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A stabilized mixed finite element method for the biharmonic equation based on biorthogonal systems

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ABSTRACT

We propose a stabilized finite element method for the approximation of the biharmonic equation with a clamped boundary condition. The mixed formulation of the biharmonic equation is obtained by introducing the gradient of the solution and a Lagrange multiplier as new unknowns. Working with a pair of bases forming a biorthogonal system, we can easily eliminate the gradient of the solution and the Lagrange multiplier from the saddle point system leading to a positive definite formulation. Using a superconvergence property of a gradient recovery operator, we prove an optimal a priori estimate for the finite element discretization for a class of meshes.

1. Introduction

There are many engineering applications, where fourth order elliptic and parabolic partial differential equations appear, for example, the Stokes problem in stream function and vorticity formulation [1], thin beams and plates, strain gradient elasticity [2,3], phase separation of a binary mixture [4] and scattered data fitting with thin plate splines [5]. Standard procedure needs H^2-conforming finite elements to discretize the variational formulation of the biharmonic equation. These H^2-conforming finite elements are difficult to construct in unstructured meshes; moreover, the resulting linear systems are also difficult to solve.

One approach to avoid this difficulty is to use a mixed formulation of the biharmonic equation. There are many mixed formulations of the biharmonic equations [6–8,2,9–13]. Often a mixed formulation is combined with a discontinuous Galerkin method as in [3,14,4]. The most popular mixed formulation is the Ciarlet–Raviart formulation based on the vorticity and stream function. A standard discretization of the Ciarlet–Raviart element yields a saddle point problem, and it is generally less efficient to solve a saddle point problem than a positive definite problem. We have proposed a finite element technique for discretizing the Ciarlet–Raviart formulation of the biharmonic equation in [15] using biorthogonal or quasi-biorthogonal systems. Working with biorthogonal or quasi-biorthogonal systems, all auxiliary variables can be statically condensed out from the system and a positive definite system based only on a stream function can be obtained. However, the a priori error estimates show a suboptimal convergence behavior as in the use of the standard discretization technique. Therefore, we now work with another mixed formulation to obtain an optimal a priori error estimate.

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In this contribution, we start with a mixed formulation of the biharmonic equation with clamped boundary condition proposed in [11,13]. The formulation is also obtained from Reissner–Mindlin plate equations when the plate thickness becomes zero [16,17]. If we use simple finite element spaces, we do not get the coercivity of the formulation. Then one needs to add a suitable stabilization term to obtain the coercivity as in [16]. Our goal here is to get an efficient finite element scheme to solve the biharmonic equation with the clamped boundary condition.

The formulation is obtained by introducing the gradient of the solution of the biharmonic equation as a new unknown and writing an additional variational equation in terms of a Lagrange multiplier. This gives rise to two additional vector unknowns: the gradient of the solution and the Lagrange multiplier. In order to obtain an efficient numerical scheme, we carefully choose a pair of bases for the space of the gradient of the solution and the Lagrange multiplier space in the discrete setting. Choosing the pair of bases forming a biorthogonal system for these two spaces, we can eliminate the degree of freedom associated with the gradient of the solution and the Lagrange multiplier and arrive at a positive definite formulation. The positive definite formulation involves only the degree of freedom associated with the solution of the biharmonic equation. Hence a reduced system is obtained, which is easy to solve.

We prove an optimal a priori error estimate for the finite element solution when the mesh is uniformly regular. The a priori error estimate deteriorates for irregular meshes. One essential ingredient for the proof of the optimal a priori error estimate is the use of superconvergence property of the gradient recovery technique [18]. The assumption on the mesh is also motivated from the gradient recovery technique. This is a major theoretical contribution of this paper to apply the gradient superconvergence in the proof of a priori error estimate. This idea may be applicable to other finite element techniques as well.

The structure of the rest of the paper is organized as follows. In the next section, we briefly recall a mixed formulation for the biharmonic equation suitable for our analysis. Section 3 is devoted to the numerical analysis of the approach. We also present the algebraic formulation of the problem and briefly discuss how the static condensation can be applied to get a formulation only based on the stream function. Finally, we prove an optimal a priori error estimates for a class of regular meshes in Section 4. Comparing the error estimate based on Ciarlet–Raviart formulation of the biharmonic equation, the new formulation gives better error estimate for a class of meshes, see [19,20,15].

2. A mixed formulation of biharmonic equation

In this section we introduce a mixed formulation of the biharmonic problem suitable for our purpose. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded convex domain with polygonal or polyhedral boundary $\partial\Omega$ and outward pointing normal $\mathbf{n}$ on $\partial\Omega$, The biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega \tag{1}$$

with clamped boundary condition

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \tag{2}$$

is studied extensively in [6,7,2,8–12,21,14,4,13]. Starting with the mixed formulation of the biharmonic problem presented in [11,13], a stabilization term is added as proposed in [16] for the Reissner–Mindlin plate equations so that the variational formulation can be discretized by using a biorthogonal system. Here the variational formulation is based on the stream function, its gradient and the Lagrange multiplier. The central idea of our approach is to use a pair of bases forming a biorthogonal system for discretizing the gradient of the stream function and the Lagrange multiplier.

In the following, we make use of the standard Sobolev spaces $L^p(\Omega)$, $H^s(\Omega)$ and $W^{s,p}(\Omega)$, where $s, p \in \mathbb{R}$ with $p \geq 1$, see [22,23,2,24,25]. We will use $H_0^1(\Omega)$ and $H_0^2(\Omega)$ to denote the subspaces of $H^1(\Omega)$ and $H^2(\Omega)$, respectively, whose elements satisfy the homogeneous Dirichlet boundary condition in the sense of trace.

We consider the following variational form of the biharmonic equation (1) with the clamped boundary condition (2):

$$J(u) = \inf_{v \in H_0^2(\Omega)} J(v), \tag{3}$$

with

$$J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \, d\mathbf{x} - \int_\Omega f v d\mathbf{x}.$$

Let $V := H_0^1(\Omega) \times [H_0^1(\Omega)]^d$ and for two matrix-valued functions $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^{d\times d}$ and $\boldsymbol{\beta} : \Omega \to \mathbb{R}^{d\times d}$, the inner product be defined as

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_{H^k(\Omega)} := \sum_{i=1}^{d}\sum_{j=1}^{d} (\alpha_{ij}, \beta_{ij})_{H^k(\Omega)},$$

where $(\boldsymbol{\alpha})_{ij} = \alpha_{ij}$, $(\boldsymbol{\beta})_{ij} = \beta_{ij}$ with $\alpha_{ij}, \beta_{ij} \in H^k(\Omega)$, and the norm $\| \cdot \|_{H^k(\Omega)}$ is induced from this inner product. For $k = 0$, an equivalent notation

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_{L^2(\Omega)} := \sum_{i=1}^{d} \sum_{j=1}^{d} \int_\Omega \alpha_{ij}\beta_{ij}\, \mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{\alpha} : \boldsymbol{\beta}\, \mathrm{d}\mathbf{x}$$

for the L^2-inner product will be used and the L^2-norm $\| \cdot \|_{L^2(\Omega)}$ is induced by this inner product.

A new formulation of the functional J in (3) is obtained by introducing an auxiliary variable $\boldsymbol{\sigma} = \nabla u$ such that the minimization problem (3) is rewritten as the following constrained minimization problem [11,13]:

$$\underset{\substack{(u,\boldsymbol{\sigma})\in V\\ \boldsymbol{\sigma}=\nabla u}}{\arg\min} \left(\frac{1}{2} \|\nabla \boldsymbol{\sigma}\|^2_{L^2(\Omega)} - \int_\Omega f\, u\, \mathrm{d}\mathbf{x} \right).$$

3. Finite element approximation

Let $\mathcal{T}_h$ be a quasi-uniform partition of the domain Ω in simplices with the mesh-size h. Let $\hat{T}$ be the reference triangle or tetrahedron defined as

$$\hat{T} := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0,\ i = 1, \ldots, d,\ \text{and} \sum_{i=1}^{d} x_i < 1 \right\}.$$

Let $\mathcal{P}_1(T)$ be the space of linear functions on any element $T \in \mathcal{T}_h$. The finite element space based on the mesh $\mathcal{T}_h$ is defined as the space of continuous functions whose restrictions to an element T are linear functions:

$$S_h := \{ v_h \in H^1_0(\Omega) : v_h|_T \in \mathcal{P}_1(T),\ T \in \mathcal{T}_h \},$$

see [2,21,17].

We assume that the discrete Lagrange multiplier space $M_h \subset L^2(\Omega)$ satisfies the following assumptions.

Assumption 1. 1(i) $\dim M_h = \dim S_h$.

1(ii) There is a constant $\beta > 0$ independent of the triangulation $\mathcal{T}_h$ such that

$$\|\phi_h\|_{L^2(\Omega)} \le \beta \sup_{\mu_h \in M_h \setminus \{0\}} \frac{\int_\Omega \mu_h \phi_h\, \mathrm{d}\mathbf{x}}{\|\mu_h\|_{L^2(\Omega)}}, \quad \phi_h \in S_h.$$

1(iii) The space M_h has the approximation property:

$$\inf_{\lambda_h \in M_h} \|\phi - \lambda_h\|_{L^2(\Omega)} \le Ch|\phi|_{H^1(\Omega)}, \quad \phi \in H^1(\Omega).$$

To obtain the discrete form of the minimization problem (2), we introduce a finite element space $V_h \subset V$ as $V_h = S_h \times [S_h]^d$. Replacing the space V in (2) by our discrete space V_h, our discrete problem is to minimize

$$\underset{(u_h,\boldsymbol{\sigma}_h)\in V_h}{\arg\min} \left(\frac{1}{2} \|\nabla \boldsymbol{\sigma}_h\|^2_{L^2(\Omega)} - \int_\Omega f\, u_h\, \mathrm{d}\mathbf{x} \right) \tag{4}$$

subject to

$$\langle \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \rangle_{L^2(\Omega)} = \langle \nabla u_h, \boldsymbol{\tau}_h \rangle_{L^2(\Omega)}, \quad \boldsymbol{\tau}_h \in [M_h]^d. \tag{5}$$

Now we introduce a saddle point formulation of the minimization problem (4), see also [2,26]. Introducing a Lagrange multiplier unknown $\boldsymbol{\phi}_h$, the variational saddle point formulation of the minimization problem (4) is to find $((u_h, \boldsymbol{\sigma}_h), \boldsymbol{\phi}_h) \in V_h \times [M_h]^d$ so that

$$\begin{aligned} &\tilde{A}((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) + B(\boldsymbol{\phi}_h, (v_h, \boldsymbol{\tau}_h)) = f(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in V_h,\\ &B(\boldsymbol{\psi}_h, (u_h, \boldsymbol{\sigma}_h)) = 0, \quad \boldsymbol{\psi}_h \in [M_h]^d, \end{aligned} \tag{6}$$

where bilinear forms $\tilde{A}(\cdot,\cdot)$, $B(\cdot,\cdot)$ and $f(\cdot)$ are given by

$$\tilde{A}((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = \int_\Omega \nabla \boldsymbol{\sigma}_h : \nabla \boldsymbol{\tau}_h\, \mathrm{d}\mathbf{x},$$

$$B(\boldsymbol{\psi}_h, (v_h, \boldsymbol{\tau}_h)) = \int_\Omega \boldsymbol{\tau}_h \cdot \boldsymbol{\psi}_h\, \mathrm{d}\mathbf{x} - \int_\Omega \nabla v_h \cdot \boldsymbol{\psi}_h\, \mathrm{d}\mathbf{x}, \quad \text{and} \quad f(v_h) = \int_\Omega f\, v_h\, \mathrm{d}\mathbf{x}.$$

The mixed formulation of our problem is closely related to the mixed formulation of Mindlin–Reissner plate [26,16,27,28], and hence we use some ideas presented in [26,16] to analyze our problem. The existence and uniqueness of the solution of the saddle point problem (6) is performed by using the theory presented in [26,16]. The main difficulty here as well as in the context of Mindlin–Reissner plate is that the bilinear form $\tilde{A}(\cdot,\cdot)$ is not elliptic on the whole space V_h. However, it would be sufficient that the bilinear form $\tilde{A}(\cdot,\cdot)$ is elliptic on the space $\operatorname{Ker} B_h$ defined as

$$\operatorname{Ker} B_h := \left\{ (v_h, \boldsymbol{\tau}_h) \in V_h : \int_\Omega (\boldsymbol{\tau}_h - \nabla v_h)\cdot \boldsymbol{\psi}_h \,\mathrm{d}\mathbf{x} = 0,\ \boldsymbol{\psi}_h \in [M_h]^{\mathrm{d}} \right\}.$$

If we choose S_h as the standard finite element space and M_h satisfying Assumption 1(i)–(iii), we cannot still satisfy coercivity of $\tilde{A}(\cdot,\cdot)$ even on the space $\operatorname{Ker} B_h$. That is why we modify the bilinear form $\tilde{A}(\cdot,\cdot)$ consistently by adding a stabilization term so that we obtain the ellipticity on the space $\operatorname{Ker} B_h$. The modification of the bilinear form $\tilde{A}(\cdot,\cdot)$ is done as suggested by Arnold and Brezzi [16] for the Mindlin–Reissner plate so that our discrete saddle point problem is to find $((u_h,\boldsymbol{\sigma}_h),\boldsymbol{\phi}_h) \in V_h \times [M_h]^d$ such that

$$\begin{aligned} &A((u_h,\boldsymbol{\sigma}_h),(v_h,\boldsymbol{\tau}_h)) + B(\boldsymbol{\phi}_h,(v_h,\boldsymbol{\tau}_h)) = f(v_h), \quad (v_h,\boldsymbol{\tau}_h) \in V_h, \\ &B(\boldsymbol{\psi}_h,(u_h,\boldsymbol{\sigma}_h)) = 0, \quad \boldsymbol{\psi}_h \in [M_h]^d, \end{aligned} \tag{7}$$

where the bilinear form $A(\cdot,\cdot)$ is defined as

$$A((u_h,\boldsymbol{\sigma}_h),(v_h,\boldsymbol{\tau}_h)) = \int_\Omega \nabla\boldsymbol{\sigma}_h : \nabla\boldsymbol{\tau}_h \,\mathrm{d}\mathbf{x} + r\int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h)\cdot(\boldsymbol{\tau}_h - \nabla v_h)\,\mathrm{d}\mathbf{x}$$

with $r > 0$ being a parameter. Since the stabilization term is consistent, the parameter $r > 0$ can be arbitrary in principle. This parameter r can be utilized to accelerate the solver as in an augmented Lagrangian formulation [29]. Since we do not focus on this aspect of the problem, we simply put $r = 1$ in the rest of the paper. After putting $r = 1$, we have

$$A((u_h,\boldsymbol{\sigma}_h),(v_h,\boldsymbol{\tau}_h)) = \tilde{A}((u_h,\boldsymbol{\sigma}_h),(v_h,\boldsymbol{\tau}_h)) + \int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h)\cdot(\boldsymbol{\tau}_h - \nabla v_h)\,\mathrm{d}\mathbf{x}.$$

We note that the lowest order finite element approach proposed in [16] requires the enrichment of the finite element space S_h with element-wise-defined bubble functions and does not work for the clamped plate. Our goal is to obtain an efficient finite element approach based on the standard linear finite element space.

Here our interest is to eliminate the degree of freedom corresponding to $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$ and arrive at a formulation only depending on u_h. This will dramatically reduce the size of the system matrix, and the system matrix after elimination of these variables will be positive definite. For the solution of the reduced system, one can thus use very efficient numerical techniques. Therefore, we closely look at the algebraic formulation of the problem. In the following, we use the same notation for the vector representation of the solution and the solutions as elements in S_h, $[S_h]^d$ and $[M_h]^d$. Let $\mathtt{A}$, $\mathtt{B}$, $\mathtt{W}$, $\mathtt{K}$, $\mathtt{D}$ and $\mathtt{M}$ be the matrices associated with the bilinear forms $\int_\Omega \nabla\boldsymbol{\sigma}_h : \nabla\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x}$, $\int_\Omega \nabla u_h\cdot\psi_h\,\mathrm{d}\mathbf{x}$, $\int_\Omega \nabla u_h\cdot\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x}$, $\int_\Omega \nabla u_h\cdot\nabla v_h\,\mathrm{d}\mathbf{x}$, $\int_\Omega \boldsymbol{\sigma}_h\cdot\boldsymbol{\psi}_h\,\mathrm{d}\mathbf{x}$ and $\int_\Omega \boldsymbol{\sigma}_h\cdot\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x}$, respectively. The matrix $\mathtt{D}$ associated with the bilinear form $\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\psi}_h\,\mathrm{d}\mathbf{x}$ is often called a Gram matrix. In case of the saddle point formulation, u_h, $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$ are three independent unknowns. Letting the test functions $\boldsymbol{\tau}_h$ and v_h to be zero subsequently in the first equation of (7), we have

$$\begin{aligned} &-\int_\Omega \nabla v_h\cdot\boldsymbol{\phi}_h\,\mathrm{d}\mathbf{x} - \int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h)\cdot\nabla v_h\,\mathrm{d}\mathbf{x} = f(v_h), \quad v_h \in S_h, \\ &\int_\Omega \nabla\boldsymbol{\sigma}_h : \nabla\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x} + \int_\Omega \boldsymbol{\phi}_h\cdot\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x} + \int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h)\cdot\boldsymbol{\tau}_h\,\mathrm{d}\mathbf{x} = 0, \quad \boldsymbol{\tau}_h \in [S_h]^d. \end{aligned}$$

Then the algebraic formulation of the saddle point problem (7) can be written as

$$\begin{bmatrix} \mathtt{K} & -\mathtt{W}^T & -\mathtt{B}^T \\ -\mathtt{W} & \mathtt{A}+\mathtt{M} & \mathtt{D}^T \\ -\mathtt{B} & \mathtt{D} & 0 \end{bmatrix} \begin{bmatrix} u_h \\ \boldsymbol{\sigma}_h \\ \boldsymbol{\phi}_h \end{bmatrix} = \begin{bmatrix} f_h \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

where f_h is the vector form of discretization of the linear form $f(\cdot)$. Since our goal is to obtain an efficient numerical scheme, we want to statically condense out the degree of freedom associated with $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$. Looking closely at the linear system (8), we find that if the matrix $\mathtt{D}$ is diagonal, we can easily eliminate the degree of freedom corresponding to $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$. This then leads to a formulation involving only one unknown u_h.

Let $\{\varphi_1,\ldots,\varphi_n\}$ be the standard nodal finite element basis of S_h. We define a space M_h spanned by the basis $\{\mu_1,\ldots,\mu_n\}$, where the basis functions of S_h and M_h satisfy a condition of biorthogonality relation

$$\int_\Omega \mu_i\,\varphi_j\,\mathrm{d}\mathbf{x} = c_j\delta_{ij}, \quad c_j \neq 0,\ 1 \le i,j \le n, \tag{9}$$

where $n := \dim M_h = \dim S_h$, δ_{ij} is the Kronecker symbol, and c_j a positive scaling factor. This scaling factor c_j is chosen to be proportional to the area $|\operatorname{supp}\varphi_j|$.

In the following, we give these basis functions for linear simplicial finite elements in two and three dimensions. Here $S_h \subset H_0^1(\Omega)$ and $M_h \subset L^2(\Omega)$, but $\dim M_h = \dim S_h$. Thus there will be no degree of freedom for M_h on the boundary of Ω.

Because of this the local basis functions for M_h should be defined carefully. If an element T has no vertices on the boundary of Ω, the local basis functions for M_h are defined in a standard way. For the reference triangle $\hat{T} := \{(x,y) : 0 < x, 0 < y, x+y < 1\}$, we have

$$\hat{\mu}_1 := 3 - 4x - 4y, \quad \hat{\mu}_2 := 4x - 1, \quad \text{and} \quad \hat{\mu}_3 := 4y - 1,$$

where the basis functions $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$ are associated with three vertices $(0,0)$, $(1,0)$ and $(0,1)$ of the reference triangle. For the reference tetrahedron $\hat{T} := \{(x,y,z) : 0 < x, 0 < y, 0 < z, x+y+z < 1\}$, we have

$$\hat{\mu}_1 := 4 - 5x - 5y - 5z, \quad \hat{\mu}_2 := 5x - 1, \quad \hat{\mu}_3 := 5y - 1, \quad \text{and} \quad \hat{\mu}_4 := 5z - 1,$$

where the basis functions $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ are associated with four vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of the reference tetrahedron.

If an element T has a vertex on the boundary of Ω, we consider a local biorthogonal basis excluding this vertex. In this case, there will be only two local basis functions in the two-dimensional case and three local basis functions in the three-dimensional case. Since the local biorthogonal basis is sought in a linear polynomial space, we will have one extra degree of freedom due to one excluded vertex. This degree of freedom is utilized to guarantee that the local basis functions sum to one.

In other words, for any element T, only those vertices which are inside Ω are used to construct local biorthogonal basis functions excluding those vertices on the boundary of Ω. However, it is important to guarantee that these local basis functions sum to one. If an element T has only one vertex inside Ω, then the local basis function is set to be a constant function. A special consideration is needed if an element T has all vertices on the boundary. In such a case, the support of a biorthogonal basis function of a neighboring element is extended to this element with value one on T. We refer to [30,31] for more detail about the construction of these basis functions in the two-dimensional case. This construction is easily extended to the three-dimensional case.

The global basis functions for the test space are constructed by glueing the local basis functions together. This process is exactly as in the standard finite element method. We just need to replace the local finite element basis functions by these new local basis functions. These global basis functions then satisfy the condition of biorthogonality (9) with global finite element basis functions. As these functions in M_h are defined exactly in the same way as the finite element basis functions in S_h, they satisfy $\operatorname{supp}\mu_i = \operatorname{supp}\varphi_i$ for $i = 1, \ldots, n$.

Now we show that M_h satisfies Assumption 1(i)–(iii). As the first assumption is satisfied by construction, we consider the second assumption. Let $\phi_h = \sum_{k=1}^{n} a_k \phi_k \in S_h$ and set $\mu_h = \sum_{k=1}^{n} a_k \mu_k \in M_h$. By using the biorthogonality relation (9) and the quasi-uniformity assumption, we get

$$\int_\Omega \phi_h \mu_h \,\mathrm{d}\mathbf{x} = \sum_{i,j=1}^{n} a_i a_j \int_\Omega \phi_i \,\mu_j \,\mathrm{d}\mathbf{x} = \sum_{i=1}^{n} a_i^2 c_i \geq C \sum_{i=1}^{n} a_i^2 h_i^d \geq C \|\phi_h\|^2_{L^2(\Omega)},$$

where h_i denotes the mesh-size at $i$the vertex. Taking into account the fact that $\|\phi_h\|^2_{L^2(\Omega)} \equiv \|\mu_h\|^2_{L^2(\Omega)} \equiv \sum_{i=1}^{n} a_i^2 h_i^d$, we find that Assumption 1(ii) is satisfied. Since the sum of the local basis functions of M_h is one, Assumption 1(iii) can be proved as in [32,31].

After statically condensing out variables $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$, we arrive at a reduced system

$$(\mathtt{K} - (\mathtt{W}^T \mathtt{D}^{-1} \mathtt{B} + \mathtt{B}^T \mathtt{D}^{-1} \mathtt{W}) + \mathtt{B}^T \mathtt{D}^{-1} (\mathtt{A} + \mathtt{M}) \mathtt{D}^{-1} \mathtt{B}) u_h = f_h.$$

The variational formulation of this reduced system is given by (11).

4. An a priori error estimate

Before proceeding to establish an a priori error estimate, we want to eliminate the gradient of the smoother $\boldsymbol{\sigma}_h$ and Lagrange multiplier $\boldsymbol{\phi}_h$ from the saddle point problem (7). To this end, we introduce a quasi-projection operator: $Q_h : L^2(\Omega) \to S_h$, which is defined as

$$\int_\Omega Q_h v \,\mu_h \,\mathrm{d}\mathbf{x} = \int_\Omega v \mu_h \,\mathrm{d}\mathbf{x}, \quad v \in L^2(\Omega), \;\; \mu_h \in M_h.$$

This type of operator is introduced in [33] to obtain the finite element interpolation of non-smooth functions satisfying boundary conditions, and is used in [34] in the context of mortar finite elements. The definition of Q_h allows us to write the weak gradient as

$$\boldsymbol{\sigma}_h = Q_h(\nabla u_h),$$

where operator Q_h is applied to the vector ∇u_h component-wise. We see that Q_h is well defined due to Assumption 1(i) and (ii). Furthermore, the restriction of Q_h to S_h is the identity. Hence Q_h is a projection onto the space S_h. We note that Q_h is not the orthogonal projection onto S_h but an oblique projection onto S_h, see [35,36]. Using the biorthogonality relation between the basis functions of S_h and M_h, the action of operator Q_h on a function $v \in L^2(\Omega)$ can be written as

$$Q_h v = \sum_{i=1}^{n} \frac{\int_\Omega \mu_i \, v \,\mathrm{d}\mathbf{x}}{c_i} \varphi_i,$$

which tells that the operator Q_h is local in the sense to be given below, see also [37]. Let $S(T')$ be the patch of an element $T' \in \mathcal{T}_h$ which is the interior of the closed set

$$\bar{S}(T') = \bigcup \{\bar{T} \in \mathcal{T}_h : \partial T \cap \partial T' \neq \emptyset\}.$$

Then Q_h is local in the sense that for any $v \in L^2(\Omega)$, the value of $Q_h v$ at any point in $T \in \mathcal{T}_h$ only depends on the value of v in $S(T)$ [37]. In the following, we will use a generic constant C, which will take different values at different places but will be always independent of the mesh-size h. The stability and approximation properties of Q_h in L^2 and H^1-norm can be shown as in [31,30].

Lemma 1. *Under Assumption 1(i)–(iii)*

$$\|Q_h v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)} \quad \text{for } v \in L^2(\Omega),$$
$$|Q_h w|_{H^1(\Omega)} \leq C|w|_{H^1(\Omega)} \quad \text{for } w \in H^1(\Omega),$$

and for $0 < s \leq 1$ *and* $v \in H^{1+s}(\Omega)$

$$\|v - Q_h v\|_{L^2(\Omega)} \leq Ch^{1+s}|v|_{H^{s+1}(\Omega)},$$
$$\|v - Q_h v\|_{H^1(\Omega)} \leq Ch^{s}|v|_{H^{s+1}(\Omega)}.$$

Using the property of operator Q_h, we can eliminate the degrees of freedom corresponding to $\boldsymbol{\sigma}_h$ so that our problem is to find $u_h \in S_h$ such that

$$J(u_h) = \min_{v_h \in S_h} J(v_h), \tag{10}$$

where

$$J(v_h) = \|\nabla(Q_h(\nabla v_h))\|^2_{L^2(\Omega)} + \|Q_h(\nabla v_h) - \nabla v_h\|_{L^2(\Omega)} - 2f(v_h).$$

Let the bilinear form $a(\cdot,\cdot)$ be defined as

$$a(u_h, v_h) = \int_\Omega \nabla\boldsymbol{\sigma}_h : \nabla\boldsymbol{\tau}_h \, d\mathbf{x} + \int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h)\cdot(\boldsymbol{\tau}_h - \nabla v_h)\, d\mathbf{x}$$

with $\boldsymbol{\sigma}_h = Q_h(\nabla u_h)$ and $\boldsymbol{\tau}_h = Q_h(\nabla v_h)$. Since the bilinear form $a(\cdot,\cdot)$ is symmetric, the minimization problem (10) is equivalent to the variational problem of finding $u_h \in S_h$ such that [2,17]

$$a(u_h, v_h) = f(v_h), \quad v_h \in S_h. \tag{11}$$

Moreover, we can show that the bilinear form $A(\cdot,\cdot)$ is coercive on $V_h \times V_h$. For a proof, see [16].

Lemma 2. *The bilinear form* $A(\cdot,\cdot)$ *is coercive on* $V_h \times V_h$. *That is*

$$A((u_h, \boldsymbol{\sigma}_h), (u_h, \boldsymbol{\sigma}_h)) \geq C(\|u_h\|^2_{H^1(\Omega)} + \|\boldsymbol{\sigma}_h\|^2_{H^1(\Omega)}).$$

Lemma 3. *The bilinear form* $a(\cdot,\cdot)$ *satisfies*

$$a(u_h, u_h) \geq C(\|u_h\|^2_{L^2(\Omega)} + \|\nabla Q_h(\nabla u_h)\|^2_{L^2(\Omega)}), \quad u_h \in S_h.$$

Proof. Using Poincaré and a triangle inequality

$$\begin{aligned}\|u_h\|^2_{L^2(\Omega)} + \|\nabla Q_h(\nabla u_h)\|^2_{L^2(\Omega)} &\leq C(\|\nabla u_h - Q_h(\nabla u_h)\|^2_{L^2(\Omega)} + \|Q_h(\nabla u_h)\|^2_{L^2(\Omega)})\\ &\leq C(\|\nabla u_h - Q_h(\nabla u_h)\|^2_{L^2(\Omega)} + \|\nabla Q_h(\nabla u_h)\|^2_{L^2(\Omega)}),\end{aligned}$$

where we apply Poincaré inequality again in the last step since $Q_h(\nabla u_h) \in V_h$. □

Due to this result, we can define an inner product

$$(u_h, v_h)_a := a(u_h, v_h), \quad \text{and the corresponding norm } \|u_h\|_a := a(u_h, u_h)$$

induced by the bilinear form $a(\cdot,\cdot)$. Hence the following theorem holds.

Theorem 1. *The variational problem (11) admits a unique solution which depends continuously on the data.*

Proof. Since $u_h, v_h \in S_h$, it follows that $|a(u_h, v_h)| \leq \|u_h\|_a \|v_h\|_a$ and $|f(v_h)| \leq C\|v_h\|_a$. Moreover, using the definition of our norm $\|\cdot\|_a$, $a(v_h, v_h) = \|v_h\|_a$, and thus $a(\cdot,\cdot)$ is elliptic with respect to the norm $\|\cdot\|_a$. Hence our variational problem (11) has a unique solution by Lax–Milgram Lemma [2,25]. From the definition of the P-inner product, we have

$$a(v_h, v_h) = \|v_h\|_a^2, \quad v_h \in S_h,$$

and thus, for the solution $u_h \in S_h$, $\|u_h\|_a^2 = f(u_h)$. □

Since we have a unique solution u_h of the variational problem (11), $\boldsymbol{\sigma}_h$ is also uniquely determined. The error estimate is obtained in the *energy norm* $\|\cdot\|_A$ induced by the bilinear form $A(\cdot,\cdot)$ defined as

$$\|(u, \boldsymbol{\sigma})\|_A := \sqrt{|\boldsymbol{\sigma}|_{H^1(\Omega)}^2 + \|\boldsymbol{\sigma} - \nabla u\|_{L^2(\Omega)}^2}, \quad (u, \boldsymbol{\sigma}) \in H^1(\Omega) \times [H^1(\Omega)]^d.$$

Theorem 2. *Let u be the solution of continuous problem (3) with $u \in H^4(\Omega)$, $\boldsymbol{\sigma} = \nabla u$ and $\boldsymbol{\phi} = \Delta\boldsymbol{\sigma}$, and u_h be that of discrete problem (11) with $\boldsymbol{\sigma}_h = Q_h(\nabla u_h)$. Then there exists a constant $C > 0$ independent of the mesh-size h so that*

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_A \leq C\left(\inf_{(w_h, \boldsymbol{\theta}_h) \in \operatorname{Ker} B_h} \|(u - w_h, \boldsymbol{\sigma} - \boldsymbol{\theta}_h)\|_A + h|\boldsymbol{\phi}|_{H^1(\Omega)}\right).$$

Proof. Here u, $\boldsymbol{\sigma}$ and $\boldsymbol{\phi}$ satisfy [26]

$$\begin{aligned} &A((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) + B(\boldsymbol{\phi}, (v, \boldsymbol{\tau})) = f(v), \quad (v, \boldsymbol{\tau}) \in V, \\ &B(\boldsymbol{\psi}, (u, \boldsymbol{\sigma})) = 0, \quad \boldsymbol{\psi} \in [L^2(\Omega)]^d. \end{aligned}$$

Let $(w_h, \boldsymbol{\theta}_h) \in \operatorname{Ker} B_h$ so that $(u_h - w_h, \boldsymbol{\sigma}_h - \boldsymbol{\theta}_h) \in \operatorname{Ker} B_h$, and hence

$$\|(u_h - w_h, \boldsymbol{\sigma}_h - \boldsymbol{\theta}_h)\|_A \leq \sup_{(v_h, \boldsymbol{\tau}_h) \in \operatorname{Ker} B_h} \frac{A((u_h - w_h, \boldsymbol{\sigma}_h - \boldsymbol{\theta}_h), (v_h, \boldsymbol{\tau}_h))}{\|(v_h, \boldsymbol{\tau}_h)\|_A}$$

Since $A((u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) + B(\boldsymbol{\phi}, (v_h, \boldsymbol{\tau}_h)) = 0$ for all $(v_h, \boldsymbol{\tau}_h) \in \operatorname{Ker} B_h$, we have

$$\begin{aligned} A((u_h - w_h, \boldsymbol{\sigma}_h - \boldsymbol{\theta}_h), (v_h, \boldsymbol{\tau}_h)) &= A((u - w_h, \boldsymbol{\sigma} - \boldsymbol{\theta}_h), (v_h, \boldsymbol{\tau}_h)) + A((u_h - u, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}), (v_h, \boldsymbol{\tau}_h)) \\ &= A((u - w_h, \boldsymbol{\sigma} - \boldsymbol{\theta}_h), (v_h, \boldsymbol{\tau}_h)) + B(\boldsymbol{\phi}, (v_h, \boldsymbol{\tau}_h)). \end{aligned}$$

Denoting the orthogonal projection of $\boldsymbol{\phi}$ onto $[M_h]^d$ with respect to L^2-inner product by $\tilde{\boldsymbol{\phi}}_h$, we have

$$B(\boldsymbol{\phi}, (v_h, \boldsymbol{\tau}_h)) = \int_\Omega (\boldsymbol{\tau}_h - \nabla v_h) \cdot (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}_h)\, d\mathbf{x} \leq Ch\|\boldsymbol{\tau}_h - \nabla v_h\|_{L^2(\Omega)} |\boldsymbol{\phi}|_{H^1(\Omega)}.$$

The result then follows by using the continuity of $A(\cdot,\cdot)$. □

The theoretical proof of the approximation is based on the superapproximation of a gradient recovery operator recently proposed in [18]. First we need an assumption on our mesh similar to mesh conditions in [18]. Let $\mathcal{N}_h = \{\mathbf{x}_i\}_{i=1}^{n_v}$ be the set of all interior vertex nodes in $\mathcal{T}_h$, and S_i be the support of the finite element basis function ϕ_i at $\mathbf{x}_i \in \mathcal{N}_h$. We impose the following assumption on our mesh.

Assumption 2. Choosing $\mathbf{x}_i \in \mathcal{N}_h$ as the origin of local coordinates,

$$\sum_{T \in S_i} \frac{|T|}{|S_i|}(\mathbf{z}_T) = O(h^{1+\alpha})\mathbf{1}, \quad \mathbf{x}_i \in \mathcal{N}_h,$$

where $\mathbf{z}_T$ is the coordinate vector of the barycenter of element T, $\alpha > 0$, and $\mathbf{1}$ is the d-dimensional vector having each component 1.

This assumption holds with $\alpha = \infty$ for uniform meshes of the regular pattern, the Union Jack pattern and the criss-cross pattern. The assumption allows $O(h^{1+\alpha})$ deviation from those meshes. If two adjacent triangles in $\mathcal{T}_h$ form an $O(h^{1+\alpha})$ parallelogram, this assumption is satisfied [18]. The two triangles are adjacent when they share a common edge, and the two adjacent triangles form an $O(h^{1+\alpha})$ parallelogram if the lengths of any two opposite edges differ only by $O(h^{1+\alpha})$. This assumption guarantees that the relative positions of the barycenters of the elements are controlled. We refer to [18,38] for further discussion on such meshes.

Let $(\nabla I_h u)_{|T}$ be the restriction of $\nabla I_h u$ to an element $T \in \mathcal{T}_h$. Then

$$Q_h(\nabla I_h u)(\mathbf{x}_i) = \sum_{T \in S_i} \frac{|T|}{|S_i|}(\nabla I_h u)_{|T}.$$

The following theorem can be proved exactly as in [18].

Theorem 3. *Under Assumption 2, if* $u \in W^{3,\infty}(S_i)$*, for any* $\mathbf{x}_i \in \mathcal{N}_h$

$$|(Q_h(\nabla I_h u))(\mathbf{x}_i) - (\nabla u)(\mathbf{x}_i)| \leq Ch^{1+\alpha} \|u\|_{W^{3,\infty}(S_i)}.$$

Our goal is to prove a superapproximation property of the gradient recovery operator Q_h as in [18].

Theorem 4. *Let* $u \in W^{3,\infty}(\Omega)$*, and* $I_h u$ *be the Lagrange interpolation of* u *with respect to vertex nodes in* $\mathcal{T}_h$*. Assume that the triangulation satisfies Assumption 2. Then*

$$\|\nabla u - Q_h(\nabla I_h u)\|_{0,\Omega} \leq C(h^2 |u|_{H^3(\Omega)} + h^{1+\alpha} \|u\|_{W^{3,\infty}(\Omega)}).$$

Proof. Since the Lagrange interpolation operator reproduces all piecewise linear polynomials with respect to the mesh $\mathcal{T}_h$,

$$\|u - I_h u\|_{0,\Omega} \leq Ch^2 |u|_{H^3(\Omega)}.$$

Now we decompose

$$\nabla u - Q_h(\nabla I_h u) = \nabla u - I_h \nabla u + I_h \nabla u - Q_h(\nabla I_h u). \tag{12}$$

The approximation property of I_h yields

$$\|\nabla u - I_h \nabla u\|_{0,\Omega} \leq Ch^2 |u|_{H^3(\Omega)}.$$

Under Assumption 2, we have Theorem 3, and hence

$$\begin{aligned} \|Q_h(\nabla I_h u) - I_h \nabla u\|_{L^2(\Omega)} &\leq C \left(\sum_{T \in \mathcal{T}_h} |T| \sum_{\mathbf{z} \in \mathcal{N}_h \cap \bar{T}} |(Q_h(\nabla I_h u))(\mathbf{z}) - (\nabla u)(\mathbf{z})|^2 \right)^{1/2} \\ &\leq Ch^{1+\alpha} \|u\|_{W^{3,\infty}(\Omega)}. \quad \square \end{aligned}$$

The reason for getting a better estimate than in [18] is that we have homogeneous Dirichlet boundary condition for the gradient. If the finite element mesh is uniformly regular, then we get $\alpha = 1$, and hence we have the following corollary.

Corollary 1. *Assuming that the mesh* $\mathcal{T}_h$ *satisfies Assumption 2 with* $\alpha = 1$*. Then for* $\nabla u \in [H_0^1(\Omega)]^d$*, we have the following superapproximation property*

$$\|\nabla u - Q_h(\nabla I_h u)\|_{0,\Omega} \leq Ch^2 \|u\|_{W^{3,\infty}(\Omega)}.$$

The following theorem guarantees the suboptimal convergence rate of the finite element approximation under Assumption 2.

Theorem 5. *Under the assumptions of Theorem 2, there exists* $(v_h, \boldsymbol{\tau}_h) \in \operatorname{Ker} B_h$ *such that*

$$\|(u - v_h, \boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|_A \leq C(h|u|_{H^3(\Omega)} + h|u|_{H^2(\Omega)} + h^\alpha \|u\|_{W^{3,\infty}(\Omega)}).$$

Proof. Let v_h be the Lagrange interpolation of u with respect to the mesh $\mathcal{T}_h$. Then it is well known that

$$\|u - v_h\|_{H^k(\Omega)} \leq h^{2-k} |u|_{H^2(\Omega)}, \quad k = 0, 1.$$

Let us recall the definition of the error in the energy norm

$$\|(u - v_h, \boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|_A = \sqrt{|\boldsymbol{\sigma} - \boldsymbol{\tau}_h|^2_{H^1(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h - \nabla u + \nabla v_h\|^2_{L^2(\Omega)}}.$$

Let $\boldsymbol{\tau}_h = Q_h(\nabla v_h)$ so that $(v_h, \boldsymbol{\tau}_h) \in \operatorname{Ker} B_h$. The approximation property of operator Q_h given by Theorem 4 yields

$$\|\nabla u - Q_h(\nabla v_h)\|_{L^2(\Omega)} \leq C(h^2 |u|_{H^3(\Omega)} + h^{1+\alpha} \|u\|_{W^{3,\infty}(\Omega)}).$$

Hence, it suffices to show that

$$\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H^1(\Omega)} \leq C(h|u|_{H^3(\Omega)} + h^\alpha \|u\|_{W^{3,\infty}(\Omega)}).$$

Since $\boldsymbol{\sigma} = \nabla u$ and $\boldsymbol{\tau}_h = Q_h(\nabla v_h)$,

$$\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H^1(\Omega)} \leq \|\boldsymbol{\sigma} - Q_h \boldsymbol{\sigma}\|_{H^1(\Omega)} + \|Q_h \boldsymbol{\sigma} - Q_h(\nabla v_h)\|_{H^1(\Omega)}. \tag{13}$$

The first term in the right-hand side of (13) has the correct approximation from Lemma 1. To estimate the second term, we use an inverse estimate

$$\|Q_h \boldsymbol{\sigma} - Q_h(\nabla v_h)\|_{H^1(\Omega)} \leq \frac{C}{h} \|Q_h \boldsymbol{\sigma} - Q_h(\nabla v_h)\|_{L^2(\Omega)},$$

and apply the projection property and L^2-stability of Q_h to write

$$\|Q_h\boldsymbol{\sigma} - Q_h(\nabla v_h)\|_{H^1(\Omega)} \leq \frac{C}{h}\|\nabla u - Q_h(\nabla v_h)\|_{L^2(\Omega)}.$$

Since Theorem 4 gives

$$\|\nabla u - Q_h(\nabla I_h u)\|_{0,\Omega} \leq C(h^2|u|_{H^3(\Omega)} + h^{1+\alpha}\|u\|_{W^{3,\infty}(\Omega)}),$$

we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H^1(\Omega)} \leq C(h|u|_{H^3(\Omega)} + h^{\alpha}\|u\|_{W^{3,\infty}(\Omega)}). \quad \square$$

We combine the result of Theorems 2 and 5 to get the final result.

Theorem 6. *Let u be the solution of continuous problem (3) with $u \in H^4(\Omega)$, $\boldsymbol{\sigma} = \nabla u$ and $\boldsymbol{\phi} = \Delta\boldsymbol{\sigma}$, and u_h be that of discrete problem (11) with $\boldsymbol{\sigma}_h = Q_h(\nabla u_h)$. Then under Assumption 2 there exists a constant $C > 0$ independent of the mesh-size h so that*

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_A \leq C(h\|u\|_{H^3(\Omega)} + h^{\alpha}\|u\|_{W^{3,\infty}(\Omega)} + h|\boldsymbol{\phi}|_{H^1(\Omega)}).$$

If the mesh $\mathcal{T}_h$ satisfies Assumption 2 with $\alpha = 1$, we get the optimal estimate:

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_A \leq Ch(\|u\|_{W^{3,\infty}(\Omega)} + |\boldsymbol{\phi}|_{H^1(\Omega)}).$$

5. Conclusion

A mixed finite element method is presented for approximating the biharmonic equation with clamped boundary condition. Two additional vector variables are introduced to obtain the mixed formulation: the gradient of the stream function and the Lagrange multiplier. Working with a pair of finite element bases forming a biorthogonal system for the gradient of the stream function and the Lagrange multiplier, we can eliminate these two vector variables from the algebraic system and arrive at a formulation involving only the stream function. This yields an efficient discretization scheme. The superapproximation property of the gradient recovery operator allows us to show that the finite element approximation is optimal for uniformly regular meshes.

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