On a Free Boundary Problem for the \( p \)-Laplacian

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Submitted by Avner Friedman

Received February 12, 1997

1. INTRODUCTION

In this paper we consider the free boundary problem

\[
\text{div}(\frac{1}{|Du|^2} Du) = 0 \quad \text{in} \quad \Omega_0 \setminus \bar{\Omega}_1 = \Omega \subset \mathbb{R}^n, \quad 1 < p < \infty, \quad p \neq n \quad (1.1)
\]

\[
u \to 1, \quad |Du| \to c_1 \quad \text{uniformly a.e. as} \quad x \to \partial \Omega_1
\]

\[
u \to 0, \quad |Du| \to c_0 \quad \text{uniformly a.e. as} \quad x \to \partial \Omega_0. \quad (1.2)
\]

\( \Omega_1 \) and \( \Omega_0 \) are assumed to be bounded, connected, separate domains such that \( \Omega_0 \supset \Omega_1 \); they are also assumed to be starshaped with respect to the origin, which is taken inside \( \Omega_1 \). We consider a weak solution to (1.1), that is, a function \( u \in W^{1,p}(\Omega) \) such that

\[
\int_{\Omega} < |Du|^{-2} Du, D\phi > dx = 0 \quad \forall \phi \in C_0^0(\Omega) \quad (1.3)
\]

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and the boundary conditions are assumed to hold in the following way. Given \( \epsilon > 0 \), there are open sets \( U_{1, \epsilon} \) containing \( \partial \Omega_1 \), \( U_{0, \epsilon} \) containing \( \partial \Omega_0 \) such that
\[
\begin{align*}
|1 - u| < \epsilon, & \quad ||Du| - c_i| < \epsilon & \text{for a.e. } x \in U_{1, \epsilon} \cap \Omega \\
|u| < \epsilon, & \quad ||Du| - c_0| < \epsilon & \text{for a.e. } x \in U_{0, \epsilon} \cap \Omega
\end{align*}
\] (1.4)

with respect to the \( n \)-dimensional Lebesque measure \( L^n \).

We prove the following result:

**Theorem 1.1.** Problem (1.1), (1.2) has a weak solution in \( \Omega \) if and only if \( \Omega_1 \) and \( \Omega_0 \) are concentric spheres whose radii are given by
\[
R_i = \left( \frac{n - p}{p - 1} \right) \left[ c_1^{i-((p-n)/(1-n))} - c_0^{i-((p-n)/(1-n))} \right]^{-1} c_i^{i-((p-1)/(1-n))}.
\] (1.5)

The solution is then radially symmetric and given by
\[
u = \left( \frac{p - 1}{n - p} \right)^{((p-n)/(p-1))} \left[ c_1^{\nu-((p-n)/(n-1))} - c_0^{\nu-((p-n)/(n-1))} \right]^{((1-n)/(p-1))} \\
\times \left[ R_0^{\nu-((p-n)/(p-1))} - R_1^{\nu-((p-n)/(p-1))} \right]. \] (1.6)

The proof of Theorem 1.1 is based on the following maximum principle:

**Lemma 1.1.** If \( u \) is a weak solution to
\[
\begin{align*}
\text{div}(|Du|^p Du) = 0 & \quad \text{in } \Omega_0 \setminus \Omega_1 \\
u \to 1, & \quad u \to 0 \text{ uniformly a.e. as } x \to \partial \Omega_1, x \to \partial \Omega_0, \text{ respectively}
\end{align*}
\] (1.7)

then the following auxiliary function
\[
P(u, x) = \frac{|Du|^p}{(u + \alpha)^{p((n-1)/(n-p))}}, \] (1.8)

where \( \alpha \) is an arbitrary nonnegative constant, does not have maximum inside \( \Omega \), unless it is constant in \( \Omega \).

From Lemma 1.1 it follows:

**Lemma 1.2.** Let \( u \) be a weak solution to problem (1.1), (1.2). Then \( P(u, x) \) is constant in \( \Omega \).

An easy consequence of Lemma 1.2 is that \( \Omega_1 \) and \( \Omega_0 \) are spheres and that \( u \) is radially symmetric in \( \Omega \), so that \( \Omega_1 \) and \( \Omega_0 \) are concentric and (1.5), (1.6) follow.
In [12, 13] Payne and Philippin obtained the fundamental result that a maximum principle holds for particular combinations of \( u \) and \( |Du| \), where \( u \) is a harmonic function or more generally a classical solution to \( \text{div}(g(|Du|^2)Du) = 0 \), \( g \) being a nonnegative \( C^1 \) function of its argument, satisfying the condition \( G(s) = g(s) + 2g'(s) > 0. \) They used such \( P \)-functions to obtain various interesting results in potential theory (see, e.g., [14–16]). In particular the correspondent of our result for the laplacian was given in [11] and for the \( p \)-laplacian with \( p = n \) in [13]. In both cases the boundaries were assumed to be starshaped with respect to the origin and such that there was a normal vector defined at each point; the solutions were assumed to be \( C^2(\Omega) \).

After this paper was accepted, Professor G. Alessandrini kindly communicated to us that for the case of a smooth domain a different proof of Theorem 1.1, based on the method of moving planes, is contained in his paper. “A Symmetry Theorem for Condensers,” Math. Meth. Appl. Sci. 15 (1992), 315–320.

2. PROOFS OF THE RESULTS

2.1. Proof of Lemma 1.1

We prove the lemma by contradiction, following an argument similar to the ones in [3, 6].

We recall that a weak solution to (1.1) is locally \( C^{1,\alpha} \) (see, e.g., [4]). Moreover, due to the comparison principle and the Harnack inequality (see, e.g., [8]), for a solution to (1.7) it must be \( 0 < u < 1 \) in \( \Omega \). To see that \( u > 0 \) in \( \Omega \), suppose that there exists \( \bar{x} \in \Omega \) such that \( u(\bar{x}) < 0 \); from boundary conditions, given \( \epsilon > 0 \) such that \( u(\bar{x}) < -\epsilon \), there would be \( U_\epsilon \supset \partial \Omega \) such that \( u > -\epsilon \) on \( U_\epsilon \cap \Omega \). Then \( u \geq -\epsilon \) on \( \partial U_\epsilon \cap \Omega \) and by the comparison principle and the Harnack inequality it would be \( u > -\epsilon \) on \( \Omega \setminus U_\epsilon \), so in particular \( u(\bar{x}) > -\epsilon \). Then \( u(\bar{x}) \geq 0 \) and so \( u(x) \geq 0 \) for every \( x \in \Omega \); by the Harnack inequality we have in fact \( u(x) > 0 \) in \( \Omega \). In the same way we obtain \( u < 1 \) in \( \Omega \). On a solution to (1.7), \( P \) cannot take an interior maximum at a point where \( Du = 0 \). In fact, since \( P \geq 0 \), if \( Du = 0 \) we would have \( P = 0 \). Therefore if \( x_0 \in \Omega \) is such that

\[
\sup_{x \in \Omega} P(u, x) = P(u, x_0)
\]

then we must have \( Du(x_0) \neq 0 \). Hence the equation is nondegenerate in a neighborhood \( V \) of \( x_0 \) and so the solution is \( C^\infty(V) \) (see, e.g., [7, 10]). We can then compute classical second derivatives of \( u \) in \( x_0 \).

In the following we will use the summation convention on repeated indices and the abbreviation \( q = |Du|^2 \).
Equation (1.1) can be written in nondivergence form as
\[ a_{ij}(Du)u_{ij} = 0 \]  
(2.10)

with
\[ a_{ij}(Du) = (p - 2)|Du|^{p-4}u_ju_i + |Du|^{p-2}\delta_{ij}. \]  
(2.11)

We define
\[ d_{ij}(Du) = \frac{a_{ij}(Du)}{|Du|^{p-2}} = (p - 2)|Du|^{-2}u_ju_i + \delta_{ij} \]  
(2.12)

and we prove that, if in \( x_0 \in \Omega \) (2.9) holds, then we have
\[ (d_{ij}(Du)P_j)_j(x_0) \geq 0. \]  
(2.13)

By an orthogonal transformation we can assume
\[ Du(x_0) = (0, \ldots, 0, |Du(x_0)|). \]  
(2.14)

Computing from (1.8) we have for \( i = 1, \ldots, n \)
\[ P_{in} = \frac{pq^{(p-2)/2}u_{ji}u_{ji}}{(u + \alpha)^{p((n-1)/(n-p))}} - p\left(\frac{n-1}{n-p}\right)q^{p/2}u_{ji} \]  
\[ \quad \frac{(u + \alpha)^{p((n-1)/(n-p)) + 1}}{\alpha^{p((n-1)/(n-p)) + 1}} \]  
(2.15)

and therefore at \( x_0 \) from \( P_{in} = 0 \) we obtain
\[ u_{in} = \left(\frac{n-1}{n-p}\right)q \]  
\[ \frac{u_{ji}}{(u + \alpha)}. \]  
(2.16)

By computations from (2.12) and (2.15), by using (2.10), and the following identity, obtained differentiating (2.10) with respect to \( x_i \),
\[ (a_{ij}(Du)(u_{ji})_j = 0, \quad l = 1, \ldots, n \]  
(2.17)

we obtain
\[ (d_{ij}(Du)P_j)_j = p\frac{a_{ij}(Du)u_{ji}u_{ji}}{(u + \alpha)^{p((n-1)/(n-p))}} \]
\[ - 2p^{2}\left(\frac{n-1}{n-p}\right)\frac{a_{ij}(Du)u_{ji}u_{ji}}{(u + \alpha)^{p((n-1)/(n-p)) + 1}} \]
\[ + p\left(\frac{n-1}{n-p}\right)\left[p\left(\frac{n-1}{n-p} - 1\right)\right]q^{p/2}u_{ji}u_{ji}d_{ij}(Du) \]  
\[ \quad \frac{(u + \alpha)^{p((n-1)/(n-p)) + 2}}{(u + \alpha)^{p((n-1)/(n-p)) + 1}}, \]  
(2.18)
In \( x_0 \) we have

\[
a_{ij}(Du) = (p - 2)q^{(p-4)/2}u_{,i}u_{,j} + q^{(p-2)/2}\delta_{ij}
\]

\[
= \begin{pmatrix}
q^{(p-2)/2} & 0 & 0 & \ldots & 0 \\
0 & q^{(p-2)/2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & (p - 1)q^{(p-2)/2}
\end{pmatrix}.
\tag{2.19}
\]

Then

\[
a_{ij}u_{,i}u_{,j} = \sum_{l=1}^{n} \sum_{j=1}^{n} u_{,j}u_{,j}a_{ij}
\]

\[
= \sum_{l=1}^{n} \left[ \sum_{i=1}^{n-1} q^{(p-2)/2}u_{,j}^2 + (p - 1)q^{(p-2)/2}u_{,jn}^2 \right]
\]

\[
= q^{(p-2)/2} \sum_{i=1}^{n-1} \sum_{l=1}^{n} u_{,j}^2 + (p - 1)q^{(p-2)/2} \sum_{l=1}^{n} u_{,jn}^2.
\tag{2.20}
\]

Now the Schwarz inequality gives

\[
\left( \sum_{i=1}^{n-1} u_{,i} \right)^2 \leq (n - 1) \sum_{i=1}^{n-1} u_{,i}^2 \leq (n - 1) \sum_{i=1}^{n-1} \sum_{l=1}^{n} u_{,j}^2.
\tag{2.21}
\]

On the other hand (2.10) and (2.19) give at \( x_0 \)

\[
a_{ij}u_{,ij} = q^{(p-2)/2} \sum_{i=1}^{n-1} u_{,j} + (p - 1)q^{(p-2)/2}u_{,nn} = 0.
\tag{2.22}
\]

Hence from (2.16)

\[
\sum_{i=1}^{n-1} u_{,ij} = -(p - 1)u_{,nn} = -\frac{(p - 1)(n - 1)q}{(n-p)(u + \alpha)}.
\tag{2.23}
\]

Using (2.20), (2.21), and (2.23) we get

\[
a_{ij}u_{,ji}u_{,jj} \geq \frac{(n - 1)(p - 1)(p + n - 2)q^{(p+2)/2}}{(n-p)^2(u + \alpha)^2}.
\tag{2.24}
\]
Next, (2.16) and (2.19) give at $x_0$

$$a_{ij}u_{ij}u_{ij} = a_{nn}u_{nn} = \frac{(p - 1)(n - 1)q^{(p+2)/2}}{(n-p)(u + \alpha)}. \tag{2.25}$$

We also have at $x_0$

$$d_{ij}u_{ij} = d_{nn}q = (p - 1)q. \tag{2.26}$$

Finally we compute at $x_0$

$$d_{nj,i}u_{n} = \frac{a_{nj,i}u_{n}}{q^{(p-2)/2}} - (p - 2)\frac{d_{nn}u_{nn}}{q^{(p-2)/2}}. \tag{2.27}$$

By differentiating (2.11) with respect to $x_i$ we obtain at $x_0$

$$a_{nj,i}u_{n} = (p - 2)q^{(p-2)/2}u_{nn} + (p - 2)q^{(p-2)/2}u_{i} - \frac{(p - 2)(p - 1)(n - 1)q}{(n-p)(u + \alpha)}. \tag{2.28}$$

so that by (2.23)

$$d_{ij}u_{i} = d_{nj,i}u_{n} = (p - 2)\sum_{i=1}^{n-1}u_{ij} = - \frac{(p - 2)(p - 1)(n - 1)q}{(n-p)(u + \alpha)}. \tag{2.29}$$

We obtain (2.13) by inserting (2.24), (2.25), (2.26), and (2.29) into (2.18). \hfill \blacksquare

### 2.2. Proof of Lemma 1.2

From conditions (1.4) we have that there exist two open sets, $U_1 \supset \partial \Omega_1$, $U_2 \supset \partial \Omega_2$, such that on a solution to (1.1), (1.2), $Du \neq 0$ on $U_1 \cap \Omega$ and on $U_2 \cap \Omega$. This means that $u \in C^2(U_1 \cap \Omega)$, $u \in C^2(U_2 \cap \Omega)$ and by the results in [1, 2, 18] the boundary $\partial \Omega$ is $C^2$. So in a neighborhood of $\partial \Omega$, $u$ is $C^2$ and classically satisfies the equation (see, e.g., [7, 10]).

We recall the following Rellich integral identity, valid for a function $u \in C^2(\overline{\Omega})$ such that $Du \neq 0$,

$$\int_{\partial \Omega} |Du|^p \langle x, \eta \rangle dS = (n-p) \int_{\Omega} |Du|^p dx$$

$$- p\int_{\Omega} \langle x, Du \rangle \text{div}(|Du|^{p-2}Du) dx + p\int_{\partial \Omega} |Du|^{p-2} \langle Du, x \rangle \frac{\partial u}{\partial \eta} dS, \tag{2.30}$$

where $\eta$ is the normal vector defined at each point of $\partial \Omega$, pointing outside $\Omega$. 
In order to apply (2.30) to a solution to (1.1), (1.2) we proceed as in [3] and we consider for \( \varepsilon > 0 \) the regularized boundary problem in \( \Omega \):

\[
\begin{align*}
\text{div} \left[ (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} Du_\varepsilon \right] &= 0 \\
u_\varepsilon |_{\partial \Omega_1} &= 1, \quad u_\varepsilon |_{\partial \Omega_0} = 0.
\end{align*}
\tag{2.31}
\]

By the results in [9], \( u_\varepsilon \in C^{1, \alpha}(\overline{\Omega}) \) for some \( \alpha > 0 \) and \( u_\varepsilon \to u, Du_\varepsilon \to Du \) uniformly on \( \overline{\Omega} \) as \( \varepsilon \to 0 \). The operator in (2.31) is uniformly elliptic in \( \Omega \) and so, due to the \( C^2 \) regularity of \( \partial \Omega \), for every \( \varepsilon \to 0 \) problem (2.31) admits a unique solution \( u_\varepsilon \in C^2(\overline{\Omega}) \). By the Rellich identity we have

\[
\begin{align*}
\int_{\partial \Omega} \left[ (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} - \varepsilon^{\frac{p}{2}} \right] \langle x, \eta \rangle \, dS \\
= p \int_{\partial \Omega} (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \langle Du_\varepsilon, x \rangle \frac{\partial u_\varepsilon}{\partial \eta} \, dS \\
+ n \int_{\Omega} \left[ (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} - \varepsilon^{\frac{p}{2}} \right] \, dx \\
- p \int_{\Omega} (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |Du_\varepsilon|^2 \, dx. \tag{2.32}
\end{align*}
\]

Passing to the limit as \( \varepsilon \to 0 \) in (2.32) we obtain

\[
(1 - p) \int_{\partial \Omega} |Du|^p \langle x, \eta \rangle \, dS = (n - p) \int_{\Omega} |Du|^p \, dx. \tag{2.33}
\]

For a solution to (2.31) we have

\[
\int_{\Omega} (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |Du_\varepsilon|^2 \, dx = \int_{\partial \Omega} u (|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \langle Du_\varepsilon, \eta \rangle \, dS \tag{2.34}
\]

so that, passing to the limit as \( \varepsilon \to 0 \) we get

\[
\int_{\Omega} |Du|^p \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \eta} |Du|^{p-2} \, dS. \tag{2.35}
\]

Then for a solution to (1.1), (1.2) we have

\[
(p - 1)c_\Omega^p |\Omega_1| - (p - 1)c_\Omega^p n|\Omega_0| = (n - p)c_\Omega^{p-1} |\partial \Omega_1|. \tag{2.36}
\]
Equation (1.1) on the boundary of $\Omega$ can be written as

$$0 = \text{div}(|Du|^{p-2}Du) = -|Du|^{p-1}(n-1)H(x)$$

$$+ (p-1)|Du|^{p-2}\langle Du, \text{Hess}(Du) \rangle, \quad (2.37)$$

where $H(x)$ is the average curvature of the boundary at $x_0$. So

$$H_1(x) = \frac{(p-1)}{(n-1)c_1} \langle Du, \text{Hess}(Du) \rangle |_{\partial \Omega_1}, \quad (2.38)$$

$$H_0(x) = \frac{(p-1)}{(n-1)c_0} \langle Du, \text{Hess}(Du) \rangle |_{\partial \Omega_0}. \quad (2.39)$$

Now we select constant $\alpha$ in (1.8) such that $P |_{\partial \Omega_0} = P |_{\partial \Omega_1}$; it must be

$$\frac{c_0}{c_1} = \left(1 + \frac{1}{\alpha}\right)^{(1-n)/(n-p)}. \quad (2.40)$$

With this choice of $\alpha$ it follows from Lemma 1.1 and Hopf’s maximum principle (see, e.g., [17]) that

$$\frac{\partial P}{\partial \eta} \geq 0 \text{ on } \partial \Omega_0, \quad \frac{\partial P}{\partial \eta} \geq 0 \text{ on } \partial \Omega_1. \quad (2.41)$$

So

$$\frac{\partial P}{\partial \eta} |_{\partial \Omega_1} = \left\langle \frac{DP}{|Du|}, Du \right\rangle |_{\partial \Omega_1}$$

$$= \frac{c_1^{p-2}p \langle \text{Hess}(Du), Du \rangle |_{\partial \Omega_1}}{(1 + \alpha)^{p(n-1)/(n-p)}}$$

$$= \frac{p(n-1)c_1^{p-1}}{(n-p)(1 + \alpha)^{p(n-1)/(n-p)+1} \geq 0} \quad (2.42)$$

which combined with (2.38) gives

$$H_1(x) \geq \frac{(p-1)c_1}{(n-p)(1 + \alpha)}. \quad (2.43)$$

In the same way we obtain

$$H_0(x) \leq \frac{(p-1)c_0}{(n-p)\alpha}. \quad (2.44)$$
We can now multiplicate (2.43) and (2.44) by \( x \), \( h \) and integrate over \( V_1 \), \( V_0 \), as \( V_1 \) and \( V_0 \) are starshaped:

\[
|\partial \Omega_1| = -\int_{\partial \Omega_1} H_1(x) \langle x, \eta \rangle \, dS \geq \frac{(p - 1)c_1n|\Omega_1|}{(n - p)(1 + \alpha)} \tag{2.45}
\]

\[
|\partial \Omega_0| = \int_{\partial \Omega_0} H_0(x) \langle x, \eta \rangle \, dS \leq \frac{(p - 1)c_0n|\Omega_0|}{(n - p)\alpha}. \tag{2.46}
\]

Using (2.37), (2.45), and (2.46) we get

\[
|\partial \Omega_0|c_0 \leq \frac{(p - 1)c_0^2n|\Omega_0|}{(n - p)\alpha} = \frac{(p - 1)c_0^2n|\Omega_2| - (n - p)c_0^{p-1}|\partial \Omega_1|}{(n - p)\alpha c_0^{p-2}}
\leq \frac{(p - 1)c_0^2n|\Omega_2|}{(n - p)\alpha c_0^{p-2}} \cdot \frac{c_0^{p-1}|\partial \Omega_1|}{c_0^{p-2}}. \tag{2.47}
\]

By the divergence theorem applied to Eq. (1.1), working on the approximations and passing to the limit as in (2.32), (2.34) we see that actually we must have equality in (2.47). Then we must have equality in all the relations before (2.47) and so we obtain that \( P \) is constant in \( \Omega \).\]

From the proof of Lemma 1.2 we obtain that the boundaries of \( \Omega_1 \) and \( \Omega_0 \) are spheres, since their average curvature is constant.

We easily see that also level surfaces of \( u \) in \( \Omega \) are spheres: in fact, as it was proven in [5], since \( \Omega_1 \) and \( \Omega_0 \) are starshaped, so are the level surfaces; on the other hand, since \( P \) is constant in \( \Omega \), \( |Du| \) is constant on level surfaces; so we can use the Rellich identity in the set between any level surface and the boundary of \( \Omega_1 \) or \( \Omega_0 \) and obtain the result by proceeding as in the proof of Lemma 1.2.

So the solution to problem (1.1), (1.2) is such that all its level sets are spheres and such that \( |Du| \) is constant on level sets; then it must be radially symmetric. So \( \Omega_1 \) and \( \Omega_0 \) are concentric and (1.5), (1.6) come from the boundary conditions.

**ACKNOWLEDGMENT**

Part of this work was done while visiting the Department of Mathematics at Purdue University. The author gratefully acknowledges its hospitality.
REFERENCES