Towards a mathematical theory of machine discovery from facts

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Abstract

This paper intends to give a theoretical foundation of machine discovery from facts. We point out that the essence of a computational logic of scientific discovery or a logic of machine discovery is the refutability of the entire spaces of hypotheses. We discuss this issue in the framework of inductive inference of length-bounded elementary formal systems (EFSs), which are a kind of logic programs over strings of characters and correspond to context-sensitive grammars in Chomsky hierarchy.

First we present some characterization theorems on inductive inference machines that can refute hypothesis spaces. Then we show differences between our inductive inference and some other related inferences such as in the criteria of reliable identification, finite identification and identification in the limit. Finally we show that for any $n$, the class, i.e. hypothesis space, of length-bounded EFSs with at most $n$ axioms is inferable in our sense, that is, the class is refutable by a consistently working inductive inference machine. This means that sufficiently large hypothesis spaces are identifiable and refutable.

1. Introduction

In the middle of this century the logic of scientific discovery was deeply discussed by philosophers [26, 27]. Recently in Artificial Intelligence, especially in Cognitive Science, researchers are extensively discussing frameworks for scientific discovery from various viewpoints [36]. They have obtained a lot of rich results on the components of scientific behavior such as scientific knowledge structures and scientific activities. However, they look little dependent on the philosophical results.
Before going into such detailed discussions, we need to setup a computational logic of scientific discovery in a mathematical way so that we can precisely discuss what kinds of machine discovery can help. One of the best ways to do this should be to reexamine the philosophical results from computational viewpoints. In the present paper, we start with making the Popperian logic of scientific discovery computational.

The logic of scientific discovery is essentially a triad of problem discovery, solution invention and critical test [38]. Popper was mainly interested in the final stage but not so much in the first two stages. More exactly the Popperian logic of scientific discovery concentrated on the testability, falsifiability or refutability of hypotheses or scientific theories. He also asserted that scientific theory should have been refuted by observed facts and any such theory could by no means be verified [26]. Thus, we tentatively believe the current theory until we face with an observation which is inconsistent with the theory.

His assertion had an influence on the studies of inductive inference started with Gold's identification in the limit, some of which is called Popperian induction [4]. A Popperian inductive inference machine [13] produces only recursive programs as its guesses, which makes each hypothesis testable and refutable. Then the consistent and conservative inductive inference can be viewed as a computational realization of the Popperian notion of refutability. In inductive inference, the inference machine requires data or facts from time to time and produces hypotheses from time to time. The hypotheses produced by the machine are to be consistent with the facts read so far, and each of them is to be refuted when the machine faces with inconsistent data or facts. Here we note that there still exist other possibilities to realize Popper's refutation principle in the context of inductive inference, even when an inductive inference machine produces nonrecursive programs.

Thus the Popperian logic of scientific discovery can be viewed as a basis of modern inductive inference studies. And inductive inference is a mathematical basis of machine learning. Then what should be a logic of machine discovery or a computational logic of scientific discovery?

The machine discovery we are concerned with in this paper is to make computers discover some scientific theories from given data or facts. Hence, machine learning should be a key technology for machine discovery. In fact, by using machine learning techniques many results have been reported by many authors [36]. In machine learning, first we must select a hypothesis space from which the learning machine proposes theories or hypotheses. The space is naturally required to be large, but to make the learning efficient it is required to be small. As far as data or facts are presented according to a hypothesis that is unknown but guaranteed to be in the space as in the ordinary inductive inference, the machine will eventually identify the hypothesis, and hence no problem may arise. In machine discovery, however, we cannot assume this. God knows whether or not a hypothesis behind the data or facts belongs to the space.

If the hypothesis is not in the space, most learning machines will continue for ever to search the space for a new hypothesis. Usually, we cannot know the time when to stop.
such an effective searching. This is the most crucial problem we must solve in realizing machine discovery systems. In machine discovery the sequences of data or facts are given at first independently of the space. We cannot give in advance the space that includes the desired theory. As stated above, however, we have to keep the space as small as possible without any guarantee that the theory is in it. If the learning machine can explicitly tell us that there are no theories in the space which explain the given sequence, the machine will work for machine discovery.

Hence, the essence of a computational logic of scientific discovery should be that the entire hypothesis space is refutable by a sequence of observed data or facts. If there exist rich hypothesis spaces that can be refuted, we can give a space and a sequence to the machine, and then we can just wait for an output from it. The machine will discover a hypothesis which is producing the sequence if it is in the space, otherwise it will refute the whole space and stop. When the space is refuted, we may give another space to the machine and try to make such a discovery in the new space.

In this paper we choose inductive inference as the framework for machine learning. Then the machine discovery system is an inductive inference machine that can refute hypothesis spaces. In order to make our discussion clearer we also choose, as the hypothesis spaces, classes of elementary formal systems (EFSs) which are a kind of logic programs over strings of characters. Thus, we are concerned with formal languages in this paper. Moreover, we assume that every class in question be an indexed family of recursive languages, which is of special interest with respect to potential applications. This assumption is quite natural to make a grammar, i.e. a hypothesis or theory, refutable by an observation and also to generate grammars as hypotheses automatically and successively. Note here that any indexed family of recursive languages is identifiable in the limit from complete data, i.e., positive and negative data, but the class of all recursive languages is not such a family. Hence, it is reasonable to choose some subclasses of the recursive languages as the hypothesis spaces for machine learning and machine discovery.

If the class is a finite set of recursive languages, it is trivially refutable from complete data. Also if the class contains all finite languages, it is easily shown not to be refutable. Then are there any meaningful classes, i.e., hypothesis spaces, that are identifiable and refutable? We give a positive answer to this question. We will call such classes to be refutably inferable.

First we show some characterizations of such inductive inference. Then we show that some sufficiently large classes of normal languages are refutably inferable from complete data, while the classes that are refutably inferable only from positive data are very small.

This paper is organized as follows: Section 2 presents basic notions and definitions necessary for our discussion. In Section 3, we discuss some conditions on refutable inferability from positive data or complete data. Concerning refutable inferability from positive data, we present some necessary and sufficient conditions and reveal that the power is very small. Then in Section 4, we show the differences between the inferable classes in the criteria of refutable identification, reliable identification, finite
identification and identification in the limit. Among them the reliable identification is the only inference that deals with sequences from the hypotheses not in the given spaces. However, as we will see in Section 2, the reliable inference machine does not tell us that the sequence is not in the space, but it just does not converge to any hypothesis in the space. In Sections 5 and 6, we present some natural classes that are refutably inferable from complete data. First we show that a class which consists of unions of at most \( n \) concepts from \( n \) classes is refutably inferable from complete data if each class satisfies a certain condition. Then we show our main result: The classes definable by length-bounded EFSs with at most \( n \) axioms are refutably inferable from complete data, and reveal that there are sufficiently large classes that are refutably inferable from complete data.

2. Preliminaries

We start with basic definitions and notions on inductive inference of indexed families of recursive concepts.

Let \( U \) be a recursively enumerable set to which we refer as a universal set. Then we call \( \text{LE } U \) a concept. In case the universal set \( U \) is the set \( \Sigma^+ \) of all nonnull finite strings over a finite alphabet \( \Sigma \), we also call \( \text{LE } U \) a language.

Definition. Let \( \mathbb{N} = \{1, 2, \ldots \} \) be the set of all natural numbers. A class \( \mathcal{C} = \{L_i\}_{i \in \mathbb{N}} \) of concepts is said to be an indexed family of recursive concepts if there is a recursive function \( f: \mathbb{N} \times U \rightarrow \{0, 1\} \) such that

\[
f(i, w) = \begin{cases} 1 & \text{if } w \in L_i, \\ 0 & \text{otherwise.} \end{cases}
\]

In what follows, we assume that a class of concepts is an indexed family of recursive concepts without any notice, and identify a class with hypothesis space.

Definition. A positive presentation, or a text, of a nonempty concept \( L \) is an infinite sequence \( w_1, w_2, \ldots \) of elements in the universal set \( U \) such that \( \{w_1, w_2, \ldots \} = L \). A complete presentation, or an informant, of a concept \( L \) is an infinite sequence \( (w_1, t_1), (w_2, t_2), \ldots \) of elements in \( U \times \{+, -, \} \) such that \( \{w_i | t_i = +, i \geq 1 \} = L \) and \( \{w_i | t_i = -, i \geq 1 \} = L^c = U \setminus L \). In what follows, \( \sigma \) or \( \delta \) denotes a positive or complete presentation, and \( \sigma [n] \) denotes the \( \sigma \)'s initial segment of length \( n \geq 1 \). For a positive or complete presentation \( \sigma \), each element in \( \sigma \) is called a fact. For a positive presentation \( \sigma \), \( \sigma [n]^+ \) denotes the set of all facts in \( \sigma [n] \). For a complete presentation \( \sigma \), \( \sigma [n]^+ \) (resp., \( \sigma [n]^− \)) denotes the set of all elements in the universal set \( U \) that appear in \( \sigma [n] \) with the sign + (resp., the sign −), that is, \( \sigma [n]^+ = \{w_i | \sigma(w_i, +) \in \sigma [n]\} \) and \( \sigma [n]^− = \{w_i | \sigma(w_i, -) \in \sigma [n]\} \).
A set $T$ is said to be consistent with a concept $L$ if $T \subseteq L$. A pair $(T, F)$ of sets is said to be consistent with a concept $L$, if $T \subseteq L$ and $F \subseteq L^\complement$. For a positive presentation $\sigma$ and for $n \geq 1$, the finite sequence $\sigma[n]$ is said to be consistent with a concept $L$ if $\sigma[n]^+ \subseteq L$. For a complete presentation $\sigma$ and for $n \geq 1$, the finite sequence $\sigma[n]$ is said to be consistent with a concept $L$ if $\sigma[n]^+ \subseteq L$ and $\sigma[n]^- \subseteq L^\complement$.

For two sequences $\psi_1$ and $\psi_2$, the sequence which is obtained by concatenating $\psi_1$ with $\psi_2$ is denoted by $\psi_1 \cdot \psi_2$.

Here we note that for a class $\mathcal{C} = \{L_i\}_{i \in \mathbb{N}}$ and explicitly given finite set $T, F \subseteq U$, whether or not $T \subseteq L_i$ and whether or not $(T, F)$ is consistent with $L_i$ are recursively decidable for any index $i$, because for any $w \in U$, whether or not $w \in L_i$ is recursively decidable.

The notion of consistency was introduced by Birzdin [11] and Blum and Blum [12].

**Definition.** An inductive inference machine (IIM) is an effective procedure, or a certain type of Turing machine, which requests inputs from time to time and produces positive integers from time to time. An inductive inference machine that can refute hypothesis spaces (RIIM) is an effective procedure, or a certain type of Turing machine, which requests inputs from time to time and either (i) produces positive integers from time to time or (ii) refutes the class and stops after producing some positive integers. The outputs produced by a machine are called guesses.

For an IIM or an RIIM $M$ and a nonempty initial segment $\sigma[n] = w_1, w_2, \ldots, w_n$ of a positive or complete presentation, we define $M(\sigma[n])$ as follows: Initialize and start $M$. If it requests a fact for the $i$th time with $1 \leq i \leq n$, then feed $w_i$ and continue the execution.

(I) In case $M$ requests the $(n+1)$st fact, or it stops after it requested the $n$th fact. If it produces a positive integer or the 'refutation' sign after it requested the $n$th fact, then let $M(\sigma[n])$ be the last integer or the 'refutation' sign produced by $M$, otherwise, let $M(\sigma[n]) = 0$.

(II) In case $M$ stops before requesting the $n$th fact, let $M(\sigma[n]) = 0$.

(III) Otherwise, leave $M(\sigma[n])$ undefined.

To denote the latest valid output of an IIM or an RIIM on input $\sigma[n]$, we introduce the notation of $\bar{M}(\sigma[n])$.

For an IIM or an RIIM $M$, we define $\bar{M}(\sigma[n])$ as follows.

(I) Suppose $M(\sigma[n])$ is defined: (i) If there is the 'refutation' sign in the sequence $M(\sigma[1]), M(\sigma[2]), \ldots, M(\sigma[n])$, then let $\bar{M}(\sigma[n])$ be the 'refutation' sign. (ii) Otherwise, if there is a positive integer in the sequence $M(\sigma[1]), M(\sigma[2]), \ldots, M(\sigma[n])$, then let $\bar{M}(\sigma[n])$ be the last positive integer in the sequence, otherwise let $\bar{M}(\sigma[n]) = 0$.

(II) Otherwise, leave $\bar{M}(\sigma[n])$ undefined.

For an IIM or RIIM $M$, it is easy to see that for any $n \geq 1$, if $M(\sigma[n])$ is defined, then $M(\sigma[1]), M(\sigma[2]), \ldots, M(\sigma[n-1])$ are also defined.
The intended interpretation is as follows: (i) In case $M(\sigma[n])$ is defined as the 'refutation' sign, $M$ refutes the class concerned. (ii) In case $M(\sigma[n])$ is defined as a positive integer $i$, $M$ guesses the $i$th concept in the class. (iii) In case $M(\sigma[n])$ is defined as the integer 0, $M$ makes no guess. (iv) In case $M([n])$ is undefined, $M$ is out of control.

For two nonempty finite sequences $\psi_1$ and $\psi_2$, we write $M(\psi_1) = M(\psi_2)$, if (i) both $M(\psi_1)$ and $M(\psi_2)$ are undefined, or (ii) both $M(\psi_1)$ and $M(\psi_2)$ are defined and their values are identical. For a nonempty finite sequence $\psi$ and an integer $i$, we write $M(\psi) = i$ (resp., $M(\psi) > i$) if $M(\psi)$ is defined as an integer and the value of $M(\psi)$ is equal to $i$ (resp., greater than $i$). In a similar way, we also define the relations $\bar{M}(\psi_1) = \bar{M}(\psi_2), \bar{M}(\psi) = i$ and $\bar{M}(\psi) > i$.

Hereafter, for a concept $L \subseteq U$, we write $L \subseteq \emptyset$, if there is an $L_i \subseteq \emptyset$ such that $L_i = L$.

In the present paper, we assume that any subset of the universal set can be the target concept, when we are considering reliable inference and refutable inference, respectively. Thus an IIM and an RIIM have to deal with any sequence. This assumption comes from the main motivation of this work. Here we note that, however, the choice of the family from which a target concept expected to come may have great influence on the learning power (cf. [12]).

**Definition.** An IIM or an RIIM $M$ is said to **converge to an index $i$ for a positive or complete presentation $\sigma$**, if there is an $n \geq 1$ such that for any $m \geq n, M(\sigma[m]) = i$.

An RIIM $M$ is said to **refute a class $C$ from a positive or complete presentation $\sigma$**, if there is an $n \geq 1$ such that $M(\sigma[n])$ is the 'refutation' sign. In this case we also say that $M$ refutes the class $C$ from $\sigma[n]$.

Let $C = \{L_i\}_{i \in \mathbb{N}}$ be a class. For a concept $L_i \subseteq \emptyset$ and a positive or complete presentation $\sigma$ of $L_i$, an IIM or an RIIM $M$ is said to **infer the concept $L_i$ w.r.t. $C$ in the limit from $\sigma$** if $M$ converges to an index $j$ with $L_j = L_i$ for $\sigma$.

(a) An IIM $M$ is said to **infer a class $C = \{L_i\}_{i \in \mathbb{N}}$ in the limit from positive data (resp., complete data)**, if for any $L_i \subseteq \emptyset$, $M$ infers $L_i$ w.r.t. $C$ in the limit from any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L_i$. A class $C$ is said to be **inferable in the limit from positive data (resp., complete data)**, if there is an IIM $M$ which infers the class $C$ in the limit from positive data (resp., complete data).

(b) An IIM $M$ is said to **reliably infer a class $C$ from positive data (resp., complete data)** if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$) and any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$, (i) if $L \subseteq \emptyset$, then $M$ infers $L$ w.r.t. $C$ in the limit from $\sigma$, (ii) otherwise, $M$ does not converge to any index for $\sigma$. A class $C$ is said to be **reliably inferable from positive (resp., complete data)**, if there is an IIM $M$ which reliably infers the class $C$ from positive data (resp., complete data).

(c) An RIIM $M$ is said to **refutably infer a class $C$ from positive data (resp., complete data)** if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$) and any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$, (i) if $L \subseteq \emptyset$, then $M$ infers $L$ w.r.t. $C$ in the limit from $\sigma$, (ii) otherwise, $M$ refutes the class.
A class $\mathcal{C}$ is said to be refutably inferable from positive data (resp., complete data) if there is an RIIM $M$ which refutably infers the class $\mathcal{C}$ from positive data (resp., complete data).

Note that when we consider inductive inference from positive data, we restrict every concept to a nonempty concept, because we cannot make any positive presentation of the empty concept.

The notion of reliable inference was introduced by Minicozzi [22] and Blum and Blum [12] for function learning, and it was adapted to language learning by Osherson et al. [25] and Sakurai [29]. By definition, it is easy to see that if a class $\mathcal{C}$ is refutably inferable from positive data (resp., complete data), then $\mathcal{C}$ is reliably inferable from positive data (resp., complete data). However, the converse does not hold as shown in Section 4.

Note that if an inference machine $M$ does not converge to any index for a positive or complete presentation $\sigma$, then $M(\sigma[n])$ may be undefined for some $n \geq 1$. On the other hand, we implicitly use the following proposition in showing some properties on inferability.

**Proposition 1.** (a) Assume that an IIM $M$ infers a class $\mathcal{C}$ in the limit from positive data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U$ if there exists an $L \in \mathcal{C}$ such that all elements in $\psi$ are in $L$, then $M(\psi)$ is always defined [14].

(b) Assume that an RIIM $M$ refutably infers a class $\mathcal{C}$ from positive data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U$, $M(\psi)$ is always defined (based on [12]).

Similar statements are also valid for the case of complete data.

The above proposition claims that as far as we feed facts that are from a certain concept in the class, an IIM either (i) successively requests another facts in a finite time forever or (ii) stops in a finite time after producing some guesses. On the other hand, even when we feed any facts that may not be from any concept in the class, an RIIM either (i) successively requests another facts in a finite time forever or (ii) stops in a finite time after producing some guesses.

Since we are considering an indexed family of recursive concepts, every class can be inferred from complete data by a simple enumeration method. However, we cannot take the class of all recursive concepts as a hypothesis space, because the following proposition holds (cf. e.g. [28]).

**Proposition 2.** The class $\mathcal{C}$ of all recursive concepts is not an indexed family of recursive concepts.

In case an RIIM $M$ is fed a positive or complete presentation of a nonrecursive concept, $M$ should refute the class. Therefore, even if we could take the class of all recursive concepts, it would be still significant to consider refutable inferability.

For reliable inferability, the following theorem holds.
Theorem 3 (Osherson [25], Sakurai [29]). A class $\mathcal{C}$ is reliably inferable from positive data if and only if $\mathcal{C}$ contains no infinite concept.

(b) Every class is reliably inferable from complete data.

In this paper, an effective procedure is said to recursively generate a finite set $T$ if it enumerates all elements in $T$ and then halts (cf. [18]). An effective procedure $P$ is said to be uniformly and recursively generate a finite-set-valued function $F$ with parameters $x_1, \ldots, x_n$ if $P$ on any input $(x_1, \ldots, x_n)$ in the domain of $F$ recursively generates the finite set $F(x_1, \ldots, x_n)$. A finite-set-valued function $F$ with parameters $x_1, \ldots, x_n$ is said to be uniformly and recursively generable if there is an effective procedure which uniformly and recursively generates the function $F$.

3. Characterizations

In order to characterize the refutable inferability, we need the following lemma.

Lemma 4. Let $M$ be an RIIM which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data). Then for a nonempty concept $L$ (resp., a concept $L$), for a positive presentation $\sigma$ (resp., a complete presentation $\sigma$) of $L$ and for $n \geq 1$, if $M$ refutes the class $\mathcal{C}$ from $\sigma[n]$, then $\sigma[n]$ is not consistent with any $L \in \mathcal{C}$.

Proof. Assume that an RIIM $M$ refutes a class $\mathcal{C}$ from $\sigma[n]$. Then suppose that there is an $L \in \mathcal{C}$ such that $\sigma[n]$ is consistent with $L$. Let $\delta$ be a positive presentation (resp., a complete presentation) of $L$. Then the infinite sequence $\sigma[n] \cdot \delta$ becomes a positive presentation (resp., a complete presentation) of $L$. Therefore, $M$ cannot infer $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma[n] \cdot \delta$, which contradicts the assumption. $\Box$

By Lemma 4, we obtain the following proposition.

Proposition 5. (a) If a class $\mathcal{C}$ is refutably inferable from positive data, then

(3.1) for any nonempty concept $L \notin \mathcal{C}$, there is a finite set $T \subseteq L$ such that $T$ is not consistent with any $L \in \mathcal{C}$.

(b) If a class $\mathcal{C}$ is refutably inferable from complete data, then

(3.2) for any concept $L \notin \mathcal{C}$, there are finite sets $T \subseteq L$ and $F \subseteq L^*$ such that $(T, F)$ is not consistent with any $L \in \mathcal{C}$.

Proof. We give only the proof of (a). The proof of (b) can be given in a similar way.

Assume that an RIIM $M$ refutably infers a class $\mathcal{C}$ from positive data. Let $L \in \mathcal{C}$ be a nonempty concept, and let $\sigma$ be an arbitrary positive presentation of $L$. By definition, there is an $n \geq 1$ such that $M$ refutes the class $\mathcal{C}$ from $\sigma[n]$. Let $T = \sigma[n]^+$. Then by Lemma 4, $T$ is not consistent with any $L \in \mathcal{C}$. $\Box$
Corollary 6. If a class $\mathcal{C}$ contains all nonempty finite concepts, then $\mathcal{C}$ is not refutably inferable from positive data and complete data, respectively.

Proof. We only give the proof of the case of complete data. The proof for positive data can be given in a similar way.

Assume that a class $\mathcal{C}$ contains all nonempty finite concepts. Then let $L \not\in \mathcal{C}$ be a concept. Let $T \subseteq L$ and $F \subseteq L^c$ be finite sets. (i) In case $T$ is not empty, there is an $L_i \in \mathcal{C}$ with $T = L_i$, and it follows that $(T, F)$ is consistent with $L_i$. (ii) In case $T$ is empty, since $F$ is a finite set, there is an $L_i \in \mathcal{C}$ such that $F \subseteq I_i^c$, which means $(T, F)$ is consistent with $L_i$. Therefore, by Proposition 5, we see that $\mathcal{C}$ is not refutably inferable from complete data.

In characterizing the refutable inferability, the notion of consistency plays an important role.

Definition (Based on [11, 12]). Let $\mathcal{C} = \{L_i\}_{i \in \mathbb{N}}$ be a class. An RIIM $M$ which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data) is said to be consistent if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$), any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$ and any $n \geq 1$, (i) if $M(\sigma[n])$ is the 'refutation' sign, then $\sigma[n]$ is not consistent with any $L_i \in \mathcal{C}$, (ii) if $M(\sigma[n]) > 0$, then $\sigma[n]$ is consistent with $L_{M(\sigma[n])}$.

An RIIM $M$ which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data) is said to be responsive if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$), any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$ and any $n \geq 1$ if $M$ does not refute the class $\mathcal{C}$ from $\sigma[n]$, then $M(\sigma[n]) > 0$ holds, that is, while $M$ does not refute the class, $M$ produces a guess between any two input requests in the computation of $M$ on input $\sigma$ (cf. [2]).

A class $\mathcal{C}$ is said to be refutably, consistently and responsively inferable from positive data (resp., complete data) if there is a consistently and responsively working RIIM which refutably infers the class $\mathcal{C}$ from positive data (resp., complete data).

In a similarly way, we can also define a consistently or responsibly working IIM (cf. [2]).

Here we note that a consistently and responsively working RIIM refutes a class immediately after the observed data become not consistent with any concept in the class.

Since we are considering an indexed family of recursive concepts, we can easily show that if a class $\mathcal{C}$ is inferable in the limit from positive data or complete data, then it can be achieved by a consistently and responsively working IIM. Furthermore, as shown later, if a class $\mathcal{C}$ is refutably inferable from positive data or complete data, then it can be achieved by a consistently and responsively working RIIM.
Definition. For a finite set $T \subseteq U$, let
\[
econs_p(T) = \begin{cases} 
1 & \text{if there exists an } L_i \in \mathcal{C} \text{ such that } T \text{ is consistent with } L_i, \\
0 & \text{otherwise}.
\end{cases}
\]

For finite sets $T, F \subseteq U$, let
\[
econs_c(T, F) = \begin{cases} 
1 & \text{if there exists an } L_i \in \mathcal{C} \text{ such that } (T, F) \text{ is consistent with } L_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Proposition 7. If a class $C$ is refutably inferable from positive data, then
\[(3.3)\] the function $econs_p$ for $\mathcal{C}$ is recursive.

Proof. Assume that an RIIM $M$ refutably infers $\mathcal{C}$ from positive data. Let $T = \{w_1, \ldots, w_n\} \subseteq U$ be a nonempty finite set, and put $\sigma = w_1, w_2, \ldots, w_n, w_1, w_2, \ldots$. Clearly, the infinite sequence $\sigma$ is a positive presentation of the concept $T$.

Thus when we successively feed $\sigma$, $M$ either refutes the class $\mathcal{C}$ or produces an index $i$ with $T = L_i$ after producing some positive integers. Therefore, we can recursively compute the function $econs_p(T)$ for $\mathcal{C}$ as follows: Simulate $M$ with presenting $\sigma$.

During the simulation, (i) if $M$ refutes the class $\mathcal{C}$, then output 0 and stop, (ii) if $M$ produces an index $i$ with $T \subseteq L_i$, then output 1 and stop, (iii) otherwise, continue the simulation. We note that whether or not $T \subseteq L_i$ is recursively decidable. By Lemma 4, it is clear that the above output agrees with the $econs_p(T)$. □

Theorem 8 (Based on [16]). If a class $\mathcal{C}$ is refutably inferable from complete data, then
\[(3.4)\] the function $econs_p$ for $\mathcal{C}$ is recursive.

Proof. Assume that an RIIM $M$ refutably infers $\mathcal{C}$ from complete data. Let $T = \{w_1, \ldots, w_n\} \subseteq U$ and $F = \{w_{n+1}, \ldots, w_m\} \subseteq U$ be finite sets.

It is easy to see that if $T \cap F \neq \emptyset$, then $econs_c(T, F) = 0$. Thus in what follows, we assume $T \cap F = \emptyset$.

Let $\psi_0 = (w_1, +), (w_n, +), (w_{n+1}, -), \ldots, (w_m, -)$, and let $u_1, u_2, \ldots$ be an effective enumeration of $U \setminus (T \cup F)$. Then let $\mathcal{F}$ be the set of all initial segments of $u_1, u_2, \ldots$ coupled with + and −, that is, $\mathcal{F} = \{\emptyset; (u_1, +); (u_1, -); (u_1, +); (u_1, -); (u_1, +); (u_2, +); (u_1, +); (u_2, -); (u_1, -); (u_2, +); (u_1, -); (u_2, -); \ldots\}$. We define the binary relation $\subseteq$ over $\mathcal{F}$ as follows: $\psi_1 \subseteq \psi_2$ if and only if $\psi_1$ is an initial segment of $\psi_2$. This gives a partial ordering of $\mathcal{F}$, and it becomes a binary tree, which can be diagramed in Fig. 1.

Then we define a subtree $\mathcal{S}$ of $\mathcal{F}$ as follows:
\[
\mathcal{S} = \{\psi \in \mathcal{F} | \bar{M}(\psi_0 \cdot \psi) \neq \text{‘refutation’}\}.
\]

Here we note that if $\psi_1 \subseteq \psi_2$ and $\bar{M}(\psi_0 \cdot \psi_1) \neq \text{‘refutation’}$, then $\bar{M}(\psi_0 \cdot \psi_2) = \text{‘refutation’}$.

Claim A. The subtree $\mathcal{S}$ is finite if and only if there is no concept $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. 
Proof of the claim. (I) The ‘if’ part. Assume that there is no concept $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. Then suppose that the subtree $\mathcal{S}$ has an infinite branch, say $\psi_1, \psi_2, \ldots$. By the construction of the subtree $\mathcal{S}$, there is an infinite sequence $t_1, t_2, \ldots \in \{+,-\}$ such that for any $i \geq 1$, $\psi_i = (u_1, t_1), (u_2, t_2), \ldots, (u_i, t_i)$. Put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \ldots$, and let $L = \{u_i \in U \mid (u_i, +) \in \sigma, i \geq 1\}$ be a concept. Then $(T, F)$ is consistent with $L$, and $\sigma$ is a complete presentation of $L$. By assumption, $L$ is not in $\mathcal{C}$, and it follows that $M$ refutes $\mathcal{C}$ from $\sigma$ for some $n \geq 1$. However, by the construction, there is a $j \geq 1$ such that $\sigma \uparrow n = \sigma_j$. This contradicts the assumption of $\psi_j \in \mathcal{S}$.

Thus we see that the subtree $\mathcal{S}$ has no infinite branch, and it follows by Endlichkeitslerma for trees with finite branching (cf. [28, Exercise 9.40]) that the subtree $\mathcal{S}$ is finite.

(II) The ‘only if’ part. Assume that the subtree $\mathcal{S}$ is finite. Therefore, the subtree $\mathcal{S}$ has no infinite branch. Then suppose that there is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. For $j \geq 1$, let $t_j = +$ if $u_j \in L_i$, otherwise let $t_j = -$. Then put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \ldots$. By the construction, $\sigma$ is a complete presentation of $L_i$. Thus $M$ does not refute $\mathcal{C}$ from $\sigma$. It is easy to see that this contradicts the assumption. $\square$

Claim B. There is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$ if and only if there is a $\psi \in \mathcal{S}$ such that $\bar{M}(\psi_0, \psi) > 0$ and $(T, F)$ is consistent with $L_{\bar{M}(\psi_0, \psi)}$.

Proof of the claim. The ‘if’ part is obvious. Thus we only give the proof of the ‘only if’ part. Assume that there is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. For $j \geq 1$, let $t_j = +$ if $u_j \in L_i$, otherwise let $t_j = -$. Then put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \ldots$. By the construction, $\sigma$ is a complete presentation of $L_i$. Since $M$ infers $L_i$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, it follows that there is an $n \geq 1$ such that $M(\sigma \uparrow n) > 0$ and $L_i = L_{\bar{M}(\sigma \uparrow n)}$. It is easy to see that there is a $\psi \in \mathcal{S}$ such that $\psi_0 \cdot \psi = \sigma \uparrow n$. $\square$
Therefore, we can compute $\text{econs}_c(T,F)$ as follows: Search for a node $\psi \in \mathcal{P}$ such that $\bar{M}(\psi_0 \cdot \psi) > 0$ and $(T,F)$ is consistent with $L_{M(\psi_0 \cdot \psi)}$. By Claims A and B, we see that one of the following two cases must happen:

1. such a node is found,
2. the subtree $\mathcal{P}$ is confirmed to be finite.

In the case (1), put $\text{econs}_c(T,F) = 1$ and in the case (2), put $\text{econs}_c(T,F) = 0$. It is easy to see that the result obtained agrees with the definition of $\text{econs}_c(T,F)$. This completes the proof. $\square$

We note that the above proof heavily depends on the fact that any RIIM which refutably infers a class has to handle any sequence over the universal set (cf. Proposition 1).

**Theorem 9.** (a) If a class $\mathcal{C}$ satisfies the following three conditions, then $\mathcal{C}$ is refutably, consistently and responsively inferable from positive data.

1. For any nonempty concept $L \notin \mathcal{C}$, there is a finite set $T \subseteq L$ such that $T$ is not consistent with any $L_i \in \mathcal{C}$.
2. The function $\text{econs}_p$ for $\mathcal{C}$ is recursive.
3. The class $\mathcal{C}$ is inferable in the limit from positive data.

(b) If a class $\mathcal{C}$ satisfies the following two conditions, then $\mathcal{C}$ is refutably, consistently and responsively inferable from complete data.

1. For any concept $L \notin \mathcal{C}$, there are finite sets $T \subseteq L$ and $F \subseteq L^c$ such that $(T,F)$ is not consistent with any $L_i \in \mathcal{C}$.
2. The function $\text{econs}_c$ for $\mathcal{C}$ is recursive.

**Proof.** We only give the proof of (a). The proof of (b) can be given in a similar way, where we note that every indexed family of recursive concepts is consistently and responsively inferable in the limit from complete data.

Assume that a class $\mathcal{C}$ satisfies the above three conditions (3.1), (3.3) and (3.5). Let $M$ be an IIM which infers $\mathcal{C}$ in the limit from positive data. Without loss generality, we can assume that it works consistently and responsively. Then let us consider the procedure in Fig. 2.

Assume that we feed a positive presentation $\sigma$ of a nonempty concept $L$ to the procedure.

1. In case $L \notin \mathcal{C}$, by condition (3.1), there is a finite set $T \subseteq L$ such that $\text{econs}_p(T) = 0$, and by the definition of a positive presentation, we see that there is an $n \geq 1$ such that $T \subseteq \sigma[n]^+$. Therefore, the procedure refutes the class $\mathcal{C}$ from $\sigma[n]$ and stops.
2. (II) In case $L \in \mathcal{C}$, as easily seen, for any finite set $T \subseteq L$, the value of $\text{econs}_p(T)$ never becomes 0. Since the IIM $M$ infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, it follows that the procedure infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

Furthermore, it is easy to see that the procedure works consistently and responsively. $\square$

By Propositions 5 and 7, Theorems 8 and 9, we have the following corollary.
Procedure RIIM \( M \)
begin
\( T = \emptyset; \)
repeat
read the next fact \( w \) and store it in \( T; \)
if \( \text{econs}_p(T) = 0 \) then
refute the class \( C \) and stop;
else begin
simulate \( M \) with presenting the fact \( w \) until requesting the next fact;
if \( M \) produces a guess then output it;
end;
forever;
end.

Fig. 2. An inference machine that can refute hypothesis spaces from positive data.

Corollary 10. (a) For a class \( C \), the following three statements are equivalent:

(i) \( C \) is refutably inferable from positive data;
(ii) \( C \) is refutably, consistently and responsively inferable from positive data;
(iii) \( C \) satisfies conditions (3.1), (3.3) and (3.5).

(b) For a class \( C \), the following three statements are equivalent:

(i) \( C \) is refutably inferable from complete data;
(ii) \( C \) is refutably, consistently and responsively inferable from complete data;
(iii) \( C \) satisfies conditions (3.2) and (3.4).

The above conditions heavily depend on each concept rather than the properties of the class concerned. Thus we need to investigate another conditions concerned with the properties of the class itself.

Definition. A class \( C \) is said to be closed under the subset operation if for any \( L_1 \in C \), all nonempty subsets of \( L_1 \) are also in the class \( C \).

A class \( C \) is said to be of finite hierarchy if there is no infinite sequence of concepts \( L_{i_1}, L_{i_2}, \ldots \in C \) such that \( L_{i_1} \not\subset L_{i_2} \not\subset \ldots \).

As easily seen, if a class \( C \) has the so-called finite elasticity [40], then \( C \) is of finite hierarchy. Moreover, if a class \( C \) is closed under the subset operation and it is of finite hierarchy, then \( C \) contains no infinite concept, as shown in the proof of Lemma 12.

Lemma 11. If a class \( C \) is refutably inferable from positive data, then \( C \) satisfies the following two conditions:

(3.6) \( C \) is closed under the subset operation,
(3.7) \( C \) is of finite hierarchy.
Proof. (I) Suppose that condition (3.6) does not hold, that is, there is a nonempty concept \( L \) such that \( L \not\in \mathcal{C} \) and \( L \subseteq L_i \) for some \( L_i \in \mathcal{C} \). Then every subset of \( L \) is consistent with \( L_i \). Therefore, by Proposition 5, \( \mathcal{C} \) is not refutably inferable from positive data.

(II) Assume that a class \( \mathcal{C} \) is refutably inferable from positive data. Then by above (I), condition (3.6) holds.

Claim. The class \( \mathcal{C} \) contains no infinite concept.

Proof of the claim. Suppose that \( \mathcal{C} \) contains an infinite concept \( L_i \). Then we see that \( \mathcal{C} \) contains all nonempty finite subset \( L_i \), because \( \mathcal{C} \) is assumed to be closed under the subset operation. We can show that any RIIM does not infer \( L_i \) w.r.t. \( \mathcal{C} \) in the limit from a certain positive presentation of \( L_i \) as in [14]. This is a contradiction. □

Lemma 12. If a class \( \mathcal{C} \) satisfies the following three conditions, then \( \mathcal{C} \) is refutably, consistently and responsively inferable from positive data.

(3.6) \( \mathcal{C} \) is closed under the subset operation.

(3.7) \( \mathcal{C} \) is of finite hierarchy.

(3.3) The function \( \text{econs}_p \) for \( \mathcal{C} \) is recursive.

Proof. Assume that a class \( \mathcal{C} \) satisfies the above three conditions. Then by Theorem 9, it suffices for us to show that \( \mathcal{C} \) satisfies conditions (3.1) and (3.5).

Claim. The class \( \mathcal{C} \) contains no infinite concept.

Proof of the claim. Suppose that \( \mathcal{C} \) contains an infinite concept \( L_i \). Then all subsets of \( L_i \) are also in \( \mathcal{C} \), because \( \mathcal{C} \) is assumed to be closed under the subset operation. Therefore, there is an infinite sequence of concepts \( L_{i_1}, L_{i_2}, \ldots \in \mathcal{C} \) such that \( L_{i_1} \subseteq L_{i_2} \subseteq \cdots \), which contradicts condition (3.7). □

(I) By this claim, we see that \( \mathcal{C} \) is inferable in the limit from positive data, that is, condition (3.5) is satisfied (cf. [2]).

(II) Let \( L \not\in \mathcal{C} \) be a nonempty concept. (i) In case \( L \) is a finite concept, suppose there exists an \( L_i \in \mathcal{C} \) such that \( L \subseteq L_i \). Then \( L \in \mathcal{C} \) holds, because \( \mathcal{C} \) is assumed to be closed under the subset operation. This contradicts the assumption. Therefore, \( L \) is not consistent with any \( L_i \in \mathcal{C} \). Hence, it suffices for us to take \( L \) itself as \( T \) in condition
(3.1). (ii) In case L is an infinite concept, suppose that condition (3.1) does not hold. Then for any finite set T \subseteq L, there is an \( L_i \in \mathcal{G} \) such that \( T \) is consistent with \( L_i \). However, this implies that the above T’s themselves are in \( \mathcal{G} \), because \( \mathcal{G} \) is assumed to be closed under the subset operation. To sum up, every finite set \( T \subseteq L \) is in \( \mathcal{G} \), and it follows that there is an infinite sequence \( L_{i_1}, L_{i_2}, \ldots \) such that \( L_{i_1} \subseteq L_{i_2} \subseteq \cdots \), because \( L \) is an infinite concept. This contradicts condition (3.7). \( \square \)

By Corollary 10 and Lemmas 11 and 12, we have the following theorem.

**Theorem 13.** For a class \( \mathcal{C} \), the following four statements are equivalent:

(i) \( \mathcal{C} \) is refutably inferable from positive data;
(ii) \( \mathcal{C} \) is refutably, consistently and responsively inferable from positive data;
(iii) \( \mathcal{C} \) satisfies conditions (3.1), (3.3) and (3.5);
(iv) \( \mathcal{C} \) satisfies conditions (3.6), (3.7) and (3.3).

**Example 1.** Let \( \mathcal{F}_n \) be the class of all nonempty finite concepts each of which cardinality is just \( n \). Then this class is not refutably inferable from positive data for any \( n \geq 2 \), because it is not closed under the subset operation.

In contrast with the above class, let \( \mathcal{F}_{\leq n} \) be the class of all nonempty finite concepts each of which cardinality is at most \( n \). We note that \( \mathcal{F}_{\leq 1} = \mathcal{F}_1 \). As easily seen, the function \( \text{econs}_p \) for \( \mathcal{F}_{\leq n} \) is recursive, because \( \text{econs}_p(T) = 1 \) if and only if the cardinality of \( T \) is not greater than \( n \). Furthermore, this class is closed under the subset operation and of finite hierarchy. Therefore, \( \mathcal{F}_{\leq n} \) is refutably inferable from positive data.

Lastly let \( \mathcal{F}_\infty \) be the class of all nonempty finite concepts (cf. Corollary 6). The function \( \text{econs}_p \) for \( \mathcal{F}_\infty \) is recursive, because \( \text{econs}_p(T) = 1 \) for any finite set \( T \subseteq U \). This class is closed under the subset operation but is not of finite hierarchy. Therefore, \( \mathcal{F}_\infty \) is not refutably inferable from positive data.

Here we present a sufficient condition for a class to be refutably inferable from complete data, which is very strict but widely applicable as shown in Sections 5 and 6.

The following lemma is basic.

**Lemma 14.** Let \( n \geq 1 \) be an integer, let \( L_1, \ldots, L_n \subseteq U \) be concepts, and let \( L \subseteq U \) be a concept which differs from \( L_1, \ldots, L_n \). Then for any complete presentation \( \sigma \) of \( L \), there is an \( m \geq 1 \) such that \( \sigma[m] \) is not consistent with any concept \( L_i \) with \( 1 \leq i \leq n \).

**Definition.** Let \( \mathcal{C} = \{ L_i \}_{i \in \mathbb{N}} \) be a class, and let \( \mathcal{S} \) be a subclass of \( \mathcal{C} \). A set \( I \) of indices is said to be a cover-index set of \( S \), if the collection of all concepts each of which has an index in \( I \) is equal to \( \mathcal{S} \), that is, \( \mathcal{S} = \{ L_i \in \mathcal{C} \mid i \in I \} \).

**Theorem 15.** If a class \( \mathcal{C} \) satisfies the following two conditions, then \( \mathcal{C} \) is refutably inferable from complete data:

(3.8) For any \( w \in U \), there is a uniformly and recursively generable finite cover-index set of the subclass \( \{ L_i \in \mathcal{C} \mid w \in L_i \} \) of \( \mathcal{C} \).

(3.9) The class \( \mathcal{C} \) contains the empty concept as its member.
Proof. Assume that a class $\mathcal{C}$ satisfies conditions (3.8) and (3.9). Then let us consider the procedure in Fig. 3.

Roughly speaking, this algorithm works as follows: The while-loop (1) is devoted to waiting for the first positive fact with searching for an index of the empty concept by an enumeration method. When the first positive fact appears, it recursively generates a finite set of indices whose concepts can explain the fact. Then the for-loop (2) is devoted to striking off the indices whose concepts contradict observed facts. When all of the indices are struck off, the algorithm refutes the hypothesis space.

Assume that we feed a complete presentation $\sigma$ of a concept $L$ to the procedure.

(I) In case $L = \emptyset$, it is easy to see that the while-loop (1) never terminates and that the procedure infers the empty concept w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

(II) In case $L \neq \emptyset$, the procedure terminates the while-loop (1) in a finite time.

(i) In case $L \in \mathcal{C}$, it is easy to see that $L$ is in the subclass $\{L_i \in \mathcal{C} | w \in L_i\}$, and it follows that there is an index $j \in I$ such that $L_j = L$. Since $I$ is a finite set, we see by Lemma 14 that the for-loop (2) is eventually executed with $j \in I$ such that $L_j = L$, and

```
Procedure RIIM M;
begin
  T = \emptyset; F = \emptyset; i = 1;
  read_store (T, F);
  while T = \emptyset do begin
    while F \not\subseteq L_i do i = i + 1;
    output i;
    read_store (T, F);
  end;
  let \{w\} = T;
  recursively generate a cover-index set of \{L_i \in \mathcal{C} | w \in L_i\}, and set it to I;
  for each j \in I do
    while (T, F) is consistent with $L_j$ do begin
      output j;
      read_store (T, F);
    end;
    refute the class $\mathcal{C}$ and stop;
end;
Procedure read_store (T, F);
begin
  read the next fact (w, t);
  if t = '-' then $T = T \cup \{w\}$ else $F = F \cup \{w\}$;
end.
```

Fig. 3. An inference machine that can refute hypothesis spaces from complete data.
the while-loop (3) never terminates. That is, the procedure infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

(ii) In case $L \notin \mathcal{C}$, since $I$ is an explicitly given finite set, we see by Lemma 14 that the procedure refutes the class $\mathcal{C}$ from $\sigma$. \hfill \Box

Before proceeding to the next example, we briefly recall a pattern and a pattern language. For more details, please refer to [1, 2].

Fix a nonempty finite alphabet. A pattern is a nonnull finite string of constant and variable symbols. The pattern language $L(\pi)$ generated by a pattern $\pi$ is the set of all strings obtained by substituting nonnull strings of constant symbols for the variables in $\pi$. Since two patterns that are identical except for renaming of variables generate the same pattern language, we do not distinguish one from the other. We can enumerate all patterns recursively and whether or not $w \in L(\pi)$ for any $w$ and $\pi$ is recursively decidable. Therefore, we can consider the class of pattern languages as an indexed family of recursive concepts, where the pattern itself is considered to be an index.

**Example 2.** We consider the class $\mathcal{PdF}$ of pattern languages. As easily seen, the empty concept $L = \emptyset$ is not in $\mathcal{PdF}$. Furthermore, for any finite set $F \subseteq U$, there is an $L_i \in \mathcal{PdF}$ such that $(\emptyset, F)$ is consistent with $L_i$. In fact, let $l$ be the length of the longest string in $F$. Then $(\emptyset, F)$ is consistent with the language of the pattern $x_1 x_2 \ldots x_{l+1}$. Therefore, by Proposition 5, we see that $\mathcal{PdF}$ is not refutably inferable from complete data.

However, the class $\mathcal{PdF}$ satisfies condition (3.8). In fact, fix an arbitrary constant string $w$. As easily seen, if $w \in L(\pi)$, then $\pi$ is not longer than $w$. The set of all patterns shorter than a fixed length is a recursively generable finite set, and whether or not $w \in L(\pi)$ for any $w$ and $\pi$ is recursively decidable. Therefore, the set $\{\pi \mid w \in L(\pi)\}$ is a recursively generable finite set.

Thus, by Theorem 15, we see that if we add the empty concept to the class of pattern languages, then the obtained class is refutably inferable from complete data.

4. **Comparisons with other identifications**

In this section, by some distinctive examples of classes, we compare the criterion of refutable inference with some other criteria. This is motivated by the following question: What should we do if we face with facts that are inconsistent with a finitely inferred hypothesis?

In what follows, for the criterion of finite identification for a class of recursive concepts, please refer to [23, 18, 15]. For the purpose of comparing the inferability from positive data with the inferability from complete data, we assume that all concepts are *nonempty* throughout this section.
The following proposition is obvious, because for a complete presentation $\sigma$ of a nonempty concept $L$, we can effectively obtain a positive presentation of $L$ by getting rid of all negative facts of $\sigma$ and repeating a positive fact.

**Proposition 16.** If a class $C$ is inferable in the limit from positive data, then $C$ is also inferable in the limit from complete data.

Furthermore, the above assertion is still valid if we replace the phrase ‘inferable in the limit’ with the phrase ‘finitely inferable’, ‘reliably inferable’ and ‘refutably inferable’, respectively.

The following theorem presents an interesting relation between mind changes and information presentations.

**Theorem 17 (Lange and Zeugmann [17]).** If a class $C$ is finitely inferable from complete data, then $C$ is also conservatively inferable in the limit from positive data, and thus it is inferable in the limit from positive data.

In [19], more detailed results on mind changes and information presentations have been obtained.

In the following examples, we assume appropriate universal sets and indexing of the class.

**Example 3.** Again we consider the classes $\mathcal{F}_n$, $\mathcal{F}_{\leq n}$ and $\mathcal{F}_*$ defined in Example 1 (cf. Theorem 3).

(A.1) For any $n \geq 2$, the class $\mathcal{F}_n$ is not refutably inferable from complete data.

In fact, let $L \subseteq U$ be a concept with cardinality 1. It is easy to see that there are no finite sets $T \subseteq L$ and $F \subseteq L^c$ that satisfy condition (3.2) in Proposition 5.

(A.2) For any $n \geq 1$, the class $\mathcal{F}_n$ is reliably inferable from positive data.

(A.3) For any $n \geq 1$, the class $\mathcal{F}_n$ is finitely inferable from positive data.

(B.1) For any $n \geq 1$, the class $\mathcal{F}_{\leq n}$ is refutably inferable from positive data.

(B.2) For any $n \geq 1$, the class $\mathcal{F}_{\leq n}$ is reliably inferable from positive data.

(B.3) For any $n \geq 2$, the class $\mathcal{F}_{\leq n}$ is not finitely inferable from complete data.

(C.1) $\mathcal{F}_*$ is not refutably inferable from complete data (cf. Corollary 6).

(C.2) $\mathcal{F}_*$ is reliably inferable from positive data.

(C.3) $\mathcal{F}_*$ is not finitely inferable from complete data.

**Example 4.** Let $\Sigma = \{a\}$, $L_1 = \{a^j | j \geq 1\}$, and $L_i = \{a^j | 1 \leq j \leq i - 1\}$ for $i \geq 2$. Then let $\mathcal{CLC} = \{L_i\}_{i \in \mathbb{N}}$ be the class of interest.

(D.1) The class $\mathcal{CLC}$ is refutably inferable from complete data.

In fact, let $L \not\in \mathcal{CLC}$ be a nonempty concept. Then as easily seen, there is a $j \geq 1$ such that $a^j \not\in L$ but $a^{j+1} \in L$. Therefore, $T = \{a^{j+1}\}$ and $F = \{a^j\}$ satisfy condition (3.2). It is easy to see that condition (3.4) is also satisfied (cf. Theorem 9).

(D.2) The class $\mathcal{CLC}$ is not inferable in the limit from positive data.
We note that, in [19], this class was shown to be inferable within one mind change from complete data but not inferable in the limit from positive data.

**Example 5.** Let $\mathcal{PR}_{\leq n}$ be the class of concepts each of which consists of all multiples of at most $n$ prime numbers. For example, the concept $\{2, 4, 6, \ldots, 7, 14, 21, \ldots\}$ is in $\mathcal{PR}_{\leq i}$ with $i \geq 2$.

(E.1) For any $n \geq 1$, the class $\mathcal{PR}_{\leq n}$ is refutably inferable from complete data.

In fact, let $L \in \mathcal{PR}_{\leq n}$ be a nonempty concept. (i) In case $L$ contains more than $n$ prime numbers, let $T$ be a set of some $n + 1$ prime numbers in $L$, and let $F = \emptyset$. (ii) In case $L$ contains no prime number, let $m$ be the least integer in $L$. Then let $T = \{m\}$ and $F = \{1, \ldots, m - 1\}$. (iii) Otherwise, let $p_1, \ldots, p_k$ be all prime numbers in $L$. As easily seen, the following (1) or (2) holds. (1) There is an $m \in L$ which is not a multiple of any $p_i$ with $1 \leq i \leq k$. Then let $T = \{m\}$, and let $F$ be the finite set of all prime numbers less than $m$ each of which differs from $p_1, \ldots, p_k$. (2) There is an $m \notin L$ which is a multiple of some $p_i$. Then let $T = \{p_1, \ldots, p_k\}$ and $F = \{m\}$. It is easy to see that the above defined $T$ and $F$ satisfy the condition (3.2). Furthermore, it is easy to see that the condition (3.4) is satisfied.

(E.2) For any $n > 1$, the class $\mathcal{PR}_{\leq n}$ is not reliably inferable from positive data.

(E.3) For any $n \geq 2$, the class $\mathcal{PR}_{\leq n}$ is not finitely inferable from complete data.

(E.4) For any $n \geq 1$, the class $\mathcal{PR}_{\leq n}$ is inferable in the limit from positive data.

Let $\mathcal{PR}_n$ be the class of concepts each of which consists of all multiples of $n$ distinct prime numbers. For example, the concept $\{2, 4, 6, \ldots, 7, 14, 21, \ldots\}$ is in $\mathcal{PR}_2$, but not in $\mathcal{PR}_1$ with $i = 1$ or $i \geq 3$. We note that $\mathcal{PR}_1 = \mathcal{PR}_{\leq 1}$.

(F.1) For any $n \geq 2$, the class $\mathcal{PR}_n$ is not refutably inferable from complete data.

In fact, let $L = \{2\}$. Then for any finite set $F \subseteq L$, there is a prime number which is greater than any integer in $F$. Therefore, there are no finite sets $T \subseteq L$ and $F \subseteq L^c$ that satisfy condition (3.2).

(F.2) For any $n \geq 1$, the class $\mathcal{PR}_n$ is not reliably inferable from positive data.

(F.3) For any $n \geq 1$, the class $\mathcal{PR}_n$ is finitely inferable from positive data.

**Example 6.** Again we consider the class $\mathcal{PA}\mathcal{T}$ of pattern languages. As easily seen from Theorem 15 and Example 2, the empty concept is the only concept that does not satisfy condition (3.2). Since all concepts are assumed to be nonempty in this section, the class of pattern languages is shown to be refutably inferable from complete data.

(G.1) $\mathcal{PA}\mathcal{T}$ is refutably inferable from complete data.

(G.2) $\mathcal{PA}\mathcal{T}$ is not reliably inferable from positive data.

(G.3) $\mathcal{PA}\mathcal{T}$ is finitely inferable from complete data but not finitely inferable from positive data. (cf. [23, 17]).

(G.4) $\mathcal{PA}\mathcal{T}$ is inferable in the limit from positive data (cf. [22]).

We can summarize the above comparisons in Fig. 4.
data' or 'from complete data', respectively. For example, REF-TXT is the collection of all classes that are refutably inferable from positive data.

The classes \( \mathcal{C}_1, \mathcal{C}_n \) and \( \mathcal{C}_{\leq n} \) consist of all finite concepts each of which cardinality is just 1, \( n \geq 2 \), at most \( n \) and unrestricted finite, respectively (cf. Example 3). The class \( \mathcal{C}_{\mathcal{P}} \) has been shown in Example 4. The classes \( \mathcal{P}_{\mathcal{R}}, \mathcal{P}_{\mathcal{R}_n} \) and \( \mathcal{P}_{\mathcal{R}_{\leq n}} \) consist of all multiples of a prime number, \( n \geq 2 \) distinct prime numbers and at most \( n \) prime numbers, respectively (cf. Example 5). The class \( \mathcal{P}_{\mathcal{A}} \) is the class of a pattern languages (cf. Example 6). The class \( \mathcal{S}_{\mathcal{F}} \) is the so-called superfinite class [14], that is, a class contains all finite concepts and at least one infinite concept.

In Fig. 4, we see that a subclass of a refutably inferable class is not always refutably inferable.

5. Unions of some classes

In this section we consider two type of union classes. First we take a class as the collection of all concepts from \( n \) classes.

**Definition.** Let \( n \geq 1 \) be an integer, and let \( \mathcal{C}_1, \ldots, \mathcal{C}_n \) be classes. For \( i \) with \( 1 \leq i \leq n \) and \( j \geq 1 \), the \( j \)th concept \( L_{ij} \) of the class \( \mathcal{C}_i \) is denoted by \( L_{i(j)} \). Then the union class of \( \mathcal{C}_1, \ldots, \mathcal{C}_n \) is represented as

\[
\bigcup_{i=1}^n \mathcal{C}_i = \{L_{i(j)}\}_{1 \leq i \leq n, j \in \mathbb{N}}.
\]

By assuming a bijective coding from \( \{1, \ldots, n\} \times \mathbb{N} \) to \( \mathbb{N} \), the new class above becomes an indexed family of recursive concepts.
Theorem 18. Let \( n \geq 1 \) be an integer, and let \( \mathcal{C}_1, \ldots, \mathcal{C}_n \) be classes each of which is refutably inferable from positive data (resp., complete data). Then the class \( \bigcup_{i=1}^{n} \mathcal{C}_i \) is refutably inferable from positive data (resp., complete data).

Proof. We only give the proof of the case of positive data. The proof for complete data can be given in a similar way.

For any \( i \) with \( 1 \leq i \leq n \), let \( M_i \) be an RIIM which refutably infers \( \mathcal{C}_i \) from positive data. Then let us consider the procedure in Fig. 5.

Assume that we feed a positive presentation \( \sigma \) of a nonempty concept \( L \) to the procedure.

(I) In case \( L \notin \bigcup_{i=1}^{n} \mathcal{C}_i \), then for any \( i \) with \( 1 \leq i \leq n \), \( L \notin \mathcal{C}_i \) holds, and it follows that \( M_i \) refutes the class \( \mathcal{C}_i \) from \( \sigma \). Thus the procedure refutes the class \( \bigcup_{i=1}^{n} \mathcal{C}_i \) from \( \sigma \).

(II) In case \( L \in \bigcup_{i=1}^{n} \mathcal{C}_i \), let \( i_0 \) be the least integer such that \( L \in \mathcal{C}_{i_0} \). Then for any \( i \) with \( 1 \leq i < i_0 \), \( L \notin \mathcal{C}_i \) holds, and it follows that \( M_i \) refutes the class \( \mathcal{C}_i \) from \( \sigma \). Therefore, the for-loop in the procedure reaches the case of \( i = i_0 \). Since \( M_{i_0} \) infers \( L \) w.r.t. \( \mathcal{C}_{i_0} \) in the limit from \( \sigma \), it follows that \( M_{i_0} \) converges to an index \( j \) with \( L(i_{0},j) = L \) for \( \sigma \). Thus the procedure converges to the coding of \((i_{0},j) \) for \( \sigma \). That is, the procedure infers \( L \) w.r.t. \( \bigcup_{i=1}^{n} \mathcal{C}_i \) in the limit from \( \sigma \).

Thus the procedure refutably infers the class \( \bigcup_{i=1}^{n} \mathcal{C}_i \) from positive data. \( \square \)

A similar theorem is also valid for reliable inference (cf. [22, 12]). As stated in [12], it can be generalized to effective infinite unions for reliable inference. As easily seen, however, it cannot be done for refutable inference. The class \( \mathcal{E}_{\#} \) is a good example for this difference (cf. Example 3).

We now consider a class of concepts each of which is a union of at most \( n \) concepts from \( n \) class (cf. [40]).

---

**Procedure RIIM**

begin
  for \( i = 1 \) to \( n \) do
    while \( M_i \) does not refute the class \( \mathcal{C}_i \) do begin
      simulate \( M_i \) with presenting facts read so far;
      during the simulation,
      if \( M_i \) requests another fact then
        read the next fact and present it to \( M_i \);
      if \( M_i \) produces a guess \( j \) then
        output the coding of \((i,j)\);
      end;
    refute the class \( \bigcup_{i=1}^{n} \mathcal{C}_i \) and stop;
  end.

Fig. 5. An RIIM for unions by \( \bigcup \).
Definition. Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be classes. For $i$ with $1 \leq i \leq n$ and $j \geq 0$, $L_{(i,j)}$ denotes the empty concept if $j = 0$, otherwise the $j$th concept $L_j$ of the class $\mathcal{C}_i$. Then we define a class generated by $\mathcal{C}_1, \ldots, \mathcal{C}_n$ as follows:

$$\bigcup_{i=1}^n \mathcal{C}_i = \left\{ \bigcup_{i=1}^n L_{(i,j)} \Big| (j_1, \ldots, j_n) \in \mathcal{N}^n \right\},$$

where $\mathcal{N}^n$ is the set of all $n$-tuples of nonnegative integers, that is, $\mathcal{N}^n = \{0, 1, 2, \ldots \}^n$.

By assuming a bijective coding from $\mathcal{N}^n$ to $\mathbb{N}$, the new class above becomes an indexed family of recursive concepts.

Each class satisfies the condition (3.8), then the above class is shown to be refutably inferable from complete data.

Theorem 19. Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be classes each of which satisfies condition (3.8). Then the class $\bigcup_{i=1}^n \mathcal{C}_i$ is refutably inferable from complete data.

Proof. Let us consider the procedure in Fig. 6, where the procedure read-store is the same one as in Fig. 3 and $\lambda$ is a special element not in the universal set $U$.

Assume that we feed a complete presentation $\sigma$ of a concept $L_{\text{base}}$ to the procedure.

(A) In case $L_{\text{base}} \in \bigcup_{i=1}^n \mathcal{C}_i$, let $NE(m) = \{(j_1, \ldots, j_n) \in \mathcal{N}^n |$ there are just $m$ nonempty concepts among $L_{1,j_1}, \ldots, L_{n,j_n}\}$. Then $(T_{m-1}, F_{m-1})$ is not consistent with $(L_{(1,k_1)} \cup \cdots \cup L_{(n,k_n)})$ for any $(k_1, \ldots, k_n) \in NE(m)$.

Claim. In the procedure, for any $m$ with $0 \leq m \leq n$, if $T_m$ and $F_m$ are defined, then $(T_m, F_m)$ is not consistent with $(L_{(1,j_1)} \cup \cdots \cup L_{(n,j_n)})$ for any $(j_1, \ldots, j_n) \in NE(m)$.

Proof of the claim. This proof is given by mathematical induction on $m$.

(I) In case $m = 0$, it is clear because $T_0$ is nonempty and the union of $n$ empty concepts is empty.

(II) In case $m \geq 1$, we assume the claim for $m - 1$, and assume that $T_m$ and $F_m$ are defined. Then suppose that there is an $n$-tuple $(k_1, \ldots, k_n) \in NE(m)$ such that $(T_m, F_m)$ is consistent with $(L_{(1,k_1)} \cup \cdots \cup L_{(n,k_n)})$. Then $(T_{m-1}, F_{m-1})$ is consistent with $(L_{(1,k_1)} \cup \cdots \cup L_{(n,k_n)})$, because $T_{m-1} \subseteq T_m$ and $F_{m-1} \subseteq F_m$ hold.

Now suppose that there is an $i \geq 1$ such that $L_{(i,k_i)} \neq \emptyset$ and $(L_{(i,k_i)} \cap T_{m-1}) = \emptyset$. Then $T_m \subseteq (L_{(1,k_1)} \cup \cdots \cup L_{(i-1,k_{i-1})} \cup L_{i+1,k_{i+1}} \cup \cdots \cup L_{(n,k_n)})$ and $F_m \subseteq (L_{(1,k_1)} \cup \cdots \cup L_{(n,k_n)})$ hold. This means that $(T_{m-1}, F_{m-1})$ is consistent with $(L_{(1,k_1)} \cup \cdots \cup L_{(i-1,k_{i-1})} \cup L_{(i+1,k_{i+1})} \cup \cdots \cup L_{(n,k_n)})$. This contradicts the induction hypothesis, because $(k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n) \in NE(m-1)$.

Thus we have $(L_{(i,k_i)} \cap T_{m-1}) \neq \emptyset$ for any $i$ with $L_{(i,k_i)} \neq \emptyset$. Therefore, we can take $u_i$'s as follows: If $L_{(i,k_i)} \neq \emptyset$, then $u_i \in (T_{m-1} \cup \{\lambda\})$ and the number of $u_i$'s other than $\lambda$ is just $m$, the $n$-tuple $(u_1, \ldots, u_n)$ is in $\mathcal{G}_m$. Thus there is a case where the for-loop (2) is executed with $(u_1, \ldots, u_n)$. In this case there is an $n$-tuple $(k'_1, \ldots, k'_n) \in S_1 \times \cdots \times S_n$ such that for any $i$ with $1 \leq i \leq n$, $L_{k'_i} = L_{k_i}$. 

Procedure RIIM \( M \):
begin
\( T = \emptyset , \ F = \emptyset \);
read store \((T, F)\);
while \( T = \emptyset \) do begin
output the coding of \((0, \ldots , 0)\);
read store \((T, F)\);
end;
\( T_0 = T \); \( F_0 = F \);
for \( m = 1 \) to \( n \) do begin
let \( \mathcal{T}_m = \{ (w_1, \ldots , w_n) \mid \) the number of \( w_i \)’s \( \in (T_{m-1} \cup \{ \lambda \})^n \) other than \( \lambda \) is just \( m \}; \)
for each \((w_1, \ldots , w_n) \in \mathcal{T}_m\) do begin
for \( i = 1 \) to \( n \) do if \( w_i = \lambda \) then let \( S_i = \{ 0 \} \)
else recursively generate a cover-index set w.r.t. \( \mathcal{C}_i \)
of \( \{ L_{\langle i,j \rangle} \in \mathcal{C}_i \mid w_i \in L_{\langle i,j \rangle} \} \), and set it to \( S_i \);
if all \( S_i \)’s are nonempty then
for each \((j_1, \ldots , j_n) \in S_1 \times \cdots \times S_n\) do begin
while \((T, F)\) is consistent with \((L_{\langle 1,j_1 \rangle} \cup \cdots \cup L_{\langle n,j_n \rangle})\) do begin
output the coding of \((j_1, \ldots , j_n)\);
read store \((T, F)\);
end;
end;
end;
end;
end;
end;
end;
refute the class \( \bigcup_{i=1}^n \mathcal{C}_i \) and stop;
end.

Fig. 6. An RIIM for unions by \( \bigcup \).

Thus there is a case where the for-loop (3) is executed with \((k'_1, \ldots , k'_n)\). Since \((T_m, F_m)\) is consistent with \((L_{\langle 1,k'_1 \rangle} \cup \cdots \cup L_{\langle n,k'_n \rangle})\), it follows that for any finite sets \( T \subseteq T_m \) and \( F \subseteq F_m \), \((T, F)\) is consistent with \((L_{\langle 1,k'_1 \rangle} \cup \cdots \cup L_{\langle n,k'_n \rangle})\). However, the while-loop (4) should terminate with \( T \subseteq T_m \) and \( F \subseteq F_m \), because \( T_m \) and \( F_m \) are assumed to be defined. This is a contradiction. □

Since \( L_{\text{base}} \in \bigcup_{i=1}^n \mathcal{C}_i \), there are an \( m_0 \) with \( 0 \leq m_0 \leq n \) and an \( n \)-tuple \((j_1, \ldots , j_n)\) such that \((j_1, \ldots , j_n) \in \mathcal{N}(E(m_0)) \) and \( L_{\text{base}} = (L_{\langle 1,j_1 \rangle} \cup \cdots \cup L_{\langle n,j_n \rangle}) \). By this claim, we see that \( T_m \) and \( F_m \) are never defined. By Lemma 14, this means that the procedure outputs the coding of an \( n \)-tuple \((j_1, \ldots , j_n)\) with \( L_{\text{base}} = (L_{\langle 1,j_1 \rangle} \cup \cdots \cup L_{\langle n,j_n \rangle}) \) and the while-loop (1) or (4) never terminates.
(B) In case $L_{\text{base}} \notin \bigcup_{i=1}^{n} \mathcal{G}_i$, by using Lemma 14 $n$ times, we see that the procedure refutes the class $\bigcup_{i=1}^{n} \mathcal{G}_i$ from $\sigma$. □

Example 7. We consider the class $\mathcal{P} \mathcal{A} \mathcal{F}$ of pattern languages. As shown in Example 2, the class satisfies condition (3.8). Therefore, by Theorem 19, for any $n \geq 1$, the class of unions of at most $n$ pattern languages is refutably inferable from complete data.

By Corollary 6, we see that if the number of patterns is not bounded by a constant number, then the class is not refutably inferable from complete data, because it contains all nonempty finite languages.

We note that for any $n \geq 1$, the class of unions of at most $n$ pattern languages is shown to be inferable in the limit from positive data (cf. [40, 32, 33]).

6. EFS definable classes

In this section, we consider the so-called model inference (cf. [30]) and language learning using elementary formal systems (EFSs).

The EFSs were originally introduced by Smullyan [37] to develop his recursion theory. In a word, EFSs are a kind of logic programming language which uses patterns instead of terms in first order logic [41], and they are shown to be natural devices to define languages [5].

In this paper, we briefly recall EFSs. For detailed definitions and properties of EFSs, please refer to [37, 5, 8, 9, 41].

Let $\Sigma$, $X$, and $\Pi$ be mutually disjoint nonempty sets. We assume that $\Sigma$ is finite, and fix it throughout this section. Elements in $\Sigma$, $X$, and $\Pi$ are called constant symbols, variables, and predicate symbols, respectively. By $p, q, p_1, p_2, \ldots$, we denote predicate symbols. Each predicate symbol is associated with a positive integer which we call an arity.

In general, for a set $S$, $S^+$ denotes the set of all nonempty finite strings over $S$, and $\# S$ denotes the cardinality of $S$.

Definition. A term, or a pattern, is an element in $(\Sigma \cup X)^+$, that is, it is a nonnull string over $(\Sigma \cup X)$. By $\pi, \pi_1, \pi_2, \ldots$, we denote terms. A term $\pi$ is said to be ground if $\pi \in \Sigma^+$. By $w, w_1, w_2, \ldots$, we denote ground terms.

An atomic formula (atom) is an expression of the form $p(\pi_1, \ldots, \pi_n)$, where $p$ is a predicate symbol with arity $n$, and $\pi_1, \ldots, \pi_n$ are terms. By $A, B, A_1, A_2, \ldots$, we denote atoms. An atom $p(\pi_1, \ldots, \pi_n)$ is said to be ground if $\pi_1, \ldots, \pi_n$ are ground terms.

We define the well-formed formulas and clauses in the ordinary ways [20].

Definition. A definite clause is a clause of the form

$$A \leftarrow B_1, \ldots, B_n,$$
where \( n \geq 0 \), and \( A, B_1, \ldots, B_n \) are atoms. The atom \( A \) above is called the head of the clause, and the sequence \( B_1, \ldots, B_n \) is called the body of the clause. By \( C, D, C_1, C_2, \ldots \), we denote definite clauses. Then an EFS is a finite set of definite clauses, each of which is called an axiom.

A substitution is a homomorphism from terms to terms which maps each symbol \( a \in \Sigma \) to itself.

In the world of EFSs, the Herbrand base (HB) is the set of all ground atoms. A subset \( I \) of \( HB \) is called an Herbrand interpretation. We also define Herbrand model, and the last Herbrand model in the ordinary ways [20].

For an EFS \( \Gamma \), the least Herbrand model is denoted by \( M(\Gamma) \). For an EFS \( \Gamma \) and a predicate symbol \( p \) with arity \( n \), we define the set of \( n \)-tuples of ground terms as follows:

\[
L(\Gamma, p) = \{ (w_1, \ldots, w_n) \in (\Sigma^*)^n \mid p(w_1, \ldots, w_n) \in M(\Gamma) \}.
\]

In case the arity of \( p \) is 1, i.e. \( p \) is unary, we regard \( L(\Gamma, p) \) as a language over \( \Sigma \).

We now put a syntactical restriction on EFSs, because the least Herbrand model \( M(\Gamma) \) for an unrestricted EFS \( \Gamma \) may not be recursive, that is, for a ground atom \( A \), we cannot recursively decide whether or not \( A \in M(\Gamma) \).

For a term \( \pi \), \( \| \pi \| \) denotes the length of \( \pi \), and \( o(x, \pi) \) denotes the number of all occurrences of a variable \( x \) in \( \pi \). For an atom \( p(\pi_1, \ldots, \pi_n) \), we define the length of the atom and the number of variable's occurrences in the atom as follows:

\[
\| p(\pi_1, \ldots, \pi_n) \| = \| \pi_1 \| + \cdots + \| \pi_n \|, \\
o(x, p(\pi_1, \ldots, \pi_n)) = o(x, \pi_1) + \cdots + o(x, \pi_n).
\]

**Definition.** A clause \( A \leftarrow B_1, \ldots, B_n \) is said to be length-bounded if

\[
\| A \theta \| \geq \| B_1 \theta \| + \cdots + \| B_n \theta \|
\]

for any substitution \( \theta \).

An EFS \( \Gamma \) is said to be length-bounded if all axioms of \( \Gamma \) are length-bounded.

The notion of length-bounded clauses is characterized by the following lemma.

**Lemma 20** (Arikawa et al. [8, 9]). A clause \( A \leftarrow B_1, \ldots, B_n \) is length-bounded, if and only if \( \| A \| \geq \| B_1 \| + \cdots + \| B_n \| \) and \( o(x, A) \geq o(x, B_1) + \cdots + o(x, B_n) \) hold for any variable \( x \).

From now on, we only consider length-bounded EFSs. For length-bounded EFSs, the following theorem holds.

**Theorem 21** (Arikawa et al. [8, 9], Yamamoto [41]). For a length-bounded EFS \( \Gamma \), the least Herbrand model \( M(\Gamma) \) is recursive, that is, for any ground atom \( A \), whether or not \( A \in M(\Gamma) \) is recursively decidable.

Furthermore, the following theorem shows the power of length-bounded EFSs.
Theorem 22 (Arikawa et al. [8, 9]). A language \( L \subseteq \Sigma^+ \) is context-sensitive if and only if \( L \) is definable by a length-bounded EFS.

We devote the rest of this section to investigating the refutable inferability of length-bounded EFS definable classes. We adjust the definitions of inferability to the case of EFSs as follows, but as easily seen, the essential part is kept unchanged from definitions in Section 2. In what follows, we assume that outputs from an IIM or an RIIM are EFSs.

Definition. For an atom \( A \), \( \text{pred}(A) \) denotes the predicate symbol of \( A \). For a set \( \Pi_0 \subseteq \Pi \) and a set \( S \) of atoms, \( S|_{\Pi_0} \) denotes the set of atoms restricted to \( \Pi_0 \), that is,

\[
S|_{\Pi_0} = \{ A \in S \mid \text{pred}(A) \in \Pi_0 \}.
\]

A predicate-restricted complete presentation of a set \( I \subseteq HB \) w.r.t. \( \Pi_0 \subseteq \Pi \) is an infinite sequence \((A_1, t_1), (A_2, t_2), \ldots \) of elements in \( HB|_{\Pi_0} \times \{ +, - \} \) such that \( \{ A_i | t_i = +, i \geq 1 \} = I|_{\Pi_0} \) and \( \{ A_i | t_i = -, i \geq 1 \} = HB|_{\Pi_0} \setminus I|_{\Pi_0} \).

An IIM \( M \) or an RIIM \( M \) is said to converge to an EFS \( r \) for a presentation \( \sigma \), if there is an \( n_0 \geq 1 \) such that for any \( m \geq n_0 \), \( M(\sigma[m]) = \Gamma \).

Let \( \mathcal{C} \) be a class of EFSs. For an EFS \( r \in \mathcal{C} \) and a predicate-restricted complete presentation \( \sigma \) of \( M(\Gamma) \) w.r.t. \( \Pi_0 \subseteq \Pi \), an IIM \( M \) or an RIIM \( M \) is said to infer the EFS \( \Gamma \) w.r.t. \( \mathcal{C} \) in the limit from \( \sigma \) if \( M \) converges to an EFS \( \Gamma' \in \mathcal{C} \) with \( M(\Gamma')|_{\Pi_0} = M(\Gamma)|_{\Pi_0} \) for \( \sigma \).

A class \( \mathcal{C} \) is said to be theoretical-term-freely inferable in the limit from complete data, if for any nonempty finite subset \( \Pi_0 \) of \( \Pi \), there is an IIM \( M \) which infers \( \Gamma \) w.r.t. \( \mathcal{C} \) in the limit from \( \sigma \) for any EFS \( \Gamma \in \mathcal{C} \) and any predicate-restricted complete presentation \( \sigma \) of \( M(\Gamma) \) w.r.t. \( \Pi_0 \).

A class \( \mathcal{C} \) is said to be theoretical-term-freely and refutably inferable from complete data, if for any nonempty finite subset \( \Pi_0 \) of \( \Pi \), there is an RIIM \( M \) which satisfies the following condition: For any set \( I \subseteq HB \) and any predicate-restricted complete presentation \( \sigma \) of \( I \) w.r.t. \( \Pi_0 \), (i) if there is an EFS \( \Gamma \in \mathcal{C} \) such that \( M(\Gamma)|_{\Pi_0} = I|_{\Pi_0} \), then \( M \) infers \( \Gamma \) w.r.t. \( \mathcal{C} \) in the limit from \( \sigma \), (ii) otherwise \( M \) refutes the class \( \mathcal{C} \) from \( \sigma \).

Theoretical terms are supplementary predicates that are necessary for defining some goal predicates. In the above definition, the phrase 'theoretical-term-freely inferable' means that from the only facts on the goal predicates, an IIM or an RIIM generates some suitable predicates and infers an EFS which explains the goal predicates.

Definition. For two EFSs \( \Gamma \) and \( \Gamma' \), and a set \( \Pi_0 \subseteq \Pi \), we write \( \Gamma \equiv_{\Pi_0} \Gamma' \), if we can make \( \Gamma' \) identical to \( \Gamma \) by renaming predicate symbols other than those in \( \Pi_0 \) and by renaming variables in each axiom. We assume some canonical form of an EFS w.r.t. \( \Pi_0 \), and \( \text{canon}(\Gamma, \Pi_0) \) denotes the representative EFS of the set of EFSs \( \{ \Gamma' \mid \Gamma \equiv_{\Pi_0} \Gamma' \} \).
For an EFS $\Gamma$, $\text{HPRED}(\Gamma)$ (resp., $\text{BPRED}(\Gamma)$) denotes the set of all predicate symbols appearing in the heads (resp., the bodies) of the axioms of $\Gamma$, and $\text{PRED}(\Gamma)$ denotes the set $\text{HPRED}(\Gamma) \cup \text{BPRED}(\Gamma)$. We also define various sets and classes as follows:

$\mathcal{LB}^n = \{ \Gamma \mid \Gamma$ is a length-bounded EFS with $n$ axioms $\}$, 

$\mathcal{LB}^n[\Pi_0] = \{ \text{canon}(\Gamma, \Pi_0) \mid \Gamma \in \mathcal{LB}^n \}$, 

$\mathcal{MLB}^n[\Pi_0] = \{ \Gamma \in \mathcal{LB}^n[\Pi_0] \mid \text{BPRED}(\Gamma) \subseteq \text{HPRED}(\Gamma) \}$, 

$\mathcal{MLB}^n[\Pi_0](l) = \left\{ \Gamma \in \mathcal{MLB}^n[\Pi_0] \mid \text{the head's length of each axiom of } \Gamma \text{ is not greater than } l. \right\}$, 

$\mathcal{Y}_W = \{ r \mid r \text{ is a length-bounded EFS with } n \text{ axioms} \}$, 

$\mathcal{Z}_P[\Pi, \mathcal{Y}_W] = \{ \text{canon}(T, ZZ) \mid r \in \mathcal{Y}_W \}$, 

$\mathcal{Z}_Y[\Pi, \mathcal{Z}_P] = \{ kP \mid kP \in \mathcal{Z}_P \}$, 

$\mathcal{ML}^n[\Pi_0](l) = \bigcup_{i=0}^{n} \mathcal{MLB}^n[\Pi_0](l)$, 

$$\mathcal{MLB}^n[\Pi_0] = \bigcup_{i=0}^{n} \mathcal{MLB}^n[\Pi_0](l),$$

where $l, n \geq 0$, and $\Pi_0 \subseteq \Pi$.

We note that for any EFSs $\Gamma$ and $\Gamma'$, and any $\Pi_0 \subseteq \Pi$, whether or not $\Gamma' \equiv_{\Pi_0} \Gamma$ is recursively decidable, and that we can effectively obtain the EFS canon$(\Gamma, \Pi_0)$.

We prepare some basic lemmas.

Lemma 23 (Shinohara [33]). Let $\Gamma$ be a length-bounded EFS, and let $A \in M(\Gamma)$ be a ground atom. Then if $\Gamma$ has an axiom $C$ whose head is longer than $A$, then $A \in M(\Gamma \setminus \{ C \})$ holds.

Lemma 24. For any EFS $\Gamma \in \mathcal{LB}^{\leq n}$ and any $\Pi_0 \subseteq \Pi$, there is an EFS $\Gamma' \in \mathcal{MLB}^{\leq n}[\Pi_0]$ such that $M(\Gamma')|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$. 

Proof. Let $\Gamma' \in \mathcal{LB}^{\leq n}$ and $\Gamma_1 = \text{canon}(\Gamma, \Pi_0)$. Then $M(\Gamma_1)|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$ holds. Therefore, if $\text{BPRED}(\Gamma_1) \subseteq \text{HPRED}(\Gamma_1)$, then $\Gamma_1$ itself is in $\mathcal{MLB}^{\leq n}[\Pi_0]$. Otherwise, let $\Gamma_2$ be the EFS which is obtained from $\Gamma_1$ by subtracting all axioms that have predicate symbols in $\text{BPRED}(\Gamma_1) \setminus \text{HPRED}(\Gamma_1)$ in their bodies, and let $\Gamma_3 = \text{canon}(\Gamma_2, \Pi_0)$. Then clearly, $M(\Gamma_3)|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$ and $\Gamma_3 \in \mathcal{MLB}^{\leq n}[\Pi_0]$ hold. 

Lemma 25. Let $n \geq 1$ be an integer, let $\Pi_0 \subseteq \Pi$ be a set of predicate symbols, let $T \subseteq HB|_{\Pi_0}$ be a nonempty finite set, and let $F \subseteq HB|_{\Pi_0}$ be a finite set. Assume that $(T, F)$ is not consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{n-1}[\Pi_0]$. Then for any EFS $\Gamma \in \mathcal{MLB}^n[\Pi_0] \setminus \mathcal{MLB}^n[\Pi_0](l)$, $(T, F)$ is not consistent with $M(\Gamma)|_{\Pi_0}$, where $l = \max \{ \| A \| \mid A \in T \}$.

Proof. Let $\Gamma \in \mathcal{MLB}^n[\Pi_0] \setminus \mathcal{MLB}^n[\Pi_0](l)$. Then there is a length-bounded clause $C \in \Gamma$ whose head's length is greater than $l$. Let $\Gamma_1 = \Gamma \setminus \{ C \}$. Suppose that $(T, F)$ is
consistent with $M(\Gamma)|_{\Pi_0}$. Since the head of $C$ is longer than every ground atom in $T$, it follows by Lemma 23 that $T \subseteq M(\Gamma)|_{\Pi_0}$. Furthermore, since $\Gamma_1 \subseteq \Gamma$, it follows that $M(\Gamma_1) \subseteq M(\Gamma)$ by monotonicity of the least Herbrand model. Therefore, $F \subseteq HB|_{\Pi_0} \setminus M(\Gamma)|_{\Pi_0} \subseteq HB|_{\Pi_0} \setminus M(\Gamma_1)|_{\Pi_0}$ holds. Thus $(T, F)$ is consistent with $M(\Gamma_1)|_{\Pi_0}$. Let $\Gamma_2 = \text{canon}(\Gamma_1, \Pi_0)$. Then $M(\Gamma_2)|_{\Pi_0} = M(\Gamma_1)|_{\Pi_0}$ holds, and it follows that $(T, F)$ is consistent with $M(\Gamma_1)|_{\Pi_0}$. This contradicts the assumption, because $\Gamma_2 \not\subseteq M(\Gamma_1)|_{\Pi_0}$. 

**Lemma 26.** For any $l, n \geq 0$ and any finite subset $\Pi_0$ of $\Pi$, the set $\mathcal{MLB}^{[n]}[\Pi_0](l)$ is a uniformly and recursively generable finite set.

**Proof.** As easily seen, for any EFS $\Gamma \in \mathcal{MLB}^{[n]}[\Pi_0](l)$, the number of predicate symbols appearing in $\Gamma$ is at most $n$ and the arities of those predicate symbols are at most $l$.

Put $PA(l, n) = \{\{q_1^{(j_1)}, \ldots, q_k^{(j_k)}\} | 0 \leq k \leq m \text{ and } 1 \leq j_1 \leq \cdots \leq j_k \leq l\}$, where for a predicate symbol $q^{(t)}$, the superscript $t$ represents its arity. Then put $PR(l, n, \Pi_0) = \{\{\Pi' \cup \Pi'' | \Pi' \subseteq \Pi_0 \text{ and } \Pi'' \in PA(l, n - \# \Pi')\}$. By the above observation, it is sufficient for us to generate EFSs $\Gamma$ with $\text{PRED}(\Gamma) \in PR(l, n, \Pi_0)$, because we do not distinguish two EFSs that are identical except for renaming of predicate symbols other than those in $\Pi_0$.

As easily seen, the above $PR(l, n, \Pi_0)$ is a uniformly and recursively generable finite set. Furthermore, the set of all terms shorter than a fixed length is a uniformly and recursively generable finite set, where we do not distinguish two terms that are identical except for renaming of variables.

Roughly speaking, we recursively generate EFSs in $\mathcal{MLB}^{[n]}[\Pi_0](l)$ as follows: We combine sets of predicate symbols in $PR(l, n, \Pi_0)$ and terms whose lengths are not greater than $l$, rearrange variables in each axiom, make canonical form w.r.t. $\Pi_0$ of them and check whether or not each obtained EFS is in $\mathcal{MLB}^{[n]}[\Pi_0](l)$ by using Lemma 20.

**Theorem 27.** For any $n \geq 0$, the class $\mathcal{L}[\leq n]$ is theoretical-term-freely and refutably inferable from complete data.

**Proof.** Let us consider the procedure in Fig. 7, where the procedure read_store is the same one as in Fig. 3.

Let $\Pi_0$ be a nonempty finite subset of $\Pi$, and let $I_{base} \subseteq HB$ be a set of ground atoms. Then assume that we feed a predicate-restricted complete presentation $\sigma$ of $I_{base}$ w.r.t. $\Pi_0$ to the procedure.

(A) In case there is an EFS $\Gamma_{base} \in \mathcal{L}[\leq n]$ such that $I_{base}|_{\Pi_0} = M(\Gamma_{base})|_{\Pi_0}$.

**Claim.** In the procedure, for any $m$ with $0 \leq m \leq n$ if $T_m$ and $F_m$ are defined, then $(T_m, F_m)$ is not consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0]$.  


**Procedure** RIIM $M(n, \Pi_0)$;  
begin  
$T = \emptyset$; $F = \emptyset$;  
read_store ($T; F$);  
while $T = \emptyset$ do begin  
output the empty EFS;  
read_store ($T; F$);  
end;  
$T_0 = T$; $F_0 = F$;  
for $m = 1$ to $n$ do begin  
$L_m = \max \{ \| A \| \mid A \in T_{m-1} \}$;  
recursively generate $\mathcal{MLB}^{[m]}[\Pi_0](l_m)$, and set it to $S$;  
for each $\Gamma \in S$ do  
while ($T; F$) is consistent with $M(\Gamma)|_{\Pi_0}$ do begin  
output $\Gamma$;  
read_store ($T; F$);  
end;  
$T_m = T$; $F_m = F$;  
end;  
refute the class $\mathcal{E}[\leq n]$ and stop;  
end.

Fig. 7. An RIIM for the class $\mathcal{E}[\leq n]$.

**Proof of the claim.** This proof is given by mathematical induction on $m$.

(I) In case $m = 0$, it is clear because $T_0$ is nonempty and the least Herbrand model of the empty EFS is empty.

(II) In case $m \geq 1$, we assume the claim for $m - 1$, and assume that $T_m$ and $F_m$ are defined. Then we see that $T_{m-1}$ and $F_{m-1}$ are also defined, and by the induction hypothesis, $(T_{m-1}, F_{m-1})$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m-1]}[\Pi_0]$. Therefore, by Lemma 25, $(T_{m-1}, F_{m-1})$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0] \setminus \mathcal{MLB}^{[m]}[\Pi_0](l_m)$. Thus $(T_m, F_m)$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0] \setminus \mathcal{MLB}^{[m]}[\Pi_0](l_m)$, because $T_{m-1} \subseteq T_m$ and $F_{m-1} \subseteq F_m$ hold.

Furthermore, since the for-loop (2) terminates, we see that $(T_m, F_m)$ is also *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0](l_m)$.

Hence we have the claim for $m$. □

By Lemma 24, there is an EFS $\Gamma \in \mathcal{MLB}^{[\leq n]}[\Pi_0]$ such that $M(\Gamma)|_{\Pi_0} = M(\Gamma_{\text{base}})|_{\Pi_0}$ ($= I_{\text{base}}|_{\Pi_0}$). Therefore, we see by the above claim that $T_n$ and $F_n$ are never defined. By Lemma 14, this means that the procedure outputs an EFS $\Gamma$ with $M(\Gamma)|_{\Pi_0} = I_{\text{base}}|_{\Pi_0}$ and the while-loop (1) or (3) never terminates.
(B) In case there is no EFS $\Gamma \in \mathcal{LA}^{\leq n}$ such that $I_{\text{base}}|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$. By using Lemma 14 $n$ times, we see that the procedure refutes the class $\mathcal{LA}^{\leq n}$ from $\sigma$. □

By Corollary 6, we see that if the number of axioms is not bounded by a constant number, then this class is not refutably inferable, because it contains all nonempty finite concepts on $HB|_{\Pi_0}$.

The following corollary is obvious from Theorem 27.

**Corollary 28.** For any $n \geq 0$, the class of languages definable by EFSs in $\mathcal{LA}^{\leq n}$ is refutably inferable from complete data.

We note that Shinohara [33] showed that the classes in Theorem 27 and Corollary 28 are inferable in the limit from positive data.

7. Concluding remarks

We have pointed out that the essence of the computational logic of scientific discovery or the logic of machine discovery should be the refutability of the whole space of hypotheses by observed data or given facts. Then we have shown a series of such sufficiently large hypothesis spaces in terms of the elementary formal systems. More exactly, for any $n$ the class of length-bounded EFSs with at most $n$ axioms are refutably identifiable. The argument to prove this is also valid for the classes of weakly reducing EFSs, and linear or weakly reducing logic programs [31, 10] with at most $n$ axioms.

The refutability we proposed here forms an interesting contrast to the original one: In the logic of scientific discovery for scientists each theory in a hypothesis space is to be refutable, while in the computational version of the logic for machines or the logic of machine discovery the space itself is to be refutable by an observation. Popper contributed to the modern theory of inductive inference. In his books [26, 27], however, Popper strongly denied the induction so far developed by, for example, Mill [21], which had two stages of mechanical creation of a hypothesis from observations and proof of its validity. In fact, he said that the induction, i.e., inference based on many observations was a myth and it was neither a psychological fact, nor a fact of ordinary life, nor one of scientific procedure [26]. What he wanted to assert in those books was that scientific theory should have been refuted by observed data or facts and any such theory could by no means be verified. He could not agree with the assertion that the induction should have proved the validity of theory. However, we think there were no reasons to deny the stage of mechanical creation of hypotheses.

The inductive inference machine that can refute the hypothesis space itself works as an automatic system for scientific discovery. If the machine for scientific discovery cannot refute the whole space of hypotheses, it can just work for computer-aided scientific discovery. That is, we need to check from time to time whether the machine is still searching for a possible hypothesis.
As a future work we have left the refutability of hypothesis spaces in another two major frameworks of PAC and MAT learning [39, 3], which we will discuss elsewhere. As for the PAC learning we obtained very practical results in the area of Molecular Biology based on our theoretical studies on learnability [6, 7, 35]. Our learning system succeeded in discovering some simple and accurate knowledge, i.e., hypotheses, in a very short time. However, it was still an application of PAC learning, but not machine discovery. In such a system we could not have any way to stop the learning algorithm even when there remains no possibility to find any good hypotheses to explain the data or facts given so far. We were lucky, because the hypotheses our learning system found out happened to be in the spaces we initially gave. The spaces were natural subclasses of the refutably inferable classes we have shown in the last section.

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References


