A Class of Three-Engel Groups*

Hermann Heineken

Mathematisches Institut der Universität, 852 Erlangen, Bismarckstr. 1 1/2, Germany

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In this note groups are considered all of whose cyclic subgroups are subnormal of defect two (i.e., are normal subgroups of normal subgroups of the group). It is easily seen that these groups are $E_3$ groups and that all cyclic subgroups of any $E_3$ group are subnormal of defect two. It is also clear that the groups all of whose subgroups have normal normalizers (called “$N$ groups” by Hobby [3]) belong to the class of groups considered.

We obtain the following results: If all cyclic subgroups of the group $G$ are subnormal of defect two, then $x \circ (y^{(2)} \circ z) = 1$ for all $x, y, z \in G$. If, furthermore, $G$ is not a torsion group, then $x^{(2)} \circ y = 1$ for all $x, y \in G$. Using the results of Levi [5] it is possible to derive a bound for the nilpotency class of $G$.

Notations.

\[
\{x, y, z, \ldots\} = \text{subgroup generated by } x, y, z, \ldots,
\]

\[
x^{(1)} \circ y = x \circ y = x^{-1}y^{-1}xy,
\]

\[
x^{(n)} \circ y = x \circ (x^{(n-1)} \circ y),
\]

\[
A \circ B = \{a \circ b \mid a \in A, b \in B\},
\]

\[
G_1 = G, \quad G_n = G \circ G_{n-1},
\]

\[
Z(G) = \{x \mid x \in G, g \circ x = 1 \text{ for all } g \in G\} = \text{centre of } G,
\]

$G$ is an $E_i$ group if $x^{(i)} \circ y = 1$ for all $x, y \in G$.

Commutator identities like (10.2.1.1)-(10.2.1.4) in M. Hall [1; p. 150] are frequently used without special reference.

1. In this section we assume that the group $G$ with all its cyclic subgroups subnormal of defect two possesses an element of infinite order.

**Theorem 1.** Assume that the group $G$ satisfies the following two conditions:

(a) Every cyclic subgroup of $G$ is subnormal of defect two;

(b) There is an element of infinite order in $G$.

* Dedicated to Ernst Witt on his sixtieth birthday
Then

(I) \( x^{(2)} \circ y = 1 \) for all \( x, y \in G \);

(II) \( G_4 = (G_3)^3 = 1 \).

Proof. Let \( u \) be an element of infinite order in \( G \) and let \( g \) be any element of \( G \). It is a consequence of (a) that

\[ u^{(2)} \circ g = u^k \]

for some integer \( k \). This shows in particular that \( G \) is an \( E_3 \) group, and we obtain from [2; Lemma 1, p. 683], that

\[ 1 = (u \circ g) \circ ((u \circ g) \circ u) = (u \circ g) \circ u^{-k} = (u^{(2)} \circ g)^k = u^{k^3}, \]

and from the infinite order of \( u \) we conclude \( k = 0 \). Thus

\[ u^{(2)} \circ g = 1 \quad \text{for all } g \in G \text{ and all } u \in G \text{ of infinite order.} \quad (1) \]

As a next step we want to show \( g^{(2)} \circ u = 1 \) for all \( g \in G \). If \( g \) itself is an element of infinite order we obtain this equation from (1). If \( g \) is of finite order, then \( gu \) is of infinite order (since \( E_3 \) groups are locally nilpotent and the set of elements of finite order is a (characteristic) subgroup). By application of (1),

\[ 1 = (gu)^{(2)} \circ g = (gu) \circ ((gu) \circ g) = (gu) \circ (g \circ u) = ((gu) \circ (u \circ g))^{-1} \]

and we combine this with (1) again to obtain

\[ g^{(2)} \circ u = 1 \quad \text{for all } g \in G \text{ and all } u \in G \text{ of infinite order.} \quad (2) \]

By Kappe [4; Folgerung 2.2, p. 190], all elements \( x \) of a group \( Y \) satisfying \( x^{(a)} \circ x = 1 \) for all \( x \in Y \) form a (characteristic) subgroup of \( Y \). We want to show that all elements of \( G \) belong to this set of elements, and by (2) all elements of infinite order belong to it. If \( v \) is an element of finite order and \( u \) is an element of infinite order of \( G \), then \( uv \) is of infinite order. So, by (2),

\[ g^{(2)} \circ u = g^{(2)} \circ (uv) = 1 \quad \text{for all } g \in G \]

and

\[ g^{(2)} \circ (u^{-1}(uv)) = g^{(2)} \circ v = 1 \quad \text{for all } g \in G \]

2. Now we turn to the general case.

**Theorem 2.** If all cyclic subgroups of the group $G$ are subnormal of defect two, then

1. $x \circ (y^{(2)} \circ x) = 1$ for all $x, y, z \in G$;
2. $G_5 = (G_4)_3 = 1$.

**Proof.** If $G$ is not a torsion group, the statements of Theorem 2 hold by Theorem 1. From now on we assume that $G$ is a torsion group. Since $G$ is an $E_3$ group, $G$ is locally nilpotent and the direct product of its primary components. By [2; Lemma 4, p. 687] we obtain that

\[
(x \circ (y^{(2)} \circ x))^3 = 1
\]

and

\[
 x \circ (y^{(2)} \circ x) = y \circ (x \circ (y \circ x)) = y \circ (x^{(2)} \circ y) = x \circ (y \circ (x \circ y))
\]

for all $x, y \in G$. Assume the existence of a pair of elements $x, y$ in $G$ such that $x \circ (y^{(2)} \circ x) \neq 1$. By the local nilpotency $x$ and $y$ may be chosen such that $\{x, y\}$ is a 2-group. By hypothesis on $G$ there are integers $m, n$ such that

\[
y^{(2)} \circ x = y^n
\]

and

\[
x \circ (y \circ x) = x^m.
\]

Since $x \circ (y^{(2)} \circ x) = y \circ (x \circ (y \circ x)) \neq 1$ we have $y^n \neq 1$ and $x^m \neq 1$. Thus

\[
x \circ (y^{(2)} \circ x) = y \circ (x \circ (y \circ x)) \in \{x\} \cap \{y\}
\]

and

\[
x \circ (y^{(2)} \circ x) \in \{x^m\} \cap \{y^n\} \subseteq \{x\} \cap \{y\}.
\]

Assume that the order of $x^m$ does not exceed the order of $y^n$. Then we may put

\[
\{x^m\} \cap \{y^n\} = \{x^{m2^k}\} \quad \text{and} \quad x^{m2^k} = y^{an2^k}.
\]

Then

\[
\{y^n, x^m\} = \{y^n, y^{-an}x^m\}, \quad \{x, y\} = \{y^{-an}x, y\}
\]

and

\[
\{y^n\} \cap \{y^{-an}x^m\} = 1.
\]
We compute the new basic commutators of length three,
\[
y(2) \circ (y \circ x) = y(2) \circ x = y^n,
\]
\[
(y^a x) \circ (y \circ (y^a x)) = (y^a x) \circ (y \circ x)
\]
\[
= (x \circ (y \circ x))(y^a \circ (y \circ x))(x \circ (y^a \circ (y \circ x)))^{-1}
\]
\[
= x^m y^n c^a,
\]
where \( c = x \circ (y(2) \circ x) = y \circ (y(2) \circ y) \). If \( x^m y^n - 1 \), then
\[
c = y \circ (x \circ (y \circ x)) = y \circ ((y^a x) \circ (y \circ (y^a x))) = y \circ (c^a),
\]
which is impossible in a nilpotent group unless \( c = 1 \). If \( x^m y^n - 1 \), we find
\[
(x^m y^n - 1) = (x^m y^n)^2 = (x^m y^n)^2 \in \{ y^n \} \text{ if and only if } (x^m y^n)^2 = 1 \text{ by choice of } a.
\]
So we find
\[
c = y \circ ((y^a x) \circ (y \circ (y^a x))) \in \{ y^n \} \cap \{ x^m y^n c^a \} = 1.
\]
Thus
\[
y \circ (x \circ (y \circ x)) = 1 \quad \text{for all } x, y \in G. \quad (3)
\]
Consider now a primary finitely generated subgroup \( A \) of \( G \). By [2; Hauptsatz 2, p. 682] \( A \) is nilpotent, and we consider \( \bar{A} = A/A_5 \). The hypothesis on \( G \) in Theorem 2 is inherited by subgroups and by quotient groups, so \( \bar{A} \) is a group all of whose cyclic subgroups are subnormal of defect two. Hence \( a \circ (b^{(2)} \circ a) = 1 \) for all \( a, b \in \bar{A} \). Consider the elements \( u, v, b \) of \( \bar{A} \). By hypothesis there are integers \( r, s \) such that
\[
b^{(2)} \circ u = b^r \quad \text{and} \quad b^{(2)} \circ v = b^s.
\]
Since \( \bar{A} \) is primary we may assume without loss of generality that \( b^r \) is a power of \( b^s \),
\[
b^r = b^{sk}.
\]
We conclude that
\[
v \circ (b^{(2)} \circ u) = v \circ (b^{(2)} \circ v)^k = 1.
\]
If we substitute \( b \) for \( x \) and \( uv \) for \( y \) in (1), we obtain
\[
1 = (uv) \circ (b^{(2)} \circ (uv))
\]
\[
= (u \circ (b^{(2)} \circ u))(v \circ (b^{(2)} \circ u))(u \circ (b^{(2)} \circ v))(v \circ (b^{(2)} \circ v))
\]
\[
= u \circ (b^{(2)} \circ v).
\]
Thus
\[
u \circ (w^{(2)} \circ v) = 1 \quad \text{for all } u, v, w \in \bar{A}. \quad (4)
\]
Assume now that $A$ is not a 3-group. Then the quotient group $\bar{A}/Z(\bar{A})$ is an $E_3$ group and $(\bar{A}/Z(\bar{A}))^3 = 1$ by Levi [5]. Therefore $A_4 = 1$ and $A_4 = A_5$; we conclude that $A_4 = 1$. If $A$ is a 3-group then $A$ is a 3-group and $A_5 = 1$ by [2; Hauptsatz 3, p. 682]. Thus $\bar{A} = A$ and $A/Z(A)$ is an $E_2$ group. By Levi [5], $(A/Z(A))^3 = 1$ and we conclude that $A_3^3 = 1$. As a locally nilpotent group is nilpotent of class $m$ if all its finitely generated subgroups are, we obtain (II) of Theorem 2. Then (I) follows from (4).

REFERENCES