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Difference Equations for Generalized Meixner Polynomials

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In this paper is introduced a system of polynomials orthogonal with respect to the classical discrete weight function for Meixner polynomials with an extra point mass added at $x=0$. A difference operator of infinite order is constructed for which these new polynomials are eigenfunctions and a second-order difference equation is given with polynomial coefficients, n-dependent and of at most degree 2, which these polynomials satisfy.

1. Introduction

For the polynomials $\{L_n^{\alpha,N}(x)\}_{n=0}^{\infty}$ that are orthogonal on $[0, \infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x} + N\,\delta(x), \qquad \alpha > -1,\ N \geqslant 0$$

(see Koornwinder [7]), Koekoek and Koekoek [6] found a differential equation of the form

$$N \sum_{i=0}^{\infty} a_i(x)\, y^{(i)}(x) + xy''(x) + (\alpha+1-x)\, y'(x) + ny(x) = 0,$$

where the coefficients $a_i(x)$, $i \in \{1, 2, 3, ...\}$, are independent of n and $a_0(x) = a_0(n, \alpha)$ depends on n but is independent of x. For a more constructive approach to this differential equation see [2].

At a conference held in Erice (May 1990), Askey [1] posed the problem of finding difference equations of a similar form for generalizations of the discrete orthogonal polynomials that are orthogonal with respect to a

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classical weight function at which a point mass at the point $x=0$ is added. In [3] a solution to this problem for Charlier polynomials is given and in the present paper we deal with the more complicated case of Meixner polynomials. Moreover, we construct a second-order difference equation with polynomial coefficients, n-dependent and of degree at most 2, which the generalized Meixner polynomials satisfy.

2. Meixner Polynomials

Taking a normalization slightly different from the one used in [4], we define the classical Meixner polynomials $M_n(x;\beta,c)$ by the generating function

$$\sum_{n=0}^{\infty} M_n(x;\beta,c)\,t^n=\left(1-\frac{t}{c}\right)^x (1-t)^{-x-\beta}, \tag{2.1}$$

from which it easily follows that

$$M_n(x;\beta,c)=(-1)^n\sum_{k=0}^{n}\binom{x}{k}\binom{-x-\beta}{n-k}c^{-k} \tag{2.2}$$

$$=\frac{(\beta)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, & -x\\ & \beta\end{matrix}\,\middle|\,1-\frac{1}{c}\right),\qquad n=0,1,2,\ldots. \tag{2.3}$$

Note that (2.2) can be used for all values of β and c except $c=0$. Formula (2.3) is not defined for $\beta=0,-1,-2,\ldots$, and for $c=0$. Obviously

$$M_n(0;\beta,c)=\frac{(\beta)_n}{n!},\qquad n=0,1,2,\ldots. \tag{2.4}$$

Meixner polynomials are closely related to Jacobi polynomials. In fact, we have

$$M_n(x;\beta,c)=(-c)^{-n}P_n^{(x-n,\beta-1)}(1-2c),\qquad n=0,1,2,\ldots. \tag{2.5}$$

For $\beta>0$ and $0<c<1$ the Meixner polynomials satisfy the orthogonality relation

$$(1-c)^{\beta}\sum_{x=0}^{\infty} M_n(x;\beta,c)\,M_p(x;\beta,c)\frac{c^x(\beta)_x}{x!}$$
$$=\frac{c^{-n}}{n!}(\beta)_n\,\delta_{np},\qquad n,p=0,1,2,\ldots \tag{2.6}$$

and the second-order difference equation

$$x\,\Delta\nabla y(x)+[\beta c-x(1-c)]\,\Delta y(x)+n(1-c)\,y(x)=0, \tag{2.7}$$

where $\Delta y(x)=y(x+1)-y(x)$ and $\nabla y(x)=y(x)-y(x-1)$.

Direct consequences of (2.1) are the following formulae which are valid for all real x, β, c (except $c=0$), and v, and for all $n\in\{0, 1, 2, ...\}$:

$$M_n(x+v;\beta-v,c)=\sum_{k=0}^{n}\binom{v}{k}\left(-\frac{1}{c}\right)^k M_{n-k}(x;\beta,c), \tag{2.8}$$

$$M_n(x;\beta-v,c)=\sum_{k=0}^{n}\binom{v}{k}(-1)^k M_{n-k}(x;\beta,c), \tag{2.9}$$

$$\Delta M_n(x;\beta,c)=\left(\frac{c-1}{c}\right)M_{n-1}(x;\beta+1,c). \tag{2.10}$$

In the sequel we always take $\beta>0$, $0<c<1$, and since c is kept fixed during the whole paper we simplify the notation putting $M_n(x;\beta)$ instead of $M_n(x;\beta,c)$.

3. Generalized Meixner Polynomials

Let P denote the space of all polynomials with real coefficients. We consider the inner product

$$\langle f(x), g(x)\rangle=(1-c)^\beta\sum_{x=0}^{\infty}\frac{c^x(\beta)_x}{x!}f(x)\,g(x)$$
$$+Nf(0)\,g(0),\qquad N\geqslant 0, f, g\in P. \tag{3.1}$$

We show that coefficients A_n and B_n can be chosen in such a way that the polynomials $M_n^N(x;\beta)=M_n^N(x;\beta,c)$ that are orthogonal with respect to the inner product (3.1) can be written as

$$M_n^N(x;\beta)=A_nM_n(x;\beta)+B_nM_{n-1}(x-1;\beta+1).$$

Suppose that $n\geqslant 2$ and $p(x)=xq(x)$ with degree $[q(x)]\leqslant n-2$. Then we obtain using (2.6)

$$\langle p(x), M_n^N(x;\beta)\rangle$$
$$=B_n(1-c)^\beta\sum_{x=0}^{\infty}\frac{c^x(\beta)_x}{x!}xq(x)\,M_{n-1}(x-1;\beta+1)$$
$$=B_n(1-c)^\beta c\beta\sum_{x=1}^{\infty}\frac{c^{x-1}(\beta+1)_{x-1}}{(x-1)!}q(x)\,M_{n-1}(x-1;\beta+1)=0.$$

Hence for $n \geqslant 1$ the coefficients A_n and B_n have to fulfill only the following condition:

$$\begin{aligned} 0 &= \langle 1, M_n^N(x;\beta)\rangle \\ &= B_n(1-c)^\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} M_{n-1}(x-1;\beta+1) \\ &\quad + NA_n M_n(0;\beta) + NB_n M_{n-1}(-1;\beta+1). \end{aligned}$$

By using (2.8) with $v=-1$ and (2.6) it follows that

$$(1-c)^{\beta} \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} M_{n-1}(x-1;\beta+1) = c^{-n+1}.$$

So a possible choice for A_n and B_n is $A_n = Nc^{n-1}M_{n-1}(-1;\beta+1)+1$ and $B_n = -Nc^{n-1}M_n(0;\beta)$, and we put

$$\begin{aligned} M_n^N(x;\beta) &= [Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\, M_n(x;\beta) \\ &\quad - Nc^{n-1}M_n(0;\beta)\, M_{n-1}(x-1;\beta+1), \qquad n=0,1,2,\ldots. \end{aligned} \tag{3.2}$$

Here and in the sequel we use $M_{-k}(x;\beta,c)=0$, $k=1,2,\ldots$. Note that $M_n^0(x;\beta)=M_n(x;\beta)$ and that $M_n^N(0;\beta)=M_n(0;\beta)$.

4. The Difference Equation

We are looking for a difference equation of the form

$$\begin{aligned} N \sum_{i=0}^{\infty} a_i(x)\, \Delta^i y(x) + x\, \Delta\nabla y(x) + [\beta c - x(1-c)] \\ \times \Delta y(x) + n(1-c)\, y(x) = 0 \end{aligned} \tag{4.1}$$

for the polynomials $\{M_n^N(x;\beta)\}_{n=0}^{\infty}$ given by (3.2), where the coefficients $\{a_i(x)\}_{i=1}^{\infty} = \{a_i(x,\beta,c)\}_{i=1}^{\infty}$ are functions of x, β, and c, but are independent of the degree n. Moreover, since we want the polynomials $\{M_n^N(x;\beta)\}_{n=0}^{\infty}$ to be eigenfunctions of a difference operator, we assume $a_0(x)=a_0(n,\beta,c)$ to be independent of x but dependent on n. So we insert

$$\begin{aligned} y(x) &= M_n^N(x;\beta) \\ &= [Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\, M_n(x;\beta) \\ &\quad - Nc^{n-1}M_n(0;\beta)\, M_{n-1}(x-1;\beta+1) \end{aligned}$$

into (4.1). Using the difference equation (2.7) for the classical Meixner polynomials, (2.10), and (2.8) in the case $v=1$, we find

$$N[Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_n(x;\beta)$$
$$-N^2c^{n-1}M_n(0;\beta)\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_{n-1}(x-1;\beta+1)$$
$$-Nc^{n-1}M_n(0;\beta)(1-c)\,M_{n-1}(x-2;\beta+2)=0.$$

This relation has to hold for all values of n, β, c, and $N>0$. Its left-hand side is a polynomial in N, so each coefficient must be zero. Thus we find

$$M_{n-1}(-1;\beta+1)\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_n(x;\beta)$$
$$-M_n(0;\beta)\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_{n-1}(x-1;\beta+1)=0 \tag{4.2}$$

and

$$\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_n(x;\beta)-c^{n-1}(1-c)\,M_n(0;\beta)\,M_{n-1}(x-2;\beta+2)=0.$$

This leads to the following systems of equations:

$$\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_n(x;\beta)=c^{n-1}(1-c)\,M_n(0;\beta)\,M_{n-1}(x-2;\beta+2), \tag{4.3}$$

$$\sum_{i=0}^{\infty}a_i(x)\,\Delta^iM_{n-1}(x-1;\beta+1)=c^{n-1}(1-c)$$
$$\times M_{n-1}(-1;\beta+1)\,M_{n-1}(x-2;\beta+2). \tag{4.4}$$

Formula (4.2) can be rewritten as

$$\sum_{i=1}^{\infty}a_i(x)\,[M_{n-1}(-1;\beta+1)\,\Delta^iM_n(x;\beta)$$
$$-M_n(0;\beta)\,\Delta^iM_{n-1}(x-1;\beta+1)]$$
$$=a_0(n,\beta,c)\,[M_{n-1}(x-1;\beta+1)\,M_n(0;\beta)$$
$$-M_n(x;\beta)\,M_{n-1}(-1;\beta+1)].$$

The right-hand side is 0 for $x=0$ and since this holds for all values of n, β, and c, we conclude step by step that $a_i(0)=a_i(0,\beta,c)=0$ for all $i\in\{1,2,...\}$. Therefore, setting $x=0$ in formula (4.3), we obtain

$$a_0(x)=a_0(0)=a_0(n,\beta,c)=c^{n-1}(1-c)\,M_{n-1}(-2;\beta+2), \qquad (4.5)$$

and Eqs. (4.3) and (4.4) can be rewritten as

$$\sum_{i=1}^{\infty} a_i(x)\,\Delta^i M_n(x;\beta)=c^{n-1}(1-c)\,[M_n(0;\beta)\,M_{n-1}(x-2;\beta+2) - M_n(x;\beta)\,M_{n-1}(-2;\beta+2)], \qquad (4.6)$$

and

$$\sum_{i=1}^{\infty} a_i(x)\,\Delta^i M_{n-1}(x-1;\beta+1) = c^{n-1}(1-c)\,[M_{n-1}(-1;\beta+1)\,M_{n-1}(x-2;\beta+2) - M_{n-1}(x-1;\beta+1)\,M_{n-1}(-2;\beta+2)]. \qquad (4.7)$$

We now show that any solution of the system (4.7) also satisfies (4.6). Since by (2.8) with $v=1$ we have

$$\sum_{i=1}^{\infty} a_i(x)\,\Delta^i M_n(x;\beta)=\sum_{i=1}^{\infty} a_i(x)\,\Delta^i M_n(x-1;\beta+1) - \frac{1}{c}\sum_{i=1}^{\infty} a_i(x)\,\Delta^i M_{n-1}(x-1;\beta+1),$$

it remains to be proved that for $n\geqslant 1$,

$$\begin{aligned}&c^{n-1}(1-c)\,[M_n(0;\beta)\,M_{n-1}(x-2;\beta+2)-M_n(x;\beta)\,M_{n-1}(-2;\beta+2)]\\ &\quad=c^n(1-c)\,[M_n(-1;\beta+1)\,M_n(x-2;\beta+2)-M_n(x-1;\beta+1)\\ &\qquad\times M_n(-2;\beta+2)]-c^{n-2}(1-c)\,[M_{n-1}(-1;\beta+1)\\ &\qquad\times M_{n-1}(x-2;\beta+2)-M_{n-1}(x-1;\beta+1)\,M_{n-1}(-2;\beta+2)].\end{aligned}$$

This can be shown by combining terms and applying the formula (2.8) with $v=1$ for different values of x and β.

5. A Formula for the Coefficients $a_i(x)$

We now solve the system (4.7). Writing n instead of $n-1$ and using (2.10), we get

$$\sum_{i=1}^{\infty} a_i(x)\left(\frac{c-1}{c}\right)^i M_{n-i}(x-1;\beta+i+1)$$
$$=c^n(1-c)\,[M_n(-1;\beta+1)\,M_n(x-2;\beta+2)$$
$$-M_n(x-1;\beta+1)\,M_n(-2;\beta+2)],\ n=1,2,\ldots. \tag{5.1}$$

If we consider $a_i(x)(1-1/c)^i$ as unknown, the matrix T of the system (5.1) is triangular with entries t_{ij} for which we have

$$t_{ij}=M_{i-j}(x-1;\beta+j+1),\qquad \text{for}\quad i,j=1,2,\ldots.$$

We show that the entries u_{ij} of the inverse matrix are

$$u_{ij}=M_{i-j}(-x+1;-\beta-i),\qquad \text{for}\quad i,j=1,2,\ldots. \tag{5.2}$$

In order to prove (5.2) we use the generating function (2.1) to find that

$$\sum_{n=0}^{\infty} M_n(-x+1;-\beta-i)\,t^n \sum_{n=0}^{\infty} M_n(x-1;\beta+j+1)\,t^n=(1-t)^{i-j-1}.$$

Equating the coefficients of t^{i-j} $(i\geqslant j)$ on both sides we obtain

$$\sum_{k=j}^{i} M_{i-k}(-x+1;-\beta-i)\,M_{k-j}(x-1;\beta+j+1)=\delta_{ij}.$$

We conclude that the unique solution of the system (5.1) reads

$$a_i(x)=\left(\frac{c-1}{c}\right)^{-i}\sum_{k=1}^{i} M_{i-k}(-x+1;-\beta-i)$$
$$\times c^k(1-c)\,[M_k(-1;\beta+1)\,M_k(x-2;\beta+2)$$
$$-M_k(x-1;\beta+1)\,M_k(-2;\beta+2)],\qquad i=1,2,\ldots. \tag{5.3}$$

The difference equation (4.1), with $a_0(x)$ given by (4.5) and $a_i(x)$ by (5.3) for $i=1,2,\ldots$, is of infinite order for all values of β and c $(0<c<1)$ and $N>0$. We prove this by evaluating the coefficient $k_i=k_i(\beta,c)$ of x^i in $a_i(x)$, $i\geqslant 1$. From (2.3) we derive that

$$M_n(x;\beta,c)=\frac{1}{n!}\left(\frac{c-1}{c}\right)^n x^n+\text{terms with lower powers of } x. \tag{5.4}$$

Furthermore, in the case $v=n$, formula (2.8) can be rewritten as

$$(-c)^n M_n(x+n;\beta-n,c)=\sum_{k=0}^{n}\binom{n}{k}(-c)^k M_k(x;\beta,c). \tag{5.5}$$

Hence we can write, using (5.3), (5.4), (5.5), and (2.8) with $v=1$,

$$k_i=\left(\frac{c}{c-1}\right)^i \sum_{k=1}^{i} \frac{1}{(i-k)!}\left(-\frac{c-1}{c}\right)^{i-k} c^k(1-c)$$
$$\times\frac{1}{k!}\left(\frac{c-1}{c}\right)^k [M_k(-1;\beta+1)-M_k(-2;\beta+2)]$$
$$=\frac{(-1)^i}{i!}(1-c)\sum_{k=0}^{i}\binom{i}{k}(-c)^k [M_k(-1;\beta+1)-M_k(-2;\beta+2)]$$
$$=\frac{c^i(1-c)}{i!}[M_i(i-1;\beta-i+1)-M_i(i-2;\beta-i+2)]$$
$$=\frac{c^{i-1}(c-1)}{i!}M_{i-1}(i-2;\beta-i+2). \tag{5.6}$$

Hence using (2.5) we obtain

$$k_i=(-1)^i\frac{1-c}{i!}P_{i-1}^{(-1,\beta-i+1)}(1-2c)$$
$$=\frac{c-1}{i!}P_{i-1}^{(\beta-i+1,-1)}(2c-1).$$

In particular, $k_1=c-1$. Expressing the Jacobi polynomials in terms of ${}_2F_1$'s and using Euler's transformation,

$${}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix}\middle| z\right)=(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a, & c-b\\ & c\end{matrix}\middle| z\right),$$

we obtain the relation

$$2nP_n^{(\beta,-1)}(z)=(n+\beta)(z+1)\,P_{n-1}^{(\beta,1)}(z).$$

Hence

$$k_i(\beta,c)=\frac{\beta c(c-1)}{i!\,(i-1)}P_{i-2}^{(\beta-i+1,1)}(2c-1), \qquad \text{for} \quad i\geqslant 2. \tag{5.7}$$

In order to show that the difference equation (4.1) is of infinite order we prove that no pair of consecutive coefficients (k_i, k_{i+1}) can vanish simultaneously. This even holds for arbitrary complex values of β and c, provided that $\beta c(c-1)\neq 0$.

We start with two known relations for Jacobi polynomials (see [5, p. 173, formula (33), with n replaced by $n-1$, and formula (35)]):

$$
\begin{aligned}
&nP_n^{(\alpha,\beta)}(z) \\
&\qquad = -(\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z) + \tfrac{1}{2}(\alpha+\beta+2n)(1+z)\,P_{n-1}^{(\alpha,\beta+1)}(z), \\
&(\alpha+\beta+2n)\,P_n^{(\alpha-1,\beta)}(z) \\
&\qquad = (\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(z) - (\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z).
\end{aligned}
$$

We eliminate $P_n^{(\alpha,\beta)}(z)$ from these two relations to find

$$
\begin{aligned}
(\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z) + nP_n^{(\alpha-1,\beta)}(z) &= \frac{1}{2}(\alpha+\beta+n)(1+z)\,P_{n-1}^{(\alpha,\beta+1)}(z) \\
&= (1+z)\frac{d}{dz}P_n^{(\alpha-1,\beta)}(z),
\end{aligned}
$$

and we take the special case $\beta := 1$, $n := i-1$, $\alpha := \beta - i + 1$:

$$
iP_{i-2}^{(\beta-i+1,1)}(z) + (i-1)\,P_{i-1}^{(\beta-i,1)}(z) = (1+z)\frac{d}{dz}P_{i-1}^{(\beta-i,1)}(z). \qquad (5.8)
$$

Now suppose that $k_i(\beta, c) = k_{i+1}(\beta, c) = 0$ for some $i \geqslant 2$. Putting for convenience $2c-1 = z$, we then get from (5.7)

$$
P_{i-2}^{(\beta-i+1,1)}(z) = P_{i-1}^{(\beta-i,1)}(z) = 0 \qquad \text{for some} \quad i \geqslant 2.
$$

Hence from (5.8) we conclude that both $P_{i-1}^{(\beta-i,1)}(z)$ and its derivative vanish. Since the Jacobi polynomial satisfies a second-order differential equation, this is impossible if z is a regular point of the differential equation, for then *all* its derivatives would vanish. The only singular points are $z=-1$ and $z=1$ corresponding to $c=0$ and $c=1$.

It is well known that the Charlier polynomials $C_n^{(a)}(x)$ can be regarded as limits of Meixner polynomials:

$$
C_n^{(a)}(x) = \lim_{\beta\to\infty}\left(-\frac{a}{\beta}\right)^n M_n\!\left(x;\beta,\frac{a}{\beta}\right).
$$

By using this relation in (4.5), (5.3), and (5.6), we retrieve the results of [3] for generalized Charlier polynomials.

6. A Second-Order Difference Equation

In this section we show that the polynomials $M_n^N(x;\beta,c)$ satisfy a second-order difference equation with polynomial coefficients, n-dependent, and of at most second degree. We construct this difference equation in a way similar to the method derived in [7, Prop. 6.1], for obtaining differential equations for systems of orthogonal polynomials.

First, using (2.10) twice, we write (3.2) in the form

$$M_n^N(x;\beta)=[Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\,M_n(x-1;\beta)$$
$$+\left[1-N\frac{c^n}{c-1}M_n(-1;\beta)\right]\Delta M_n(x-1;\beta),\qquad n=0,1,2,..., \tag{6.1}$$

and from (2.7) it is not difficult to derive that

$$c(x+\beta)\,\Delta^2 M_n(x-1;\beta)+[(c-1)(x-n)+c\beta]\,\Delta M_n(x-1;\beta)$$
$$+(1-c)\,nM_n(x-1;\beta)=0,\qquad n=0,1,2,.... \tag{6.2}$$

If we put

$$u:=M_n^N(x;\beta),\qquad p:=1+Nc^{n-1}M_{n-1}(-1;\beta+1),$$
$$q:=1-N\frac{c^n}{c-1}M_n(-1;\beta),\qquad y:=M_n(x-1;\beta), \tag{6.3}$$

(6.1) and (6.2) can be rewritten as

$$u=py+q\,\Delta y, \tag{6.4}$$

$$(cx+c\beta)\,\Delta^2 y+[(c-1)x+c(\beta-n)+n]\,\Delta y+(1-c)\,ny=0. \tag{6.5}$$

We assume $q\neq 0$. We eliminate Δy and $\Delta^2 y$ from (6.4), (6.5), and the equation obtained by taking the difference of (6.4). This leads to the relation

$$q(cx+c\beta)\,\Delta u+(ax+b)u+(dx+f)\,y=0, \tag{6.6}$$

with

$$a:=q(c-1)-pc,\qquad b:=\beta c(q-p)+qn(1-c),\qquad d:=p^2c-pq(c-1),$$
$$f:=p^2c\beta-pq(c\beta-cn+n)+q^2(1-c)n. \tag{6.7}$$

Next we eliminate y and Δy from (6.4), (6.6), and the first difference of (6.6). We finally obtain

$$q^2c(dx+f)(x+\beta+1)\,\Delta^2 u$$
$$+[q(ax+a+b+qc)(dx+f)+qc(x+\beta)(dpx+dp+fp-dq)]\,\Delta u$$
$$+[(dx+aq+d+f)(dx+f)+(ax+b)(dpx+dp+fp-dq)]\,u=0,$$

where u, p, and q are given in (6.3) and a, b, d, and f in (6.7).

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