,3,5]$, therefore $w_3^{-1} = s_2s_3$. Check that $w_2 = 1$ and, finally, $w_1 = s_1$. Thus $\sigma = w_1 \cdots w_4 = (s_1)(1)(s_3s_2)(s_4s_3s_2)$.

The uniqueness in Theorem 3.1 follows by cardinality, since the number of canonical words in S_n is at most

$$\prod_{j=1}^{n-1} \operatorname{card}\big(R_j^S\big) = |S_n|.$$

This proves Theorem 3.1. □

3.2. *A generating set for A_n*

We turn now to A_n. As was already mentioned in Section 1.3.1, we let

$$a_i := s_1s_{i+1} \quad (1 \leqslant i \leqslant n-1).$$

The set

$$A := \{a_i \mid 1 \leqslant i \leqslant n-1\}$$

generates the alternating group on n letters A_{n+1}. This generating set and its following properties appear in [13].

Proposition 3.3 [13, Proposition 2.5]. *The defining relations of A are*

$$(a_ia_j)^2 = 1 \quad (|i-j| > 1); \qquad (a_ia_{i+1})^3 = 1 \quad (1 \leqslant i < n-1);$$
$$a_1^3 = 1 \quad \textit{and} \quad a_i^2 = 1 \quad (1 < i \leqslant n-1).$$

The general braid-relation $(a_ia_{i+1})^3 = 1$ implies the following braid-relations:

(1) $a_2a_1a_2 = a_1^{-1}a_2a_1^{-1}$,

(2) $a_2a_1^{-1}a_2 = a_1a_2a_1$,

(3) $a_{i+1}a_ia_{i+1} = a_ia_{i+1}a_i$ if $i \geqslant 2$ (since $a_i^{-1} = a_i$).

Let

$$\overline{A} := A \cup \big\{a_1^{-1}\big\},$$

where A is defined as above. Clearly, $\overline{A}$ is a generating set for A_{n+1}.

3.3. The canonical presentation

For each $1 \leqslant j \leqslant n-1$, define

$$R_j^A = \{1, a_j, a_j a_{j-1}, \ldots, a_j \cdots a_2, a_j \cdots a_2 a_1, a_j \cdots a_2 a_1^{-1}\} \tag{7}$$

and note that $R_1^A, \ldots, R_{n-1}^A \subseteq A_{n+1}$.

Theorem 3.4. *Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R_j^A$, $1 \leqslant j \leqslant n-1$, such that $v = v_1 \cdots v_{n-1}$.*

Definition 3.5. Call the above $v = v_1 \cdots v_{n-1}$ in Theorem 3.4 the A-canonical presentation of v.

Proof of Theorem 3.4. Let $v = w_1 \cdots w_n$, $w_j \in R_j^S$, be the S-canonical presentation of v. Rewrite that presentation explicitly as

$$v = (s_{i_1} s_{i_2}) \cdots (s_{i_{2r-1}} s_{i_{2r}}). \tag{8}$$

Note that $s_i s_j = (s_i s_1)(s_1 s_j) = a_{i-1}^{-1} a_{j-1}$ (denote $a_0 = 1$). Thus each s_i in (8) is replaced by a corresponding $a_{i-1}^{\pm 1}$. It follows that for each $2 \leqslant j \leqslant n$, w_j is replaced by $v_{j-1} \in R_{j-1}^A$ and $v = v_1 \cdots v_{n-1}$. This proves the existence of such a presentation.

A second proof of the existence follows from the following A-procedure.

The A-procedure. It is similar to the S-procedure. We describe its first step, which is also its inductive step.

Let $\sigma \in A_{n+1}$, $\sigma = [\ldots, n+1, \ldots]$. As in the S-procedure, pull $n+1$ to the right: $\sigma s_r s_{r+1} \cdots s_n = [b_1, b_2, \ldots, n+1]$. The ($S$-) length of $s_r s_{r+1} \cdots s_n$ is $n-r+1$; if it is odd, use $\sigma s_r s_{r+1} \cdots s_n s_1 = [b_2, b_1, \ldots, n+1]$. Thus

$$v_{n-1} = \begin{cases} s_n s_{n-1} \cdots s_r, & \text{if } n-r+1 \text{ is even;} \\ s_1 s_n s_{n-1} \cdots s_r, & \text{if } n-r+1 \text{ is odd.} \end{cases}$$

Case $r \geqslant 2$. Then $s_1 s_j = s_j s_1$ for all $j \geqslant r+1$, hence

$$v_{n-1} = \begin{cases} (s_1 s_n)(s_1 s_{n-1}) \cdots (s_1 s_r) = a_{n-1} \cdots a_{r-1}, & \text{if } n-r+1 \text{ is even;} \\ (s_1 s_1)(s_1 s_n) \cdots (s_1 s_r) = a_{n-1} \cdots a_{r-1}, & \text{if } n-r+1 \text{ is odd.} \end{cases}$$

Case $r = 1$. If $n-r+1 = n$ is even,

$$v_{n-1} = s_n \cdots s_2 s_1 = (s_1 s_n) \cdots (s_1 s_3)(s_2 s_1) = a_{n-1} \cdots a_2 a_1^{-1},$$

and similarly if $n-r+1$ is odd.

This completes the first step. In the next step, pull n to the nth position (i.e., second from the right), etc. This proves the existence of such a presentation $v = v_1 \cdots v_{n-1}$.

Example. Let $\sigma = [3,5,4,2,1]$, so $n+1=5$. Now $\sigma s_2 s_3 s_4 = [3,4,2,1,5]$ and since $s_2 s_3 s_4$ is of odd length ($=3$), permute 3 and 4: $\sigma s_2 s_3 s_4 s_1 = [4,3,2,1,5]$. Thus $v_3 = s_1 s_4 s_3 s_2 = (s_1 s_1)(s_1 s_4)(s_1 s_3)(s_1 s_2) = a_3 a_2 a_1$. Similarly, $v_2 = a_2 a_1^{-1}$ and $v_1 = a_1$, hence $[3,5,4,2,1] = (a_1)(a_2 a_1^{-1})(a_3 a_2 a_1)$.

Uniqueness follows by cardinality: note that for all $1 \leqslant j \leqslant n-1$, $|R_j^A| = j+2$, hence the number of such words $v_1 \cdots v_{n-1}$ in A_{n+1} is at most

$$\prod_{j=1}^{n-1} (j+2) = |A_{n+1}|.$$

Since each element in A_{n+1} does have such a presentation, this implies the uniqueness—and the proof of Theorem 3.4 is complete. □

Given $w \in S_n$, we say that s_i occurs ℓ times in w if it occurs ℓ times in the canonical presentation of w. Similarly, for the number of occurrences of a_i, or of a_1^{-1}, in $v \in A_{n+1}$. The number of occurrences of s_1, as well as those of $a_1^{\pm 1}$, are of particular importance in this paper.

Lemma 3.6. *Let $w \in S_n$, then the number of occurrences of s_i in w equals the number of occurrences of s_i in w^{-1}. Similarly for A_{n+1} and $a_1^{\pm 1}$.*

This is an obvious corollary of the following lemma.

Lemma 3.7. *Let $w = s_{i_1} \cdots s_{i_p}$ be the canonical presentation of $w \in S_n$. Then the canonical presentation of w^{-1} is obtained from the presentation $w^{-1} = s_{i_p} \cdots s_{i_1}$ by commuting moves only—without any braid moves.*

Similarly for $v,\ v^{-1} \in A_{n+1}$.

Proof. We prove for S_n. The proof is by induction on n. Write $w = w_1 \cdots w_{n-1}$, $w_j \in R_j^S$. If $w_{n-1} = 1$ then $w \in S_{n-1}$ and the proof follows by induction.

Let $w_{n-1} = s_{n-1} s_{n-2} \cdots s_k$ where $1 \leqslant k \leqslant n-1$. Now either $w_{n-2} = 1$ or $w_{n-2} = s_{n-2} s_{n-3} \cdots s_\ell$ for some $1 \leqslant \ell \leqslant n-2$, and similarly for w_{n-3}, w_{n-4}, etc. The case $w_{n-2} = 1$ is similar to the case $w_{n-2} \neq 1$ and is left to the reader, so let $w_{n-2} \neq 1$ and

$$w^{-1} = w_{n-1}^{-1} w_{n-2}^{-1} \cdots = (s_k \cdots s_{n-1})(s_\ell \cdots s_{n-2}) w_{n-3}^{-1} w_{n-4}^{-1} \cdots.$$

Notice that $s_{n-1}(s_\ell \cdots s_{n-3}) = (s_\ell \cdots s_{n-3}) s_{n-1}$, hence

$$w^{-1} = (s_k \cdots s_{n-2})(s_\ell \cdots s_{n-3})(s_{n-1} s_{n-2}) w_{n-3}^{-1} w_{n-4}^{-1} \cdots.$$

Next, move $s_{n-1}s_{n-2}$ to the right, similarly, by commuting moves. Continue by similarly pulling s_{n-3}—in w_{n-3}^{-1}—to the right, etc. It follows that by such commuting moves we obtain

$$w^{-1} = \overline{w}^{\,-1}(s_{n-1}s_{n-2}\cdots s_d)$$

for some d, where $\overline{w} = s_{j_r}\cdots s_{j_1} \in S_{n-1}$, and is in canonical form. By induction, transform $\overline{w}^{\,-1}$ to its canonical form by commuting moves—and the proof is complete. □

4. The length statistics

The canonical presentations of the previous sections allow us to introduce the S- and A-lengths.

Definition 4.1 (*The length statistics*).

(1) Let $w \in S_n$ with $w = s_{i_1}\cdots s_{i_r}$ its S-canonical presentation, then its S-length is $\ell_S(w) = r$.
(2) Let $v \in A_{n+1}$ with $v = a_{i_1}^{\epsilon_1}\cdots a_{i_r}^{\epsilon_r}$ ($\epsilon_i = \pm 1$) its A-canonical presentation, then its A-length is $\ell_A(v) = r$.

For example, $\ell_A(a_1) = 1$ and $\ell_S(a_1) = \ell_S(s_1 s_2) = 2$.

Remark 4.2. An analogue of Fact 2.1 holds: All irreducible expressions of $v \in A_{n-1}$ are of length $\ell_A(v)$. This fact will not be used in the paper.

Definition 4.3. (1) Let $w \in S_n$. The number of times that s_1 occurs in the S-canonical presentation of w is denoted by $\mathrm{del}_S(w)$.

(2) Let $v \in A_{n-1}$. The number of times that $a_1^{\pm 1}$ occurs in the A-canonical presentation of v is denoted by $\mathrm{del}_A(v)$.

For example, $\mathrm{del}_S(s_1s_2s_1s_3) = 2$ and $\mathrm{del}_A(a_1^{-1}a_2a_1a_3a_2a_1^{-1})) = 3$.

A combinatorial characterization of del_S (del_A) is given in Section 7.

Relations between del_S and the S- and A-lengths of $v \in A_{n+1}$ are given by the following proposition.

Proposition 4.4. *Let $w \in A_{n+1}$; then*

$$\ell_A(w) = \ell_S(w) - \mathrm{del}_S(w).$$

Moreover, let

$$w = s_{i_1}\cdots s_{i_{2r}} = w_1\cdots w_n, \quad w_i \in R_i^S, \tag{9}$$

be its S-canonical presentation and

$$w = a_{j_1}^{\epsilon_1} \cdots a_{j_t}^{\epsilon_t} = v_1 \cdots v_{n-1}, \quad v_i \in R_i^A, \tag{10}$$

its A-canonical presentation. Then

$$\ell_A(v_i) = \begin{cases} \ell_S(w_{i+1}) & \text{if } s_1 \text{ does not occur in } w_{i+1}, \\ \ell_S(w_{i+1}) - 1 & \text{if } s_1 \text{ occurs in } w_{i+1}. \end{cases} \tag{11}$$

Proof. As in the proof of Theorem 3.4, the proof easily follows from (8) by replacing $s_i s_j$ by $(s_i s_1)(s_1 s_j)$. $\square$

The S-lengths $\ell_S(w_{i+1})$ and the A-lengths $\ell_A(v_i)$ in (11) can be calculated directly from $w = [b_1, \ldots, b_{n+1}]$ as follows.

Proposition 4.5. *Let $w \in S_{n+1}$ as above. For each $2 \leqslant j \leqslant n$, let $T_j(w)$ denote the set of indices i such that $i < j$ and $w = [\ldots, j, \ldots, i, \ldots]$ (i.e. $w^{-1}(i) > w^{-1}(j)$); denote $t_j(w) = |T_j(w)|$. Keeping the notations of Proposition* 4.4*, we have*:

(1) *$\ell_S(w_j) = t_{j+1}(w)$. Moreover, $T_{j+1}(w)$ is the full set $\{1, \ldots, j\}$ (i.e. $t_{j+1}(w) = j$) if and only if s_1 occurs in w_j.*
(2) *$\ell_A(v_k)$ equals $|T_k(w)|$, provided that $T_k(w)$ is not the full set $\{1, \ldots, k-1\}$, and it equals $|T_k(w)| - 1$ otherwise.*

Proof. By an easy induction on n, prove that

$$\big(\ell_S(w_1), \ldots, \ell_S(w_n)\big) = \big(t_2(w), \ldots, t_{n+1}(w)\big).$$

This follows since

$$[b_1, \ldots, b_n, n+1] s_n s_{n-1} \cdots s_r = [b_1, \ldots, b_{r-1}, n+1, b_r, \ldots, b_n].$$

Here are the details: Write $w = w_1 \cdots w_n$, let $\sigma = w_1 \cdots w_{n-1}$, so $\sigma = [d_1, \ldots, d_n, n+1]$. If $w_n = 1$, the claim follows by induction. Let $w_n = s_n s_{n-1} \cdots s_r$ for some $r \geqslant 1$. Then $w = \sigma w_n = [d_1, \ldots, d_{r-1}, n+1, d_r, \ldots d_n]$. Thus $t_{n+1}(w) = n - r + 1 = \ell_S(w_n)$. Also, for $2 \leqslant j \leqslant n$, $t_j(w) = t_j(\sigma)$, and the proof of part (1) follows by induction. Part (2) now follows from (11). $\square$

5. f-pairs of statistics

5.1. The covering map

Theorems 3.1 and 3.4 allow us to introduce the following definition.

Definition 5.1. Define $f : A_{n+1} \to S_n$ as follows:

$$f(a_1) = f\bigl(a_1^{-1}\bigr) = s_1 \quad \text{and} \quad f(a_i) = s_i, \quad 2 \leqslant i \leqslant n-1.$$

Now extend $f : R_j^A \to R_j^S$ via

$$f(a_j a_{j-1} \cdots a_\ell) = s_j s_{j-1} \cdots s_\ell, \qquad f(a_j \cdots a_1) = f\bigl(a_j \cdots a_1^{-1}\bigr) = s_j \cdots s_1.$$

Finally, let $v \in A_{n+1}$, $v = v_1 \cdots v_{n-1}$ its unique A-canonical presentation, then

$$f(v) = f(v_1) \cdots f(v_{n-1}),$$

which is clearly the S-canonical presentation of $f(v)$.

Notice that for $v \in A_{n+1}$, $\ell_A(v) = \ell_S(f(v))$. We therefore say that the pair of the length statistics (ℓ_S, ℓ_A) is an f-pair. More generally, we have

Definition 5.2. Let m_S be a statistics on the symmetric groups and m_A a statistics on the alternating groups. We say that (m_S, m_A) is an f-pair (of statistics) if for any n and $v \in A_{n+1}$, $m_A(v) = m_S(f(v))$.

Examples of f-pairs are given in Proposition 5.4.

Proposition 5.3. *For every* $\pi \in A_{n+1}$

$$\mathrm{Des}_A(\pi) = D_S\bigl(f(\pi)\bigr).$$

Proof. It is left to the reader. □

It follows that the descent statistics are f-pairs. By Definition 4.3, $(\mathrm{del}_S, \mathrm{del}_A)$ is an f-pair. We summarize:

Proposition 5.4. *The following pairs* (ℓ_S, ℓ_A), $(\mathrm{des}_S, \mathrm{des}_A)$, $(\mathrm{maj}_S, \mathrm{maj}_A)$, $(\mathrm{rmaj}_{S_n}, \mathrm{rmaj}_{A_{n+1}})$ *and* $(\mathrm{del}_S, \mathrm{del}_A)$ *are f-pairs.*

5.2. *The 'del' statistics*

The following basic properties of del_S play an important role in this paper.

Proposition 5.5. (1) *For each* $w \in S_n$, $|f^{-1}(w)| = 2^{\mathrm{del}_S(w)}$.
(2) *For each* $w \in S_n$ *and* $v \in A_{n+1}$,

$$\mathrm{del}_S(w) = \mathrm{del}_S\bigl(w^{-1}\bigr) \quad \textit{and} \quad \mathrm{del}_A(v) = \mathrm{del}_A\bigl(v^{-1}\bigr). \tag{12}$$

Proof. Part (1) follows since each occurrence of s_1 can be replaced by an occurrence of either a_1 or a_1^{-1}. Part (2) follows from Lemma 3.6. □

We have the following general proposition.

Proposition 5.6. *Let (m_S, m_A) be an f-pair of statistics, then for all n*

$$\sum_{v\in A_{n+1}} q^{m_A(v)} t^{\operatorname{del}_A(v)} = \sum_{w\in S_n} q^{m_S(w)} (2t)^{\operatorname{del}_S(w)}.$$

Proof. Since $A_{n+1} = \bigcup_{w\in S_n} f^{-1}(w)$, a disjoint union, we have:

$$\begin{aligned}\sum_{v\in A_{n+1}} q^{m_A(v)} t^{\operatorname{del}_A(v)} &= \sum_{w\in S_n} \sum_{v\in f^{-1}(w)} q^{m_A(v)} t^{\operatorname{del}_A(v)}\\ &= \sum_{w\in S_n} \sum_{v\in f^{-1}(w)} q^{m_S(f(v))} t^{\operatorname{del}_S(f(v))} = \sum_{w\in S_n} \sum_{v\in f^{-1}(w)} q^{m_S(w)} t^{\operatorname{del}_S(w)}\\ &= \sum_{w\in S_n} 2^{\operatorname{del}_S(w)} q^{m_S(w)} t^{\operatorname{del}_S(w)}. \qquad \square\end{aligned}$$

A refinement of Proposition 5.6 is given in Proposition 5.10.

Proposition 5.7. *With the above notations, we have*:

(1) $$\sum_{\sigma\in S_n} q^{\ell_S(\sigma)} t^{\operatorname{del}_S(\sigma)} = (1+qt)\big(1+q+q^2t\big)\cdots\big(1+q+\cdots+q^{n-1}t\big).$$

(2) $$\sum_{w\in A_{n+1}} q^{\ell_A(w)} t^{\operatorname{del}_A(w)} = (1+2qt)\big(1+q+2q^2t\big)\cdots\big(1+q+\cdots+q^{n-2}+2q^{n-1}t\big).$$

Proof. (1) The proof of part (1) is similar to the proof of Corollary 1.3.10 in [17]. Let $w_j \in R_j^S$, then $\operatorname{del}_S(w_j) = 1$ if $w_j = s_j \ldots s_1$ and $= 0$ otherwise. Let $w \in S_n$ and let $w = w_1 \cdots w_{n-1}$ be its S-canonical presentation, then $\operatorname{del}_S(w) = \operatorname{del}_S(w_1) + \cdots + \operatorname{del}_S(w_{n-1})$ and $\ell_S(w) = \ell_S(w_1) + \cdots + \ell_S(w_{n-1})$. Thus

$$\sum_{w\in S_n} q^{\ell_S(w)} t^{\operatorname{del}_S(w)} = \prod_{j=1}^{n-1} \Bigg(\sum_{w_j\in R_j^S} q^{\ell_S(w_j)} t^{\operatorname{del}_S(w_j)} \Bigg).$$

The proof now follows since

$$\sum_{w_j\in R_j^S} q^{\ell_S(w_j)} t^{\operatorname{del}_S(w_j)} = 1 + q + q^2 + \cdots + q^{j-2} + q^{j-1}t.$$

(2) By Proposition 5.6, part (2) follows from part (1). □

5.3. Connection with the Stirling numbers

Recall that $c(n,k)$ is the number of permutations in S_n with exactly k cycles, $1 \leqslant k \leqslant n$: $c(n,k)$ are *the sign-less Stirling numbers of the first kind*. Let $w_S(n,\ell)$ denote the number of S-canonical words in S_n with ℓ appearances of s_1. Similarly, let $w_A(n+1,\ell)$ denote the number of A-canonical words in A_{n+1} with ℓ appearances of $a_1^{\pm 1}$.

We prove

Proposition 5.8. *Let* $0 \leqslant \ell \leqslant n-1$, *then*

$$(1)\quad \sum_{\ell \geqslant 0} w_S(n,\ell)t^\ell = (t+1)(t+2)\cdots(t+n-1),$$

hence $w_S(n,\ell) = c(n,\ell+1)$;

$$(2)\quad \sum_{\ell \geqslant 0} w_A(n,\ell)t^\ell = (2t+1)(2t+2)\cdots(2t+n-1),$$

hence $w_A(n+1,\ell) = 2^\ell \cdot c(n,\ell+1)$.

Proof. Substitute $q=1$ in Proposition 5.7 and, in part (1), apply Proposition 1.3.4 of [17], which states that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1). \qquad \square$$

Further connections with the Stirling numbers are given below (Propositions 5.11, 5.12, and 7.10) and in [14].

5.4. A multivariate refinement

Definition 5.9. Let $w \in S_n$, $w = w_1 \cdots w_{n-1}$ its S-canonical presentation and let $1 \leqslant j \leqslant n-1$. Denote $\epsilon_{S,j}(w) = 1$ if s_1 occurs in w_j, and $=0$ otherwise*;* also denote

$$\bar{\epsilon}_S(w) = \big(\epsilon_{S,1}(w),\ldots,\epsilon_{S,n-1}(w)\big) \quad \text{and} \quad t^{\bar{\epsilon}_S(w)} = t_1^{\epsilon_{S,1}(w)} \cdots t_{n-1}^{\epsilon_{S,n-1}(w)}.$$

Similarly for $v = v_1 \cdots v_{n-1} \in A_{n+1}$: $\epsilon_{A,j}(v) = 1$ if $a_1^{\pm 1}$ occurs in v_j, and $=0$ otherwise, and define $\bar{\epsilon}_A(w)$ similarly. Clearly, $\mathrm{del}_S(w) = \sum_j \epsilon_{S,j}(w)$ and $\mathrm{del}_A(v) = \sum_j \epsilon_{A,j}(v)$.

Proposition 5.6 admits the following generalization.

Proposition 5.10. *Let* (m_S, m_A) *be an* f*-pair of statistics*; *then for all* n,

$$\sum_{v \in A_{n+1}} q^{m_A(v)} \prod_{j=1}^{n-1} t_j^{\epsilon_{A,j}(v)} = \sum_{w \in S_n} q^{m_S(w)} \prod_{j=1}^{n-1} (2t_j)^{\epsilon_{S,j}(w)}.$$

Proof. It is a slight generalization of the proof of Proposition 5.6—and is left to the reader. □

We end this section with another two multivariate generalizations, which will not be used in the rest of the paper. Proposition 5.7 generalizes as follows.

Proposition 5.11. *Let ℓ_S, ℓ_A be the length statistics*; *then*

$$(1)\quad \sum_{w\in S_n} q^{\ell_S(w)}\prod_{j=1}^{n-1}(t_j)^{\epsilon_{S,j}(w)} = (1+qt_1)\big(1+q+q^2t_2\big)\cdots\big(1+q+\cdots+q^{n-1}t_{n-1}\big).$$

$$(2)\quad \sum_{v\in A_{n+1}} q^{\ell_A(v)}\prod_{j=1}^{n-1}t_j^{\epsilon_{A,j}(v)} = (1+2qt_1)\cdots\big(1+q+\cdots+q^{n-2}+2q^{n-1}t_{n-1}\big).$$

One can generalize Proposition 5.8 as follows. Let $w=w_1\cdots w_{n-1}\in S_n$, a canonical presentation, with $\epsilon_{S,j}(w)$ and $\bar\epsilon_S(w)$ as in Definition 5.9. Given $\bar\epsilon=(\epsilon_1,\dots,\epsilon_{n-1})$ with all $\epsilon_i\in\{0,1\}$, denote $w_S(n,\bar\epsilon)=\operatorname{card}\{w\in S_n \mid \bar\epsilon_S(w)=\bar\epsilon\}$. Also denote $|\bar\epsilon|=\sum_j\epsilon_j$ and $t^{\bar\epsilon}=\prod_j t_j^{\epsilon_j}$. Note that

$$\sum_{|\bar\epsilon|=\ell} w_S(n,\bar\epsilon)=w_\ell(n,\ell)=c(n,\ell+1).$$

Similarly, introduce the analogous notations for A_{n+1}.

Proposition 5.8 now generalizes as follows.

Proposition 5.12. *With the above notations*:

$$(1)\quad \sum_{\bar\epsilon} w_S(n,\bar\epsilon)t^{\bar\epsilon}=(t_1+1)\cdots(t_{n-1}+n-1)',$$

$$(2)\quad \sum_{\bar\epsilon} w_A(n,\bar\epsilon)t^{\bar\epsilon}=(2t_1+1)\cdots(2t_{n-1}+n-1).$$

6. The major index and the delent number

Recall the definitions of $\operatorname{rmaj}_{S_n}$ and $\operatorname{rmaj}_{A_{n+1}}$ from Sections 1.2 and 1.3. In this section, we prove

Theorem 6.1.

$$(1)\quad \sum_{\sigma\in S_n} q^{\ell_S(\sigma)}t^{\operatorname{del}_S(\sigma)} = \sum_{\sigma\in S_n} q^{\operatorname{rmaj}_{S_n}(\sigma)}t^{\operatorname{del}_S(\sigma)} = (1+qt)\big(1+q+q^2t\big)\cdots\big(1+q+\cdots+q^{n-1}t\big).$$

(2) $$\sum_{w\in A_{n+1}} q^{\ell_A(w)}t^{\mathrm{del}_A(w)} = \sum_{w\in A_{n+1}} q^{\mathrm{rmaj}_{A_{n+1}}(w)}t^{\mathrm{del}_A(w)}$$
$$= (1+2qt)\bigl(1+q+2q^2t\bigr)\cdots\bigl(1+q+\cdots+q^{n-2}+2q^{n-1}t\bigr).$$

Note that Theorem 6.1 follows from our Main Theorem 9.1. However, the proof of Theorem 9.1 applies the machinery required for the proof of Theorem 6.1 combined with additional, more elaborate arguments—therefore we prove the latter here.

The proof of Theorem 6.1 follows from the lemmas below. Recall that the descent set—hence also the major indices maj_S and rmaj_{S_n}—are defined for any sequence of integers, not necessarily distinct. Here n denotes the number of letters in the sequence.

Lemma 6.2. *Let $x_1,\ldots,x_n$ and y be integers, not necessarily distinct, such that $x_i < y$ for $1\leqslant i\leqslant n$. Let u be the n-tuple $u=[x_1,\ldots,x_n]$, and let*

$$v_i=[x_1,\ldots,x_{i-1},y,x_i,\ldots,x_n],\qquad 1\leqslant i\leqslant n+1$$

(thus $v_1=[y,x_1,\ldots,x_n]$ and $v_{n+1}=[x_1,\ldots,x_n,y]$). Then

(1) $$\sum_{i=1}^{n+1} q^{\mathrm{maj}_S(v_i)} = q^{\mathrm{maj}_S(u)}\bigl(1+q+\cdots+q^n\bigr)\quad \textit{and} \tag{13}$$

$$\sum_{i=1}^{n} q^{\mathrm{maj}_S(v_i)} = q^{\mathrm{maj}_S(u)}\bigl(q+q^2+\cdots+q^n\bigr); \tag{14}$$

(2) $$\sum_{i=1}^{n+1} q^{\mathrm{rmaj}_{S_{n+1}}(v_i)} = q^{\mathrm{rmaj}_{S_n}(u)}\bigl(1+q+\cdots+q^n\bigr)\quad \textit{and} \tag{15}$$

$$\sum_{i=2}^{n+1} q^{\mathrm{rmaj}_{S_{n+1}}(v_i)} = q^{\mathrm{rmaj}_{S_n}(u)}\bigl(1+q+\cdots+q^{n-1}\bigr). \tag{16}$$

The proof of Lemma 6.2 is by a rather straight-forward induction, hence is omitted.

Lemma 6.3. *Recall that $R_n^S=\{1,s_n,\ldots,s_ns_{n-1}\cdots s_1\}\subseteq S_{n+1}$ and let $w\in S_n$ (so $w\in S_{n+1}$, where $w(n+1)=n+1$). Then*

$$\sum_{\tau\in R_n^S} q^{\mathrm{maj}_S(w\tau)} = q^{\mathrm{maj}_S(w)}\bigl(1+q+\cdots+q^n\bigr),$$

and

$$\sum_{\tau\in R_n^S} q^{\mathrm{rmaj}_{S_{n+1}}(w\tau)} = q^{\mathrm{rmaj}_{S_n}(w)}\bigl(1+q+\cdots+q^n\bigr).$$

Proof. Write $w \in S_n$ as $w = [w(1), \ldots, w(n)]$ ($= u$ in Lemma 6.2). Similarly write $w \in S_n \subseteq S_{n+1}$ as $w = [w(1), \ldots, w(n), n+1]$ ($= v_{n+1}$, in Lemma 6.2, where $y = n+1$). Thus

$$ws_n = \bigl[w(1), \ldots, n+1, w(n)\bigr] \quad (= v_n),$$

$$ws_n s_{n-1} = \bigl[w(1), \ldots, n+1, w(n-1), w(n)\bigr] \quad (= v_{n-1}),$$

etc., and the proof follows by the previous lemma. □

Remark 6.4. Let $\widetilde{R}_n^S = R_n^S - \{s_n s_{n-1} \cdots s_1\} \subseteq S_{n+1}$, and let $\sigma \in S_n$. It follows from Eq. (16) that

$$\sum_{\tau \in \widetilde{R}_n^S} q^{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau)} = q^{\mathrm{rmaj}_{S_n}(\sigma)}\bigl(1 + q + \cdots + q^{n-1}\bigr).$$

Lemma 6.5. *For every $\sigma \in S_n$,*

$$\sum_{\tau \in R_n^S} q^{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau)} t^{\mathrm{del}_S(\sigma\tau)} = q^{\mathrm{rmaj}_{S_n}(\sigma)} t^{\mathrm{del}_S(\sigma)}\bigl(1 + q + \cdots + q^{n-1} + tq^n\bigr).$$

Proof. By Lemma 6.3

$$\bigl\{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau) \mid \tau \in R_n^S\bigr\} = \bigl\{\mathrm{rmaj}_{S_n}(\sigma) + i \mid 0 \leqslant i \leqslant n\bigr\}.$$

Let $\eta = s_n s_{n-1} \cdots s_1$ and note that $\mathrm{rmaj}_{S_n}(\sigma) + n = \mathrm{rmaj}_{S_{n+1}}(\sigma\eta)$ (this is the statement "$\mathrm{rmaj}_{S_{n+1}}(v_1) = \mathrm{rmaj}_{S_n}(u) + n$" in the proof of Lemma 6.2).

Let $\tau \in R_n^S$. If $\tau \neq \eta$ then $\mathrm{del}_S(\sigma\tau) = \mathrm{del}_S(\sigma)$ since both σ and $\sigma\tau$ have the same number of occurrences of s_1. By a similar reason, $\mathrm{del}_S(\sigma\eta) = \mathrm{del}_S(\sigma) + 1$. Thus

$$\begin{aligned}
&\bigl\{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau)\mathrm{del}_S(\sigma\tau) \mid \tau \in R_n^S\bigr\} \\
&\quad = \bigl\{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau)\mathrm{del}_S(\sigma\tau) \mid \tau \in R_n^S,\ \tau \neq \eta\bigr\} \cup \bigl\{\mathrm{rmaj}_{S_{n+1}}(\sigma\eta)\mathrm{del}_S(\sigma\eta)\bigr\} \\
&\quad = \bigl\{\bigl(\mathrm{rmaj}_{S_n}(\sigma) + i\bigr)\mathrm{del}_S(\sigma) \mid 0 \leqslant i \leqslant n-1\bigr\} \cup \bigl\{\bigl(\mathrm{rmaj}_{S_n}(\sigma) + n\bigr)(\mathrm{del}_S(\sigma) + 1)\bigr\}
\end{aligned}$$

(disjoint unions with no repetitions in the sets) which translates to

$$\sum_{\tau \in R_n^S} q^{\mathrm{rmaj}_{S_{n+1}}(\sigma\tau)} t^{\mathrm{del}_S(\sigma\tau)} = q^{\mathrm{rmaj}_{S_n}(\sigma)} t^{\mathrm{del}_S(\sigma)}\bigl(1 + q + \cdots + q^{n-1} + tq^n\bigr). \qquad \square$$

Proposition 6.6. *For all n,*

$$\sum_{\sigma \in S_n} q^{\mathrm{rmaj}_{S_n}(\sigma)} t^{\mathrm{del}_S(\sigma)} = (1 + tq)\bigl(1 + q + tq^2\bigr) \cdots \bigl(1 + q + \cdots + q^{n-2} + tq^{n-1}\bigr).$$

Proof. Follows from Lemma 6.5 by induction on n, since

$$S_{n+1} = \bigcup_{\tau \in R_n^S} S_n \tau. \qquad \square$$

Proof of Theorem 6.1. Part (1) clearly follows by comparing part (1) of Proposition 5.7 with Proposition 6.6.

Part (2) follows from part (1) by Proposition 5.6. $\square$

7. Additional properties of the delent number

We show first that $\mathrm{del}_S(w)$ is the number of left-to-right minima of w.

Definition 7.1. Let $w \in S_n$. Call $2 \leqslant j \leqslant n$ l.t.r.min (left-to-right minima) of w if $w(i) > w(j)$ for all $1 \leqslant i < j$.

Define $\mathrm{Del}_S(w)$ as the set of l.t.r.min of w:

$$\mathrm{Del}_S(w) := \bigl\{2 \leqslant j \leqslant n \mid \forall i < j \;\; w(i) > w(j)\bigr\}.$$

For example, let $w = [3, 2, 7, 8, 4, 6, 1, 5]$, then $\{2, 7\}$ are the l.t.r.min.

Proposition 7.2. *Let $w \in S_n$, then $\mathrm{del}_S(w)$ equals the number of l.t.r.min of w^{-1}. Since by Lemma 3.6 $\mathrm{del}_S(w) = \mathrm{del}_S(w^{-1})$, this also equals the number of l.t.r.min of w. In particular,*

$$\bigl|\mathrm{Del}_S(w)\bigr| = \mathrm{del}_S(w) = \mathrm{del}_S\bigl(w^{-1}\bigr).$$

Proof. By induction on $n \geqslant 2$. First, $S_2 = \{1, s_1\}$ and $s_1 = [2, 1]$ has one l.t.r.min. Proceed now with the inductive step. Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of w, let $\sigma = w_1 \cdots w_{n-2}$ (so $\sigma \in S_{n-1} \subseteq S_n$) and assume that the assertion is true for σ. Write $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$. If $w_{n-1} = 1$, the proof is given by the induction hypothesis. Otherwise, $w_{n-1}^{-1} = s_k s_{k+1} \cdots s_{n-1}$ for some $1 \leqslant k \leqslant n-1$. Denoting $s_{[k,n-1]} = s_k s_{k+1} \cdots s_{n-1}$, we see that $w^{-1} = s_{[k,n-1]}\sigma^{-1}$. Comparing σ^{-1} with $w^{-1} = s_{[k,n-1]}\sigma^{-1}$, we see that

(1) the (position containing) n in σ^{-1} is replaced in w^{-1} by k;
(2) each j in σ^{-1}, $k \leqslant j \leqslant n-1$, is replaced by $j+1$ in w^{-1};
(3) each j, $1 \leqslant j \leqslant k-1$, is unchanged.

Thus $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$, $w^{-1} = [c_1, \ldots, c_{n-1}, k]$, and the tuples $(b_1, \ldots, b_{n-1})$ and $(c_1, \ldots, c_{n-1})$ are order-isomorphic. This implies that if $k > 1$ then σ^{-1} and w^{-1} have the same left-to-right minima. Let $k = 1$, then w^{-1} has $i = n$ as an additional left-to-right minima, and the proof is complete. $\square$

Remark 7.3. The above proof implies a bit more: Note that the above case $k = 1$ is equivalent to both $n \in \mathrm{Del}_S(w^{-1})$ and to $\epsilon_{S,n-1}(w) = 1$, where $\epsilon_{S,i}(w)$ are given by Definition 5.9. By induction on n, the above proof implies that $\mathrm{Del}_S(w^{-1}) = \{i + 1 \mid \epsilon_{S,i}(w) = 1\}$. Let now $D \subseteq [n - 1]$ and let $\pi \in S_n$. The condition $D = \mathrm{Del}_S(\pi^{-1})$ implies that $D = \{i + 1 \mid \epsilon_{S,i}(\pi) = 1\}$; this determines $\bar{\epsilon}_S(\pi)$ uniquely, and hence determines a unique value $t^{\epsilon_D} := t^{\bar{\epsilon}_S(\pi)}$: if $D \neq H$ then $t^{\epsilon_D} \neq t^{\epsilon_H}$. We shall apply this observation in the proof of Theorem 9.1.

The definition of l.t.r.min can be extended as follows.

Definition 7.4. Let $w = [b_1, \dots, b_n] \in S_n$. Then $3 \leqslant j \leqslant n$ is an a.l.t.r.min (almost-left-to-right-minima) if there is at most one b_i smaller than b_j and left of b_j: $\mathrm{card}\{1 \leqslant i \leqslant j \mid b_i < b_j\} \leqslant 1$.

For $w \in A_{n+1}$ define $\mathrm{Del}_A(w)$ to be the set of a.l.t.r.min of w.

Remark 7.5. (1) Without the restriction $3 \leqslant j$ in Definition 7.4, $j \in \{1, 2\}$ is an a.l.t.r.min.

(2) If $b_i = 1$ and $b_j = 2$ are interchanged in $w = [b_1, \dots, b_n]$, this does not change the set of a.l.t.r.min indices. Also, if b_1 and b_2 are interchanged this would not change the set of a.l.t.r.min indices. Thus, $s_1 w$ and $w s_1$ have the same a.l.t.r.min as w itself.

Proposition 7.6. *Let $w \in S_n$, then the number of occurrences of s_2 in* (*the canonical presentation of*) *w equals the number of a.l.t.r.min of w^{-1}. Lemma* 3.6 *implies that this is also the number of a.l.t.r.min of w.*

Proof. By induction on n. This is easily verified for $n = 2$, and we proceed with the inductive step.

Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of w, and denote $\sigma = w_1 \cdots w_{n-2}$, so that $w^{-1} = w_{n-1}^{-1} \sigma^{-1}$. If $w_{n-1} = 1$ we are done by induction. Otherwise, by the S-procedure, $w_{n-1} = s_{n-1} \cdots s_k x$ where $k \geqslant 2$ and $x \in \{1, s_1\}$.

Write $w^{-1} = x s_k \cdots s_{n-1} \sigma^{-1} = x s_{[k,n-1]} \sigma^{-1}$. By Remark 7.5, $s_{[k,n-1]} \sigma^{-1}$ and $x s_{[k,n-1]} \sigma^{-1}$ have the same number of a.l.t.r.min. Therefore it suffices to show:

1. If $k \geqslant 3$ then σ^{-1} has equal number of a.l.t.r.min as $s_{[k,n-1]} \sigma^{-1}$.
2. If $k = 2$, $s_{[2,n-1]} \sigma^{-1}$ has one more a.l.t.r.min than σ^{-1}.

Let $\sigma^{-1} = [b_1, \dots, b_{n-1}, n]$, then $s_{[k,n-1]} \sigma^{-1} = [c_1, \dots, c_{n-1}, k]$, and as in the proof of Proposition 7.2, $(b_1, \dots, b_{n-1})$ and $(c_1, \dots, c_{n-1})$ are order isomorphic. If $k \geqslant 3$, the last position (with k) is not an a.l.t.r.min, while if $k = 2$, it is an additional a.l.t.r.min, and this implies the proof. $\square$

By essentially the same argument, we have

Proposition 7.7. *Let $v \in A_{n+1}$, then* $\mathrm{del}_A(v)$ *equals the number of a.l.t.r.min of v^{-1}. In particular,* $|\mathrm{Del}_A(v)| = \mathrm{del}_A(v) = \mathrm{del}_A(v^{-1})$.

Proof. Again, by induction on n. This is easily verified for $n+1=3$, so proceed with the inductive step.

Let $v = v_1 \cdots v_{n-1}$ be the A-canonical presentation of v, and denote $\sigma = v_1 \cdots v_{n-2}$, so that $v^{-1} = v_{n-1}^{-1}\sigma^{-1}$. If $v_{n-1} = 1$ we are done by induction. Otherwise, by the A-procedure, $v_{n-1} = xs_n \cdots s_k y$ where $k \geqslant 2$ and $x, y \in \{1, s_1\}$; moreover, $k = 2$ if and only if either a_1 or a_1^{-1} occurs in v_{n-1}.

Write $v^{-1} = ys_k \cdots s_n x\sigma^{-1} = ys_{[k,n]}x\sigma^{-1}$ and proceed as in the proof of Proposition 7.6, applying Remark7.5(2). □

Remark 7.8. Given $w \in S_n$, one can define a.a.l.t.r.min, a.a.a.l.t.r.min, etc., then one can prove the corresponding propositions, which are analogues of Proposition 7.6. For example, we have

Definition 7.9. Let $w = [b_1, \ldots, b_n] \in S_n$. Then $1 \leqslant i \leqslant n$ is an a.a.l.t.r.min (almost-almost-left-to-right-minima) if $\mathrm{card}\{1 \leqslant j \leqslant i \mid b_j < b_i\} \leqslant 2$ and

(1) $i \neq 1, 2, 3$ (which is Definition 7.9.1 of a.a.l.t.r.min), or
(2) $b_i \neq 1, 2, 3$ (which is Definition 7.9.2 of a.a.l.t.r.min).

One can then prove that, with either definition of a.a.l.t.r.min, the number of a.a.l.t.r.min of $w \in S_n$ equals the number of occurrences of s_3 in w. Similarly for the occurrences of the other s_i's.

Similarly to Proposition 5.8, we define $w_S(n, \ell, k)$ to be the number of S-canonical words in S_n with ℓ occurrences of s_k (define $w_A(n+1, \ell, k)$ similarly), and we have

Proposition 7.10. *Let $k \leqslant n-1$, then*

$$\sum_{\ell=0}^{n-k} w_S(n, \ell, k)t^\ell = k!\,(kt+1)(kt+2)\cdots(kt+n-k),$$

hence $w_S(n, \ell, k) = k!\,k^\ell c(n-k+1, \ell+1)$, and similarly for $w_A(n+1, \ell, k)$.

Proof. It is omitted. □

8. Lemmas on shuffles

In this section we prove lemmas which will be used in the next section to prove the main theorem.

8.1. Equi-distribution on shuffles

The following result follows from Theorem 2.6.

Proposition 8.1. *Let $i \in [n-1]$, and let $\pi \in S_n$ with $\mathrm{supp}(\pi) \subseteq [i]$. Then*

$$\sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{i\}} q^{\mathrm{rmaj}_{S_n}(\pi r) - \mathrm{rmaj}_{S_i}(\pi)} = \sum_{\mathrm{Des}(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi r) - \ell_S(\pi)} = \begin{bmatrix} n \\ i \end{bmatrix}_q.$$

Proof. Let $\rho_n := (1,n)(2,n-1)\cdots \in S_n$ and $\rho_i := (1,i)(2,i-1)\cdots \in S_i$. By (4),

$$\sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{i\}} q^{\mathrm{rmaj}_{S_n}(\pi r) - \mathrm{rmaj}_{S_i}(\pi)} = \sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{i\}} q^{\mathrm{maj}_S(\rho_n \pi r \rho_n) - \mathrm{maj}_S(\rho_i \pi \rho_i)}$$
$$= \sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{i\}} q^{\mathrm{maj}_S(\rho_n \pi \rho_n \rho_n r \rho_n) - \mathrm{maj}_S(\rho_i \pi \rho_i)} = \sum_{\mathrm{Des}_S(\hat{r}^{-1}) \subseteq \{n-i\}} q^{\mathrm{maj}_S(\rho_n \pi \rho_n \hat{r}) - \mathrm{maj}_S(\rho_i \pi \rho_i)}.$$

The last equality follows from (5).

Note that $\mathrm{supp}(\rho_n \pi \rho_n) \subseteq [n-i+1, n]$ and verify that $\nu_{n-i}^{-1} \rho_n \pi \rho_n \nu_{n-i} = \rho_i \pi \rho_i$, where $\nu_{n-i} := (1, n-i+1)(2, n-i+2)\cdots$. Indeed, let $j \leqslant i$, then $\nu_{n-i}(j) = j+n-i$, hence $\rho_n \nu_{n-i}(j) = \rho_n(j+n-i) = n-(j+n-i)+1 = i-j+1 = \rho_i(j)$. Similarly, if $k \leqslant i$, also $\nu_{n-i}^{-1} \rho_n(k) = \rho_i(k)$. This implies the above equality. Now, obviously $\mathrm{supp}(1) \subseteq [n-i]$ and $\mathrm{maj}_S(1) = 0$. Thus by Garsia–Gessel's Theorem (Theorem 2.6) (taking $\pi_1 = 1$ and $\pi_2 = \rho_n \pi \rho_n$) the right-hand side is equal to

$$\sum_{\mathrm{Des}_S(\hat{r}^{-1}) \subseteq \{n-i\}} q^{\mathrm{maj}_S(1 \cdot \rho_n \pi \rho_n \cdot \hat{r}) - \mathrm{maj}_S(1) - \mathrm{maj}_S(\nu_{n-i}^{-1} \rho_n \pi \rho_n \nu_{n-i})} = \begin{bmatrix} n \\ i \end{bmatrix}_q.$$

The equality

$$\sum_{\mathrm{Des}(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi r) - \ell_S(\pi)} = \sum_{\mathrm{Des}(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi \cdot 1 \cdot r) - \ell_S(\pi) - \ell_S(1)} = \begin{bmatrix} n \\ i \end{bmatrix}_q$$

is an immediate consequence of Fact 2.5, combined with (3). □

Note 8.2. Let r be an $\{i\}$-shuffle and let $\mathrm{supp}(\pi) \subseteq [i]$ as above. If $r(1) \neq 1$, necessarily $r(1) = i+1$, hence also $\pi r(1) = i+1$. It follows that

$$\pi r(1) \in \{\pi(1), i+1\}.$$

The next lemma requires some preparations.

Fix $1 \leqslant i \leqslant n-1$ and define $g_i : S_n \to S_{n-1}$ as follows: Let $\sigma = [a_1, \ldots, a_n] \in S_n$, then $g_i(\sigma) = [a'_1, \ldots, a'_{n-1}]$ is defined as follows: delete $a_j = i+1$, leave $a'_k = a_k$ unchanged if $a_k \leqslant i$, and change $a'_t = a_t - 1$ if $a_t \geqslant i+2$. Denote $g_i(\sigma) = \sigma'$. For example, let $\sigma = [5,2,3,6,1,4]$ and $i = 2$, then $g_2(\sigma) = \sigma' = [4,2,5,1,3]$. Let $\mathrm{supp}(\pi) \subseteq [i]$, then $g_i(\pi) = \pi : \pi' = \pi$. Moreover, since π only permutes $1, \ldots, i$, the following basic property of g_i is rather obvious, since $\mathrm{supp}(\pi) \subseteq [i]$.

Fact 8.3. (1) *Let $\sigma \in S_n$, then $\pi(g_i\sigma) = g_i(\pi\sigma)$, namely, $(\pi\sigma)' = \pi'\sigma' = \pi\sigma'$.*

(2) *g_i is a bijection between the $\{i\}$-shuffles $r \in S_n$ satisfying $r(1) = i + 1$, and all the $\{i\}$-shuffles $r' \in S_{n-1}$:*

$$g_i : \{r \in S_n \mid \mathrm{Des}_S(r^{-1}) \subseteq \{i\},\ r(1) = i + 1\} \to \{r' \in S_{n-1} \mid \mathrm{Des}_S(r^{-1}) \subseteq \{i\}\}$$

is a bijection.

Lemma 8.4. *Let r be an $\{i\}$-shuffle, let $1 \leqslant i \leqslant n - 2$, $\mathrm{supp}(\pi) \subseteq [i]$ and assume $r(1) = i + 1$. Also let $g_i(\pi) = \pi'$ and $g_i(r) = r'$.*

(1) *If $r(2) = i + 2$ then $\mathrm{rmaj}_{S_n}(\pi r) = \mathrm{rmaj}_{S_{n-1}}(\pi' r')$.*
(2) *If $r(2) = 1$ then $\mathrm{rmaj}_{S_n}(\pi r) = n - 1 + \mathrm{rmaj}_{S_{n-1}}(\pi' r')$.*

Proof. By Note 8.2, $\pi r = [i + 1, a_2, \ldots, a_n]$; then, applying g_i, we have $\pi' r' = [a'_2, \ldots, a'_n]$, and it is easy to check that for all $2 \leqslant k \leqslant n - 1$, $a_k > a_{k+1}$ if and only if $a'_k > a'_{k+1}$. Thus, for $2 \leqslant k \leqslant n - 1$, $k \in \mathrm{Des}(\pi r)$ if and only if $k - 1 \in \mathrm{Des}(\pi' r')$; note also that such k contributes $n - k = (n - 1) - (k - 1)$ to both $\mathrm{rmaj}_{S_n}(\pi r)$ and to $\mathrm{rmaj}_{S_{n-1}}(\pi' r')$.

(1) If $r(2) = i + 2$ then $a_2 = \pi r(2) = i + 2$, hence $1 \notin \mathrm{Des}(\pi r)$, and the descents of πr occur only for (some) $2 \leqslant k \leqslant n - 1$, and the above argument implies the proof.

(2) If $r(2) = 1$ then $a_2 = \pi r(2) = \pi(1) < i + 1$, hence 1 is a descent of πr, contributing $n - 1$ to $\mathrm{rmaj}_{S_n}(\pi r)$, and again, the above argument completes the proof. $\square$

Lemma 8.5. *With the notations of Proposition* 8.1*,*

$$(1) \qquad \sum_{\substack{\mathrm{Des}_S(r^{-1}) \subseteq \{i\} \\ \pi r(1) = i+1}} q^{\mathrm{rmaj}_{S_n}(\pi r) - \mathrm{rmaj}_{S_i}(\pi)} = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q;$$

$$(2) \qquad \sum_{\substack{\mathrm{Des}_S(r^{-1}) \subseteq \{i\} \\ \pi r(1) = \pi(1)}} q^{\mathrm{rmaj}_{S_n}(\pi r) - \mathrm{rmaj}_{S_i}(\pi)} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q.$$

Proof. By induction on $n - i$. For $n - i = 1$, the $\{n - 1\}$-shuffles are $[1, \ldots, j - 1, n, j, \ldots, n - 1] = [1, \ldots, n]s_{n-1}s_{n-2} \cdots s_j$, $1 \leqslant j \leqslant n - 1$. Thus the summation in (2) is over $r \in R^S_{n-1} - \{s_{n-1}s_{n-2} \cdots s_1\}$ and Eq. (2) follows from Remark 6.4 (with $n - 1$ replacing n). Now,

$$\mathrm{sum}(1) + \mathrm{sum}(2) = \sum_{\mathrm{Des}_S(r^{-1}) \subseteq \{n-1\}} q^{\mathrm{rmaj}_{S_n}(\pi r) - \mathrm{rmaj}_{S_{n-1}}(\pi)}.$$

Hence, by Proposition 8.1,

$$\mathrm{sum}(1) + \mathrm{sum}(2) = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q,$$

so

$$\mathrm{sum}(1) = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q - \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q = q^{n-1},$$

which verifies (1) in that case.

Let now $n - i \geqslant 2$ and assume the lemma holds for $n - 1 - i$.

(1) Since $\mathrm{Des}(r^{-1}) \subseteq \{i\}$ and $r(1) = i + 1$, either $r(2) = i + 2$ (then $\pi r(2) = i + 2$), or $r(2) = 1$ (then $\pi r(2) = \pi(1)$). Thus, the sum in (1) equals $\mathrm{sum}[r(2) = i + 2] + \mathrm{sum}[r(2) = 1]$. Apply g_i to the permutations in these sums, and apply Lemma 8.4(1) and Fact 8.3; then, by induction on n,

$$\mathrm{sum}\big[r(2) = i + 2\big] = \sum_{\substack{\mathrm{Des}_S(r'^{-1}) \subseteq \{i\} \\ \pi' r'(1) = i+1}} q^{\mathrm{rmaj}_{S_{n-1}}(\pi' r') - \mathrm{rmaj}_{S_i}(\pi')} = q^i \begin{bmatrix} n-2 \\ i \end{bmatrix}_q.$$

Similarly, by Lemma 8.4(2) and Fact 8.3,

$$\mathrm{sum}\big[r(2) = 1\big] = \sum_{\substack{\mathrm{Des}_S(r'^{-1}) \subseteq \{i\} \\ \pi' r'(1) = \pi'(1)}} q^{n-1+\mathrm{rmaj}_{S_{n-1}}(\pi' r') - \mathrm{rmaj}_{S_i}(\pi')} = q^{n-1} \begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q.$$

Adding the last two sums, we conclude:

$$\begin{aligned} \sum_{\substack{\mathrm{Des}_S(r^{-1}) \subseteq \{i\} \\ \pi r(1) = i+1}} q^{\mathrm{rmaj}_{S_{n-1}}(\pi r) - \mathrm{rmaj}_{S_i}(\pi)} &= q^i \begin{bmatrix} n-2 \\ i \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q \\ &= q^i \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_q + q^{n-1-i} \begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q \right) = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \end{aligned}$$

(2) is an immediate consequence of Proposition 8.1 and part (1), since

$$\begin{bmatrix} n \\ i \end{bmatrix}_q - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q. \qquad \square$$

We have an analogous lemma for length.

Lemma 8.6. *With the notation of Proposition* 8.1,

$$(1) \qquad \sum_{\substack{\mathrm{Des}_S(r^{-1}) \subseteq \{i\} \\ \pi r(1) = i+1}} q^{\ell_S(\pi r) - \ell_S(\pi)} = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q;$$

$$(2) \qquad \sum_{\substack{\mathrm{Des}_S(r^{-1})\subseteq\{i\} \\ \pi r(1)=\pi(1)}} q^{\ell_S(\pi r)-\ell_S(\pi)} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q.$$

Proof. The case $n-i=0$ is obvious (the sum in (1) is empty while in (2), $r=1$), so assume $i \leqslant n-1$. Recall that in general, $\ell_S(\sigma)$ equals the number $\mathrm{inv}_S(\sigma)$ of inversions of σ.

We prove (1) first, so let $\pi r(1) = i+1$. As in Lemma 8.4, write

$$\pi r = [i+1, a_2, \ldots, a_n] \quad \text{and} \quad g_i(\pi r) = \pi' r' = \bigl[a_2', \ldots, a_n'\bigr],$$

and compare their inversions. Clearly, $i+1$ contributes i inversions to $\mathrm{inv}_S(\pi r)$. Also, as in the proof of Lemma 8.4, there is a bijection between the inversions among $\{a_2, \ldots, a_n\}$ and those among $\{a_2', \ldots, a_n'\}$. Thus $\mathrm{inv}_S(\pi r) = i + \mathrm{inv}_S(\pi' r')$. Also, since $\mathrm{supp}(\pi) \subseteq [i]$, $\mathrm{inv}_S(\pi) = \mathrm{inv}_S(\pi')$. Induction, Fact 8.3 and Proposition 8.1 imply the proof of (1). Now, by Proposition 8.1, (1) implies the proof of (2). □

8.2. Canonical presentation of shuffles

Observation 8.7. *Let $1 \leqslant i < n$. Every $\{i\}$-shuffle has a unique canonical presentation of the form $w_i w_{i+1} \cdots w_{n-1}$, where $\ell(w_j) \geqslant \ell(w_{j+1})$ for all $j \geqslant i$.*

Proof. Apply the S-procedure that follows Theorem 3.1. Note that after pulling $n, n-1, \ldots, i+1$ to the right, an $\{i\}$-shuffle is transformed into the identity permutation. □

Let $\bar\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1})$, then denote $t^{\bar\epsilon} = t_1^{\epsilon_1} \cdots t_{n-1}^{\epsilon_{n-1}}$.

Corollary 8.8. *Recall Definition* 5.9. *For an $\{i\}$-shuffle w,*

$$\mathrm{del}_S(w) = \begin{cases} 1, & \textit{if } w(1) = i+1, \\ 0, & \textit{otherwise}, \end{cases}$$

and therefore

$$t^{\bar\epsilon_S(w)} = t_i^{\mathrm{del}_S(w)} = \begin{cases} t_i, & \textit{if } w(1) = i+1, \\ 1, & \textit{otherwise}. \end{cases}$$

Proof. Write $w = w_i w_{i+1} \cdots w_{n-1}$ (the canonical presentation) with $\ell_S(w_i) \geqslant \cdots \geqslant \ell_S(w_{n-1})$, then $\epsilon_{S,j}(w) = 0$ for $j > i$. Thus $\mathrm{del}_S(w)$ is either 1 or 0, and is 1 exactly when $w_i = s_i \cdots s_1$, in which case $w(1) = i+1$. □

Remark 8.9. Let $r, \pi \in S_n$, r an $\{i\}$-shuffle and $\mathrm{supp}(\pi) \subseteq [i]$. Then the corresponding canonical presentations are: $\pi = w_1 \cdots w_i$, $r = w_{i+1} \cdots w_{n-1}$, hence also $\pi r = w_1 \cdots w_{n-1}$ is canonical presentation. In particular, $\bar\epsilon_S(\pi r) = \bar\epsilon_S(\pi) + \bar\epsilon_S(r)$.

We generalize: Let $B = \{i_1, i_2\}$ and let $w \in S_n$ be a B-shuffle. Then w shuffles the three subsets $\{1, \ldots, i_1\}$, $\{i_1+1, \ldots, i_2\}$, and $\{i_2+1, \ldots, n\}$. Clearly, w has a unique presentation as a product $w = \tau_1\tau_2$ where $\tau_2 \in S_n$ shuffles $\{1, \ldots, i_2\}$ with $\{i_2+1, \ldots, n\}$, and $\tau_1 \in S_{i_2}$ shuffles $\{1, \ldots, i_1\}$ with $\{i_1+1, \ldots, i_2\}$. By Observation 8.7, $\tau_1 = w_{i_1} w_{i_1+1} \cdots w_{i_2-1}$ and $\tau_2 = w_{i_2} w_{i_2+1} \cdots w_{n-1}$, where each $w_j \in R_j^S$. Thus

$$w = w_{i_1} \cdots w_{i_2-1} w_{i_2} \cdots w_{n-1}$$

is the S-canonical presentation of w,

$$\mathrm{del}_S(w) = \mathrm{del}_S(\tau_1) + \mathrm{del}_S(\tau_2) \quad \text{and} \quad t^{\bar{\epsilon}_S(w)} = t_{i_1}^{\mathrm{del}_S(\tau_1)} t_{i_2}^{\mathrm{del}_S(\tau_2)}.$$

This easily generalizes to an arbitrary $B = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n-1\}$, which proves the following proposition.

Proposition 8.10. *Let $B = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n-1\}$ and let $i_{k+1} := n$. Every B-shuffle $\pi \in S_n$ has a unique presentation*

$$\pi = \tau_1 \cdots \tau_k$$

where τ_j is an $\{i_j\}$-shuffle in $S_{i_{j+1}}$ (for $1 \leqslant j \leqslant k$). Moreover,

$$\mathrm{del}_S(\pi) = \sum_{j=1}^{k} \mathrm{del}_S(\tau_j) \quad \textit{and} \quad t^{\bar{\epsilon}_S(\pi)} = t_{i_1}^{\mathrm{del}_S(\tau_1)} \cdots t_{i_k}^{\mathrm{del}_S(\tau_k)}.$$

9. The main theorem

Recall the definitions of the A-descent set Des_A and the A-descent number des_A (Definition 1.5). Let $B \subseteq [n-1]$ and $\pi \in S_n$. Recall from Fact 2.4 that $\mathrm{Des}_S(\pi^{-1}) \subseteq B$ if and only if π is a B-shuffle.

The following is our main theorem, which we now prove.

Theorem 9.1. *For every subsets $D_1 \subseteq [n-1]$ and $D_2 \subseteq [n-1]$,*

$$(1) \quad \sum_{\left\{\pi \in S_n \,\middle|\, \substack{\mathrm{Des}_S(\pi^{-1}) \subseteq D_1 \\ \mathrm{Del}_S(\pi^{-1}) \subseteq D_2}\right\}} q^{\mathrm{rmaj}_{S_n}(\pi)} = \sum_{\left\{\pi \in S_n \,\middle|\, \substack{\mathrm{Des}_S(\pi^{-1}) \subseteq D_1 \\ \mathrm{Del}_S(\pi^{-1}) \subseteq D_2}\right\}} q^{\ell_S(\pi)} \quad \textit{and}$$

$$(2) \quad \sum_{\left\{\sigma \in A_{n+1} \,\middle|\, \substack{\mathrm{Des}_A(\sigma^{-1}) \subseteq D_1 \\ \mathrm{Del}_A(\sigma^{-1}) \subseteq D_2}\right\}} q^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} = \sum_{\left\{\sigma \in A_{n+1} \,\middle|\, \substack{\mathrm{Des}_A(\sigma^{-1}) \subseteq D_1 \\ \mathrm{Del}_A(\sigma^{-1}) \subseteq D_2}\right\}} q^{\ell_A(\sigma)}.$$

An immediate consequence of Theorem 9.1 is

Corollary 9.2.

$$(1)\quad \sum_{\pi\in S_n} q_1^{\mathrm{rmaj}_{S_n}(\pi)} q_2^{\mathrm{des}_S(\pi^{-1})} q_3^{\mathrm{del}_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{\ell_S(\pi)} q_2^{\mathrm{des}_S(\pi^{-1})} q_3^{\mathrm{del}_S(\pi^{-1})}.$$

$$(2)\quad \sum_{\sigma\in A_n} q_1^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} q_2^{\mathrm{des}_A(\sigma^{-1})} q_3^{\mathrm{del}_A(\sigma^{-1})} = \sum_{\sigma\in A_n} q_1^{\ell_A(\sigma)} q_2^{\mathrm{des}_A(\sigma^{-1})} q_3^{\mathrm{del}_A(\sigma^{-1})}.$$

9.1. A lemma

Lemma 9.3. *Let $i \in [n]$, and let σ be a permutation in S_n, such that* $\mathrm{supp}(\sigma) \subseteq [i]$. *Then*

$$(1)\quad \sum_{\mathrm{Des}(r^{-1})\subseteq\{i\}} q^{\ell_S(\sigma r)} t^{\bar\epsilon_S(\sigma r)} = q^{\ell_S(\sigma)} t^{\bar\epsilon_S(\sigma)} \cdot \left(\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right) \quad \textit{and}$$

$$(2)\quad \sum_{\mathrm{Des}(r^{-1})\subseteq\{i\}} q^{\mathrm{rmaj}_{S_n}(\sigma r)} t^{\bar\epsilon_S(\sigma r)} = q^{\mathrm{rmaj}_{S_i}(\sigma)} t^{\bar\epsilon_S(\sigma)} \cdot \left(\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right).$$

Proof. By Definition 5.9 and Remark 8.9,

$$t^{\bar\epsilon_S(\sigma r)} = t^{\bar\epsilon_S(\sigma)+\bar\epsilon_S(r)},$$

and by Corollary 8.8,

$$t^{\bar\epsilon_S(r)} = \begin{cases} t_i, & \text{if } r(1) = i+1, \\ 1, & \text{otherwise.} \end{cases}$$

Noting that $r(1) = i+1$ if and only if $\sigma r(1) = i+1$, and recalling that $\sigma r(1) \in \{\sigma(1), i+1\}$, we obtain

$$t^{\bar\epsilon_S(\sigma r)} = \begin{cases} t^{\bar\epsilon_S(\sigma)} t_i, & \text{if } \sigma r(1) = i+1, \\ t^{\bar\epsilon_S(\sigma)}, & \text{if } \sigma r(1) = \sigma(1). \end{cases}$$

Combining this with Lemmas 8.5 and 8.6 gives the desired result. For example, concerning length,

$$\begin{aligned} \sum_{\mathrm{Des}(r^{-1})\subseteq\{i\}} q^{\ell_S(\sigma r)} t^{\bar\epsilon_S(\sigma r)} &= \sum_{\substack{\mathrm{Des}(r^{-1})\subseteq\{i\} \\ \sigma r(1)=\sigma(1)}} q^{\ell_S(\sigma r)} t^{\bar\epsilon_S(\sigma r)} + \sum_{\substack{\mathrm{Des}(r^{-1})\subseteq\{i\} \\ \sigma r(1)=i+1}} q^{\ell_S(\sigma r)} t^{\bar\epsilon_S(\sigma r)} \\ &= q^{\ell_S(\sigma)} t^{\bar\epsilon_S(\sigma)} \cdot \left(\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right). \end{aligned}$$

This proves part (1). A similar argument proves (2). $\square$

9.2. Proof of the main theorem

Proof of Theorem 9.1(1). By the principle of inclusion and exclusion, we may replace $\mathrm{Del}_S(\pi^{-1}) \subseteq D_2$ by $\mathrm{Del}_S(\pi^{-1}) = D_2$ in both sides of Theorem 9.1(1). By Remark 7.3, $\{\pi \in S_n \mid \mathrm{Del}_S(\pi^{-1}) = D_2\}$ (i.e. the set D_2) determines the unique value $t^{\epsilon_{D_2}} := t^{\bar{\epsilon}_S(\pi)}$.

Hence, Theorem 9.1(1) is equivalent to the following statement:

For every subset $B \subseteq [n-1]$

$$\sum_{\{\pi \in S_n \mid \mathrm{Des}_S(\pi^{-1}) \subseteq B\}} q^{\mathrm{rmaj}_{S_n}(\pi)} t^{\bar{\epsilon}_S(\pi)} = \sum_{\{\pi \in S_n \mid \mathrm{Des}_S(\pi^{-1}) \subseteq B\}} q^{\ell_S(\pi)} t^{\bar{\epsilon}_S(\pi)}.$$

This statement is proved by induction on the cardinality of B. If $|B| = 1$ then $B = \{i\}$ for some $i \in [n-1]$, and Theorem 9.1(1) is given by Lemma 9.3 (with $\sigma = 1$). Assume that the theorem holds for every $B \subseteq [n-1]$ of cardinality less than k. Let $B = \{i_1, \ldots, i_k\} \subseteq [n-1]$ and denote $\bar{B} := \{i_1, \ldots, i_{k-1}\}$. By Proposition 8.10, for every $\pi \in S_n$ with $\mathrm{Des}_S(\pi^{-1}) \subseteq B$ there is a unique presentation

$$\pi = \bar{\pi}\tau_k,$$

where $\bar{\pi}$ is a $\bar{B}$-shuffle in S_{i_k} and τ_k is an $\{i_k\}$-shuffle in S_n. Moreover, $\mathrm{Des}_S(\pi^{-1}) \subseteq B$ if and only if π has such a presentation. Hence

$$\begin{aligned}
&\sum_{\{\pi \in S_n \mid \mathrm{Des}_S(\pi^{-1}) \subseteq B\}} q^{\mathrm{rmaj}_{S_n}(\pi)} t^{\bar{\epsilon}_S(\pi)} \\
&\quad = \sum_{\{\bar{\pi} \in S_{i_k},\, \tau_k \in S_n \mid \mathrm{Des}_S(\bar{\pi}^{-1}) \subseteq \bar{B},\, \mathrm{Des}_S(\tau_k^{-1}) \subseteq \{i_k\}\}} q^{\mathrm{rmaj}_{S_n}(\bar{\pi}\tau_k)} t^{\bar{\epsilon}_S(\bar{\pi}\tau_k)} \\
&\quad = \sum_{\{\bar{\pi} \in S_{i_k} \mid \mathrm{Des}_S(\bar{\pi}) \subseteq \bar{B}\}} \ \sum_{\{\tau_k \in S_n \mid \mathrm{Des}_S(\tau_k^{-1}) \subseteq \{i_k\}\}} q^{\mathrm{rmaj}_{S_n}(\bar{\pi}\tau_k)} t^{\bar{\epsilon}_S(\bar{\pi}\tau_k)}.
\end{aligned}$$

By Lemma 9.3(2), this equals

$$\sum_{\{\bar{\pi} \in S_{i_k} \mid \mathrm{Des}_S(\bar{\pi}^{-1}) \subseteq \bar{B}\}} q^{\mathrm{rmaj}_{S_{i_{k-1}}}(\bar{\pi})} t^{\bar{\epsilon}_S(\bar{\pi})} \cdot \left(\begin{bmatrix} n-1 \\ i_k - 1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right),$$

which, by induction, equals

$$\sum_{\{\bar{\pi} \in S_{i_k} \mid \mathrm{Des}_S(\bar{\pi}^{-1}) \subseteq \bar{B}\}} q^{\ell_S(\bar{\pi})} t^{\bar{\epsilon}_S(\bar{\pi})} \cdot \left(\begin{bmatrix} n-1 \\ i_k - 1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right).$$

Now by a similar argument, this time applying Lemma 9.3(1),

$$\sum_{\{\pi\in S_n \mid \mathrm{Des}_S(\pi^{-1})\subseteq B\}} q^{\ell_S(\pi)} t^{\bar{\epsilon}_S(\pi)}$$
$$= \sum_{\{\bar{\pi}\in S_{i_k} \mid \mathrm{Des}_S(\bar{\pi}^{-1})\subseteq \bar{B}\}} q^{\ell_S(\bar{\pi})} t^{\bar{\epsilon}_S(\bar{\pi})} \cdot \left(\begin{bmatrix} n-1 \\ i_k-1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right),$$

and the proof follows. □

Proof of Theorem 9.1(2). By the principle of inclusion and exclusion and Remark 7.3, Theorem 9.1(2) is equivalent to the following statement:

For every subset $B \subseteq [n-1]$,

$$\sum_{\{\sigma\in A_{n+1} \mid \mathrm{Des}_A(\sigma^{-1})\subseteq B\}} q^{\mathrm{rmaj}_{A_{n+1}}(\sigma)} t^{\bar{\epsilon}_A(\sigma)} = \sum_{\{\sigma\in A_{n+1} \mid \mathrm{Des}_A(\sigma^{-1})\subseteq B\}} q^{\ell_A(\sigma)} t^{\bar{\epsilon}_A(\sigma)}.$$

By Proposition 5.10, this part is reduced to Theorem 9.1(1). □

Appendix A

In this section we present another pair of statistics, leading to a different analogue of MacMahon's Theorem.

For $1 \leqslant i < n$, define a map $h_i : S_n \mapsto S_n$ as follows:

$$h_i(\pi) := \begin{cases} s_i\pi, & \text{if } i \in \mathrm{Des}_S(\pi^{-1}), \\ \pi, & \text{if } i \notin \mathrm{Des}_S(\pi^{-1}). \end{cases}$$

For every permutation $\pi \in S_n$, define

$$\hat{\ell}_i(\pi) := \ell_S(h_i(\pi)), \quad \text{and} \quad \widehat{\mathrm{maj}}_i(\pi) := \mathrm{maj}_S(h_i(\pi)).$$

Then $\hat{\ell}_i$ and $\widehat{\mathrm{maj}}_i$ are equi-distributed over the even permutations in S_n (i.e. over the alternating group A_n).

Theorem A.1. *Let $n \geqslant 2$, then*

$$\sum_{\pi\in A_n} q^{\hat{\ell}_i(\pi)} = \sum_{\pi\in A_n} q^{\widehat{\mathrm{maj}}_i(\pi)} = \prod_{i=3}^{n} (1 + q + \cdots + q^{i-1}).$$

Proof. By definition,

$$\text{Image}(h_i) = \{\pi \in S_n \mid i \notin \text{Des}_S(\pi^{-1})\} = \{\pi \in S_n \mid \pi^{-1} \text{ is an } ([n] \setminus \{i\})\text{-shuffle}\}.$$

Also, for each $\sigma \in \text{Image}(h_i)$, $h_i^{-1}(\sigma) = \{\sigma, s_i\sigma\}$, and exactly one element in the set $\{\sigma, s_i\sigma\}$ is even.

Thus, by Garsia–Gessel's Theorem (Theorem 2.6),

$$\sum_{\pi \in A_n} q^{\widehat{\text{maj}}_i(\pi)} = \sum_{\{\pi \in S_n \mid \pi^{-1} \text{is an } ([n]\setminus\{i\})\text{-shuffle}\}} q^{\text{maj}(\pi)} = \begin{bmatrix} n \\ 2, 1, \ldots, 1 \end{bmatrix}_q$$
$$= \prod_{i=3}^{n} (1 + q + \cdots + q^{i-1}),$$

and similarly for $\hat{\ell}_i$. $\square$

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