



On stabilization and control for the critical Klein–Gordon equation on a 3-D compact manifold

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Abstract

In this article, we study the internal stabilization and control of the critical nonlinear Klein–Gordon equation on 3-D compact manifolds. Under a geometric assumption slightly stronger than the classical geometric control condition, we prove exponential decay for some solutions bounded in the energy space but small in a lower norm. The proof combines profile decomposition and microlocal arguments. This profile decomposition, analogous to the one of Bahouri and Gérard (1999) [2] on \mathbb{R}^3 , is performed by taking care of possible geometric effects. It uses some results of S. Ibrahim (2004) [21] on the behavior of concentrating waves on manifolds.

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Résumé

Dans cet article, on étudie la stabilisation et le contrôle interne de l'équation de Klein–Gordon critique sur des variétés de dimension 3. Sous des conditions géométriques légèrement plus fortes que la condition de contrôle géométrique classique, on prouve la décroissance exponentielle de solutions bornées dans l'espace d'énergie mais petites dans des normes plus faibles. La preuve combine la décomposition en profils et des arguments microlocaux. Cette décomposition, analogue à celle de Bahouri et Gérard (1999) [2] sur \mathbb{R}^3 , nécessite l'analyse de certains effets dus à la géométrie. Elle utilise des résultats de S. Ibrahim (2004) [21] sur le comportement d'ondes de concentration sur les variétés.

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0. Introduction

In this article, we study the internal stabilization and exact controllability for the defocussing critical nonlinear Klein–Gordon equation on some compact manifolds:

$$\begin{cases} \square u = \partial_t^2 u - \Delta u = -u - |u|^4 u & \text{on } [0, +\infty[\times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{E}, \end{cases} \quad (1)$$

where Δ is the Laplace–Beltrami operator on M and \mathcal{E} is the energy space $H^1(M) \times L^2(M)$. The solution displays a conserved energy

$$E(t) = \frac{1}{2} \left(\int_M |\partial_t u|^2 + \int_M |u|^2 + \int_M |\nabla u|^2 \right) + \frac{1}{6} \int_M |u|^6. \quad (2)$$

This problem was already treated in the subcritical case by B. Dehman, G. Lebeau and E. Zuazua [11]. The problem is posed in a different geometry but their proof could easily be transposed in our setting. Yet, their result fails to apply to the critical problem for two main reasons, as explained in their paper:

- (a) The bootstrap argument they employed to improve the regularity of solutions vanishing in the zone of control ω so that the existing results on unique continuation apply, does not work for this critical exponent.
- (b) They cannot use the linearizability results by P. Gérard [19] to deduce that the microlocal defect measure for the nonlinear problem propagates as in the linear case.

In this paper, we propose a strategy to avoid the second difficulty at the cost of an additional condition for the subset ω . It was already performed by B. Dehman and P. Gérard [8] in the case of \mathbb{R}^3 with a flat metric. In fact, in that case, this defect of linearizability is described by the profile decomposition of H. Bahouri and P. Gérard [2]. The purpose of this paper is to extend a part of this proof to the case of a manifold with a variable metric. This more complicated geometry leads to extra difficulties, in the profile decomposition and the stabilization argument. We also mention the recent result of L. Aloui, S. Ibrahim and K. Nakanishi [1] for \mathbb{R}^d . Their method of proof is very different and uses Morawetz-type estimates. They obtain uniform exponential decay for a damping around spatial infinity for any nonlinearity, provided the solution exists globally. This result is stronger than ours, but their method does not seem to apply to the more complicated geometries we deal with.

We will need some geometrical condition to prove controllability. The first one is the classical geometric control condition of Rauch and Taylor [33] and Bardos, Lebeau and Rauch [3], while the second one is more restrictive.

Assumption 0.1 (*Geometric Control Condition*). There exists $T_0 > 0$ such that every geodesic travelling at speed 1 meets ω in a time $t < T_0$.

Definition 0.1. We say that $(x_1, x_2, t) \in M^2 \times \mathbb{R}_+^*$ is a couple of focus at distance t if the set

$$F_{x_1, x_2, t} := \{ \xi \in S_{x_1}^* M \mid \exp_{x_1} t \xi = x_2 \}$$

of directions of geodesics stemming from x_1 and reaching x_2 in a time t has a positive surface measure.

We denote by T_{focus} the infimum of the $t > 0$ such that there exists a couple of focus at distance t .

If M is compact, we have necessarily $T_{focus} > 0$.

Assumption 0.2 (*Geometric control before refocusing*). The open set ω satisfies the Geometric Control Condition in a time $T_0 < T_{focus}$.

For example, for \mathbb{T}^3 , there is no refocusing and the geometric assumption is the classical Geometric Control Condition. Yet, for the sphere S^3 , our assumption is stronger. For example, it is fulfilled if ω is a neighborhood of $\{x_4 = 0\}$. We can imagine some geometric situations where the Geometric Control Condition is fulfilled while our condition is not, for example if we take only a neighborhood of $\{x_4 = 0, x_3 \geq 0\}$ (see Remark 0.1 and Fig. 1 for S^2). We do not know if the exponential decay is true in this case.

The main result of this article is the following theorem.

Theorem 0.1. *Let $R_0 > 0$ and ω satisfy Assumption 0.2. Then, there exist $T > 0$ and $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\begin{aligned} \|(u_0, u_1)\|_{H^1 \times L^2} &\leq R_0; & \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1 \times L^2} &\leq R_0, \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq \delta; & \|(\tilde{u}_0, \tilde{u}_1)\|_{L^2 \times H^{-1}} &\leq \delta \end{aligned}$$

there exists $g \in L^\infty([0, T], L^2)$ supported in $[0, T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \square u + u + |u|^4 u = g & \text{on } [0, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

Let us discuss the assumptions on the size. In some sense, our theorem is a high frequency controllability result and expresses in a rough physical way that we can control some “small noisy data”. In the subcritical case, two similar kind of results were proved: in Dehman, Lebeau and Zuazua [11] similar results were proved for the nonlinear wave equation but without the smallness assumption in $L^2 \times H^{-1}$ while in Dehman and Lebeau [10], they obtained similar high frequency controllability results for the subcritical equation but in a uniform time which is actually the time of linear controllability (see also the work of the author [30] for the Schrödinger equation). Actually, this smallness assumption is made necessary in our proof because we are not able to prove the following unique continuation result.

Missing theorem. $u \equiv 0$ is the unique strong solution in the energy space of

$$\begin{cases} \square u + u + |u|^4 u = 0 & \text{on } [0, T] \times M, \\ \partial_t u = 0 & \text{on } [0, T] \times \omega. \end{cases}$$

In the subcritical case, this kind of theorem can be proved with Carleman estimates under some additional geometrical conditions and once the solution is known to be smooth. Yet, in the critical case, we are not able to prove this propagation of regularity. Note also that H. Koch and D. Tataru [27] managed to prove some unique continuation result in the critical case, but in the case $u = 0$ on ω instead of $\partial_t u = 0$. In the case of \mathbb{R}^3 with flat metric and ω the complementary of a ball, B. Dehman and P. Gérard [8] proved this theorem using the existence of the scattering operator proved by K. Nakanishi [32], which is not available on a manifold.

Moreover, as in the subcritical case, we do not know if the time of controllability does depend on the size of the data. This is actually still an open problem for several nonlinear evolution equations such as nonlinear wave or Schrödinger equation (even in the subcritical case). For the nonlinear wave equation, by finite speed of propagation, we do not expect the controllability in arbitrary short time, but at least to compare with the time of geometric control condition of the linear equation. Note that for certain nonlinear parabolic equations, it has been proved that we cannot have controllability in arbitrary short time, while it is the case for the linear equation, see [15] or [14].

The strategy for proving Theorem 0.1 consists in proving a stabilization result for a damped nonlinear Klein–Gordon equation and then, by a perturbative argument using the linear control, to bring the solution to zero once the energy of the solution is small enough. Namely, we prove

Theorem 0.2. *Let $R_0 > 0$, ω satisfy Assumption 0.2 and $a \in C^\infty(M)$ satisfy $a(x) > \eta > 0$ for all $x \in \omega$. Then, there exist $C, \gamma > 0$ and $\delta > 0$ such that for any (u_0, u_1) in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; \quad \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta,$$

the unique strong solution of

$$\begin{cases} \square u + u + |u|^4 u + a(x)^2 \partial_t u = 0 & \text{on } [0, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases} \tag{3}$$

satisfies $E(u)(t) \leq C e^{-\gamma t} E(u)(0)$.

This theorem is false for the classical nonlinear wave equation (see Section 3.1.1) and it is why we have chosen the Klein–Gordon equation instead.

Let us now discuss the proof of Theorem 0.2, following B. Dehman and P. Gérard [8] for the case of \mathbb{R}^3 . We have the energy decay

$$E(u)(t) = E(u)(0) - \int_0^t \int_M |a(x) \partial_t u|^2.$$

So, the exponential decay is equivalent to an observability estimate for the nonlinear damped equation. We prove it by contradiction. We are led to proving the strong convergence to zero of a normalized sequence u_n of solutions contradicting observability. In the subcritical case, the argument consisted in two steps

- to prove that the limit is zero by a unique continuation argument,
- to prove that the convergence is actually strong by linearization and linear propagation of compactness thanks to microlocal defect measures of P. Gérard [18] and L. Tartar [36].

By linearization, we mean (according to the terminology of P. Gérard [19]) that we have $\|u_n - v_n\| \xrightarrow{n \rightarrow \infty} 0$ where v_n is a solution of the linear Klein–Gordon equation with same initial data:

$$\begin{cases} \square v_n + v_n = 0 & \text{on } [0, T] \times M, \\ (v_n(0), \partial_t v_n(0)) = (u_n(0), \partial_t u_n(0)). \end{cases}$$

In our case, the smallness assumption in the lower regularity $L^2 \times H^{-1}$ makes that the limit is automatically zero, which allows to skip the first step. In the subcritical case, any sequence weakly convergent to zero is linearizable. Yet, for critical nonlinearity, there exist nonlinearizable sequences. Hopefully, in the case of \mathbb{R}^3 , this defect can be precisely described. It is linked to the non-compact action of the invariants of the equation: the dilations and translations. More precisely, the work of H. Bahouri and P. Gérard [2] states that any bounded sequence u_n of solutions to the nonlinear critical wave equation can be decomposed into an infinite sum of: the weak limit of u_n , a sequence of solutions to the free wave equation and an infinite sum of profiles which are translations–dilations of fixed nonlinear solutions. This decomposition was used by the authors of [8] to get the expected result in \mathbb{R}^3 . Therefore, we are led to make an analog of this profile decomposition for compact manifolds. We begin by the definition of the profiles.

Definition 0.2. Let $x_\infty \in M$ and $(f, g) \in \mathcal{E}_{x_\infty} = (\dot{H}^1 \times L^2)(T_{x_\infty} M)$. Given $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E}_{x_\infty} \times (\mathbb{R}_+^* \times M)^\mathbb{N}$ such that $\lim_n (h_n, x_n) = (0, x_\infty)$. We call the associated concentrating data the class of equivalence, modulo sequences convergent to 0 in \mathcal{E} , of sequence in \mathcal{E} that take the form

$$h_n^{-\frac{1}{2}} \Psi_U(x) \left(f, \frac{1}{h_n} g \right) \left(\frac{x - x_n}{h_n} \right) + o(1)\mathcal{E} \tag{4}$$

in some coordinate patch $U_M \approx U \subset \mathbb{R}^d$ containing x_∞ and for some $\Psi_U \in C_0^\infty(U)$ such that $\Psi_U(x) = 1$ in a neighborhood of x_∞ . (Here we have identified x_n, x_∞ with its image in U .)

We will prove later (Lemma 1.3) that this definition does not depend on the coordinate charts and on Ψ_U : two sequences defined by (4) in different coordinate charts are in the same class. In what follows, we will often call concentrating data associated to $[(f, g), \underline{h}, \underline{x}]$ an arbitrary sequence in this class.

Definition 0.3. Let (t_n) be a bounded sequence in \mathbb{R} converging to t_∞ and (f_n, g_n) a concentrating data associated to $[(f, g), \underline{h}, \underline{x}]$. A damped linear concentrating wave is a sequence v_n solution of

$$\begin{cases} \square v_n + v_n + a(x) \partial_t v_n = 0 & \text{on } \mathbb{R} \times M, \\ (v_n(t_n), \partial_t v_n(t_n)) = (f_n, g_n). \end{cases} \tag{5}$$

The associated damped nonlinear concentrating wave is the sequence u_n solution of

$$\begin{cases} \square u_n + u_n + a(x) \partial_t u_n + |u_n|^4 u_n = 0 & \text{on } \mathbb{R} \times M, \\ (u_n(0), \partial_t u_n(0)) = (v_n(0), \partial_t v_n(0)). \end{cases} \tag{6}$$

If $a \equiv 0$, we will only write linear or nonlinear concentrating wave.

It might seem counterintuitive to take the initial data at $t = 0$ (and not at $t = t_n$) for the nonlinear concentrating wave. This will, actually, be more convenient in the nonlinear profile decomposition where we want to compare the linear and nonlinear solution with same initial data.

Energy estimate yields that two representatives of the same concentrating data have the same associated concentrating wave modulo strong convergence in $L_{loc}^\infty(\mathbb{R}, \mathcal{E})$. This is not obvious for the nonlinear evolution but will be a consequence of the study of nonlinear concentrating waves.

It can be easily seen that this kind of nonlinear solutions are not linearizable. Actually, it can be shown that this concentration phenomenon is the only obstacle to linearizability. We begin with the linear decomposition.

Theorem 0.3. *Let (v_n) be a sequence of solutions to the damped Klein–Gordon equation (5) with initial data, at time $t = 0$, (φ_n, ψ_n) bounded in \mathcal{E} . Then, up to extraction, there exist a sequence of damped linear concentrating waves $(p_n^{(j)})$, as defined in Definition 0.3, associated to concentrating data $[(\varphi^{(j)}, \psi^{(j)}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$, such that for any $l \in \mathbb{N}^*$,*

$$v_n(t, x) = v(t, x) + \sum_{j=1}^l p_n^{(j)}(t, x) + w_n^{(l)}(t, x), \tag{7}$$

$$\forall T > 0, \quad \overline{\lim}_{n \rightarrow \infty} \|w_n^{(l)}\|_{L^\infty([-T, T], L^6(M)) \cap L^5([-T, T], L^{10})} \xrightarrow{l \rightarrow \infty} 0, \tag{8}$$

$$\|(v_n, \partial_t v_n)\|_{\mathcal{E}}^2 = \sum_{j=1}^l \|(p_n^{(j)}, \partial_t p_n^{(j)})\|_{\mathcal{E}}^2 + \|(w_n^{(l)}, \partial_t w_n^{(l)})\|_{\mathcal{E}}^2 + o(1), \quad \text{as } n \rightarrow \infty, \tag{9}$$

where $o(1)$ is uniform for $t \in [-T, T]$.

The nonlinear flow map follows this decomposition up to an error term in the strong following norm

$$\|u\|_I = \|u\|_{L^\infty(I, H^1(M))} + \|\partial_t u\|_{L^\infty(I, L^2(M))} + \|u\|_{L^5(I, L^{10}(M))}.$$

Theorem 0.4. *Let $T < T_{focus}/2$. Let u_n be the sequence of solutions to damped nonlinear Klein–Gordon equation (6) with initial data, at time 0, (φ_n, ψ_n) bounded in \mathcal{E} . Let $p_n^{(j)}$ be the linear damped concentrating waves (resp. v the weak limit) given by Theorem 0.3 and $q_n^{(j)}$ the associated nonlinear damped concentrating wave (resp. u the associated solution of the nonlinear equation with $(u, \partial_t u)_{t=0} = (v, \partial_t v)_{t=0}$). Then, up to extraction, we have*

$$u_n(t, x) = u + \sum_{j=1}^l q_n^{(j)}(t, x) + w_n^{(l)}(t, x) + r_n^{(l)}, \tag{10}$$

$$\overline{\lim}_{n \rightarrow \infty} \|r_n^{(l)}\|_{[-T, T]} \xrightarrow{l \rightarrow \infty} 0 \tag{11}$$

where $w_n^{(l)}$ is given by Theorem 0.3.

The same theorem remains true if M is the sphere S^3 and $a \equiv 0$ (undamped equation) without any assumption on the time T .

The more precise result we get for the sphere S^3 will not be useful for the proof of our controllability result. Yet, we have chosen to give it because it is the only case where we are able to describe what happens when some refocusing occurs.

This profile decomposition has already been proved for the critical wave equation on \mathbb{R}^3 by H. Bahouri and P. Gérard [2] and on the exterior of a convex obstacle by I. Gallagher and P. Gérard [17]. The same decomposition has also been performed for the Schrödinger equation by S. Keraani [26] and quite recently for the wave maps by Krieger and Schlag [28]. Note that such decomposition has proved to be useful in different contexts: the understanding of the precise behavior near the threshold for well-posedness for focusing nonlinear wave see Kenig and Merle [25] and Duyckaerts and Merle [13], the study of the compactness of Strichartz estimates and maximizers for Strichartz estimates, (see Keraani [26]), the global existence for wave maps [28], for NLS in the hyperbolic space [23]. . . Maybe our decomposition on manifolds could be useful in one of these contexts. Let us also mention that, this kind of decomposition appears for a long time in the context of Palais–Smale sequences for critical elliptic equation and optimal constant for Sobolev embedding, but with a finite number of profiles, see Brezis and Coron [4], the book [12] and the references therein. . .

Let us describe quickly the proof of the decomposition. The linear decomposition of Theorem 0.3 is made in two steps: first, we decompose our sequence in a sum of an infinite number of sequences oscillating at different rate $h_n^{(j)}$. Then, for each part oscillating at a fixed rate, we extract the possible concentration at certain points. We only have to prove that this process produces a rest w_n^l that gets smaller in the norm $L^\infty L^6$ at each stage. Once the linear decomposition is established, Theorem 0.4 says, roughly speaking, that the nonlinear flow map acts almost linearly on the linear decomposition. To establish the nonlinear decomposition we have to prove that each element of the decomposition do not interact with the others. For each element of the linear decomposition, we are able to describe the nonlinear solution arising from this element as initial data. The linear rest w_n^l is small in $L^\infty([-T, T], L^6)$ for l large enough and so the associated nonlinear solution with same initial data is very close to the linear one. The behavior of nonlinear concentrating waves is described in [21] (see Section 2.2.1 for a short review). Before the concentration, linear and nonlinear waves are very close. For times close to the time of concentration, the nonlinear rescaled solution behaves as if the metric was flat and is subject to the scattering of \mathbb{R}^3 . After concentration, the solution is close to a linear concentrating wave but with a new profile obtained by the scattering operator on \mathbb{R}^3 .

We finish this introduction by a discussion on the geometric conditions we imposed to get our main theorem. For the linear wave equation, the controllability is known to be equivalent to the so-called Geometric Control Condition (Assumption 0.1). This was first proved by Rauch and Taylor [33] in the case of a compact manifold and by Bardos, Lebeau and Rauch [3] for boundary control (see Burq and Gérard [5] for the necessity). For the nonlinear subcritical problem, the result of [10] only requires the classical Geometric Control Condition. Our assumption is stronger and we can naturally wonder if it is really necessary. It is actually strongly linked with the critical behavior and nonlinear concentrating waves. Removing this stronger assumption would require a better understanding of the scattering operator of the nonlinear equation on \mathbb{R}^3 (see Remark 0.1). However, we think that the same result could be obtained with the following weaker assumption.

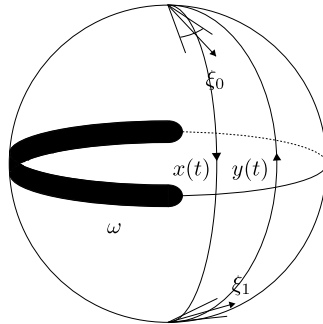


Fig. 1. Possible situation on the sphere.

Assumption 0.3. ω satisfies the Geometric Control Condition. Moreover, for every couple of focus (x_1, x_2, t) at distance t , according to Definition 0.1, each geodesic starting from x_1 in direction ξ such that $\exp_{x_1} t\xi = x_2$ meets ω in a time $0 \leq s < t$.

Finally, we note that our theorem can easily be extended to the case of \mathbb{R}^3 with a metric flat at infinity. In this case, our stabilization term $a(x)$ should fulfill the two assumptions:

- there exist $R > 0$ and $\rho > 0$ such that $a(x) > \rho$ for $|x| > R$,
- $a(x) > \rho$ for $x \in \omega$ where ω satisfies Assumption 0.2.

The proof would be very similar. The only difference would come from the fact that the domain is not compact. So the profile decomposition would require the “compactness at infinity” (see property (1.6) of [2]). Moreover, the equipartition of the energy could not be made only with measures but with an explicit computation (see (3.14) of [8]).

Remark 0.1. In order to know if our stronger Assumptions 0.2 or 0.3 are really necessary compared to the classical Geometric Control Condition, we need to prove that the following scenario cannot happen. We take the example of S^3 with ω a neighborhood of $\{x_4 = 0, x_3 \geq 0\}$.

Take some data concentrating on the north pole, with a Fourier transform (on the tangent plane) supported around a direction ξ_0 . The nonlinear solution will propagate linearly as long as it does not concentrate: at time t it will be supported in a neighborhood of the point $x(t)$ where $x(t)$ follows the geodesic stemming from the north pole at time 0 in direction ξ_0 . Then, if ξ_0 is well chosen, it can avoid ω during that time. Yet, at time π , the solution will concentrate again in the south pole. According to the description of S. Ibrahim [21], in a short time, the solution will be transformed following the nonlinear scattering operator on \mathbb{R}^3 . So, at time $\pi + \varepsilon$ the solution is close to a linear concentrating wave but it concentrates with a new profile which is obtained with the nonlinear scattering operator on \mathbb{R}^3 . This operator is strongly nonlinear and we do not know whether the new profile will be supported in Fourier near a new direction ξ_1 . If it happens, the solution will then be supported near the point $y(t)$ where $y(t)$ follows the geodesic stemming from the south pole at time π in direction ξ_1 . In this situation, it will be possible that the trajectory $y(t)$ still avoids ω . If this phenomenon happens several times, we would have a sequence that concentrates periodically on the north and south pole but always avoiding the region ω (which in that case satisfies Geometric Control Condition).

We are led to the following informal question. If S is the scattering operator on \mathbb{R}^3 , then it is possible that for some data $(f, g) \in \dot{H}^1 \times L^2$ supported in Fourier near a direction ξ_0 , the Fourier transform of $S(f, g)$ is supported near another direction ξ_1 . In other words, can the nonlinear wave operator change the direction of the light?

Note, that in this specific example of S^3 , the use of the momentum on \mathbb{R}^3 should allow to conclude. This will be detailed in a forthcoming article.

The structure of the article is as follows. The first section contains some preliminaries that will be used all along the article: the existence theorem for damped nonlinear equation, the description of the main properties of concentrating waves and the useful properties of the scales necessary for the linear decomposition. The second section contains the proof of the profile decomposition of Theorems 0.3 and 0.4. It is naturally divided in two steps corresponding to the linear decomposition and the nonlinear one. We close this section by some useful consequences of the decomposition. The third section contains the proof of the main theorems: the control and stabilization.

Note that the main argument for the proof of stabilization is contained in the last Section 3: in Proposition 3.1 we apply the linearization argument to get rid of the profiles while Theorem 3.1 contains the proof of the weak observability estimates. We advise the hurried reader to have a first glance at these two proofs in order to understand the global argument.

0.1. Notation

For an interval I , denote

$$\|u\|_I = \|u\|_{L^\infty(I, H^1(M))} + \|\partial_t u\|_{L^\infty(I, L^2(M))} + \|u\|_{L^5(I, L^{10}(M))}.$$

Moreover, when we work in local coordinate, we will need the similar norm (except for \dot{H}^1 instead of H^1)

$$\|u\|_{I \times \mathbb{R}^3} = \|u\|_{L^\infty(I, \dot{H}^1(\mathbb{R}^3))} + \|\partial_t u\|_{L^\infty(I, L^2(\mathbb{R}^3))} + \|u\|_{L^5(I, L^{10}(\mathbb{R}^3))}.$$

Note that if $I = \mathbb{R}$, then $\|u\|_{I \times \mathbb{R}^3}$ is invariant by the translation and scaling $u \mapsto \frac{1}{\sqrt{h}} u(\frac{t-t_0}{h}, \frac{x-x_0}{h})$.

The energy spaces are denoted by

$$\begin{aligned} \mathcal{E} &= H^1(M) \times L^2(M), \\ \mathcal{E}_{x_\infty} &= \dot{H}^1(T_{x_\infty} M) \times L^2(T_{x_\infty} M) \end{aligned}$$

with the respective norms

$$\begin{aligned} \|(f, g)\|_{\mathcal{E}}^2 &= \|f\|_{L^2(M)}^2 + \|\nabla f\|_{L^2(M)}^2 + \|g\|_{L^2(M)}^2, \\ \|(f, g)\|_{\mathcal{E}_\infty}^2 &= \|\nabla f\|_{L^2(T_{x_\infty} M)}^2 + \|g\|_{L^2(T_{x_\infty} M)}^2. \end{aligned}$$

We will denote by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{E}_\infty}$ the associated scalar products.

When dealing with solutions of nonlinear wave equations on M (or on $T_{x_\infty}M$), “the unique strong solution” will mean the unique solution in the Strichartz space $L^5_{loc}(\mathbb{R}, L^{10}(M))$ (or $L^5_{loc}(\mathbb{R}, L^{10}(T_{x_\infty}M))$) such that $(u, \partial_t u) \in C(\mathbb{R}, \mathcal{E})$ (or $C(\mathbb{R}, \mathcal{E}_{x_\infty})$).

All along the article, for a point $x \in M$, we will sometimes not distinguish x with its image in a coordinate patch and will write \mathbb{R}^3 instead of $T_{x_\infty}M$. M will always be smooth, compact and the number of coordinate charts we use is always assumed to be finite. We also assume that all the charts are relatively compact. In all the article, C will denote any constant, possibly depending on the manifold M and the damping function a . We will also write \lesssim instead of $\leq C$ for a constant C .

$B^s_{2,\infty}(M)$ denotes the Besov space on M defined by

$$\|u\|_{B^s_{2,\infty}(M)} = \|\mathbf{1}_{[0,1]}(\sqrt{-\Delta_M})u\|_{L^2(M)} + \sup_{k \in \mathbb{N}} \|\mathbf{1}_{[2^k, 2^{k+1}]}(\sqrt{-\Delta_M})u\|_{H^s(M)}.$$

We use the same definition for $B^s_{2,\infty}(\mathbb{R}^3)$ with Δ_M replaced by $\Delta_{\mathbb{R}^3}$ which can be expressed using the Fourier transform and the Littlewood–Paley decomposition. Of course, $B^s_{2,\infty}(M)$ is linked with $B^s_{2,\infty}(\mathbb{R}^3)$ by the expression in coordinate charts. This will be precised in Lemma 2.1.

From now on, $a = a(x)$ will always denote a smooth real-valued function defined on M .

1. Preliminaries

1.1. Existence theorem

The existence of solutions to our equation is proved using two tools: Strichartz and Morawetz estimates. Strichartz estimates take the following form.

Proposition 1.1 (Strichartz and energy estimates). *Let $T > 0$ and (p, q) satisfy*

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad p > 2.$$

Then, there exists $C > 0$ such that any solution u of

$$\begin{cases} \square v + v + a(x)\partial_t v = f & \text{on } [-T, T] \times M, \\ (v(0), \partial_t v(0)) = (u_0, u_1) \end{cases}$$

satisfies the estimate

$$\|(v, \partial_t v)\|_{L^\infty([-T, T], \mathcal{E})} + \|v\|_{L^p([-T, T], L^q(M))} \leq C(\|(u_0, u_1)\|_{\mathcal{E}} + \|f\|_{L^1([-T, T], L^2)}).$$

Proof. The case with $a \equiv 0$ for the wave equation can be found in L.V. Kapitanski [24]. To treat the case of damped Klein–Gordon, we only have to absorb the additional terms and get the desired estimate for T small enough. We can then reiterate the operation to get the result for large times. \square

Then, we are going to prove the global existence for the equation

$$\begin{cases} \square u + u + |u|^4 u = a(x)\partial_t u + g & \text{on } [-T, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{E} \end{cases} \tag{12}$$

with $g \in L^1([-T, T], L^2(M))$ and $a \in C^\infty(M)$.

The proof is now very classical, see for example [37] for a survey of the subject. The critical defocussing nonlinear wave equation on \mathbb{R}^3 was proved to be globally well posed by Shatah and Struwe [35,34] using Morawetz estimates. Later, S. Ibrahim and M. Majdoub managed to apply this strategy in the case of variable coefficients in [22], but without damping and forcing term. In this subsection, we extend this strategy to the case with these additional terms. We also refer to the appendix of [2] where the computation of Morawetz estimates on \mathbb{R}^3 is made with a forcing term. We also mention the result of N. Burq, G. Lebeau and F. Planchon [7] in the case of 3-D domains.

We only have to check that the two additional terms do not create any trouble. Actually, the main difference is that the energy in the light cones is not decreasing, but it is locally “almost decreasing” (see formula (13)) and this will be enough to conclude with the same type of arguments.

As usual in critical problems, the local problem is well understood thanks to Strichartz estimates while we have to prove global existence. We only consider Shatah–Struwe solutions, that are satisfying Strichartz estimates and we have uniqueness for local solutions in this class. We assume that there is a maximal time of existence t_0 and we want to prove that it is infinite. The solution considered will be limit of smooth solutions of the nonlinear equation with smoothed initial data and nonlinearity. Therefore, the integrations by part are licit by a limiting argument.

We need some notations. To simplify the notations, the space–time point where we want to extend the solution will be $z_0 = (t_0, x_0) = (0, 0)$. φ is the geodesic distance on M to $x_0 = 0$ defined in a neighborhood U of 0. Denote for some small $\alpha < \beta < 0$ by

$$\begin{aligned} K_\alpha^\beta &:= \{z = (t, x) \in [\alpha, \beta] \times U \mid \varphi \leq |t|\} && \text{backward truncated cone,} \\ M_\alpha^\beta &:= \{z = (t, x) \in [\alpha, \beta] \times U \mid \varphi = |t|\} && \text{mantle of the truncated cone,} \\ D(t) &:= \{x \in U \mid \varphi \leq |t|\} && \text{spacelike section of the cone at time } t. \end{aligned}$$

In what follows, the gradient, norm, density are computed with respect to the Riemannian metric on M (for example, we have $|\nabla\varphi| = 1$). We also define by

$$\begin{aligned} e(u)(t, x) &:= \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) + \frac{1}{6}|u|^6 && \text{local energy,} \\ E(u, D(t)) &:= \int_{D(t)} e(u)(t, x) dx && \text{energy at time } t \text{ in the section of the cone,} \\ Flux(u, M_\alpha^\beta) &:= \frac{1}{\sqrt{2}} \int_{M_\alpha^\beta} \frac{1}{2}|\partial_t u \nabla\varphi - \nabla u|^2 + \frac{1}{6}|u|^6 d\sigma && \text{flux getting out of the truncated cone.} \end{aligned}$$

Lemma 1.1. *Let u be a solution of Eq. (12). The function $E(u, D(t))$ satisfies for $\alpha < \beta < 0$*

$$E(u, D(\beta)) + Flux(u, M_\alpha^\beta) = E(u, D(\alpha)) + \int \int_{K_\alpha^\beta} a(x)|\partial_t u|^2 - \Re \int \int_{K_\alpha^\beta} u \partial_t \bar{u} + \Re \int \int_{K_\alpha^\beta} g \partial_t \bar{u}$$

and it has a left limit in $t = 0$ as a function of t .

Proof. The identity is obtained by multiplying the equation by $\partial_t \bar{u}$ to get $\partial_t e(u) - \Re \operatorname{div}(\partial_t \bar{u} \nabla_x u) = a(x)|\partial_t u|^2 - u \partial_t \bar{u} + \Re g \partial_t \bar{u}$, then, we integrate over the truncated cone K_α^β and use Stokes formula. Denote $f(t) = E(u, D(t))$. Using the positivity of the flux and the Hölder inequality, we estimate

$$\begin{aligned} \|f\|_{L^\infty([\alpha, \beta])} &\leq f(\alpha) + C(\beta - \alpha)\|f\|_{L^\infty([\alpha, \beta])} + C|\alpha|(\beta - \alpha)\|f\|_{L^\infty([\alpha, \beta])}^{2/3} \\ &\quad + \|g\|_{L^1([\alpha, \beta], L^2)}\|f\|_{L^\infty([\alpha, \beta])}^{1/2}. \end{aligned}$$

Using $C|\alpha|(\beta - \alpha)\|f\|_{L^\infty([\alpha, \beta])}^{2/3} \leq C(\beta - \alpha)(\|f\|_{L^\infty([\alpha, \beta])}^{1/2} + \|f\|_{L^\infty([\alpha, \beta])})$, we get for $\beta - \alpha$ small enough

$$f(\beta)^{1/2} \leq \frac{1}{1 - 2C(\beta - \alpha)} [f(\alpha)^{1/2} + C(\beta - \alpha) + \|g\|_{L^1([\alpha, \beta], L^2)}]. \tag{13}$$

This property will replace the decrease of the energy that occurs without damping and forcing term in all the rest of the proof. It easily implies that f has a left limit. \square

Lemma 1.2. *For u and g being a strong solution of*

$$\square u + |u|^4 u = g \quad \text{on } [-T, 0] \times M$$

we have the estimate

$$\begin{aligned} \int_{D(\alpha)} |u|^6 &\leq C \left(\frac{\beta}{\alpha} (f(\beta) + f(\beta)^{1/3}) + |f(\beta) - f(\alpha)| + \|g\|_{L^1 L^2(K_\alpha^\beta)} \|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} \right. \\ &\quad + (|f(\beta) - f(\alpha)| + \|g\|_{L^1 L^2(K_\alpha^\beta)} \|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)})^{1/3} \\ &\quad + \|g\|_{L^1 L^2(K_\alpha^\beta)} (\|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} + \|\nabla u\|_{L^\infty L^2(K_\alpha^\beta)} + \|u\|_{L^\infty L^6(K_\alpha^\beta)}) \\ &\quad \left. + (\beta - \alpha) \sup_{t \in [\alpha, \beta]} [f(t) + f(t)^{1/3}] \right) \end{aligned}$$

where we have used the notation $f(t) = E(u, D(t))$.

Proof. It is a consequence of Morawetz estimates. The only difference is the presence of the forcing term g and the metric. The case of flat metric is treated in [2]. The metric leads to the same estimates with an additional term $(\beta - \alpha) \sup_{t \in [\alpha, \beta]} f(t) + f(t)^{1/3}$ as treated in [22]. Another minor difference is that in the presence of a forcing term, the energy does not decrease

and $f(\beta) + f(\beta)^{1/3}$ have to be replaced by the supremum on the interval. Note also that our estimate is made in the backward cone while the computation is made in the future cone in these references. We leave the easy modifications to the reader. \square

The previous estimates will be the main tools of the proof. It will be enough to prove some non-concentration property in the light cone for $L^\infty L^6$, $L^5 L^{10}$ and finally in energy space. It is the object of the following three corollaries.

Corollary 1.1.

$$\int_{D(\alpha)} |u(\alpha, x)|^6 dx \xrightarrow{\alpha \rightarrow 0} 0.$$

Proof. We are going to use the previous Lemma 1.2, replacing g by $g - u + a(x)\partial_t u$ and with $\beta = \varepsilon\alpha$, $0 < \varepsilon < 1$. Denote by L the limit of $f(t)$ as t tends to 0 given by Lemma 1.1. So for α small enough, we have for a constant $C > 0$

$$\|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} + \|\nabla u\|_{L^\infty L^2(K_\alpha^\beta)} + \|u\|_{L^\infty L^6(K_\alpha^\beta)} \leq 1 + C(L^{1/2} + L^{1/6}).$$

We also use

$$\begin{aligned} & \|g - u + a(x)\partial_t u\|_{L^1 L^2(K_\alpha^\beta)} \\ & \leq \|g\|_{L^1 L^2(K_\alpha^\beta)} + C(\beta - \alpha)\|u\|_{L^\infty L^2(K_\alpha^\beta)} + C(\beta - \alpha)\|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} \\ & \leq \|g\|_{L^1 L^2(K_\alpha^\beta)} + C(\beta - \alpha)(1 + L^{1/6} + L^{1/2}) \end{aligned}$$

which tends to 0 as β tends to 0. This yields

$$\overline{\lim}_{\alpha \rightarrow 0} \int_{D(\alpha)} |u(\alpha, x)|^6 dx \leq C\varepsilon(L + L^{1/3}). \quad \square$$

Corollary 1.2.

$$u \in L^5 L^{10}(K_{-T}^0).$$

Proof. Localized Strichartz estimates in cones (see Proposition 4.4 of [22]) give

$$\begin{aligned} & \|u\|_{L^4 L^{12}(K_s^0)} \\ & \leq CE(u, D(s))^{1/2} + \|u\|_{L^5 L^{10}(K_s^0)} + \|a(x)\partial_t u - u + g\|_{L^1 L^2(K_s^0)} \\ & \leq CE(u, D(s))^{1/2} + \|u\|_{L^\infty L^6(K_s^0)}(1 + \|u\|_{L^4 L^{12}(K_s^0)}) + \|\partial_t u\|_{L^\infty L^2(K_s^0)} + \|g\|_{L^1 L^2(K_s^0)}. \end{aligned}$$

A bootstrap argument and Corollary 1.1 give that for s sufficiently close to 0, $\|u\|_{L^4 L^{12}(K_s^0)}$ is bounded. We get the announced result by interpolation between $L^4 L^{12}$ and $L^\infty L^6$. \square

Corollary 1.3.

$$E(u, D(s)) \xrightarrow{s \rightarrow 0} 0.$$

Proof. Let $\varepsilon > 0$. Corollary 1.2 allows to fix $s < 0$ close to 0 so that $\|u\|_{L^5 L^{10}(K_s^0)} \leq \varepsilon$. Denote by v_s the solution to the linear equation

$$\square v_s + v_s + a(x)\partial_t v_s = 0, \quad (v_s, \partial_t v)_{t=s} = (u, \partial_t u)_{t=s}$$

then, the difference $w_s = u - v_s$ is a solution of

$$\square w_s + w_s + a(x)\partial_t w_s = -|u|^4 u, \quad (w_s, \partial_t w_s)_{t=s} = (0, 0).$$

Then, for $s < t < 0$, the linear energy estimates give

$$E_0(w_s, D(t))^{1/2} \leq C \|u\|_{L^5 L^{10}(K_s^0)}^5 \leq C\varepsilon^5$$

where we have set

$$E_0(w_s, D(t)) = \frac{1}{2} \int_{D(t, z_0)} [|\nabla w_s|^2 + |\partial_t w_s|^2] dx.$$

The triangular inequality yields

$$E_0(u, D(t))^{1/2} \leq E_0(v_s, D(t))^{1/2} + C\varepsilon^5.$$

Since v_s is a solution of the free damped linear equation, we have $E_0(v_s, D(t)) \xrightarrow{t \rightarrow 0} 0$. This yields the result with E_0 instead of E . The final result is obtained thanks to Corollary 1.1. \square

We can now finish the proof of the global existence.

Let $\varepsilon > 0$ be chosen later. By Corollary 1.3, $E(u, D(s)) \leq \varepsilon$ for s close enough to 0. By dominated convergence, for any $s < 0$ close to 0, there exists $\eta > 0$ so that

$$\int_{\varphi(x) \leq t_0 - s + \eta} e(u)(s) = E(u, D(s, \eta)) \leq 2\varepsilon$$

where $E(u, D(s, \eta))$ is the spacelike energy at time s of the cone centered at $(t_0 = \eta, x_0 = 0)$ (see Fig. 2). For s close enough to 0 and $s < s' < 0$, we apply estimate (13) in this cone. It gives

$$E(u, D(s', \eta))^{1/2} \leq C(E(u, D(s, \eta))^{1/2} + |s' - s| + \|g\|_{L^1([s, s'], L^2)}) \leq C\varepsilon^{1/2}.$$

In particular, $\|u\|_{L^\infty L^6(K)} \leq C\varepsilon^{1/2}$ on the truncated cone

$$K = \{(s', x) \mid \varphi(x) \leq \eta - s', s < s' < 0\}.$$

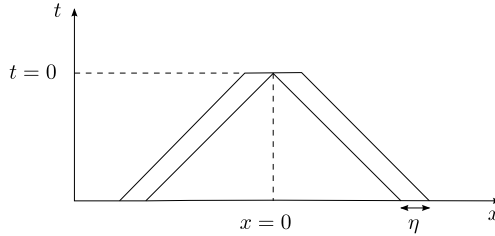


Fig. 2. The truncated cone K .

Therefore, choosing ε small enough to apply the same proof as Corollary 1.2, we get

$$\|u\|_{L^5 L^{10}(K)} < +\infty.$$

Since $x_0 = 0$ is arbitrary, a compactness argument yields one $s < 0$ such that $\|u\|_{L^5([s,0], L^{10}(M))} < +\infty$. Therefore, by the Duhamel formula, $(u(t), \partial_t u(t))$ has a limit in \mathcal{E} as t tends to 0 and u can be extended for some small $t > 0$ using local existence theory.

Remark 1.1. It is likely that the global existence can also be proved using the Kenig–Merle argument [25] and the profile decomposition below (assuming only local existence) as it is done for example in [28] for the wave maps.

1.2. Concentration waves

In this subsection, we give the details about concentrating waves that will be useful in the profile decomposition. The first lemma states that Definition 0.2 of concentrating data does not depend on the choice of coordinate patch and cut-off function Ψ_U .

Lemma 1.3. Let $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E} \times (\mathbb{R}_+^* \times M \times \mathbb{N})$ be such that $\lim_n (h_n, x_n) = (0, x_\infty)$ then, all the sequences defined by formula (4) in different coordinates charts and the cut-off function Ψ_U are equivalent, modulo convergence in \mathcal{E} .

Proof. It is very close to the one of S. Ibrahim [21] where the concentrating data are given in geodesic coordinates. So, let $V_M \approx V$ be another coordinate patch and $\Phi : V \mapsto U$ the associated transition map. Without loss of generality, we can suppose that x_∞ is represented by 0 in U and V . We have to prove that the sequences

$$h_n^{-\frac{1}{2}} \Phi^* \Psi_U(x) \left(f, \frac{1}{h_n} g \right) \left(\frac{x - \Phi(x_n)}{h_n} \right) = h_n^{-\frac{1}{2}} \Psi_U(\Phi(x)) \left(f, \frac{1}{h_n} g \right) \left(\frac{\Phi(x) - \Phi(x_n)}{h_n} \right)$$

and

$$h_n^{-1/2} \Psi_V(x) \left(f \circ D\Phi(0), \frac{1}{h_n} g \circ D\Phi(0) \right) \left(\frac{x - x_n}{h_n} \right)$$

are equivalent in the energy space associated to M or \mathbb{R}^3 (the volume form and the gradient are not the same but the energies are equivalent). By approximation, we can assume $(f, g) \in$

$(C_0^\infty(\mathbb{R}^3))^2$. We make the proof for the \dot{H}^1 part for f , the proof being simpler for g . We remark that the terms coming from derivatives hitting on $\Psi_U(x)$ tend to 0 in L^2 . Therefore, we have to prove the convergence to 0 of

$$h_n^{-3} \left\| \Psi_U(\Phi(x)) D\Phi(x) \nabla f \left(\frac{\Phi(x) - \Phi(x_n)}{h_n} \right) - \Psi_V(x) D\Phi(0) \nabla f \left(\frac{D\Phi(0)x - D\Phi(0)x_n}{h_n} \right) \right\|_{L^2(V)}^2.$$

First, we prove that the cut-off functions Ψ_U and Ψ_V can be replaced by a unique Ψ . Let δ be so that $B(0, \delta) \subset V$. Let $\Psi \in C_0^\infty(B(0, \delta))$ be such that $\Psi \equiv 1$ in a neighborhood of 0 and has a support included in the set of x such that $\Psi_V(x) = \Psi_U(\Phi(x)) = 1$, so that $\Psi\Psi_V = \Psi$ and $\Psi(\Psi_U \circ \Phi) = \Psi$. Then, on the support of $1 - \Psi$, we have $\|\Phi(x) - \Phi(x_n)\| > \varepsilon$ for some $\varepsilon > 0$ and some n large enough. Therefore, we have

$$h_n^{-3} \left\| (1 - \Psi(x)) \Psi_U(\Phi(x)) D\Phi(x) \nabla f \left(\frac{\Phi(x) - \Phi(x_n)}{h_n} \right) \right\|_{L^2(V)}^2 \leq Ch_n^{-3} \left\| \nabla f \left(\frac{\Phi(x) - \Phi(x_n)}{h_n} \right) \right\|_{L^2(\|\Phi(x) - \Phi(x_n)\| > \varepsilon)}^2$$

which is 0 for n large enough since f has compact support. Making the same proof for the other term, we are led to prove the convergence to 0 of

$$h_n^{-3} \left\| \Psi(x) D\Phi(x) \nabla f \left(\frac{\Phi(x) - \Phi(x_n)}{h_n} \right) - \Psi(x) D\Phi(0) \nabla f \left(\frac{D\Phi(0)x - D\Phi(0)x_n}{h_n} \right) \right\|_{L^2(B(0, \delta))}^2 \leq \left\| D\Phi(h_nx + x_n) \nabla f \left(\frac{\Phi(h_nx + x_n) - \Phi(x_n)}{h_n} \right) - D\Phi(0) \nabla f(D\Phi(0)x) \right\|_{L^2(\{x: |x_n + h_nx| \leq \delta\})}^2. \tag{14}$$

By the fundamental theorem of calculus, there exists $z_n(x) \in [x_n, h_nx + x_n]$ such that $|\frac{\Phi(h_nx + x_n) - \Phi(x_n)}{h_n}| = |D\Phi(z_n)x| > C|x|$ for some uniform $C > 0$. As ∇f is compactly supported, we deduce that for $|x|$ large enough, the integral is zero. So, we are led with the norm (14) with $L^2(B(0, C))$ instead of $L^2(\{x: |x_n + h_nx| \leq \delta\})$. We conclude by dominated convergence. \square

Using the previous lemma in geodesic coordinates, we get that our definition of concentrating data is the same as Definition 1.2 of S. Ibrahim [21].

Remark that for a concentrating data, $x_n - x_\infty$ cannot be defined invariantly on $T_{x_\infty}M$, we can only define the limit of $(x_n - x_\infty)/h_n$. The change of coordinates must act on x_n as an element of M and not $T_{x_\infty}M$ even if it converges to x_∞ . Yet, the functions (f, g) of a concentrating data “live” on the tangent space. Moreover, the norm in energy of a concentrating data is the one of its data.

Lemma 1.4. *Let (u_n, v_n) be a concentrating data associated to $[(\varphi, \psi), \underline{h}, \underline{x}]$, then, we have*

$$\|(u_n, v_n)\|_{\mathcal{E}} = \|(\varphi, \psi)\|_{\mathcal{E}_{x_\infty}} + o(1)$$

where ∇_{x_∞} and $L^2(T_{x_\infty} M)$ are computed with respect to the frozen metric.

The proof is a direct consequence of Lemmas 1.5 and 1.6 below or by a direct computation in coordinates.

The next definition is the tool that will be used to “track” the concentrations.

Definition 1.1. Let $x_\infty \in M$ and $(f, g) \in \mathcal{E}_{x_\infty}$. Given $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E}_{x_\infty} \times (\mathbb{R}_+^* \times M)^\mathbb{N}$ such that $\lim_n (h_n, x_n) = (0, x_\infty)$. Let (f_n, g_n) be a sequence bounded in \mathcal{E} , we set

$$D_{h_n}(f_n, g_n) \rightharpoonup (f, g)$$

if in some coordinate patch $U_M \approx U \subset \mathbb{R}^d$ containing x_∞ and for some $\Psi_U \in C_0^\infty(U)$ such that $\Psi_U(x) = 1$ in a neighborhood of x_∞ , we have

$$h_n^{\frac{1}{2}}(\Psi_U f_n, h_n \Psi_U g_n)(x_n + h_n x) \rightharpoonup (f, g) \text{ weakly in } \mathcal{E}_{x_\infty}$$

where we have identified $\Psi_U(f_n, g_n)$ with its representation on $T_{x_\infty} M$ in the local trivialization.

If this holds for one (U, Ψ_U) , it holds for any other coordinate chart with the induced transition map.

We denote $D_{h_n}^1 f_n \rightharpoonup f$ if we only consider the first part concerning \dot{H}^1 and $D_{h_n}^2 g_n \rightharpoonup g$ for the L^2 part convergence.

Of course, this definition depends on the core of concentration \underline{h} and \underline{x} . In the rest of the paper, the rate \underline{h} and \underline{x} will always be implicit. When several rates of concentration $[\underline{h}^{(j)}, \underline{x}^{(j)}]$, $j \in \mathbb{N}$, are used in a proof, we use the notation $D_h^{(j)}$ to distinguish them.

The fact that this definition is independent of the choice of a coordinate chart can be seen with the following lemma which will also be useful afterward.

Lemma 1.5. *$D_{h_n}(f_n, g_n) \rightharpoonup (f, g)$ is equivalent to*

$$\begin{aligned} \int_M \nabla_M f_n \cdot \nabla_M u_n &\xrightarrow[n \rightarrow \infty]{} \int_{T_{x_\infty} M} \nabla_{x_\infty} f \cdot \nabla_{x_\infty} \varphi, \\ \int_M g_n v_n &\xrightarrow[n \rightarrow \infty]{} \int_{T_{x_\infty} M} g \psi \end{aligned}$$

where (u_n, v_n) is any concentrating data associated with $[(\varphi, \psi), \underline{h}, \underline{x}]$.

The ∇ is computed with respect to the metric on M when the integral is over M and with respect to the frozen metric in x_∞ when the integral is over $T_{x_\infty} M$.

Proof of Lemma 1.5. We only compute the first term for the H^1 norm and assume $\varphi \in C_0^\infty(\mathbb{R}^3)$. $d\omega(y)$ denotes the Riemannian volume form at the point y , \cdot_y the scalar product at the point y and $\nabla_{h_n x + x_n} = g(h_n x + x_n)^{-1} \nabla$.

We denote $V_h = \frac{V - x_n}{h}$ and $L_{n,V} = h_n^{\frac{1}{2}} \int_{V_h} \nabla_{x_\infty} [\Psi_V f_n(x_n + h_n x)] \cdot \nabla_{x_\infty} \varphi(x) d\omega(0)$.

$$\begin{aligned} L_{n,V} &= h_n^{\frac{1}{2}} \int_{V_h} \nabla_{x_n + h_n x} [\Psi_V f_n(x_n + h_n x)] \cdot_{(x_n + h_n x)} \nabla_{x_n + h_n x} \varphi(x) d\omega(x_n + h_n x) + o(1) \\ &= h_n^{\frac{3}{2}} \int_{V_h} \Psi_V(x_n + h_n x) (\nabla_{x_n + h_n x} f_n)(x_n + h_n x) \cdot_{(x_n + h_n x)} \nabla_{x_n + h_n x} \varphi(x) d\omega(x_n + h_n x) \\ &\quad + o(1) \\ &= h_n^{-\frac{3}{2}} \int_V \nabla_y f_n(y) \cdot_y \Psi_V(y) \nabla_y \varphi\left(\frac{y - x_n}{h_n}\right) d\omega(y) + o(1) \\ &= h_n^{-\frac{1}{2}} \int_V \nabla_y f_n(y) \cdot_y \nabla_y \left[\Psi_V(y) \varphi\left(\frac{y - x_n}{h_n}\right) \right] d\omega(y) + o(1) \\ &= \int_M \nabla_M f_n \cdot \nabla_M u_n + o(1). \end{aligned}$$

Therefore, $L_{n,V}$ tends to $\int \nabla f(x) \cdot \nabla \varphi(x) d\omega(0)$ if and only if $\int_M \nabla_M f_n \cdot \nabla_M u_n$ has the same limit. \square

An easy consequence of this lemma is the link with concentrating waves.

Lemma 1.6. Let (f_n, g_n) be some concentrating data associated with $[(f, g), \underline{h}, \underline{x}]$, then, we have

$$D_{h_n}(f_n, g_n) \rightharpoonup (f, g).$$

Proof. Lemma 1.3 permits to work in geodesic coordinates so that the metric g is the identity at the point x_∞ . In this chart, we have $f_n(x_n + h_n x) = \Psi_U(x_n + h_n x) h_n^{-\frac{1}{2}} f$. So, the computation of Lemma 1.5 gives $\int \nabla_\infty f \cdot \nabla_\infty \varphi d\omega(0) = \int_M \nabla_M f_n \cdot \nabla_M u_n + o(1)$ which gives the result. \square

We conclude this subsection by a definition of orthogonality that will discriminate concentrating data.

Definition 1.2. We say that two sequences $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ and $[\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ are orthogonal if either

- $\log \left| \frac{h_n^{(1)}}{h_n^{(2)}} \right| \xrightarrow{n \rightarrow \infty} +\infty,$
- $x_\infty^{(1)} \neq x_\infty^{(2)},$ or

- $h_n^{(1)} = h_n^{(2)} = h$ and $x_\infty^{(1)} = x_\infty^{(2)} = x_\infty$ and in some coordinate chart around x_∞ , we have

$$\frac{|t_h^{(1)} - t_h^{(2)}|}{h} + \frac{|x_h^{(1)} - x_h^{(2)}|}{h} \xrightarrow{h \rightarrow 0} +\infty.$$

We note $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}] \perp [\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ and $(\underline{x}^{(1)}, \underline{t}^{(1)}) \perp_h (\underline{x}^{(2)}, \underline{t}^{(2)})$ if $\underline{h}^{(1)} = \underline{h}^{(2)} = h$.

This definition does not depend on the coordinate chart. This can be seen because we have the estimate $\frac{1}{C}|x_h^{(1)} - x_h^{(2)}| \leq |\Phi(x_h^{(1)}) - \Phi(x_h^{(2)})| \leq C|x_h^{(1)} - x_h^{(2)}|$ if Φ is the transition map.

1.3. Scales

In this subsection, we precise a few facts that will be useful in the first part of the proof of linear profile decomposition which consists of the extraction of the scales of oscillation h_n^j .

On the Hilbert space $\mathcal{E} = H^1(M) \times L^2(M)$, we define the self-adjoint operator A_M by:

$$D(A_M) = H_M^2 \times H_M^1, \\ A_M(u, v) = ((-\Delta_M)^{1/2}v, (-\Delta_M)^{1/2}u).$$

We define similarly $A_{\mathbb{R}^d}$ with the flat laplacian. We denote by $A_{\mathbb{R}^d, N}$ the obvious operator on $(H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))^N$ obtained by applying $A_{\mathbb{R}^d}$ on each ‘‘coordinate’’.

The following definition is taken from Gallagher and Gérard [17].

Definition 1.3. Let A be a self-adjoint (unbounded) operator on a Hilbert space H . Let (h_n) be a sequence of positive numbers converging to 0. A bounded sequence (u_n) in H is said to be (h_n) -oscillatory with respect to A if

$$\overline{\lim}_{n \rightarrow \infty} \|1_{|A| \geq \frac{R}{h_n}} u_n\|_H \xrightarrow{R \rightarrow \infty} 0. \tag{15}$$

(u_n) is said to be strictly (h_n) -oscillatory with respect to A if it satisfies (15) and

$$\overline{\lim}_{n \rightarrow \infty} \|1_{|A| \leq \frac{\varepsilon}{h_n}} u_n\|_H \xrightarrow{\varepsilon \rightarrow 0} 0.$$

At the contrary, (u_n) is said to be (h_n) -singular with respect to A if we have

$$\|1_{\frac{a}{h_n} |A| \leq \frac{b}{h_n}} u_n\|_H \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } 0 < a < b.$$

Remark that $1_{|x| \leq 1}$ can easily be replaced by a well-chosen function $\varphi \in C_0^\infty(\mathbb{R})$. Moreover, if a sequence (u_n) is strictly (h_n) -oscillatory while a second sequence (v_n) is (h_n) -singular, then we have the interesting property that $\langle u_n, v_n \rangle_H \xrightarrow{n \rightarrow \infty} 0$.

Proposition 1.2. Let $M = \bigcup_{i=1}^N U_i$ be a finite covering of M with some associated local coordinate patch $\Phi_i : U_i \rightarrow V_i \subset \mathbb{R}^3$. Let $1 = \sum_i \Psi_i$ be an associated partition of the unity of M with $\Psi_i \in C_0^\infty(U_i)$. Let (u_n, v_n) be a bounded sequence in the M energy space and h_n a sequence

converging to 0. Then (u_n, v_n) is (strictly) (h_n) -oscillatory with respect to A_M , if and only if all the $\Phi_{i*}\Psi_i(u_n, v_n)$ are (strictly) (h_n) -oscillatory with respect to $A_{\mathbb{R}^d}$.

Proof. First, we remark that a sequence is (strictly) (h_n) -oscillatory with respect to A if and only if it is (strictly) (h_n^2) -oscillatory with respect to A^2 . So we can replace A_M and $A_{\mathbb{R}^3}$ by $-(\Delta_M, \Delta_M)$ and $-(\Delta_{\mathbb{R}^3}, \Delta_{\mathbb{R}^3})$. We apply a proposition taken from [17] that makes the link between oscillation with different operators.

Proposition 1.3. (See Proposition 2.2.3 of [17].) Let $\Lambda : H_1 \rightarrow H_2$ be a continuous linear map between Hilbert spaces H_1, H_2 . Let A_1 be a self-adjoint operator on H_1 , A_2 be a self-adjoint operator on H_2 . Assume there exists $C > 0$ such that $\Lambda(D(A_1)) \subset D(A_2)$, $\Lambda^*(D(A_2)) \subset D(A_1)$ and for any $u \in D(A_1), v \in D(A_2)$,

$$\|A_2 \Lambda u\| \leq C(\|A_1 u\| + \|u\|), \tag{16}$$

$$\|A_1 \Lambda^* v\| \leq C(\|A_2 v\| + \|v\|). \tag{17}$$

If a bounded sequence (u_n) in H_1 is (strictly) (h_n) -oscillatory with respect to A_1 , then (Λu_n) is (strictly) (h_n) -oscillatory with respect to A_2 .

To prove the first implication, we apply the proposition with $\Lambda(u, v) = (\Phi_{1*}\Psi_1(u, v), \dots, \Phi_{N*}\Psi_N(u, v))$. We only prove the necessary estimates, the inclusions of domains being a direct consequence of the inequalities and of the density of smooth functions. To simplify the notation, we denote $(u_i, v_i) = \Phi_{i*}\Psi_i(u, v)$. The proof of (16) mainly uses the equivalent definitions of the H^s norm on a manifold.

$$\begin{aligned} \|A_{\mathbb{R}^3}^2(u_i, v_i)\|_{H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2} &= \|\Delta_{\mathbb{R}^3} u_i\|_{H_{\mathbb{R}^3}^1} + \|\Delta_{\mathbb{R}^3} v_i\|_{L_{\mathbb{R}^3}^2} \\ &\lesssim \|u_i\|_{H_{\mathbb{R}^3}^3} + \|v_i\|_{H_{\mathbb{R}^3}^2} \lesssim \|u\|_{H_M^3} + \|v\|_{H_M^2} \\ &\lesssim \|u\|_{H_M^1} + \|\Delta_M u\|_{H_M^1} + \|v\|_{L_M^2} + \|\Delta_M v\|_{L_M^2} \\ &\lesssim \|A_M^2(u, v)\|_{H_M^1 \times L_M^2} + \|(u, v)\|_{H_M^1 \times L_M^2}. \end{aligned}$$

Let us prove (17) for the duality $H^1 \times L^2$ of the scalar product. Let $(f, g) = (f_i, g_i)_{i=1, \dots, N} \in (C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3))^N$ and $(u, v) \in C^\infty(M)$.

$$\begin{aligned} &((u, v), A_M^2 \Lambda^*(f, g))_{H^1(M) \times L^2(M)} \\ &= (\Lambda A_M^2(u, v), (f, g))_{(H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))^N} \\ &= \sum_i (\Phi_{i*}\Psi_i \Delta_M u, f_i)_{H_{\mathbb{R}^3}^1} + \sum_i (\Phi_{i*}\Psi_i \Delta_M v, g_i)_{L^2} \\ &\lesssim \sum_i \|\Phi_{i*}\Psi_i \Delta_M u\|_{H_{\mathbb{R}^3}^{-1}} \|f_i\|_{H_{\mathbb{R}^3}^3} + \sum_i \|\Phi_{i*}\Psi_i \Delta_M v\|_{H_{\mathbb{R}^3}^{-2}} \|g_i\|_{H_{\mathbb{R}^3}^2} \end{aligned}$$

$$\begin{aligned} &\lesssim \|u\|_{H_M^1} \sum_i \|f_i\|_{H_{\mathbb{R}^3}^3} + \|\Delta_M v\|_{H_M^{-2}} \sum_i \|g_i\|_{H_{\mathbb{R}^3}^2} \\ &\lesssim \|(u, v)\|_{H_M^1 \times L_M^2} \left(\sum_i \|(\Delta_{\mathbb{R}^3} f_i, \Delta_{\mathbb{R}^3} g_i)\|_{H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2} + \|(f_i, g_i)\|_{H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2} \right). \end{aligned}$$

Therefore, we get $\|A_M^2 \Lambda^*(f, g)\|_{H_M^1 \times L_M^2} \leq C(\|A_{\mathbb{R}^3, N}^2(f, g)\|_{(H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2)^N} + \|(f, g)\|_{(H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2)^N})$ and Proposition 1.3 implies that (strict) (h_n) -oscillation of (u_n) with respect to A_M implies (strict) (h_n) -oscillation of Λu_n with respect to $A_{\mathbb{R}^3, N}$.

To prove the other implication, we use a quite similar operator. Denote by φ_i some other cut-off functions in $C_0^\infty(V_i) \subset C_0^\infty(\mathbb{R}^3)$ such that $\varphi_i \equiv 1$ on $Supp(\Phi_{i*} \Psi_i)$. We define by Γ the bounded operator from $(H_{\mathbb{R}^3}^1 \times L_{\mathbb{R}^3}^2)^N$ to $H_M^1 \times L_M^2$ given by

$$\Gamma(f, g) = \sum_i \Phi_{i*}^{-1} \varphi_i(f_i, g_i)$$

Then, we have $\Gamma \circ \Lambda = Id$ and we only have to prove that (strict) (h_n) -oscillation of (f_n, g_n) with respect to $A_{\mathbb{R}^3, N}$ implies (strict) (h_n) -oscillation of $\Gamma(f_n, g_n)$ with respect to A_M . The needed estimates are quite similar and we omit them. \square

Remark 1.2. Another way to prove Proposition 1.2 would have been to use the pseudodifferential operators $\varphi(h^2 \Delta_M)$ as in [6].

Now, we will prove that the (h_n) -oscillation is conserved by the equation, even with a damping term.

Proposition 1.4. *Let $T > 0$. Let (φ_n, ψ_n) be a bounded sequence of \mathcal{E} that is (strictly) (h_n) -oscillatory with respect to A_M . Let u_n be the solution of*

$$\begin{cases} \square u_n + u_n = a(x) \partial_t u_n & \text{on } [0, T] \times M, \\ (u_n(0), \partial_t u_n(0)) = (\varphi_n, \psi_n). \end{cases} \tag{18}$$

Then, $(u_n(t), \partial_t u_n(t))$ are (strictly) (h_n) -oscillatory with respect to A_M , uniformly on $[0, T]$.

At the contrary, if (φ_n, ψ_n) is (h_n) -singular with respect to A_M , $(u_n(t), \partial_t u_n(t))$ is (h_n) -singular with respect to A_M , uniformly on $[0, T]$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \chi(s) \leq 1$ and $\chi(s) = 1$ for $|s| \leq 1$. The (h_n) -oscillation (resp. strict oscillation) is equivalent to $\overline{\lim}_{n \rightarrow \infty} \|(1 - \chi)(R^2 h_n^2 \Delta)(u_n, \partial_t u_n)\|_{\mathcal{E}} \xrightarrow{R \rightarrow \infty} 0$ (resp.

$\overline{\lim}_{n \rightarrow \infty} \|\chi(\frac{h_n^2 \Delta}{R^2})(u_n, \partial_t u_n)\|_{\mathcal{E}} \xrightarrow{R \rightarrow \infty} 0$).

$v_n = (1 - \chi)(R^2 h_n^2 \Delta)u_n$ is a solution of

$$\begin{cases} \square v_n + v_n = a(x) \partial_t v_n - [\chi(R^2 h_n^2 \Delta), a] \partial_t u_n & \text{on } [0, T] \times M, \\ (v_n(0), \partial_t v_n(0)) = (1 - \chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n) \end{cases} \tag{19}$$

and energy estimates give

$$\begin{aligned} \|(v_n(t), \partial_t v_n(t))\|_{\mathcal{E}} &\leq C_T \|(1 - \chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n)\|_{\mathcal{E}} + C_T \|[a, \chi(R^2 h_n^2 \Delta)]\partial_t u_n\|_{L^1([0,t],L^2)} \\ &\leq C_T \|(1 - \chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n)\|_{\mathcal{E}} + C_T R h_n, \end{aligned}$$

where the last inequality comes from the fact that $\chi(-h^2 \Delta)$ is a semiclassical pseudodifferential operator, as proved in Burq, Gérard and Tzvetkov [6, Proposition 2.1] using the Helffer–Sjöstrand formula.

Therefore, passing to the limsup in n and using the oscillation assumption, we get the expected result uniformly in t for $0 \leq t \leq T$. The results for strict oscillation and singularity are proved similarly. \square

Proposition 1.5. *There exists $C_T > 0$ such that for every (φ_n, ψ_n) bounded sequence of \mathcal{E} weakly convergent to 0, we have the estimate*

$$\overline{\lim}_{n \rightarrow \infty} \|(u_n, \partial_t u_n)\|_{L^\infty([0,T], B_{2,\infty}^1(M) \times B_{2,\infty}^0(M))} \leq C_T \overline{\lim}_{n \rightarrow \infty} \|(\varphi_n, \psi_n)\|_{B_{2,\infty}^1(M) \times B_{2,\infty}^0(M)}$$

where u_n is the solution of (18).

Proof. Without loss of generality and since the equation is linear, we can assume that $\|(\varphi_n, \psi_n)\|_{\mathcal{E}}$ is bounded by 1. Let $\varepsilon > 0$. Let $\chi_0, \chi \in C_0^\infty(\mathbb{R})$ be so that $1 = \chi_0 + \sum_{k=1}^\infty \chi(2^{-2k} x)$. We denote $u_n^k = \chi(2^{-2k} \Delta)u_n$. Using the same estimates as in the previous lemma, we get

$$\|(u_n^k(t), \partial_t u_n^k(t))\|_{\mathcal{E}} \leq C_T \|(u_n^k(0), \partial_t u_n^k(0))\|_{\mathcal{E}} + C_T 2^{-k}.$$

Take K large enough so that $C_T 2^{-k} \leq \varepsilon$ for $k \geq K$ so that we have

$$\|(u_n^k(t), \partial_t u_n^k(t))\|_{\mathcal{E}} \leq C_T \|(\varphi_n, \psi_n)\|_{B_{2,\infty}^1(M) \times B_{2,\infty}^0(M)} + \varepsilon. \tag{20}$$

Then, for $k < K$, using again some energy estimates for the equation verified by u_n^k , we get

$$\|(u_n^k(t), \partial_t u_n^k(t))\|_{\mathcal{E}} \leq C_T \|(u_n^k(0), \partial_t u_n^k(0))\|_{\mathcal{E}} + C_T \|[a, \chi(-2^{-2k} \Delta)]\partial_t u_n\|_{L^1([0,T],L^2)}.$$

Yet, for fixed k , $[a, \chi(-2^{-2k} \Delta)]$ is an operator from L^2 into H^1 (for instance) and we conclude by the Aubin–Lions Lemma that for fixed $k \leq K$

$$\overline{\lim}_{n \rightarrow \infty} \|(u_n^k(t), \partial_t u_n^k(t))\|_{\mathcal{E}} \leq C_T \overline{\lim}_{n \rightarrow \infty} \|(u_n^k(0), \partial_t u_n^k(0))\|_{\mathcal{E}}. \tag{21}$$

We get the expected result with an additional ε by combining (20) and (21). \square

We end this subsection by two lemmas that will be useful in the nonlinear decomposition. The first one is Lemma 3.2 of [17].

Lemma 1.7. *Let h_n and \tilde{h}_n be two orthogonal scales, and let (f_n) and \tilde{f}_n be two sequences such that ∇f_n (resp. $\nabla \tilde{f}_n$) is strictly (h_n) (resp. \tilde{h}_n)-oscillatory with respect to $\Delta_{\mathbb{R}^3}$. Then, we have*

$$\overline{\lim}_{n \rightarrow \infty} \|f_n \tilde{f}_n\|_{L^3(\mathbb{R}^3)} = 0.$$

Then, we easily deduce the following result.

Lemma 1.8. *Let h_n and \tilde{h}_n be two orthogonal scales and v_n, \tilde{v}_n be two sequences that are strictly h_n (resp. \tilde{h}_n)-oscillatory with respect to Δ_M (considered on the Hilbert space H^1), uniformly on $[-T, T]$. Then, we have*

$$\|v_n \tilde{v}_n\|_{L^\infty([-T, T], L^3(M))} \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, the same result remains true if \tilde{v}_n is a constant sequence $v \in H^1$ and $\tilde{h}_n = 1$.

Proof. Using a partition of unity $1 = \sum_i \psi_i^2$ adapted to coordinate charts, we have to compute

$$\|\Phi_{i*} \psi_i v_n \psi_i \tilde{v}_n\|_{L^\infty([-T, T], L^3(\mathbb{R}^3))}.$$

Using Proposition 1.2, we infer that $\Phi_{i*} \psi_i v_n$ is strictly (h_n) -oscillatory with respect to $\Delta_{\mathbb{R}^3}$ (defined on H^1) and the same result holds for $\nabla(\Phi_{i*} \psi_i v_n)$ with respect to $\Delta_{\mathbb{R}^3}$ defined on L^2 . We conclude by applying Lemma 1.7 to $\Phi_{i*} \psi_i v_n$ and $\Phi_{i*} \psi_i \tilde{v}_n$. \square

1.4. Microlocal defect measure and energy

In this subsection, we state without proof the propagation of the measure for the damped wave equation. We refer to [18] for the definition and to [19, Section 4] or [16] in the specific context of the wave equation. It will be used several times in the article.

Lemma 1.9 (Measure for the damped equation and equicontinuity of the energy). *Let u_n, \tilde{u}_n be two sequences of solution to*

$$\square u_n + u_n = a(x) \partial_t u_n,$$

*weakly convergent to 0 in \mathcal{E} . Then, there exists a subsequence (still denoted by u_n, \tilde{u}_n) such that for any $t \in [0, T]$ there exists a (nonnegative if $u_n = \tilde{u}_n$) Radon measure μ^t on S^*M such that for any classical pseudodifferential operator B of order 0, we have with a uniform convergence in t*

$$(B(-\Delta)^{1/2} u_n(t), (-\Delta)^{1/2} \tilde{u}_n(t))_{L^2(M)} + (B \partial_t u_n(t), \partial_t \tilde{u}_n(t))_{L^2(M)} \xrightarrow{n \rightarrow \infty} \int_{S^*M} \sigma_0(B) d\mu^t. \quad (22)$$

Moreover, one can decompose

$$\mu^t = \frac{1}{2}(\mu^t_+ + \mu^t_-)$$

which satisfy the following transport equation

$$\partial_t \mu_{\pm}(t) = \pm H_{|\xi|_x} \mu_{\pm}(t) + a(x) \mu_{\pm}(t).$$

Furthermore, if $t_n \xrightarrow{n \rightarrow \infty} t$, we have the same convergence with t replaced by t_n in (22).

The microlocal defect measure of a concentrating data $[(\varphi, \psi), \underline{h}, \underline{x}]$ can be explicitly computed, as follows

$$\mu_{\pm} = (2\pi)^{-3} \delta_{x_{\infty}}(x) \otimes \int_0^{+\infty} |\hat{\psi}(r\xi) \pm i|r\xi|_{\infty} \hat{\varphi}(r\xi)|^2 r^2 dr.$$

This can be easily computed, for instance, with the next lemma.

Lemma 1.10. *Let $(\varphi_n, \psi_n) = [(\varphi, \psi), \underline{h}, \underline{x}]$ be a concentration data and $A(x, D_x), B(x, D_x)$ two polyhomogeneous pseudodifferential operators of respective order 0. Then*

$$\| (A(x, D_x)\varphi_n, B(x, D_x)\psi_n) - [(A_0(x_{\infty}, D_x)\varphi, B_0(x_{\infty}, D_x)\psi), \underline{h}, \underline{x}] \|_{H^1 \times L^2} \xrightarrow{n \rightarrow \infty} 0$$

where $A_0(x_{\infty}, D_x)$ is the Fourier multiplier of homogeneous symbol $a_0(x_{\infty}, \xi)$ defined on $T_{x_{\infty}}^* M$.

Proof. We only give a sketch of the proof for $B(x, D_x)\psi_n$. By approximation, we can assume that $\hat{\psi} \in C_0^{\infty}(\mathbb{R}^3 \setminus \{0\})$. In local coordinates centered at $x_{\infty} = 0$, we have for an $o(1)$ small in L^2

$$\begin{aligned} B(x, D_x)\psi_n &= h_n^{-\frac{3}{2}} B(x, D_x) \left[\Psi_U(x)(\chi\psi) \left(\frac{x - x_n}{h_n} \right) \right] + o(1) \\ &= h_n^{-\frac{3}{2}} [B_n(y, D_y)\psi] \left(\frac{x - x_n}{h_n} \right) + o(1) \end{aligned}$$

where $B_n(y, D_y)$ is the operator of symbol $b_n(y, \xi) = b_0(h_n y + x_n, \xi/h_n)$. Here b_0 is the principal symbol of B , homogeneous for large ξ . We write $b_0(h_n y + x_n, \xi/h_n) = b_0(x_n, \xi/h_n) + h_n y \int_0^1 (\partial_y b_0)(x_n + th_n, \xi/h_n) dt$. The first term converges to $b_0(0, \xi)$ by homogeneity while the second produces a term small in L^2 . \square

The previous lemma is made interesting when combined with the propagation of microlocal defect measure.

Lemma 1.11. *Let u_n be a sequence of solutions of $\square u_n + u_n = a(x)\partial_t u_n$ weakly convergent to 0 and $p_n = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ a linear damped concentrating wave. We assume $D_h(u_n, \partial_t u_n) \rightharpoonup 0$. Then, for any classical pseudodifferential operators $A(x, D_x)$ of order 0, we have uniformly for $t \in [-T, T]$*

$$(A(-\Delta)^{1/2} p_n(t), (-\Delta)^{1/2} u_n(t))_{L^2(M)} + (A\partial_t p_n(t), \partial_t u_n(t))_{L^2(M)} \xrightarrow{n \rightarrow \infty} 0.$$

In particular, we have

$$\nabla p_n \cdot \nabla u_n + \partial_t p_n \partial_t u_n \rightharpoonup 0 \quad \text{in } \mathcal{D}'([-T, T] \times M).$$

Proof. We first check the property for $t = t_n$. Using Lemma 1.10 several times, we are led to estimate

$$\left((-\Delta)^{1/2} \varphi_n, (-\Delta)^{1/2} u_n(t_n) \right)_{L^2(M)} + \left(\psi_n, \partial_t u_n(t_n) \right)_{L^2(M)}$$

where (φ_n, ψ_n) are the concentrating data associated with $[(A(x_\infty, D_x)\varphi, B(x_\infty, D_x)\psi), \underline{h}, \underline{x}]$. Then, the hypotheses $D_h(u_n, \partial_t u_n) \rightharpoonup 0$ and Lemma 1.5 yields the convergence to 0 for this particular case $t = t_n$. We conclude by equicontinuity and by the propagation of joint measures stated in Lemma 1.9. \square

2. Profile decomposition

2.1. Linear profile decomposition

The main purpose of this section is to establish Theorem 0.3. It is completed in two main steps: the first one is the extraction of the scales $h_n^{(j)}$ where we decompose v_n in an infinite sum of sequence $v_n^{(j)}$ which are respectively $h_n^{(j)}$ -oscillatory and the second step consists in decomposing each $v_n^{(j)}$ in an infinite sum of concentrating wave at the rate $h_n^{(j)}$. Actually, in order to perform the nonlinear decomposition, we will need that, in some sense, each profile of the decomposition do not interact with the other. It is stated in this orthogonality result.

Theorem 0.3'. *With the notation of Theorem 0.3, we have the additional following properties.*

If $2T < T_{\text{focus}}$, we have $(\underline{h}^{(k)}, \underline{x}^{(k)}, \underline{t}^{(k)}) \perp (\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)})$ for any $j \neq k$, according to Definition 1.2.

If $M = S^3$ and $a \equiv 0$ (undamped solutions), but with T eventually large, we have $(\underline{h}^{(k)}, (-1)^m \underline{x}^{(k)}, \underline{t}^{(k)} + m\pi)$ orthogonal to $(\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)})$ for any $m \in \mathbb{Z}$ and $j \neq k$.

2.1.1. Extraction of scales

Proposition 2.1. *Let $T > 0$. Let (φ_n, ψ_n) be a bounded sequence of \mathcal{E} and v_n the solution of*

$$\begin{cases} \square v_n + v_n = a(x) \partial_t v_n & \text{on } [-T, T] \times M, \\ (v_n(0), \partial_t v_n(0)) = (\varphi_n, \psi_n). \end{cases} \tag{23}$$

Then, up to an extraction, v_n can be decomposed in the following way: for any $l \in \mathbb{N}^$*

$$v_n(t, x) = v(t, x) + \sum_{j=1}^l v_n^{(j)}(t, x) + \rho_n^{(l)}(t, x),$$

where $v_n^{(l)}$ is a strictly $(h_n^{(j)})$ -oscillatory solution of the damped linear wave equation (23) on M . The scales $h_n^{(j)}$ satisfy $h_n^{(j)} \xrightarrow{n \rightarrow \infty} 0$ and are orthogonal:

$$\left| \log \frac{h_n^{(k)}}{h_n^{(j)}} \right| \xrightarrow{n \rightarrow \infty} +\infty \quad \text{if } j \neq k. \tag{24}$$

Moreover, we have

$$\overline{\lim}_{n \rightarrow \infty} \|\rho_n^{(l)}\|_{L^\infty([-T, T], L^6(M))} \xrightarrow{l \rightarrow \infty} 0, \tag{25}$$

$$\begin{aligned} \|(v_n, \partial_t v_n)(t)\|_{\mathcal{E}}^2 &= \|(v, \partial_t v)(t)\|_{\mathcal{E}}^2 + \sum_{j=1}^l \|(v_n^{(j)}, \partial_t v_n^{(j)})(t)\|_{\mathcal{E}}^2 \\ &\quad + \|(\rho_n^{(l)}, \partial_t \rho_n^{(l)})(t)\|_{\mathcal{E}}^2 + o(1)(t), \end{aligned} \tag{26}$$

where $o(1)(t) \xrightarrow{n \rightarrow \infty} 0$ uniformly for $t \in [-T, T]$.

Proof. We first make this decomposition for the initial data as done in [20] (see also [2]). Then, using the propagation of (h_n) -oscillation proved in Proposition 1.4, we extend it for all time.

More precisely, by applying the same procedure as in [20], with the operator A_M , we decompose

$$(\varphi_n, \psi_n) = (\varphi, \psi) + \sum_j^l (\varphi_n^{(j)}, \psi_n^{(j)}) + (\Phi_n^{(l)}, \Psi_n^{(l)})$$

where $(\varphi_n^{(j)}, \psi_n^{(j)})$ is $(h_n^{(j)})$ -oscillatory for A_M , $h_n^{(j)} \xrightarrow{n \rightarrow \infty} 0$, and

$$\overline{\lim}_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \|\mathbf{1}_{[2^k, 2^{k+1}[}(A_M)(\Phi_n^{(l)}, \Psi_n^{(l)})\|_{\mathcal{E}} \xrightarrow{l \rightarrow \infty} 0. \tag{27}$$

Moreover, we have the orthogonality property:

$$\|(\varphi_n, \psi_n)\|_{\mathcal{E}}^2 = \|(\varphi, \psi)\|_{\mathcal{E}}^2 + \sum_j^l \|(\varphi_n^{(j)}, \psi_n^{(j)})\|_{\mathcal{E}}^2 + \|(\Phi_n^{(l)}, \Psi_n^{(l)})\|_{\mathcal{E}}^2 + o(1), \quad n \rightarrow \infty,$$

and the $h_n^{(j)}$ are orthogonal each other as in (24). Moreover, $(\Phi_n^{(l)}, \Psi_n^{(l)})$ is $(h_n^{(j)})$ -singular for $1 \leq j \leq l$.

This decomposition for the initial data can be extended to the solution by

$$v_n(t, x) = v(t, x) + \sum_j^l v_n^{(j)}(t, x) + \rho_n^{(l)}(t, x),$$

where each $v_n^{(j)}$ is a solution of

$$\begin{cases} \square v_n^{(j)} + v_n^{(j)} = a(x) \partial_t v_n^{(j)} & \text{on } \mathbb{R}_t \times M, \\ (v_n^{(j)}(0), \partial_t v_n^{(j)}(0)) = (\varphi_n^{(j)}, \psi_n^{(j)}). \end{cases}$$

Thanks to Proposition 1.4, each $(v_n^{(j)}(t), \partial_t v_n^{(j)}(t))$ is strictly $(h_n^{(j)})$ -oscillatory and $(\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t))$ is $(h_n^{(j)})$ -singular for $1 \leq j \leq l$. So, we easily infer for instance that $\langle (\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t)), (v_n^{(j)}(t), \partial_t v_n^{(j)}(t)) \rangle_{\mathcal{E}} \xrightarrow{n \rightarrow \infty} 0$ uniformly on $[-T, T]$ where $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is the scalar product on \mathcal{E} . This is also true for the product between $v_n^{(j)}$ and $v_n^{(k)}$, $j \neq k$ thanks to the orthogonality (24). The same convergence holds for the product with v by weak convergence to 0 of the other terms. Then, we get

$$\| (v_n, \partial_t v_n) \|_{\mathcal{E}}^2 = \| (v, \partial_t v) \|_{\mathcal{E}}^2 + \sum_j^l \| (v_n^{(j)}, \partial_t v_n^{(j)}) \|_{\mathcal{E}}^2 + \| (\rho_n^{(l)}, \partial_t \rho_n^{(l)}) \|_{\mathcal{E}}^2 + o(1), \quad n \rightarrow \infty.$$

Let us now prove estimate (25) of the remaining term in $L^\infty(L^6)$. (27) gives the convergence to zero of $(\rho_n^{(l)}(0), \partial_t \rho_n^{(l)}(0))$ in $B_{2,\infty}^1(M) \times B_{2,\infty}^0(M)$. We extend this convergence for all time with Proposition 1.5 and get

$$\sup_{t \in [0, T]} \overline{\lim}_{n \rightarrow \infty} \| (\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t)) \|_{B_{2,\infty}^1 \times B_{2,\infty}^0} \xrightarrow{l \rightarrow \infty} 0.$$

The following lemma will transfer this information in local charts.

Lemma 2.1. *There exists $C > 0$ such that*

$$\begin{aligned} \frac{1}{C} \| \Lambda f \|_{B_{2,\infty}^0(\mathbb{R}^3)^N} &\leq \| f \|_{B_{2,\infty}^0(M)} \leq C \| \Lambda f \|_{B_{2,\infty}^0(\mathbb{R}^3)^N}, \\ \frac{1}{C} \| \Lambda f \|_{B_{2,\infty}^1(\mathbb{R}^3)^N} &\leq \| f \|_{B_{2,\infty}^1(M)} \leq C \| \Lambda f \|_{B_{2,\infty}^1(\mathbb{R}^3)^N} \end{aligned}$$

where Λ is the operator described in Proposition 1.2 of cut-off and transition in N local charts.

We postpone the proof of this lemma and continue the proof of the proposition. Using this lemma, we get for every coordinate patch (U_i, Φ_i) and $\Psi_i \in C_0^\infty(U_i)$

$$\overline{\lim}_{n \rightarrow \infty} \| \Phi_i^* \Psi_i \rho_n^{(l)} \|_{L^\infty([-T, T], B_{2,\infty}^1(\mathbb{R}^3))} \xrightarrow{l \rightarrow \infty} 0.$$

The refined Sobolev estimate, Lemma 3.5 of [2], yields for any $f \in H^1(\mathbb{R}^3)$

$$\| f \|_{L^6(\mathbb{R}^3)} \leq \| (-\Delta_{\mathbb{R}^3})^{1/2} f \|_{L^2}^{1/3} \| (-\Delta_{\mathbb{R}^3})^{1/2} f \|_{\dot{B}_{2,\infty}^0}^{2/3} \leq \| f \|_{H^1(\mathbb{R}^3)}^{1/3} \| f \|_{B_{2,\infty}^1(\mathbb{R}^3)}^{2/3}.$$

Therefore, we have

$$\overline{\lim}_{n \rightarrow \infty} \| \Phi_i^* \Psi_i \rho_n^{(l)} \|_{L^\infty([-T, T], L^6(\mathbb{R}^3))} \xrightarrow{l \rightarrow \infty} 0$$

and finally

$$\overline{\lim}_{n \rightarrow \infty} \| \rho_n^{(l)} \|_{L^\infty([-T, T], L^6(M))} \xrightarrow{l \rightarrow \infty} 0.$$

This completes the proof of Proposition 2.1, up to the proof of Lemma 2.1. \square

Proof of Lemma 2.1. We essentially use the following fact: see Lemma 3.1 of [2]. Let f_n be a sequence of $L^2(\mathbb{R}^3)$ weakly convergent to 0 and compact at infinity

$$\overline{\lim}_{n \rightarrow +\infty} \int_{|x| > R} |f(x)|^2 dx \xrightarrow{R \rightarrow +\infty} 0.$$

Then, f_n tends to 0 in $\dot{B}_{2,\infty}^0(\mathbb{R}^3)$ if and only if f_n is h_n singular for every scale h_n .

Actually, the same result holds for Δ_M , with the same demonstration. The compactness at infinity in \mathbb{R}^3 is only assumed to ensure

$$\overline{\lim}_{n \rightarrow +\infty} \|\mathbf{1}_{[-A,A]}(\Delta_{\mathbb{R}^3}) f_n\|_{L^2} = 0 \quad \text{for any } A > 0,$$

which is obvious in the case of Δ_M because of weak convergence and discrete spectrum.

Using Proposition 1.2, we obtain that f_n is (h_n) -singular with respect to Δ_M if and only if Λf_n is (h_n) -singular with respect to $\Delta_{\mathbb{R}^3}$. Combining the two previous results, we obtain that the two norms we consider have the same converging sequences and are therefore equivalent. \square

2.1.2. Description of linear concentrating waves (after S. Ibrahim)

In this subsection, we describe the asymptotic behavior of linear concentrating waves as described in [21] of S. Ibrahim. In [21], it is stated for the linear wave equation without damping. We give some sketch of the proof when necessary to emphasize the tiny modifications.

The following lemma yields that for times close to concentration, the linear damped concentrating wave is close to the solution of the wave equation with flat metric and without damping. It is Lemma 2.2 of [21], except that there is an additional damping term which disappears after rescaling. We do not give the proof and refer to the more complicated nonlinear case (see estimate (53)).

Lemma 2.2. Let $v_n = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ be a linear damped concentrating wave and v solution of

$$\begin{cases} \square_{\infty} v = 0 & \text{on } \mathbb{R} \times T_{x_{\infty}} M, \\ (v, \partial_t v)|_{t=0} = (\varphi, \psi). \end{cases} \tag{28}$$

Denote by \tilde{v}_n the rescaled function associated to v , that is $\tilde{v}_n = \Phi^* \Psi \frac{1}{\sqrt{h_n}} v(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n})$ where (U, Φ) is a coordinate chart around x_{∞} and $\Psi \in C_0^{\infty}(U)$ is constant equal to 1 around x_{∞} . Then, we have

$$\overline{\lim}_{n \rightarrow \infty} \|\tilde{v}_n - v_n\|_{[t_n - \Lambda h_n, t_n + \Lambda h_n] \times M} \xrightarrow{\Lambda \rightarrow \infty} 0.$$

Corollary 2.1. With the notation of the lemma, if $\tilde{t}_n = t_n + (C + o(1))h_n$, then $(v_n, \partial_t v_n)|_{t=\tilde{t}_n}$ is a concentrating data associated with $[(v(C), \partial_t v), \underline{h}, \underline{x}]$.

Moreover, Lemma 2.3 of S. Ibrahim [21] yields the “non-reconcentration” property for linear concentrating waves.

Lemma 2.3. *Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ be a linear (possibly damped) concentrating wave. Consider the interval $[-T, T]$ containing t_∞ , satisfying the following non-focusing property (see Definition 0.1)*

$$mes(F_{x, x_\infty, s}) = 0 \quad \forall x \in M \text{ and } s \neq 0 \quad \text{such that} \quad t_\infty + s \in [-T, T]. \tag{29}$$

Then, if we set $I_n^{1,\Lambda} = [-T, t_n - \Lambda h_n]$ and $I_n^{3,\Lambda} =]t_n + \Lambda h_n, T]$, we have

$$\begin{aligned} \overline{\lim}_n \|v_n\|_{L^\infty(I_n^{1,\Lambda} \cup I_n^{3,\Lambda}, L^6(M))} &\xrightarrow{\Lambda \rightarrow \infty} 0, \\ \overline{\lim}_n \|v_n\|_{L^5(I_n^{1,\Lambda} \cup I_n^{3,\Lambda}, L^{10}(M))} &\xrightarrow{\Lambda \rightarrow \infty} 0. \end{aligned}$$

Sketch of the proof of Lemma 2.3 in the damped case. To simplify the notation, we can assume $t_n = 0$. In [21], the proof is made by contradiction, assuming the existence of a subsequence (still denoted by v_n) such that $\|v_n(s_n)\|_{L^6(M)} \rightarrow C > 0$ and $\frac{|s_n|}{h_n} \rightarrow \infty$. If $s_n \rightarrow \tau \neq 0$, using the concentration-compactness principle of [31], we are led to prove that the microlocal defect measure μ associated to $v_n(t_n)$ satisfies $\mu(\{y\} \times S^2) = 0$ for any $y \in M$. We use the same argument for the damped equation except that in that case, the measure μ^t associated to $v_n(t)$ is not solution of the exact transport equation but of a damped transport equation (see Lemma 1.9). Yet, the non-focusing assumption (29) still implies $\mu^t(\{y\} \times S^2) = 0$ for all $y \in M$ and $t \neq 0$, which allows to conclude similarly.

In the case $\tau = 0$, we use in local coordinates the rescaled function $\tilde{v}_n(s, y) = \sqrt{s_n} v_n(s_n s, s_n y + x_n)$. \tilde{v}_n at time $s = 0$ is a concentrating data at scale h_n/s_n . We prove $\lim \| \tilde{v}_n(1, \cdot) \|_{L^6(\mathbb{R}^3)} = 0$. Again by concentration compactness, it is enough to prove that the microlocal defect measure μ^s of \tilde{v}_n propagates along the curves of the hamiltonian flow with constant coefficient $H_{|\xi|}$. Since v_n is a solution of $\square v_n + v_n + a(x) \partial_t v_n = 0$, \tilde{v}_n is a solution of $\square_n \tilde{v}_n + s_n^2 \tilde{v}_n + s_n a(s_n \cdot + x_n) \partial_t \tilde{v}_n = 0$ where \square_n is a suitably rescaled d'Alembert operator. Since the additional terms $s_n^2 \tilde{v}_n + s_n a(s_n \cdot + x_n) \partial_t \tilde{v}_n$ converge to 0 in $L^1 L^2$, we can finish the proof as in Lemma 2.3 of [21] by proving that μ^s propagates as if \square_n was replaced by \square_∞ , that is along the hamiltonian $H_{|\xi|}$.

The estimate in norm $L^5 L^{10}$ is obtained by interpolation of $L^\infty L^6$ with another bounded Strichartz norm. \square

In the specific case of S^3 , Lemma 4.2 of [21] allows to describe precisely the behavior of concentrating wave for large times, as follows.

Lemma 2.4. *Let \underline{p} be a sequence of solutions of*

$$\begin{cases} \square p_n = 0 & \text{on } [0, +\infty[\times M, \\ (p_n(0), \partial_t p_n(0)) = (\varphi_n, \psi_n) \end{cases}$$

where (φ_n, ψ_n) is weakly convergent to $(0, 0)$ in \mathcal{E} . Then, we have

$$p_n(t + \pi, x) = -p_n(t, -x) + o(1)(t)$$

where the $o(1)(t)$ is small in the energy space. The same holds for solutions of $\square u_n + u_n$.

In particular, if \underline{p} is a concentrating wave associated with data $[(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$, then, for any $j \in \mathbb{N}$, $p_n(t + j\pi, x)$ is a linear concentrating wave associated with $[(-1)^j(\varphi, \psi)((-1)^j \cdot), \underline{h}, (-1)^j \underline{x}, \underline{t}]$.

In the previous lemma, $-x$ refers to the embedding of S^3 into \mathbb{R}^4 . Moreover, the notation $(\varphi, \psi)(-\cdot)$ could be written more rigorously $(\varphi, \psi)(D_\infty I \cdot)$ where $D_\infty I$ is the differential at the point x_∞ of the application $I : x \mapsto -x$ defined from S^3 into itself. Actually, we are identifying the tangent plane at the south pole with the one on the north pole by the application $x \mapsto -x$ on \mathbb{R}^4 .

The fact that the result remains true for the equation $\square u + u = 0$ comes from the fact that for initial data weakly convergent to zero, the solutions of $\square u = 0$ and $\square v + v = 0$ with same data are asymptotically close in the energy space. This can be proved by observing that for a weakly convergent sequence of solutions u_n the Aubin–Lions Lemma yields that u_n converges strongly to 0 in $L^\infty([-T, T], L^2)$. So $r_n = u_n - v_n$ is a solution of $\square r_n = u_n$ and converges strongly in \mathcal{E} .

2.1.3. Extraction of times and cores of concentration

In this subsection, h_n is a fixed sequence in \mathbb{R}_+^* converging to 0. For simplicity, we will denote it by h and u_h for sequences of functions. The main purpose of this subsection is the proof of the following proposition, which is the profile decomposition for h -oscillatory sequences. It easily implies Theorem 0.3 when combined with Proposition 2.1.

Proposition 2.2. *Let (u_h) be an h -oscillatory sequence of solutions to the damped Klein–Gordon equation (23). Then, up to extraction, there exist damped linear concentrating waves p_h^k , as defined in Definition 0.3, associated to concentrating data $[(\varphi^{(k)}, \psi^{(k)}), \underline{h}, \underline{x}^{(k)}, \underline{t}^{(k)}]$, such that for any $l \in \mathbb{N}^*$, and up to a subsequence,*

$$v_h(t, x) = \sum_{j=1}^l p_h^{(j)}(t, x) + w_h^{(l)}(t, x), \tag{30}$$

$$\forall T > 0, \quad \overline{\lim}_{n \rightarrow \infty} \|w_h^{(l)}\|_{L^\infty([-T, T], L^6(M))} \xrightarrow{l \rightarrow \infty} 0, \tag{31}$$

$$\|(v_h, \partial_t v_h)\|_{\mathcal{E}}^2 = \sum_{j=1}^l \|(p_h^{(j)}, \partial_t p_h^{(j)})\|_{\mathcal{E}}^2 + \|(w_h^{(l)}, \partial_t w_h^{(l)})\|_{\mathcal{E}}^2 + o(1), \quad \text{as } h \rightarrow \infty,$$

$$\text{uniformly for } t \in [-T, T]. \tag{32}$$

Moreover, if $2T < T_{\text{focus}}$, for any $j \neq k$, we have $(\underline{x}^{(k)}, \underline{t}^{(k)}) \perp_h (\underline{x}^{(j)}, \underline{t}^{(j)})$ according to Definition 1.2.

If $M = S^3$ and $a \equiv 0$ (undamped solutions), but with T eventually large, $((-1)^m \underline{x}^{(k)}, \underline{t}^{(k)} + m\pi)$ is orthogonal to $(\underline{x}^{(j)}, \underline{t}^{(j)})$ for any $m \in \mathbb{Z}$ and $j \neq k$.

Remark 2.1. The assumptions to get the orthogonality of the cores of concentration are related to our lack of understanding of the solutions concentrating in a point x_1 where (x_1, x_2, t) is a couple of focus at distance t . We know that the solution reconcentrates after a time t in the other focus x_2 but we do not know precisely how: can it split into several concentrating waves on x_2

with different “rate of concentration”? That is to say with some different x_n converging to x_2 but which are orthogonal.

Before getting into the proof of the proposition, we state two lemmas that will be useful in the proof. Using the notation of Definition 1.1, denote

$$\delta^x(\underline{v}) = \sup_{\underline{x}} \{ \|\nabla\varphi\|_{L^2(T_{x_\infty M})}^2, D_h^1 v_h \rightharpoonup \varphi, \text{ up to a subsequence} \}$$

where the supremum is taken over all the sequences \underline{x} in M .

If $v_h \in L^\infty([-T, T], H^1(M))$, we denote

$$\delta(\underline{v}) = \sup_{\underline{x}, \underline{t}} \{ \|\nabla\varphi\|_{L^2(T_{x_\infty M})}^2, D_h^1 v_h(t_h) \rightharpoonup \varphi, \text{ up to a subsequence} \} = \sup_{\underline{t}} \delta^x(\underline{v}(t_h, \cdot))$$

where the supremum is taken over all the sequences $\underline{x} = (x_h)$ in M and $\underline{t} = (t_h)$ in $[-T, T]$.

Lemma 2.5. *Let $\Psi \in C^\infty(M)$. Then, there exists $C > 0$ such that for any \underline{v} , we have the estimate*

$$\delta^x(\Psi \underline{v}) \leq C \delta^x(\underline{v}).$$

The proof is left to the reader.

Lemma 2.6. *There exists $C > 0$ such that for any $\underline{v} = (v_h)$ a bounded strictly (h_n) -oscillatory sequence in $H^1(M)$*

$$\overline{\lim}_{n \rightarrow +\infty} \|v_h\|_{L^6} \leq C \delta^x(\underline{v})^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|v_h\|_{H^1(M)}^{1/6}.$$

Proof. This lemma is already known in the case of \mathbb{R}^3 where the definition of $\delta_{\mathbb{R}^3}^x$ is the same except that D_h^1 is only considered in the trivial coordinate chart. It is estimate (4.19) of [20] in the case of a 1-oscillatory sequence, which can be easily extended to (h_n) -oscillatory sequence by dilation.

Let $\Psi_i \in C_0^\infty(U_i)$ be associated to a coordinate patch Φ_i . By Proposition 1.2, $\Phi_i^* \Psi_i v_h$ is still (h_n) -oscillatory and we can apply the estimate on \mathbb{R}^3 . We get

$$\begin{aligned} \overline{\lim}_{n \rightarrow +\infty} \|\Phi_i^* \Psi_i v_h\|_{L^6(\mathbb{R}^3)} &\leq C \delta_{\mathbb{R}^3}^x(\Phi_i^* \Psi_i \underline{v})^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|\Phi_i^* \Psi_i v_h\|_{H^1(\mathbb{R}^3)}^{1/6} \\ &\leq C \delta_{\mathbb{R}^3}^x(\Phi_i^* \Psi_i \underline{v})^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|v_h\|_{H^1(M)}^{1/6}. \end{aligned}$$

Then, by definition of the convergence D_h , we easily get

$$\delta_{\mathbb{R}^3}^x(\Phi_i^* \Psi_i \underline{v}) \leq C \delta^x(\Psi_i \underline{v}).$$

We conclude by using Lemma 2.5 and partition of unity. \square

Lemma 2.7. *Let $T > 0$. There exists $C > 0$ such that for any sequence $\underline{v} = (v_h)$ (h_n) -oscillatory, solution of the damped linear Klein–Gordon equation on M with bounded energy, we have*

$$\overline{\lim}_{n \rightarrow +\infty} \|v_h\|_{L^\infty([-T, T], L^6(M))} \leq C \delta(\underline{v})^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|(v_h(0), \partial_t v_h(0))\|_{\mathcal{E}}^{1/6}.$$

Proof. Let t_h be an arbitrary sequence in $[-T, T]$. We apply Lemma 2.6 to the sequence $v_h(t_h)$ and get

$$\begin{aligned} \overline{\lim}_{n \rightarrow +\infty} \|v_h(t_h, \cdot)\|_{L^6} &\leq C \delta^x(\underline{v}(t_h, \cdot))^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|v_h(t_h)\|_{H^1(M)}^{1/6} \\ &\leq C \delta(\underline{v})^{1/3} \overline{\lim}_{n \rightarrow +\infty} \|(v_h(0), \partial_t v_h(0))\|_{\mathcal{E}}^{1/6} \end{aligned}$$

by definition of δ and by energy estimates. \square

Proof of Proposition 2.2. It is based on the same extraction argument as in [2] and [17]: the concentration will be tracked using our tool D_h and we will extract concentrating waves so that $\delta(\underline{v})$ decreases. We conclude with Lemma 2.7 to estimate the $L^\infty(L^6)$ norm of the remainder term.

More precisely, if $\delta(\underline{v}) = 0$, Lemma 2.7 shows that there is nothing to be proved. Otherwise, pick $(x_h^{(1)}, t_h^{(1)})$ converging to $(x_\infty^{(1)}, t_\infty^{(1)})$ and $(\varphi^{(1)}, \psi^{(1)}) \in \mathcal{E}_{x_\infty}$, such that

$$\|\nabla\varphi^{(1)}\|_{L^2(T_{x_\infty}M)}^2 + \|\psi^{(1)}\|_{L^2(T_{x_\infty}M)}^2 \geq \|\nabla\varphi^{(1)}\|_{L^2(T_{x_\infty}M)}^2 \geq \frac{1}{2}\delta(\underline{v})$$

and

$$D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \rightharpoonup_{h \rightarrow 0} (\varphi^{(1)}, \psi^{(1)}).$$

The existence of the weak limit $\psi^{(1)}$ (up to a subsequence) is ensured by the boundedness in $L^2(\mathbb{R}^3)$ of $\partial_t v_h$ (considered in a coordinate chart) by conservation of energy.

Then, we choose $p_h^{(1)}$ as the damped linear concentrating profile associated with $[(\varphi^{(1)}, \psi^{(1)}), \underline{h}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ (actually, we pick one representative in the equivalence class modulo sequences converging to 0 in the energy space as in Definition 0.2). Remark here that the assumption $t_h^{(1)} \in [-T, T]$ ensures $t_\infty^{(1)} \in [-T, T]$, which will always be the case for all the concentrating waves we consider. Then, we give a lemma that will be the main step to the orthogonality of energies.

Lemma 2.8. *Let $w_h^{(1)} = v_h - p_h^{(1)}$. Then,*

$$\|(v_h, \partial_t v_h)(t)\|_{\mathcal{E}}^2 = \|(p_h^{(1)}, \partial_t p_h^{(1)})(t)\|_{\mathcal{E}}^2 + \|(w_h^{(1)}, \partial_t w_h^{(1)})(t)\|_{\mathcal{E}}^2 + o(1)$$

where the $o(1)$ is uniform for t in bounded intervals.

Proof. We first compute the energy at time $t_h^{(1)}$. We denote by B the bilinear form associated with the energy:

$$B(a, b) = \int_M a\bar{b} + \nabla a \cdot \nabla \bar{b} + \partial_t a \partial_t \bar{b}.$$

We have to prove

$$B(p_h^{(1)}(t_h^{(1)}), w_h^{(1)}(t_h^{(1)})) = B(p_h^{(1)}(t_h^{(1)}), v_h(t_h^{(1)}) - p_h^{(1)}(t_h^{(1)})) = o(1).$$

By weak convergence to 0 in H^1 of v_h , $p_h^{(1)}$ and $w_h^{(1)}$, we can omit the term $\int_M a\bar{b}$ of B . By construction and Lemma 1.6, we have $D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \xrightarrow{h \rightarrow 0} (\varphi^{(1)}, \psi^{(1)})$ and $D_h^{(1)}(p_h^{(1)}, \partial_t p_h^{(1)}) \xrightarrow{h \rightarrow 0} (\varphi^{(1)}, \psi^{(1)})$. Therefore, $D_h^{(1)}(w_h^{(1)}, \partial_t w_h^{(1)})(t_h^{(1)}) \xrightarrow{h \rightarrow 0} (0, 0)$. Lemma 1.11 gives the expected result. Remark that if $a \equiv 0$, this is just a consequence of the conservation of scalar product for solution of linear wave equation. \square

We get the expansion of u_h announced in Proposition 2.2 by induction iterating the same process.

Let us assume that

$$v_h(t, x) = \sum_{j=1}^l p_n^{(j)}(t, x) + w_n^{(l)}(t, x),$$

$$\|(v_h, \partial_t v_h)\|_{\mathcal{E}}^2 = \sum_{j=1}^l \|(p_h^{(j)}, \partial_t p_h^{(j)})\|_{\mathcal{E}}^2 + \|(w_h^{(l)}, \partial_t w_h^{(l)})\|_{\mathcal{E}}^2 + o(1),$$

uniformly in t , as $h \rightarrow 0$, (33)

and where $p_h^{(j)}$ is a linear damped concentrating wave, associated with data $[(\varphi^{(k)}, \psi^{(k)}), \underline{h}, \underline{x}^{(k)}, \underline{t}^{(k)}]$ mutually orthogonal.

We argue as before: we can assume $\delta(\underline{w}^{(l)}) > 0$ and we can pick $(\varphi^{(l+1)}, \psi^{(l+1)}, \underline{x}^{(l+1)}, \underline{t}^{(l+1)})$ such that:

$$\begin{aligned} \|\nabla \varphi^{(l)}\|_{L^2(T_{x_\infty}^{(l+1)}M)}^2 + \|\psi^{(l)}\|_{L^2(T_{x_\infty}^{(l+1)}M)}^2 &\geq \frac{1}{2} \delta(\underline{w}^{(l)}), \\ D_h^{(l+1)}(w_h^{(l)}, \partial_t w_h^{(l)})(t_h^{(l+1)}) &\xrightarrow{h \rightarrow 0} (\varphi^{(l+1)}, \psi^{(l+1)}), \end{aligned} \tag{34}$$

and we define $p_h^{(l+1)}$ as a linear damped concentrating wave, associated with data $[(\varphi^{(l+1)}, \psi^{(l+1)}), \underline{h}, \underline{x}^{(l+1)}, \underline{t}^{(l+1)}]$. Again, Lemma 2.8 applied to $w_h^{(l)}$ and $p_h^{(l+1)}$ implies estimates (32) with $w_h^{(l+1)} = w_h^{(l)} - p_h^{(l+1)}$.

Let us now deal with estimate (31). Lemma 1.4 combined with energy estimates gives for some $C > 0$ only depending on T and a

$$\|\nabla \varphi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2 + \|\psi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2 \leq C \|(p_h^{(j)}, \partial_t p_h^{(j)})_{t=0}\|_{\mathcal{E}}^2 + o(1).$$

From this and estimate (32), we infer

$$\sum_{j=1}^l (\|\nabla\varphi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2 + \|\psi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2) \leq C \overline{\lim}_{h \rightarrow 0} \|(u_h, \partial_t u_h)\|_{\mathcal{E}}^2 \leq C.$$

So, the series of general term $(\|\nabla\varphi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2 + \|\psi^{(j)}\|_{L^2(T_{x_\infty}^{(j)}M)}^2)$ converges. Using estimate (34), we get

$$\lim_{l \rightarrow \infty} \delta(\underline{w}^{(l)}) = 0.$$

Lemma 2.7 yields

$$\overline{\lim}_{h \rightarrow 0} \|w_h^{(l)}\|_{L^\infty([-T, T], L^6(M))} \xrightarrow{l \rightarrow \infty} 0.$$

This completes the proof of the first part of Proposition 2.2. Let us now deal with the orthogonality result. We will need the following two lemmas.

Lemma 2.9. *Let $(\underline{x}^{(1)}, \underline{t}^{(1)}) \not\perp_h (\underline{x}^{(2)}, \underline{t}^{(2)})$. Let v_h be an h -oscillatory sequence solution of the damped linear wave equation such that*

$$D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \xrightarrow{h \rightarrow 0} (\varphi^{(1)}, \psi^{(1)}). \tag{35}$$

Then, there exists $(\varphi^{(2)}, \psi^{(2)})$ such that, up to a subsequence

$$D_h^{(2)}(v_h, \partial_t v_h)(t_h^{(2)}) \xrightarrow{h \rightarrow 0} (\varphi^{(2)}, \psi^{(2)}). \tag{36}$$

Moreover, we have

$$\|(\varphi^{(1)}, \psi^{(1)})\|_{\mathcal{E}_{x_\infty}} = \|(\varphi^{(2)}, \psi^{(2)})\|_{\mathcal{E}_{x_\infty}}. \tag{37}$$

Proof. First, we assume $\underline{x}^{(1)} = \underline{x}^{(2)}$. By translation in time, we can assume $\underline{t}^{(1)} = 0$. The non-orthogonality assumption yields, up to extraction, $t_h^{(2)}/h = C + o(1)$ with C constant.

Let $(\varphi, \psi) \in \mathcal{E}_\infty$ be arbitrary and p_h the linear damped concentrating wave associated with $[(\varphi, \psi), \underline{h}, \underline{x}^{(1)}, 0]$. We use the equivalent definition stated in Lemma 1.5: (35) is equivalent to

$$\begin{aligned} \int_M \nabla v_h(0) \cdot \nabla p_h(0) &\xrightarrow{n \rightarrow \infty} \int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi, \\ \int_M \partial_t v_h(0) \partial_t p_h(0) &\xrightarrow{n \rightarrow \infty} \int_{T_{x_\infty} M} \psi^{(1)} \psi. \end{aligned}$$

As both v_h and p_h are solutions of the damped wave equation on M and $t_h^{(2)} \xrightarrow{h \rightarrow 0} 0$, we have by equicontinuity (see Lemma 1.9).

$$\int_M \nabla v_h(t_h^{(2)}) \cdot \nabla p_h(t_h^{(2)}) + \int_M \partial_t v_h(t_h^{(2)}) \partial_t p_h(t_h^{(2)}) \xrightarrow{n \rightarrow \infty} \int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi + \int_{T_{x_\infty} M} \psi^{(1)} \psi.$$

Let v, w satisfy on $T_{x_\infty} M$

$$\begin{aligned} \square_\infty v &= 0, & (v, \partial_t v)|_{t=0} &= (\varphi^{(1)}, \psi^{(1)}), \\ \square_\infty w &= 0, & (w, \partial_t w)|_{t=0} &= (\varphi, \psi). \end{aligned}$$

Conservation of the scalar product yields

$$\int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi + \int_{T_{x_\infty} M} \psi^{(1)} \psi = \int_{T_{x_\infty} M} \nabla v(C) \cdot \nabla w(C) + \int_{T_{x_\infty} M} \partial_t v(C) \partial_t w(C).$$

But according to Corollary 2.1, $(p_h, \partial_t p_h)|_{t=t_h^{(2)}}$ is a concentrating data according to $[(w(C), \partial_t w(C)), \underline{h}, \underline{x}^{(1)}]$. Since the wave equation is reversible and (φ, ψ) is arbitrary, we have proved that for any concentrating data (f_h, g_h) associated with $[(\tilde{\varphi}, \tilde{\psi}), \underline{h}, \underline{x}^{(1)}]$, we have

$$\int_M \nabla v_h(t_h^{(2)}) \cdot \nabla f_h + \int_M \partial_t v_h(t_h^{(2)}) g_h \xrightarrow{n \rightarrow \infty} \int_{T_{x_\infty} M} \nabla v(C) \cdot \nabla \tilde{\varphi} + \int_{T_{x_\infty} M} \partial_t v(C) \tilde{\psi}.$$

This gives the result for $x_h^{(1)} = x_h^{(2)}$ by taking $(\varphi^{(2)}, \psi^{(2)}) = (v(C), \partial_t v(C))$ which satisfies (37) by conservation of the energy.

In the general case $\underline{x}^{(1)} \not\perp_h \underline{x}^{(2)}$, we have in a local coordinate chart and up to a subsequence $x_h^{(2)} = x_h^{(1)} + (\vec{D} + o(1))h$ where $\vec{D} \in T_{x_\infty} M$ is a constant vector. We remark that if a bounded sequence (f_h, g_h) satisfies $D_h^{(1)}(f_h, g_h) \xrightarrow{h \rightarrow 0} (\varphi, \psi)$, it also fulfills $D_h^{(2)}(f_h, g_h) \xrightarrow{h \rightarrow 0} (\varphi(\cdot + \vec{D}), \psi(\cdot + \vec{D}))$. \square

We will also need the following lemma which is the analog of Lemma 3.7 of [17]. We keep the notation of the algorithm of extraction for further use.

Lemma 2.10. *Let $\{j, j'\} \in \{1, \dots, K\}^2$ be such that*

$$(\underline{x}^{(j)}, \underline{t}^{(j)}) \not\perp_h (\underline{x}^{(K+1)}, \underline{t}^{(K+1)}) \quad \text{and} \quad (\underline{x}^{(j)}, \underline{t}^{(j)}) \perp_h (\underline{x}^{(j')}, \underline{t}^{(j')}).$$

Then, $D_h^{(K+1)}(w_h^{(K+1)}, \partial_t w_h^{(K+1)})(t_h^{(K+1)}) \rightarrow 0$ implies $D_h^{(j)}(w_h^{(K+1)}, \partial_t w_h^{(K+1)})(t_h^{(j)}) \rightarrow 0$.

Moreover, if we assume $|t_\infty^{(j)} - t_\infty^{(j')}| < T_{focus}$ (see Definition 0.1), then $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightarrow (0, 0)$ for any concentrating wave $p_h^{(j')}$ associated with $[(\varphi^{(j')}, \psi^{(j')}), \underline{h}, \underline{x}^{(j')}, \underline{t}^{(j')}]$.

Proof. The first result is a particular case of Lemma 2.9. The proof of the second part is very similar to Lemma 3.7 of [17]. To simplify the notation, we can assume by translation in time that $t_h^{(j')} = 0$. We have to distinguish two cases: time and space orthogonality.

In the case of time orthogonality, that is $|\frac{t_h^{(j)}}{h}| \xrightarrow{h \rightarrow 0} +\infty$, we first prove $D_h^{1,(j)}(p_h^{(j)})(t_h^{(j)}) \rightharpoonup 0$ (recall that the exponent 1 in $D_h^{1,(j)}$ means that we only consider the H^1 part of the weak limit). Thanks to the nonfocusing assumption, Lemma 2.3 yields

$$\|p_h^{(j')}(t_h^{(j)}, \cdot)\|_{L^6(M)} \xrightarrow{h \rightarrow 0} 0.$$

We choose (U, Φ_U) some local chart around $x_\infty^{(j)}$ and $\Psi_U \in C_0^\infty(U)$ equals to 1 around $x_\infty^{(j)}$. Then, $\|\Psi_U p_h^{(j')}(t_h^{(j)}, \cdot)\|_{L^6(M)} \xrightarrow{h \rightarrow 0} 0$ and $h^{\frac{1}{2}} \|\Psi_U p_h^{(j')}(t_h^{(j)}, x_h + hx)\|_{L^6(\mathbb{R}^3)} \xrightarrow{h \rightarrow 0} 0$ (here we have identified $\Psi_U p_h^{(j')}$ with its local representation in \mathbb{R}^3). In particular $h^{\frac{1}{2}} \Psi_U p_h^{(j')}(t_h^{(j)}, x_h + hx) \rightharpoonup 0$ and $D_h^{1,(j)}(p_h^{(j')})(t_h^{(j)}) \rightharpoonup 0$. Now, we want to prove more precisely $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup 0$. Suppose $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0, \psi)$. Take $s \in \mathbb{R}$ arbitrary. $\tilde{t}_h^{(j)} = t_h^{(j)} + sh$ fulfills the same assumption $|\frac{\tilde{t}_h^{(j)}}{h}| \xrightarrow{h \rightarrow 0} +\infty$ and the nonfocusing property $|\tilde{t}_\infty^{(j)}| < T_{focus}$. So, we conclude similarly that $D_h^{1,(j)}(p_h^{(j')})(\tilde{t}_h^{(j)}) \rightharpoonup 0$. But the proof of Lemma 2.9 gives that $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(\tilde{t}_h^{(j)}) \rightharpoonup (v, \partial_t v)(s)$ where v is a solution of

$$\square_\infty v = 0, \quad (v, \partial_t v)(0) = (0, \psi).$$

So, we have $v(s) = 0$ for any $s \in \mathbb{R}$, which gives $\psi = 0$ and $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0, 0)$.

In the case of $t_h^{(j)} \not\ll_h t_h^{(j')}$ and space orthogonality, Lemma 2.9 allows us to assume that $t_h^{(j)} = t_h^{(j')} = 0$. In local coordinates, we have

$$(p_h^{(j')}, \partial_t p_h^{(j')})(0) = h^{-\frac{1}{2}} \Psi_U(x) \left(\varphi^{(j')}, \frac{1}{h} \psi^{(j')} \right) \left(\frac{x - x_h^{(j')}}{h} \right).$$

If $x_\infty^{(j')} \neq x_\infty^{(j)}$, the conclusion is obvious. If it is not the case, take $g \in C_0^\infty(\mathbb{R}^3)$. For the first part, we have to estimate

$$\int_{\mathbb{R}^3} \Psi_U^2(x_n^{(j)} + hy) \varphi^{(j')} \left(y + \frac{x_h^{(j)} - x_h^{(j')}}{h} \right) g(y) dy$$

which goes to 0 as h tends to 0 because g is compactly supported. The same result holds for the second part for $\partial_t p_h^{(j')}$. \square

Let us come back to the proof of the orthogonality of cores in Proposition 2.2. Define

$$j_K = \max \{ j \in \{1, \dots, K\} \mid (t_h^{(j)}, x_h^{(j)}) \not\ll_h (t_h^{(K+1)}, x_h^{(K+1)}) \}$$

assuming that such an index exists.

We list a few consequences of our algorithm

$$D_h^{(l+1)}(w_h^{(l)}, \partial_t w_h^{(l)})(t_h^{(l+1)}) \xrightarrow{h \rightarrow 0} (\varphi^{(l+1)}, \psi^{(l+1)}) \quad \text{with } \varphi^{(l+1)} \neq 0 \text{ if } l \leq K, \tag{38}$$

$$w_h^{(l)} = p_h^{(l+1)} + w_h^{(l+1)}, \tag{39}$$

$$w_h^{(j_K)} = \sum_{j=j_K+1}^{K+1} p_h^{(j)} + w_h^{(K+1)}. \tag{40}$$

The definition of $p_h^{(l)}$ and Lemma 1.6 implies $D_h^{(l)}(p_h^{(l)}, \partial_t p_h^{(l)})(t_h^{(l)}) \rightharpoonup (\varphi^{(l)}, \psi^{(l)})$. Then, we get from (38) and (39) that $D_h^{(l+1)}(w_h^{(l+1)}, \partial_t w_h^{(l+1)})(t_h^{(l+1)}) \rightharpoonup (0, 0)$. We apply this to $l + 1 = j_K$ and it gives $D_h^{(K+1)}(w_h^{(j_K)}, \partial_t w_h^{(j_K)})(t_h^{(K+1)}) \rightharpoonup (0, 0)$ thanks to the first part of Lemma 2.10 and the definition of j_K .

The definition of j_K and the second part of Lemma 2.10 gives $D_h^{1,(K+1)}(p_h^{(l)}, \partial_t p_h^{(l)})(t_h^{(K+1)}) \rightharpoonup (0, 0)$ for $j_K + 1 \leq l \leq K$.

To conclude, we “apply” $D_h^{1,(K+1)}$ to equality (40) and get $D_h^{1,(K+1)} w_h^{(j_K)}(t_h^{(K+1)}) \rightharpoonup \varphi^{(K+1)}$ while we have just proved $D_h^{(K+1)}(w_h^{(j_K)}, \partial_t w_h^{(j_K)})(t_h^{(K+1)}) \rightharpoonup (0, 0)$ which is a contradiction and complete the proof of the proposition for $2T < T_{focus}$.

In the case of S^3 and large times, the orthogonality result is a consequence of the orthogonality in short times and the almost periodicity. Denote

$$j_K = \max \{ j \in \{1, \dots, K\} \mid \exists m \in \mathbb{Z} \text{ s.t. } (t_h^{(j)} + m\pi, (-1)^m x_h^{(j)}) \not\perp_h (t_h^{(K+1)}, x_h^{(K+1)}) \}.$$

Then, for any $j_K + 1 \leq j \leq K$, we can find $m^{(j)} \in \mathbb{Z}$ such that

$$\begin{aligned} |t_\infty^{(j)} + m^{(j)}\pi - t_\infty^{(j_K)}| &\leq \pi/2 < T_{focus}, \\ (t_h^{(j)} + m^{(j)}\pi, (-1)^{m^{(j)}} x_h^{(j)}) &\perp_h (t_h^{(K+1)}, x_h^{(K+1)}), \end{aligned}$$

and we denote $m^{(j_K)} \in \mathbb{Z}$ such that $(t_h^{(j_K)} + m^{(j_K)}\pi, (-1)^{m^{(j_K)}} x_h^{(j_K)}) \not\perp_h (t_h^{(K+1)}, x_h^{(K+1)})$.

We remark that $p_h^{(j)}(t_h^{(j)} + m^{(j)}\pi, \cdot)$ is still a nonzero concentrating data associated with $[(-1)^{m^{(j)}}(\varphi, \psi)((-1)^{m^{(j)}} \cdot), \underline{h}, (-1)^{j} \underline{x}]$ thanks to Lemma 2.4 (note that it is at this stage that we use $M = S^3$ and $a \equiv 0$: it is the only case where we are able to describe this phenomenon of reconcentration). So, we are in the same situation as before, and we get a contradiction.

This completes the proof of Proposition 2.2. \square

Proof of Theorem 0.3. We only have to combine the two decompositions we made. Denote by v_n^j (and the rest $\rho_n^{(l)}$) the $h_n^{(j)}$ -oscillatory component obtained by decomposition (24) and $p_n^{(j,\alpha)}$ the concentrating waves obtained from decomposition (30) (and the rest $w_n^{(j,A_j)}$). We enumerate them by the bijection σ from \mathbb{N}^2 into \mathbb{N} defined by

$$\sigma(j, \alpha) < \sigma(k, \beta) \quad \text{if } j + \alpha < k + \beta \text{ or } j + \alpha = k + \beta \text{ and } j < k.$$

For l and A_j fixed, $1 \leq j \leq l$, the rest can be written as

$$w_n^{(l, A_1, \dots, A_l)} = \rho_n^{(l)} + \sum_{j=1}^l w_n^{(j, A_j)}.$$

Let $\varepsilon > 0$. To get the result, it suffices to prove that for l_0 large enough, $\|w_n^{(l, A_1, \dots, A_l)}\|_{L^\infty(L^6)} \leq \varepsilon$ for all (l, A_1, \dots, A_l) satisfying $l \geq l_0$ and $\sigma(j, A_j) \geq \sigma(l_0, 1)$.

(9) can easily be deduced from the same orthogonality result in the two other decompositions. In particular, it gives that the series of general term $\sum_{(j, \alpha)} \overline{\lim}_{n \rightarrow \infty} \|(p_n^{(j, \alpha)}, \partial_t p_n^{(j, \alpha)})_{t=0}\|_{\mathcal{E}}^2$ is convergent. In particular, we can find l_0 large enough such that we have

$$\sum_{\sigma(j, \alpha) > \sigma(l_0, 1)} \overline{\lim}_{n \rightarrow \infty} \|(p_n^{(j, \alpha)}, \partial_t p_n^{(j, \alpha)})_{t=0}\|_{\mathcal{E}}^2 \leq \varepsilon. \tag{41}$$

Moreover, for l_0 large enough, we have for $l \geq l_0$

$$\overline{\lim}_{n \rightarrow \infty} \|\rho_n^{(l)}\|_{L^\infty(L^6)} \leq \varepsilon.$$

Then, for any $l \geq l_0$, one can find one B_l such that for any $1 \leq j \leq l$, $\tilde{A}_j \geq B_l$ implies

$$\overline{\lim}_{n \rightarrow \infty} \|w_n^{(j, \tilde{A}_j)}\|_{L^\infty(L^6)} \leq \varepsilon/l.$$

The rest can be decomposed by

$$w_n^{(l, A_1, \dots, A_l)} = \rho_n^{(l)} + \sum_{j=1}^l w_n^{(j, \max(A_j, B_l))} + S_n^{(j, A_1, \dots, A_l)},$$

where

$$S_n^{(j, A_1, \dots, A_l)} = \sum_{1 \leq j \leq l, A_j < B_l} (w_n^{(j, A_j)} - w_n^{(j, B_l)}) = \sum_{j=1}^l \sum_{A_j < \alpha \leq B_l} p_n^{j, \alpha}.$$

Since $S_n^{(j, A_1, \dots, A_l)}$ is a solution of the damped wave equation, energy estimates and Sobolev embedding give

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \|S_n^{(j, A_1, \dots, A_l)}\|_{L^\infty(L^6)}^2 &\leq C \overline{\lim}_{n \rightarrow \infty} \|(S_n^{(j, A_1, \dots, A_l)}, \partial_t S_n^{(j, A_1, \dots, A_l)})_{t=0}\|_{\mathcal{E}}^2 \\ &\leq C \sum_{j=1}^l \sum_{A_j < \alpha \leq B_l} \|(p_n^{(j, \alpha)}, \partial_t p_n^{j, \alpha})_{t=0}\|_{\mathcal{E}}^2, \end{aligned}$$

where we have used almost orthogonality in the last estimate. But the sum is restricted to some (j, α) satisfying $\sigma(j, \alpha) > \sigma(j, \alpha_j) > \sigma(l_0, 1)$ and is indeed smaller than $C\varepsilon$ thanks to (41).

Combining our estimates, we get that $\overline{\lim}_{n \rightarrow \infty} \|w_n^{(l, A_1, \dots, A_l)}\|_{L^\infty(L^6)}$ is smaller than $(2 + C)\varepsilon$ for all (l, A_1, \dots, A_l) satisfying $l \geq l_0$ and $\sigma(j, A_j) \geq \sigma(l_0, 1)$. We get the same estimates with the $L^5(L^{10})$ norm by interpolation between $L^\infty(L^6)$ and $L^4(L^{12})$. The second norm being bounded by Strichartz estimates and the fact that $w_n^{(l, A_1, \dots, A_l)}$ is uniformly bounded in the energy space. \square

We also state a few consequences of the algorithm of Theorem 0.3 that will be used below. The following two lemmas use the notation and the assumptions of Theorem 0.3.

Lemma 2.11. *Let $2T < T_{focus}$. For any $l \in \mathbb{N}$ and $1 \leq j \leq l$, we have, with the notation and assumptions of Theorem 0.3*

$$D_n^{(j)}(w_n^{(l)}, \partial_t w_n^{(l)})(t_n^{(j)}) \rightharpoonup (0, 0).$$

Proof. Assume $D_n^{(j)}(w_n^{(l)}, \partial_t w_n^{(l)})(t_n^{(j)}) \rightharpoonup (\varphi, \psi)$. We directly use the decomposition of Theorem 0.3 to write for $L > l$

$$w_n^{(l)} = \sum_{i=l+1}^L p_n^{(i)} + w_n^{(L)}.$$

In case of scale orthogonality of $h_n^{(j)}$ and $h_n^{(i)}$, for $l + 1 \leq i \leq L$, we have directly $D_n^{(j)}(p_n^{(i)}, \partial_t p_n^{(i)})(t_n^{(j)}) \rightharpoonup (0, 0)$. Otherwise, if $h_n^{(j)} = h_n^{(i)}$ and $(\underline{x}^{(j)}, \underline{t}^{(j)}) \perp_h (\underline{x}^{(i)}, \underline{t}^{(i)})$, Lemma 2.10 gives the same result. Therefore, $D_n^{(j)}(w_n^{(L)}, \partial_t w_n^{(L)})(t_n^{(j)}) \rightharpoonup (\varphi, \psi)$. Since $\overline{\lim}_{n \rightarrow \infty} \|w_n^{(L)}\|_{L^\infty([-T, T], L^6)} \xrightarrow{L \rightarrow \infty} 0$, we have $\varphi = 0$. We finish the proof as in Lemma 2.10.

We use the same argument for times $t_n^{(j)} + s h_n^{(j)}$ and get $\psi \equiv 0$ by the proof of Lemma 2.9. Remark that Lemma 2.9 requires that $w_n^{(l)}$ is strictly $h_n^{(j)}$ -oscillatory, but this can be easily avoided by decomposing $w_n^{(l)} = f_n + g_n$ with f_n ($h_n^{(j)}$)-oscillatory and g_n ($h_n^{(j)}$)-singular. \square

Lemma 2.12. *With the notation and assumptions of Theorem 0.3, we have, for any $j \in \mathbb{N}$*

$$\overline{\lim}_{n \rightarrow \infty} \|p_n^{(j)}\|_{L^5([-T, T], L^{10})} \leq C \overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5([-T, T], L^{10})}$$

where C only depends on the manifold M .

Proof. We first assume $2T < T_{focus}$. Actually, in the case of \mathbb{R}^3 , the result is proved using the fact that the $p_n^{(j)}$ are some concentration of some weak limit of a dilation of v_n . The proof for a manifold follows the same path with a little more care due to the fact that dilation only have a local meaning.

For any $\varepsilon > 0$, we prove

$$\overline{\lim}_{n \rightarrow \infty} \|p_n^{(j)}\|_{L^5([-T, T], L^{10})} \leq C \overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5([-T, T], L^{10})} + C\varepsilon.$$

We use the decomposition of Theorem 0.3 and choose $l \geq j$ large enough such that

$$\overline{\lim}_{n \rightarrow \infty} \|w_n^l\|_{L^5([-T, T], L^{10})} \leq \varepsilon.$$

Let Ψ_U be a cut-off function related to local charts (U, Φ_U) such that $\Psi_U(x) = 1$ around x_∞^j and $\Psi_U(x) = 0$ around any $x_\infty^i \neq x_\infty^j$.

For each $1 \leq i \leq l$, we decompose $[-T, T] = I_{n,i}^{1,\Lambda} \cup I_{n,i}^{2,\Lambda} \cup I_{n,i}^{3,\Lambda}$ according to Lemma 2.3.

For any i such that $x_\infty^i = x_\infty^j$, for Λ large enough, we have

$$\overline{\lim}_{n \rightarrow \infty} \|p_n^{(i)}\|_{L^5(I_{n,i}^{1,\Lambda} \cup I_{n,i}^{3,\Lambda}, L^{10})} \leq \varepsilon/l. \tag{42}$$

Moreover, Lemma 2.2 yields for Λ large enough

$$\overline{\lim}_{n \rightarrow \infty} \|p_n^{(i)} - v_n^{(i)}\|_{L^5(I_{n,i}^{2,\Lambda}, L^{10})} \leq \varepsilon/l \tag{43}$$

where $v_n^{(i)}(t, x) = \frac{1}{\sqrt{h_n^{(i)}}} \Phi_U^* \Psi_U(x) v^{(i)}\left(\frac{t-t_n^{(i)}}{h_n^{(i)}}, \frac{x-x_n^{(i)}}{h_n^{(i)}}\right)$ on a coordinate patch and $v^{(i)}$ solution of

$$\begin{cases} \square_{x_\infty^j} v^{(i)} = 0 & \text{on } \mathbb{R} \times T_{x_\infty^j} M, \\ (v^{(i)}(0), \partial_t v^{(i)}(0)) = (\varphi^{(i)}, \psi^{(i)}). \end{cases} \tag{44}$$

Thanks to (42) and (43), the conclusion of the lemma will be obtained if we prove

$$\|v^{(j)}\|_{L^5(\mathbb{R}, L^{10}(T_{x_\infty^j} M))} \leq \overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5([-T, T], L^{10})} + C\varepsilon.$$

We argue by duality. Take $f \in C_0^\infty(\mathbb{R} \times T_{x_\infty^j} M)$ with $\|f\|_{L^{5/4}(\mathbb{R}, L^{10/9})} = 1$.

From now on, we work in local coordinates around x_∞^j and we will not distinguish a function defined on $U \subset M$ with its representative in $\mathbb{R}^3 \approx T_{x_\infty^j} M$. Denote by W^j the operator defined on functions on $\mathbb{R}_t \times \mathbb{R}^3$ by

$$W^j g(s, y) := \sqrt{h_n^j} g(t_n^j + h_n^j s, t_n^j + h_n^j y).$$

The definition of $v_n^{(j)}$ in local coordinates yields

$$\int_{\mathbb{R} \times \mathbb{R}^3} (W^j 1_{[-T, T]} v_n^{(j)}) f \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R} \times \mathbb{R}^3} v^{(j)} f.$$

On the other hand

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^3} (W^j \Psi_U 1_{[-T, T]} p_n^j) f \\ &= \int_{\mathbb{R} \times \mathbb{R}^3} W^j \left[\Psi_U 1_{[-T, T]} \left(v_n - \sum_{x_\infty^i \neq x_\infty^j} p_n^i - \sum_{x_\infty^i = x_\infty^j, i \neq j} p_n^i - w_n^l \right) \right] f. \end{aligned}$$

For any $1 \leq i \leq l$, with $x_\infty^{(i)} \neq x_\infty^{(j)}$, using again Lemmas 2.3 and 2.2 and the fact that we can choose Ψ_U with $\Psi_U(x_\infty^{(i)}) = 0$, we easily get

$$\overline{\lim}_{n \rightarrow \infty} \left\| \Psi_U p_n^{(i)} \right\|_{L^5([-T, T], L^{10})} = 0.$$

So for n large enough

$$\begin{aligned} \left| \int_{\mathbb{R} \times \mathbb{R}^3} (W^j \Psi_U p_n^{(j)}) f \right| &\leq C (\|v_n\|_{L^5([-T, T], L^{10})} + 2\varepsilon) \\ &+ \left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j \left[\Psi_U 1_{[-T, T]} \sum_{x_\infty^{(i)} = x_\infty^{(j)}, i \neq j} p_n^{(i)} \right] f \right|. \end{aligned}$$

But for $i \neq j$, $x_\infty^{(i)} = x_\infty^{(j)}$, using (42) and then (43), we have for Λ and n large enough

$$\begin{aligned} \left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{[-T, T]} p_n^{(i)}] f \right| &\leq \left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{I_{n,i}^{2,\Lambda}} p_n^{(i)}] f \right| + \varepsilon/l \\ &\leq \left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{I_{n,i}^{2,\Lambda}} v_n^{(i)}] f \right| + 2\varepsilon/l. \end{aligned}$$

These terms are actually

$$\begin{aligned} &\left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{I_{n,i}^{2,\Lambda}} v_n^{(i)}] f \right| \\ &= \sqrt{\frac{h_n^j}{h_n^i}} \left| \int_{\mathbb{R} \times \mathbb{R}^3} \left[\Psi_U^2 (h_n^j x + x_n^j) 1_{\left[\frac{t_n^i - t_n^j - \Lambda h_n^i}{h_n^j}, \frac{t_n^i - t_n^j + \Lambda h_n^i}{h_n^j} \right]} v^{(i)} \left(\frac{th_n^j + t_n^j - t_n^i}{h_n^i}, \frac{xh_n^j + x_n^j - x_n^i}{h_n^i} \right) \right] f \right|. \end{aligned}$$

Since this expression is uniformly continuous in $v^i \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$, we may assume v^i in $C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$. Then, if $\frac{h_n^j}{h_n^i} \xrightarrow{n \rightarrow \infty} 0$, we have

$$\left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{I_{n,i}^{2,\Lambda}} v_n^{(i)}] f \right| = \mathcal{O} \left(\sqrt{\frac{h_n^j}{h_n^i}} \right).$$

If $\frac{h_n^j}{h_n^i} \xrightarrow{n \rightarrow \infty} \infty$, the change of variables $s = \frac{th_n^j + t_n^j - t_n^i}{h_n^j}$, $y = \frac{xh_n^j + x_n^j - x_n^i}{h_n^i}$ gives

$$\left| \int_{\mathbb{R} \times \mathbb{R}^3} W^j [\Psi_U 1_{I_{n,i}^{2,\Lambda}} v_n^i] f \right| = \mathcal{O} \left(\left(\frac{h_n^j}{h_n^i} \right)^{-7/2} \right).$$

If $h_n^j = h_n^i$, the space or time orthogonality yields that the integral is zero for n large enough.

In conclusion, for any $f \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ with $\|f\|_{L^{5/4}(\mathbb{R}, L^{10/9})} = 1$, we have proved:

$$\left| \int_{\mathbb{R} \times \mathbb{R}^3} v^j f \right| \leq C \overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5([-T, T], L^{10})} + C\varepsilon.$$

This gives the expected result by duality.

The case of S^3 is proved by considering subintervals of length smaller than T_{focus} where the former result can be applied. \square

2.2. Nonlinear profile decomposition

2.2.1. Behavior of nonlinear concentrating waves (after S. Ibrahim)

In this subsection, we recall the description of nonlinear concentrating waves. As explained in the introduction, the behavior for times close to concentration is ruled by the scattering operator on \mathbb{R}^3 with a flat metric. So, we first state the existence of the wave operator on \mathbb{R}^3 , following the notation of [2]. We state it for any constant metric on the tangent plane $T_{x_\infty} M \approx \mathbb{R}^3$.

Proposition 2.3 (Scattering operators on \mathbb{R}^3). *Let $x_\infty \in M$ and \square_∞ be the d'Alembertian operator (constant) on $T_{x_\infty} M \approx \mathbb{R}^3$ induced by the metric on M . To every solution of*

$$\begin{cases} \square_\infty v = 0 & \text{on } \mathbb{R} \times T_{x_\infty} M, \\ (v(0), \partial_t v(0)) = (\varphi, \psi) \in \mathcal{E}_{x_\infty} \end{cases}$$

there exists a unique strong solution u_\pm of

$$\begin{cases} \square_\infty u_\pm = -|u_\pm|^4 u_\pm & \text{on } \mathbb{R} \times T_{x_\infty} M, \\ \lim_{t \rightarrow \pm\infty} \|(v - u_\pm, \partial_t(v - u_\pm))(t)\|_{\mathcal{E}_{x_\infty}} = 0. \end{cases}$$

The wave operators

$$\Omega_\pm : (v, \partial_t v)_{t=0} \mapsto (u_\pm, \partial_t u_\pm)_{t=0}$$

are bijective from \mathcal{E}_{x_∞} onto itself.

The scattering operator S is defined as $S = (\Omega_+)^{-1} \circ \Omega_-$.

The analysis of nonlinear concentrating waves computed by S. Ibrahim in [21] shows that there are three different periods to be considered: before, during and after the time of concentration. Roughly speaking, for times close to the concentrating time, the solution is closed to nonlinear concentrating waves on \mathbb{R}^3 with flat metric and without damping, as described in Bahouri and Gérard [2]: in the fast time $h_n t$, it follows the scattering on \mathbb{R}^3 . Before and after the time of concentration, the nonlinear concentrating wave is “close” to some linear damped concentrating waves as defined in Table 1 below. This is precised in the following theorem whose proof can be found in S. Ibrahim [21]. Yet, in [21], the result is stated for an equation without damping and we give a sketch of the proof in the damped case in Section 2.2.2.

Table 1
Transformation of the profile through a focus.

$\lim \frac{t}{h}$	(φ_1, ψ_1)	(φ_2, ψ_2)	(φ_3, ψ_3)
$-\infty$	$\Omega_-^{-1} \circ \Omega_+(\varphi, \psi)$	$\Omega_+(\varphi, \psi)$	(φ, ψ)
0	$\Omega_-^{-1}(\varphi, \psi)$	(φ, ψ)	$\Omega_+^{-1}(\varphi, \psi)$
∞	(φ, ψ)	$\Omega_-(\varphi, \psi)$	$\Omega_+^{-1} \circ \Omega_-(\varphi, \psi)$

Theorem 2.1. Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ be a linear damped concentrating wave. We denote by \underline{u} its associated nonlinear damped concentrating wave (same data at $t = 0$). There exist three linear damped concentrating waves denoted by $[(\varphi_i, \psi_i), \underline{h}, \underline{x}, \underline{t}]$, $i = 1, 2$ or 3 such that: for all interval $[-T, T]$ containing t_∞ , satisfying the following non-focusing property (see Definition 0.1)

$$mes(F_{x, x_\infty, s}) = 0 \quad \forall x \in M \text{ and } s \neq 0 \quad \text{such that} \quad t_\infty + s \in [-T, T] \tag{45}$$

we have

$$\overline{\lim}_n \|u_n - [(\varphi_1, \psi_1), \underline{h}, \underline{x}, \underline{t}]\|_{I_n^{1,\Lambda}} \xrightarrow{\Lambda \rightarrow +\infty} 0, \tag{46}$$

$$\overline{\lim}_n \|u_n - [(\varphi_3, \psi_3), \underline{h}, \underline{x}, \underline{t}]\|_{I_n^{3,\Lambda}} \xrightarrow{\Lambda \rightarrow +\infty} 0 \tag{47}$$

where $I_n^{1,\Lambda} = [-T, t_n - \Lambda h_n]$ and $I_n^{3,\Lambda} = [t_n + \Lambda h_n, T]$.

Moreover, for times close to concentration $I_n^{2,\Lambda} = [t_n - \Lambda h_n, t_n + \Lambda h_n]$, we have

$$\forall \Lambda > 0, \quad \lim_n \|u_n - w_n\|_{I_n^{2,\Lambda}} = 0 \tag{48}$$

where $w_n(t, x) = \Psi_U(x) \frac{1}{\sqrt{h_n}} w(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n})$ on a coordinate patch and w solution of

$$\begin{cases} \square_\infty w = -|w|^4 w & \text{on } \mathbb{R} \times T_{x_\infty} M, \\ (w(0), \partial_t w(0)) = (\varphi_2, \psi_2), \end{cases} \tag{49}$$

where \square_∞ corresponds to the frozen metric on $T_{x_\infty} M$.

The different functions (φ_i, ψ_i) are defined according to Table 1, following the notation of Proposition 2.3.

Remark 2.2. Note that the transition from the first column to the third one represents the modification of profile due to the concentration and the concentrating functions are modified according to the scattering operator S . To go from the first column to the second one, we apply the operator Ω_- while we apply Ω_+^{-1} to get from the second to the third one.

Remark 2.3. The behavior for times close to concentration is not written in this way in the article [21] of S. Ibrahim, but is a byproduct of its proof. We refer to the next section which contains a sketch of the proof.

Corollary 2.2. *A nonlinear damped concentrating wave q_h is strictly (h)-oscillatory with respect to A_M and bounded in all Strichartz norms, uniformly on any bounded interval.*

Proof. The boundedness of all Strichartz norms is a consequence of the Duhamel formula and Strichartz estimates once the result is known in the case of L^5L^{10} . On the intervals $I_n^{1,\Lambda}$ and $I_n^{3,\Lambda}$ when q_h is closed to a linear concentrating wave, the result follows from Proposition 1.4 and linear Strichartz estimates. On $I_n^{2,\Lambda}$, q_h behaves like a concentration of a nonlinear solution on $T_{x_\infty}M$. The strict (h)-oscillation is obvious and the Strichartz estimates follow from the global estimates on \mathbb{R}^3 . \square

In the case of S^3 , thanks to a better knowledge of the behavior of nonlinear concentrating waves we can avoid assumption (45). This is Theorem 1.8 from [21]. It will allow us to perform the profile decomposition for large times.

Theorem 2.2. *Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ be a linear (not damped, that is $a(x) \equiv 0$) concentrating wave on S^3 . We denote by \underline{u} its nonlinear associated concentrating wave (same data at $t = 0$). We assume that $t_\infty \in]0, \pi[$. Then, for all $j \in \mathbb{Z}$, we have*

$$\overline{\lim}_n \left\| \|u_n - [\tilde{S}^{(j)} S(\varphi, \psi), \underline{h}, (-1)^j \underline{x}, \underline{t}] \right\|_{I_n + j\pi + \Lambda h_n, t_n + (j+1)\pi - \Lambda h_n} \xrightarrow{\Lambda \rightarrow +\infty} 0$$

where $\tilde{S} = S \circ A$, $\tilde{S}^{(j)} = \tilde{S} \circ \tilde{S} \circ \dots \circ \tilde{S}$, j times and $A(\varphi, \psi)(x) = -(\varphi, \psi)(-x)$.

Moreover, the cases $t_\infty \in]-\pi, 0[$ and $t_\infty = 0$ can be deduced similarly to Theorem 2.1 with some changes on the concentration data in the same spirit as Table 1.

2.2.2. *Modification of the proof of S. Ibrahim for Theorem 2.1 in the case of damped equation*

In this subsection, we give some sketch of the proof for the behavior of nonlinear damped concentrating waves announced in Section 2.2.1. These results are proved in [21] in the undamped case $a(x) \equiv 0$ and so we only briefly emphasize the main necessary modifications of the proof. To simplify, we only treat the case $\frac{t_n}{h_n} \xrightarrow{n \rightarrow +\infty} \infty$.

Sketch of the proof of estimate (46) of Theorem 2.1: Behavior before concentration. The proof is exactly the same as Corollary 3.2 of [21]. $w_n = u_n - v_n$ is a solution of

$$\begin{aligned} \square w_n + w_n + a(x)\partial_t w_n &= -|w_n + v_n|^4(w_n + v_n) \quad \text{on } I_n^{1,\Lambda} \times M, \\ (w_n, \partial_t w_n)|_{t=0} &= (0, 0). \end{aligned}$$

Using Strichartz and energy estimates, we are able to use a bootstrap argument if $\overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5(I_n^{1,\Lambda}, L^{10})}$ is small enough. This can be achieved thanks to Lemma 2.3 and gives the result. \square

Sketch of the proof of estimate (48) of Theorem 2.1: Behavior for times close to concentration. By definition of v_n and finite propagation speed, the main energy part of v_n is concentrated near x_∞ for times close to t_∞ . By estimate (46), it is also the case for u_n . Therefore, for times $t \in [t_n - \Lambda h_n, t_n + \Lambda h_n]$, we can neglect the energy outside of a fixed open set and work in local

coordinates. Moreover, in that case, we can use the norm $\|\cdot\|_{I \times \mathbb{R}^3}$ instead of $\|\cdot\|_I$ and use the fact that it is invariant by translation and scaling up to a modification of the interval of time.

Denote by \tilde{u}_n (resp. \tilde{v}_n) the rescaled function associated to u_n (resp. v_n), so that $u_n(t, x) = \frac{1}{\sqrt{h_n}} \tilde{u}_n(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n})$. We need to prove $\overline{\lim}_{n \rightarrow \infty} \|\tilde{u}_n - w\|_{[-\Lambda, \Lambda] \times \mathbb{R}^3} \xrightarrow{\Lambda \rightarrow \infty} 0$ where w is a solution of

$$\begin{cases} \square_\infty w = -|w|^4 w & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ (w, \partial_t w)|_{t=0} = (\varphi_2, \psi_2) = \Omega_-(\varphi, \psi). \end{cases}$$

By definition of Ω_- , w satisfies $\|(w - v, \partial_t(w - v))(t)\|_{\dot{H}^1 \times L^2} \xrightarrow{t \rightarrow -\infty} 0$ where v is a solution of

$$\begin{cases} \square_\infty v = 0 & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ (v, \partial_t v)|_{t=0} = (\varphi, \psi). \end{cases} \tag{50}$$

Moreover, it is known that $\Omega_-(\varphi, \psi) = \lim_{s \rightarrow -\infty} U(-s)U_0(s)(\varphi, \psi)$ where U and U_0 are the nonlinear and linear flow maps. More precisely, by Lemma 3.4 of [21], we have $\|w_\Lambda - w\|_{[-\Lambda, \Lambda] \times \mathbb{R}^3} \xrightarrow{\Lambda \rightarrow \infty} 0$ where w_Λ is the smooth solution of

$$\begin{cases} \square_\infty w_\Lambda + |w_\Lambda|^4 w_\Lambda = 0 & \text{on } [-\Lambda, \Lambda] \times \mathbb{R}^3, \\ (w_\Lambda, \partial_t w_\Lambda)|_{t=-\Lambda} = \chi_\Lambda(v, \partial_t v)|_{t=-\Lambda}, \end{cases}$$

where χ_Λ is an appropriate family of smoothing operator. So, we are left to prove $\overline{\lim}_{n \rightarrow \infty} \|\tilde{u}_n - w_\Lambda\|_{[-\Lambda, \Lambda] \times \mathbb{R}^3} \xrightarrow{\Lambda \rightarrow \infty} 0$.

We introduce the auxiliary family of functions \tilde{u}_n^Λ solution of

$$\begin{cases} \square_n \tilde{u}_n^\Lambda + h_n^2 \tilde{u}_n^\Lambda + |\tilde{u}_n^\Lambda|^4 \tilde{u}_n^\Lambda = -h_n a(h_n x + x_n) \partial_t \tilde{u}_n^\Lambda & \text{on } [-\Lambda, \Lambda] \times \mathbb{R}^3, \\ (\tilde{u}_n^\Lambda, \partial_t \tilde{u}_n^\Lambda)|_{t=-\Lambda} = (\tilde{v}_n, \partial_t \tilde{v}_n)|_{t=-\Lambda}, \end{cases}$$

where we have denoted by \square_n the dilation of the operator \square . So it can be written as $\square_n = \partial_t^2 - \sum_{i,j} g^{ij}(h_n x + x_n) \partial_{ij} + h_n V(h_n x + x_n) \cdot \nabla$ where V is a smooth vector field (note that it is only defined in an open set of size $\mathcal{O}(h_n^{-1})$ but it is also the case for $\tilde{u}_n, \tilde{u}_n^\Lambda$ and \tilde{v}_n , we omit the details). The proof is complete if we prove

$$\overline{\lim}_{n \rightarrow \infty} \|\tilde{u}_n^\Lambda - w^\Lambda\|_{[-\Lambda, \Lambda] \times \mathbb{R}^3} \xrightarrow{\Lambda \rightarrow \infty} 0 \tag{51}$$

and

$$\overline{\lim}_{n \rightarrow \infty} \|\tilde{u}_n^\Lambda - \tilde{u}_n\|_{[-\Lambda, \Lambda] \times \mathbb{R}^3} \xrightarrow{\Lambda \rightarrow \infty} 0. \tag{52}$$

We begin with (51). $r_{n,\Lambda} = \tilde{u}_n^\Lambda - w^\Lambda$ is a solution of

$$\left\{ \begin{aligned} & \square_n r_{n,\Lambda} + h_n^2 r_{n,\Lambda} + h_n a(h_n x + x_n) \partial_t r_{n,\Lambda} \\ & = |w_\Lambda|^4 w_\Lambda - |r_{n,\Lambda} + w_\Lambda|^4 (r_{n,\Lambda} + w_\Lambda) \\ & \quad - h_n^2 w_\Lambda - h_n a(h_n x + x_n) \partial_t w_\Lambda + (\square_\infty - \square_n) w_\Lambda, \\ & (r_{n,\Lambda}, \partial_t r_{n,\Lambda})|_{t=-\Lambda} = (\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))|_{t=-\Lambda}. \end{aligned} \right.$$

A quick scaling analysis easily yields that the operator $\square_n + h_n^2 + h_n a(h_n x + x_n) \partial_t$ satisfies the same Strichartz and energy estimates as $\square + 1 + a(x) \partial_t$ for some times of order Λ . Moreover, following the same argument as Lemma 2.1 of [21], we get that for fixed Λ

$$\overline{\lim}_{n \rightarrow \infty} \left\| -h_n^2 w_\Lambda - h_n a(h_n x + x_n) \partial_t w_\Lambda + (\square_\infty - \square_n) w_\Lambda \right\|_{L^1([- \Lambda, \Lambda], L^2)} = 0.$$

Thanks to Lemma 2.2, we know that $\overline{\lim}_{n \rightarrow \infty} \|(\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))(-\Lambda)\|_{\dot{H}^1 \times L^2}$ can be made arbitrary small for large Λ . Strichartz and energy estimates give for any $\eta > -\Lambda$

$$\begin{aligned} \|r_{n,\Lambda}\|_{[-\Lambda, \eta] \times \mathbb{R}^3} &\leq \|(\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))(-\Lambda)\|_{\dot{H}^1 \times L^2} \\ &\quad + \left\| -h_n^2 w_\Lambda - h_n a(h_n x + x_n) \partial_t w_\Lambda + (\square_\infty - \square_n) w_\Lambda \right\|_{L^1([- \Lambda, \eta], L^2)} \\ &\quad + \|r_{n,\Lambda}\|_{L^5([- \Lambda, \eta], L^{10})}^5 + \|r_{n,\Lambda}\|_{L^5([- \Lambda, \eta], L^{10})} \|w_\Lambda\|_{L^5([- \Lambda, \eta], L^{10})}^4. \end{aligned}$$

If $\|w_\Lambda\|_{L^5([- \Lambda, \eta], L^{10})}$ is small enough, a bootstrap gives (51) on $[-\Lambda, \eta]$. We can iterate the process by dividing $[-\Lambda, \Lambda]$ in a finite number of intervals where the bootstrap can be performed.

For (52), we observe that \tilde{u}_n^A and \tilde{u}_n are solutions of the same equation but with different initial data which satisfy thanks to estimate (46)

$$\overline{\lim}_{n \rightarrow \infty} \|(\tilde{u}_n^A - \tilde{u}_n, \partial_t(\tilde{u}_n^A - \tilde{u}_n))(-\Lambda)\|_{\mathcal{E}} = \overline{\lim}_{n \rightarrow \infty} \|(\tilde{v}_n - \tilde{u}_n, \partial_t(\tilde{v}_n - \tilde{u}_n))(-\Lambda)\|_{\mathcal{E}} \xrightarrow{\Lambda \rightarrow \infty} 0.$$

Then, Strichartz and energy estimates allow us to use a bootstrap argument on subintervals I such that $\|\tilde{u}_n^A\|_{L^5(I, L^{10})}$ is small. (51) allows to complete the proof. \square

2.2.3. Proof of the decomposition

This subsection is devoted to the proof of Theorem 0.4.

Let us define the function β in the following way:

$$\forall \omega \in \mathbb{C}, \quad \beta(\omega) \stackrel{\text{def}}{=} |\omega|^4 \omega.$$

Proposition 2.4. *Let $0 < 2T < T_{\text{focus}}$ (see Definition 0.1). Let $p_n^{(j)}, 1 \leq j \leq l$, be linear damped concentrating waves, associated with data $[(\varphi^{(j)}, \psi^{(j)}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$ (we can have $h^{(j)} = 1$ for one of it), which are orthogonal according to Definition 1.2 and such that $t_\infty^{(j)} \in [-T, T]$. Denote by $q_n^{(j)}$ the associated nonlinear damped concentrating waves (same data at $t = 0$).*

Then, we have

$$\overline{\lim}_{n \rightarrow \infty} \left\| \beta \left(\sum_{j=1}^l q_n^{(j)} \right) - \sum_{j=1}^l \beta(q_n^{(j)}) \right\|_{L^1([-T, T], L^2)} = 0. \tag{53}$$

Proof. We follow closely Lemma 4.2 of [17]

$$\left\| \beta \left(\sum_{j=1}^l q_n^{(j)} \right) - \sum_{j=1}^l \beta(q_n^{(j)}) \right\|_{L^1([-T, T], L^2)} \leq \sum_{1 \leq j_1, \dots, j_5 \leq l} \left\| \prod_{k=1}^5 q_n^{(j_k)} \right\|_{L^1([-T, T], L^2)}$$

where at least two $q_n^{(j_k)}$ are different. In the case of orthogonality of scales, we use the Hölder inequality

$$\left\| \prod_{k=1}^5 q_n^{(j_k)} \right\|_{L^1([-T, T], L^2)} \leq C \|q_n^1 q_n^2\|_{L^\infty([-T, T], L^3)} \prod_{k=3}^5 \|q_n^{(j_k)}\|_{L^3([-T, T], L^{18})}.$$

Then, Corollary 2.2 and Lemma 1.8 yield the result (note that $L^3 L^{18}$ is a pair of Strichartz norm). So now, we can assume $h_n^1 = h_n^2 = h_n$. By Hölder and Corollary 2.2, we get

$$\begin{aligned} \left\| \prod_{k=1}^5 q_n^{(j_k)} \right\|_{L^1([-T, T], L^2)} &\leq C \|q_n^1 q_n^2\|_{L^{5/2}([-T, T], L^5)} \prod_{k=3}^5 \|q_n^{(j_k)}\|_{L^5([-T, T], L^{10})} \\ &\leq C \|q_n^1 q_n^2\|_{L^{5/2}([-T, T], L^5)}. \end{aligned}$$

We apply Theorem 2.1 to q_n^1 . We obtain three couples (φ^i, ψ^i) , $i = 1, 2, 3$ and split the interval $[-T, T] = \bigcup_{j=1}^3 I_n^{j, \Lambda}$. We first deal with the interval $I_n^{1, \Lambda}$. Denote $\underline{v}_1 = [(\varphi_1, \psi_1), \underline{h}, \underline{x}, \underline{t}]$ so that

$$\|q_n^1 q_n^2\|_{L^{5/2}(I_n^{1, \Lambda}, L^5)} \leq \|q_n^1\|_{L^5(I_n^{1, \Lambda}, L^{10})} \leq C \|q_n^1 - v_{1, n}\|_{L^5(I_n^{1, \Lambda}, L^{10})} + \|v_{1, n}\|_{L^5(I_n^{1, \Lambda}, L^{10})}.$$

So, combining Theorem 2.1 and Lemma 2.3 yields

$$\overline{\lim}_n \|q_n^1 q_n^2\|_{L^{5/2}(I_n^{1, \Lambda}, L^5)} \xrightarrow{\Lambda \rightarrow \infty} 0.$$

The same result holds for $I_n^{3, \Lambda}$ and we are led with the interval $I_n^{2, \Lambda}$. In the case of time orthogonality, say $\frac{|t_n^2 - t_n^1|}{h_n} \xrightarrow{n \rightarrow \infty} +\infty$, the two intervals $[t_n^1 - \Lambda h_n, t_n^1 + \Lambda h_n]$ and $[t_n^2 - \Lambda h_n, t_n^2 + \Lambda h_n]$ have empty intersection for fixed Λ and n large enough, which yields the result by the same estimates applied to q_n^2 , once Λ is chosen large enough.

We can now assume, up to a translation in time, that $t_n^1 = t_n^2$. On $I_n^{2, \Lambda}$, Theorem 2.1 allows us to replace q_n^1 by $w_n^1(t, x) = \Psi_U^1(x) w^1(\frac{t-t_n^1}{h_n}, \frac{x-x_n^1}{h_n})$ on a coordinate patch where w^1 is a solution of a nonlinear wave equation on the tangent plane $T_{x_\infty^1} M$ and similarly for q_n^2 . In the first case of space orthogonality, that is $x_\infty^1 \neq x_\infty^2$, the result is obvious on the interval $I_n^{2, \Lambda}$ by taking Ψ_U^1 and Ψ_U^2 with empty intersection. In the case $x_\infty^1 = x_\infty^2$, we are left with the estimate of

$$\int_{I_n^2} \left(\int_{\mathbb{R}^3} |w_n^1(t, x) w_n^2(t, x)|^5 \right)^{1/2} ds \leq \int_{[-\Lambda, \Lambda]} \left(\int_{\mathbb{R}^3} \left| w^1(t, x) w^2 \left(t, x + \frac{x_n^1 - x_n^2}{h_n} \right) \right|^5 \right)^{1/2} ds.$$

This yields the result in the last case of space orthogonality by approximating w^1 and w^2 by compactly supported functions. \square

In the case of the sphere, we are able to state the same result without any restriction on the time.

Corollary 2.3. *Let $M = S^3$ and $T > 0$ (eventually large). We make the same assumptions as in Proposition 2.4, except for the time T , with the additional hypothesis:*

$[\underline{h}^{(i)}, (-1)^m \underline{x}^{(i)}, \underline{t}^{(i)} + m\pi]$ is orthogonal to $[\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$ for any $m \in \mathbb{Z}$ and $i \neq j$. Moreover, we assume $a(x) \equiv 0$ (undamped equation).

Then, the same conclusion as in Proposition 2.4 is true.

Proof. We build a covering of the interval $[-T, T]$ with a finite number of intervals of length strictly less than $T_{focus} = \pi$ so that on each of this interval $I = [\alpha, \beta]$ and for any $1 \leq i \leq l$, there exists at most one $m^{(i)} \in \mathbb{Z}$ such that $t_\infty^{(i)} + m^{(i)}\pi \in I$. Moreover, one can also impose $\alpha \neq t_\infty^{(i)} + m^{(i)}\pi$.

Therefore, $\alpha \in]t_n^{(i)} + (m^i - 1)\pi + \Lambda h_n^{(i)}, t_n^{(i)} + m^{(i)}\pi - \Lambda h_n^{(i)}[$ for large fixed Λ and n large enough. Theorem 2.2 yields $\|(q_n^{(i)} - v_n^{(i)}, \partial_t(q_n^{(i)} - v_n^{(i)}))_{t=\alpha}\|_{\mathcal{E}} \xrightarrow{n \rightarrow \infty} 0$ for a linear concentrating wave $v_n^{(i)} = [\tilde{S}^{m^{(i)}} S(\varphi^{(i)}, \psi^{(i)}), \underline{h}^{(i)}, \underline{x}^{(i)}, \underline{t}^{(i)}]$. In each interval, we are in the same situation as in Proposition 2.4 which yields the desired result. \square

Now, we are ready for the proof of the nonlinear profile decomposition. We give it in a quite sketchy way since it is very similar to the one of [2] or [17]. First, we obtain it in the particular case where the linear solution is small in Strichartz norm.

Lemma 2.13. *There exists $\delta_1 > 0$ such that if*

$$\overline{\lim}_{n \rightarrow \infty} \|v_n\|_{L^5([-T, T], L^{10})} \leq \delta_1$$

then the conclusion of Theorem 0.4 is true.

Proof. The proof is essentially the same as Lemma 4.3 of [2]. We have to estimate the rest $r_n^{(l)}$ solution of

$$\begin{cases} \square r_n^{(l)} + r_n^{(l)} + a(x)\partial_t r_n^{(l)} = \beta(u) + \sum_{j=1}^l \beta(q_n^{(j)}) - \beta\left(u + \sum_{j=1}^l q_n^{(j)} + w_n^{(l)} + r_n^{(l)}\right), \\ (r_n^{(l)}, \partial_t r_n^{(l)})_{t=0} = (0, 0). \end{cases}$$

We conclude as in [2] using Proposition 2.4 and Lemma 2.12 which is not immediate on a manifold. In the case of S^3 and $a \equiv 0$ for large T , we use Corollary 2.3 instead of Proposition 2.4. \square

Once the result is obtained when Strichartz norms are small, we divide $[-T, T]$ in a finite number of intervals where the Strichartz norms are small enough. This is done in the following lemma.

Lemma 2.14. *Let $2T < T_{focus}$. Let $\delta > 0$ and \tilde{q}_n be a sequence in $L^5([-T, T], L^{10}(M))$, such that*

$$\overline{\lim}_{n \rightarrow \infty} \|\tilde{q}_n\|_{L^5([-T, T], L^{10})} \leq \delta.$$

Fix also $l \in \mathbb{N}$ and l sequences of nonlinear concentrating waves $q_n^{(j)}$, $j = 1, \dots, l$.

Then, for any $\delta' > \delta$, there exists $L \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, we have the decomposition of $[-T, T]$ in closed intervals $I_n^{(i)}$

$$[-T, T] = \bigcup_{i=1}^L I_n^{(j)},$$

such that the sequence

$$\Gamma_n = \sum_{j=1}^l q_n^{(j)} + \tilde{q}_n$$

satisfies on each interval $I_n^{(i)}$

$$\overline{\lim}_{n \rightarrow \infty} \|\Gamma_n\|_{L^5(I_n^{(j)}, L^{10})} \leq \delta'.$$

Proof. We first treat the case $l = 1$. We divide $[-T, T] = I_n^{1, \Lambda} \cup I_n^{2, \Lambda} \cup I_n^{3, \Lambda}$ according to Theorem 2.1 (one of these intervals being possibly empty). Then, a combination of estimate (46) of Theorem 2.1 (comparison with linear concentrating wave) and Lemma 2.3 (non-reconcentration) gives for Λ large enough

$$\overline{\lim}_{n \rightarrow \infty} \|q_n^{(1)}\|_{L^5(I_n^{1, \Lambda}, L^{10})} \leq \delta' - \delta.$$

The same result holds for $I_n^{3, \Lambda}$ and we are left with the interval $I_n^{2, \Lambda}$. Once Λ is fixed, we can divide $[-\Lambda, \Lambda]$ in a finite number of intervals $I^{(i), \Lambda}$ such that $\|w\|_{L^5(I^{(i), \Lambda}, L^{10})} \leq \delta - \delta'$ where w is the function defined by Eq. (49) of Theorem 2.1. Then, we replace each $I^{(i), \Lambda}$ by $I_n^{(i), \Lambda}$ obtained by translation dilation. We conclude by the approximation (48) of $q_n^{(1)}$ by translation dilation of w on the interval $I_n^{2, \Lambda}$. \square

Note that the previous lemma also applies for large times on S^3 with $a \equiv 0$ by doing a first decomposition of $[-T, T]$ in a finite number of intervals of length strictly less than π .

End of the proof of Theorem 0.4 in the general case. We choose $l \in \mathbb{N}$ such that $\|w_n^{(l)}\| \leq \delta_1$ and use Lemma 2.14 in order to be able to apply Lemma 2.13 on each interval $I_n^{(j)}$. See [2] or in the different context of the Schrödinger equation [26]. \square

2.3. Applications

2.3.1. Strichartz estimates and Lipschitz bounds for the nonlinear evolution group

Proposition 2.5. *Let $T > 0$ be fixed. There exists a non-decreasing function, $A : [0, \infty[\rightarrow [0, \infty[$, such that any solution of*

$$\begin{cases} \square u + u + a(x)\partial_t u = -|u|^4 u & \text{on } [-T, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{E} \end{cases} \tag{54}$$

fulfills

$$\|u\|_{L^8([-T, T], L^8(M))} + \|u\|_{L^5([-T, T], L^{10}(M))} + \|u\|_{L^4([-T, T], L^{12}(M))} \leq A(\|(u_0, u_1)\|_{\mathcal{E}}).$$

Proof. The proof is exactly the same as Corollary 2 of [2]. Using Strichartz estimates, it is enough to get the result for $L^5 L^{10}$. We argue by contradiction and suppose that there exists a sequence u_n of strong solutions of Eq. (54) satisfying

$$\sup_n \|(u_{0,n}, u_{1,n})\|_{\mathcal{E}} < +\infty, \quad \|u_n\|_{L^5([-T, T], L^{10}(M))} \xrightarrow{n \rightarrow \infty} +\infty.$$

We apply the profile decomposition of Theorem 0.4 to our sequence. We get a contradiction by the fact that the $L^5([-T, T], L^{10}(M))$ norm of a nonlinear concentrating wave is uniformly bounded thanks to Corollary 2.2. This argument works for times $2T < T_{focus}$ and can be reiterated since the nonlinear energy at time T can be bounded with respect to the one at time 0 thanks to almost conservation (we can also use energy estimates once we know u is uniformly bounded in $L^5 L^{10}$). \square

Lemma 2.15. *Let $R_0 > 0$ and $T > 0$. Then, there exists $C > 0$ such any solution u satisfying*

$$\begin{cases} \square u + u + a(x)\partial_t u + |u|^4 u = 0 & \text{on } [-T, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{E}, \\ \|(u_0, u_1)\|_{\mathcal{E}} \leq R_0. \end{cases} \tag{55}$$

fulfills

$$\|(u(t), \partial_t u(t))\|_{L^2 \times H^{-1}} \leq C \|(u(0), \partial_t u(0))\|_{L^2 \times H^{-1}} \quad \forall t \in [-T, T].$$

Proof. Proposition 2.5 yields a uniform bound for u in $L^4([-T, T], L^{12}(M))$ and so for $V = |u|^4$ in $L^1([0, T], L^3(M))$. We prove uniform estimates for some solutions of the linear equation

$$\begin{cases} \square u + u + a(x)\partial_t u = Vu & \text{on } [-T, T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in L^2 \times H^{-1} \end{cases} \tag{56}$$

where V satisfies $\|V\|_{L^1([-T, T], L^3(M))} \leq A(R_0)^4$. The product of functions in $L^\infty([-T, T], L^2)$ and $L^1([-T, T], L^3)$ is in $L^1([-T, T], L^{6/5})$ and so in $L^1([-T, T], H^{-1})$ by Sobolev embedding. Standard estimates yield

$$\begin{aligned} \|(u, \partial_t u)\|_{L^\infty([0, t], L^2 \times H^{-1})} &\leq C \|(u(0), \partial_t u(0))\|_{L^2 \times H^{-1}} \\ &\quad + C(t + \|V\|_{L^1([0, t], L^3)}) \|(u, \partial_t u)\|_{L^\infty([0, t], L^2 \times H^{-1})}. \end{aligned}$$

We can divide the interval $[-T, T]$ into a finite number of intervals $[a_i, a_{i+1}]_{i=1, \dots, N}$ such that $C(t + \|V\|_{L^1([a_i, a_{i+1}], L^3(M))}) < 1/2$. N depends only on R_0 and T (not on V).

Then, on each of these intervals, we have

$$\|(u, \partial_t u)\|_{L^\infty([a_i, a_{i+1}], L^2 \times H^{-1})} \leq 2C \|(u(a_i), \partial_t u(a_i))\|_{L^2 \times H^{-1}}.$$

We obtain the expected result by iteration. The final constant C only depends on R_0 and T since it is also the case for N . \square

Corollary 2.4. *Let $R_0 > 0$ and $T > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution u satisfying (55) and $\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta$ satisfies*

$$\|(u(T), \partial_t u(T))\|_{L^2 \times H^{-1}} \leq \varepsilon.$$

We will also need the following lemma which states the local uniform continuity of the flow map. Note that it can be proved to be locally Lipschitz with a slightly more complicated argument (see Corollary 2 of [17]). We will not need this for our purpose.

Lemma 2.16. *Let u_n, \tilde{u}_n be two sequences of solutions of*

$$\begin{cases} \square u_n + u_n + |u_n|^4 u_n = g_n & \text{on } [-T, T] \times M, \\ (u_n, \partial_t u_n)_{t=0} = (u_{n,0}, u_{n,1}) & \text{bounded in } \mathcal{E}, \end{cases}$$

with $\|(u_{n,0} - \tilde{u}_{n,0}, u_{n,1} - \tilde{u}_{n,1})\|_{\mathcal{E}} + \|g_n - \tilde{g}_n\|_{L^1([-T, T], L^2)} \xrightarrow{n \rightarrow \infty} 0$. Then, we have

$$\|u_n - \tilde{u}_n\|_{[-T, T]} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. $r_n = u_n - \tilde{u}_n$ is a solution of

$$\begin{cases} \square r_n + r_n + |u_n|^4 u_n - |\tilde{u}_n|^4 \tilde{u}_n = g_n - \tilde{g}_n & \text{on } [-T, T] \times M, \\ (r_n, \partial_t r_n)_{t=0} = (u_{n,0} - \tilde{u}_{n,0}, u_{n,1} - \tilde{u}_{n,1}). \end{cases}$$

Using energy and Strichartz estimates, we get

$$\begin{aligned} \|r_n\|_{[-T, T]} &\leq C \|(u_{n,0} - \tilde{u}_{n,0}, u_{n,1} - \tilde{u}_{n,1})\|_{\mathcal{E}} + C \|g_n - \tilde{g}_n\|_{L^1([-T, T], L^2)} \\ &\quad + C \|r_n\|_{L^5([-T, T], L^{10})} (\|u_n\|_{L^5([-T, T], L^{10})}^4 + \|\tilde{u}_n\|_{L^5([-T, T], L^{10})}^4). \end{aligned}$$

Using Proposition 2.5, we can divide the interval $[-T, T]$ in a finite number of intervals $I_{i,n} = [a_{i,n}, a_{i+1,n}]$, $1 \leq i \leq N$, such that $C(\|u_n\|_{L^5(I_{i,n}, L^{10})}^4 + \|\tilde{u}_n\|_{L^5(I_{i,n}, L^{10})}^4) < 1/2$ so that the third term can be absorbed. We iterate this estimate N times, which gives the result. \square

2.3.2. Profile decomposition of the limit energy

For u solution of the nonlinear wave equation, we denote its nonlinear energy density by

$$e(u)(t, x) = \frac{1}{2} [|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 + |u(t, x)|^2] + \frac{1}{6} |u(t, x)|^6.$$

For a sequence u_n of solution with initial data bounded in \mathcal{E} , the corresponding nonlinear energy density is bounded in $L^\infty([-T, T], L^1)$ and so in the space of bounded measures on $[-T, T] \times M$. This allows to consider, up to a subsequence, its weak* limit.

The following theorem is the equivalent of Theorem 7 in [8]. It proves that the energy limit follows the same profile decomposition as u_n . It will be the crucial argument that will allow to use microlocal defect measure on each profile and then to apply the linearization argument.

Theorem 2.3. Assume $2T < T_{focus}$.

Let u_n be a sequence of solutions of

$$\square u_n + u_n + |u_n|^4 u_n = 0$$

with $(u_n, \partial_t u_n)(0)$ weakly convergent to 0 in \mathcal{E} .

The nonlinear energy density limit of u_n (up to subsequence) reads

$$e(t, x) = \sum_{j=1}^{+\infty} e^{(j)}(t, x) + e_f(t, x)$$

where $e^{(j)}$ is the limit energy limit density of $q_n^{(j)}$ (following the notation of Theorem 0.4) and

$$e_f = \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} e(w_n^{(l)})$$

where the two limits are considered up to a subsequence and in the weak* sense.

In particular, e_f can be written as

$$e_f(t, x) = \int_{\xi \in S_x^2} \mu(t, x, d\xi)$$

with

$$\mu(t, x, \xi) = \mu_-(G_t(x, \xi)) + \mu_+(G_{-t}(x, \xi))$$

where G_t is the flow map of the vector field $H_{|\xi|_x}$ on S^*M , that is the hamiltonian of the Riemannian metric.

Moreover, e is also the limit of the linear energy density

$$e_{lin}(u_n)(t, x) = \frac{1}{2} [|\partial_t u_n(t, x)|^2 + |\nabla u_n(t, x)|^2].$$

Proof. Proposition 2.5 yields $\|u_n\|_{L^8([-T, T] \times M)} \leq C$. Then, compact embedding and Lemma 2.15 yield $\|u_n\|_{L^2([-T, T] \times M)} \xrightarrow{n \rightarrow \infty} 0$ and so $\|u_n\|_{L^6([-T, T] \times M)} \xrightarrow{n \rightarrow \infty} 0$ by interpolation. Therefore, e is the limit of $b(u_n, u_n)$, with

$$b(f, g) = \partial_t f(t, x) \overline{\partial_t g(t, x)} + \nabla f(t, x) \cdot \overline{\nabla g(t, x)}.$$

Now, we have to compute the limit of $b(u_n, u_n)$ using decomposition (10) of Theorem 0.4. We set for any $l \in \mathbb{N}$

$$s_n^{(l)} = \sum_{j=1}^l q_n^{(j)}$$

and so

$$b(u_n, u_n) = b(s_n^{(l)}, s_n^{(l)}) + b(w_n^{(l)}, w_n^{(l)}) + 2b(s_n^{(l)}, w_n^{(l)}) + 2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}).$$

Because of (11), $\overline{\lim}_{n \rightarrow \infty} \|2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)})\|_{L^1([-T, T] \times M)}$ converges to zero as l tends to infinity. So, if we define $e_r^{(l)} = w^* \lim_{n \rightarrow \infty} (2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}))$, we have

$$\|e_r^{(l)}\|_{TV} \xrightarrow{l \rightarrow \infty} 0.$$

Let $\varphi \in C_0^\infty([-T, T] \times M)$. For fixed l , it remains to estimate

$$\iint_{[-T, T] \times M} \varphi b(s_n^{(l)}, w_n^{(l)}) = \sum_{j=1}^l \iint_{[-T, T] \times M} \varphi b(q_n^{(j)}, w_n^{(l)}).$$

Since $b(q_n^{(j)}, w_n^{(l)})$ is bounded in $L^\infty([-T, T], L^1)$, we can assume, up to an error arbitrary small, that φ is supported in $\{t < t_\infty^{(j)}\}$ or $\{t > t_\infty^{(j)}\}$ (replace φ by $(1 - \Psi)(t)\varphi$ with $\Psi(t_\infty^{(j)}) = 1$ and $\|\Psi\|_{L^1([-T, T])}$ small). On each interval, Theorem 2.1 allows to replace $q_n^{(j)}$ by a linear concentrating wave. Then, we combine Lemma 2.11 and Lemma 1.11 to get the weak convergence to zero of $b(s_n^{(l)}, w_n^{(l)})$ for fixed l .

Lemma 2.10 and the orthogonality of the cores of concentration give $D_h^{(j)}(p_h^{(j)})$, $\partial_t p_h^{(j')}(t_h^{(j)}) \rightharpoonup (0, 0)$ for $j \neq j'$ and $p_h^{(j')}$ a concentrating wave at rate $[h^{(j')}, t^{(j')}, x^{(j')}]$. Then, the same argument as before yields

$$b(s_n^{(l)}, s_n^{(l)}) \xrightarrow{n \rightarrow \infty} \sum_{j=1}^l e^{(j)}.$$

So we have proved that for any $l \in \mathbb{N}$

$$b(u_n, u_n) \xrightarrow{n \rightarrow \infty} e = \sum_{j=1}^l e^{(j)} + e_w^{(l)} + e_r^{(l)}$$

where $e_w^{(l)}$ is the weak* limit of $b(w_n^{(l)}, w_n^{(l)})$ and $e_r^{(l)}$ satisfies $\|e_r^{(l)}\|_{TV} \xrightarrow{l \rightarrow \infty} 0$. $e_w^{(l)}$ is the weak* limit of a sequence of solutions of the linear wave equation weakly convergent to 0 in energy space. Therefore, it has the announced form using the link with microlocal defect measure (see Lemma 1.9).

We get the final result by letting l tend to infinity. \square

Remark 2.4. The fact that $|u_n|^6$ is weakly convergent to 0 is false if we consider the limit in $\mathcal{D}'(M)$ time by time. For example, for a nonlinear concentrating wave with $t_n = 0$, the weak limit in $\mathcal{D}'([-T, T] \times M)$ of $|u_n|^6$ is of course still zero but the weak limit of $|u_n|^6(t)$ in $\mathcal{D}'(M)$ is zero if $t \neq 0$ and a multiple of a Dirac function if $t = 0$. So the limit in $\mathcal{D}'(M)$ of $e_n|_{t=0}$ is not the same as the one of $b(u_n, u_n)|_{t=0}$. This comes from the fact that the limit of $b(u_n, u_n)(t)$ is not equicontinuous as a function of t while it is the case for the nonlinear energy. Yet, in the proof, we are only interested in its limit in the space–time distributional sense which will be continuous. Actually, the discontinuity at $t = 0$ of the limit of $b(u_n, u_n)(t)$ can be described explicitly from the scattering operator. At the contrary, the fact that the nonlinear energy density $e(t)$ is continuous in time can, in this case, be seen as a consequence of the conservation of the nonlinear energy of the scattering operator.

3. Control and stabilization

3.1. Weak observability estimates, stabilization

3.1.1. Why Klein–Gordon and not the wave?

In this subsection, we prove that the expected observability estimate

$$E(u)(0) \leq C \iint_{[0, T] \times M} |a \partial_t u|^2 dt dx$$

does not hold for the nonlinear damped wave equation $\square u + \partial_t u + u^5 = 0$ (in the simpler case $a \equiv 1$), even for small data. It explains why we have chosen the Klein–Gordon equation instead. The main point is that for small data, the nonlinear solution is close to the linear one which has the constants (in space–time) as undamped solutions (which is obviously false for $\square u + u = 0$).

We take $a \equiv 1$ and initial data constant equal to $(\varepsilon, 0)$. The nonlinear wave equation takes the form of the following ODE

$$\begin{cases} \ddot{u} + \dot{u} + u^5 = 0 & \text{on } [0, T], \\ (u(0), \dot{u}(0)) = (\varepsilon, 0). \end{cases}$$

Decreasing of energy yields for any $t \geq 0$

$$E(t) = \frac{1}{2}\dot{u}^2 + \frac{1}{6}u^6(t) \leq E(0) = \frac{1}{6}\varepsilon^6$$

and so

$$|u(t)| \leq \varepsilon \quad \forall t \geq 0.$$

Then, $c = \dot{u}$ is a solution of

$$\begin{cases} \dot{c} + c + u^5 = 0 & \text{on } [0, T], \\ c(0) = 0. \end{cases}$$

Therefore,

$$c(t) = - \int_0^t e^{-(t-s)} u^5(s) ds \quad \text{and}$$

$$|\dot{u}(t)| = |c(t)| \leq \varepsilon^5.$$

For any $T > 0$, we have

$$\int_0^T |\dot{u}(s)|^2 \leq T \varepsilon^{10}.$$

Therefore, the observability estimate

$$T \varepsilon^{10} \geq \int_0^T |\dot{u}(s)|^2 \geq C E(0) = C \frac{1}{6} \varepsilon^6$$

cannot hold if ε is taken small enough.

3.1.2. Weak observability estimate

As explained in the introduction, the proof of stabilization consists in the analysis of possible sequences contradicting an observability estimate. The first step is to prove that such sequence is linearizable in the sense that its behavior is close to solutions of the linear equation.

Proposition 3.1. *Let ω satisfy Assumption 0.2 and $a \in C^\infty(M)$ satisfy $a(x) > \eta > 0$ for all $x \in \omega$. Let $T > T_0$ and u_n be a sequence of solutions of*

$$\begin{cases} \square u_n + u_n + |u_n|^4 u_n + a(x)^2 \partial_t u_n = 0 & \text{on } [0, T] \times M, \\ (u_n, \partial_t u_n)_{t=0} = (u_{0,n}, u_{1,n}) \in \mathcal{E} \end{cases} \tag{57}$$

satisfying

$$\begin{aligned}
 (u_{0,n}, u_{1,n}) &\xrightarrow{n \rightarrow \infty} 0 \quad \text{weakly in } \mathcal{E}, \\
 \iint_{[0,T] \times M} |a(x) \partial_t u_n|^2 dt dx &\xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}
 \tag{58}$$

Then, u_n is linearizable on $[0, t]$ for any $t < T - T_0$, that is

$$\|u_n - v_n\|_{[0,t]} \xrightarrow{n \rightarrow \infty} 0$$

where v_n is the solution of

$$\begin{cases} \square v_n = 0 & \text{on } [0, T] \times M, \\ (v_n, \partial_t v_n)_{t=0} = (u_{0,n}, u_{1,n}). \end{cases}$$

Proof. Denote $t_* = \sup\{s \in [0, T] \mid \lim_{n \rightarrow \infty} \|u_n - v_n\|_{[0,s]} = 0\}$ and we have to prove $t_* \geq T - T_0$. If it is not the case, we can find an interval $[t_* - \varepsilon, t_* - \varepsilon + L] \subset [0, T]$ with $T_0 < L < T_{focus}$ and $0 < 2\varepsilon < L - T_0$ (if $t_* = 0$, take the interval $[0, L] \subset [0, T]$). Then, Lemma 3.1 below gives that u_n is linearizable on $[t_* - \varepsilon, t_* + \varepsilon]$. We postpone the proof of Lemma 3.1 and finish the proof of the proposition. The definition of t_* gives $\lim_{n \rightarrow \infty} \|u_n - v_n\|_{[0, t_* - \varepsilon]} = 0$ and we have proved that $\lim_{n \rightarrow \infty} \|u_n - \tilde{v}_n\|_{[t_* - \varepsilon, t_* + \varepsilon]} = 0$ where \tilde{v}_n is a solution of

$$\square \tilde{v}_n = 0; \quad (\tilde{v}_n, \partial_t \tilde{v}_n)_{t=t_* - \varepsilon} = (u_n, \partial_t u_n)_{t=t_* - \varepsilon}.$$

Since the norm $\|\cdot\|$ controls the energy norm, this easily yields $\lim_{n \rightarrow \infty} \|u_n - v_n\|_{[0, t_* + \varepsilon]} = 0$ which is a contradiction to the definition of t_* . \square

Lemma 3.1. *With the assumptions of Proposition 3.1. Consider the profile decomposition according to Theorem 0.4 of u_n on a subinterval $[t_0, t_0 + L] \subset [0, T]$ with $T_0 < L < T_{focus}$.*

Then, for any $0 < \varepsilon < L - T_0$, this decomposition does not contain any nonlinear concentrating wave with $t_\infty^{(j)} \in [t_0, t_0 + \varepsilon]$ and u_n is linearizable on $[t_0, t_0 + \varepsilon]$.

Proof. To simplify the notation, we work on the interval $[0, L]$. Moreover, since $a(x) \partial_t u_n$ tends to 0 in $L^1 L^2$, Lemma 2.16 allows to assume with the same assumptions that u_n is a solution of the nonlinear equation without damping. Proposition 2.5 and Lemma 2.15 (with the Rellich Theorem) give that u_n is bounded in $L^8([0, T] \times M)$ and convergent to 0 in $L^2([0, T] \times M)$. Therefore, u_n tends to 0 in $L^7([0, T] \times M)$ and so $|u_n|^4 u_n$ is convergent to 0 in $L^{7/5}([0, T] \times M) \hookrightarrow L^{4/3}([0, T] \times M) \hookrightarrow H_{loc}^{-1}([0, l] \times M)$. Then, if we consider the (space-time) microlocal defect measure of u_n , the elliptic regularity and the equation verified by u_n gives that μ is supported in $\{\tau^2 = |\xi|_x^2\}$ as in the linearizable case. So, combining this with (58), we get

$$u_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } H_{loc}^1([0, L] \times \omega).$$

Using the notation of Theorem 2.3, this gives $e = 0$ on $]0, L[\times \omega$. Since all the measures in the decomposition of e are positive, we get the same result for any nonlinear concentrating wave in the decomposition of u_n , that is

$$q_n^{(j)} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } H_{loc}^1([0, L] \times \omega)$$

and if $\mu^{(j)}$ is the microlocal defect measure of $q_n^{(j)}$, we have

$$\mu^{(j)} \equiv 0 \quad \text{in } S^*(]0, L[\times \omega). \tag{59}$$

Assume that $t_\infty^{(j)} \in [0, \varepsilon]$ for one $j \in \mathbb{N}$, so that the interval $[t_\infty^{(j)}, L]$ has length greater than T_0 . Denote by $p_n^{(j)}$ the linear concentrating wave approaching $q_n^{(j)}$ in the interval $I_n^{3,A}$ according to the notation of Theorem 2.1, so that for any $t_\infty^{(j)} < t < L$ (here we use the fact that $L < T_{focus}$), we have

$$\|q_n^{(j)} - p_n^{(j)}\|_{[t,L]} \xrightarrow{n \rightarrow \infty} 0.$$

In particular, $\mu^{(j)}$ is also attached to $p_n^{(j)}$ on the time interval $]t_\infty^{(j)}, L]$. Since $p_n^{(j)}$ is a solution of the linear wave equation, its measure propagates along the hamiltonian flow. Assumption 0.2 and $|L - t_\infty^{(j)}| > T_0$ ensure that the geometric control condition is still verified on the interval $[t_\infty^{(j)}, L]$ which gives $\mu^{(j)} \equiv 0$ when combined with (59). This means that $p_n^{(j)} \equiv 0$ and so $q_n^{(j)} \equiv 0$ as expected.

Then, for the profile decomposition of u_n on the interval $[0, L]$ (here the weak limit u is necessarily zero)

$$u_n = \sum_{j=1}^l q_n^{(j)} + w_n^{(l)} + r_n^{(l)},$$

we have proved that $t_n^{(j)} \in]\varepsilon, L]$. Then Theorem 2.1 and $L < T_{focus}$ provide a linear concentrating wave $p_n^{(j)}$ such that $\overline{\lim}_{n \rightarrow \infty} \|q_n^{(j)} - p_n^{(j)}\|_{[0,\varepsilon]} = 0$ while Lemma 2.3 gives $\overline{\lim}_{n \rightarrow \infty} \|p_n^{(j)}\|_{L^5([0,\varepsilon], L^{10})} = 0$. Moreover, the conclusion of Theorem 2.1 gives $\overline{\lim}_{n \rightarrow \infty} \|w_n^{(l)} + r_n^{(l)}\|_{L^5([0,\varepsilon], L^{10})} \xrightarrow{l \rightarrow \infty} 0$. This finally yields $\overline{\lim}_{n \rightarrow \infty} \|u_n\|_{L^5([0,\varepsilon], L^{10})} = 0$ and therefore

$$\| |u_n|^4 u_n \|_{L^1([0,\varepsilon], L^2)} \xrightarrow{n \rightarrow \infty} 0.$$

This gives exactly that u_n is linearizable on $[0, \varepsilon]$. \square

We are now ready for the proof of some weak observability estimates. We recall the notation $E(u)$ for the nonlinear energy defined in (2).

Theorem 3.1. *Let ω satisfy Assumption 0.2 with T_0 and $a \in C^\infty(M)$ satisfy $a(x) > \eta > 0$ for all $x \in \omega$. Let $T > 2T_0$ and $R_0 > 0$. Then, there exists $C > 0$ such that for any u solution of*

$$\begin{cases} \square u + u + |u|^4 u + a^2(x) \partial_t u = 0 & \text{on } [0, T] \times M, \\ (u, \partial_t u)_{t=0} = (u_0, u_1) \in \mathcal{E}, \\ \|(u_0, u_1)\|_{\mathcal{E}} \leq R_0 \end{cases} \tag{60}$$

satisfies

$$E(u)(0) \leq C \left(\iint_{[0,T] \times M} |a(x)\partial_t u|^2 dt dx + \|(u_0, u_1)\|_{L^2 \times H^{-1}} E(u)(0) \right).$$

Proof. We argue by contradiction: we suppose that there exists a sequence u_n of solutions of (60) such that

$$\left(\iint_{[0,T] \times M} |a(x)\partial_t u_n|^2 dt dx + \|(u_{0,n}, u_{1,n})\|_{L^2 \times H^{-1}} E(u_n)(0) \right) \leq \frac{1}{n} E(u_n)(0).$$

Denote $\alpha_n = (E(u_n)(0))^{1/2}$. By Sobolev embedding for the L^6 norm, we have $\alpha_n \leq C(R_0)$. So, up to extraction, we can assume that $\alpha_n \rightarrow \alpha \geq 0$.

We will distinguish two cases: $\alpha > 0$ and $\alpha = 0$.

- First case: $\alpha_n \rightarrow \alpha > 0$.

The second part of the estimate gives $\|(u_{0,n}, u_{1,n})\|_{L^2 \times H^{-1}} \xrightarrow{n \rightarrow \infty} 0$ and so $(u_{0,n}, u_{1,n}) \xrightarrow{n \rightarrow \infty} 0$ in $H^1 \times L^2$. Therefore, we are in position to apply Proposition 3.1 and get that u_n is linearizable on an interval $[0, L]$ with $L > T_0$. We get a contradiction to $\alpha > 0$ by applying the following classical linear proposition, which can be easily proved using microlocal defect measure as in Lemma 3.1.

Proposition 3.2. *Let ω satisfy Assumption 0.2 with T_0 . Let $T > T_0$ and v_n be a sequence of solutions of*

$$\begin{cases} \square v_n = 0 & \text{on } [0, T] \times M, \\ (v_n(0), \partial_t v_n(0)) \xrightarrow{n \rightarrow \infty} 0 & \text{in } \mathcal{E} \end{cases}$$

satisfying

$$\iint_{[0,T] \times M} |a(x)\partial_t v_n|^2 dt dx \xrightarrow{n \rightarrow \infty} 0.$$

Then, $(v_n(0), \partial_t v_n(0)) \xrightarrow{n \rightarrow \infty} 0$ for the strong topology of $H^1 \times L^2$. The same result holds with $\square u_n$ replaced by $\square u_n + u_n$.

- Second case: $\alpha_n \rightarrow 0$.

Let us make the change of unknown $w_n = u_n/\alpha_n$. w_n is a solution of the system

$$\square w_n + a^2(x)\partial_t w_n + w_n + \alpha_n^4 |w_n|^4 w_n = 0 \tag{61}$$

and

$$\iint_{[0,T] \times M} |a(x) \partial_t w_n|^2 dt dx \leq \frac{1}{n}.$$

We have for a large constant $C > 0$ depending on R_0 and for all $t \in [0, T]$,

$$\frac{1}{C} \|(u_n, \partial_t u_n)\|_{\mathcal{E}}^2 \leq E(u_n) \leq C \|(u_n, \partial_t u_n)\|_{\mathcal{E}}^2.$$

Therefore, we have

$$\begin{aligned} \|(w_n(t), \partial_t w_n(t))\|_{\mathcal{E}} &= \frac{\|(u_n(t), \partial_t u_n(t))\|_{\mathcal{E}}}{\sqrt{E(u_n(0))}} \leq C \frac{\sqrt{E(u_n(t))}}{\sqrt{E(u_n(0))}} \leq C, \\ \|(w_n(0), \partial_t w_n(0))\|_{\mathcal{E}} &= \frac{\|(u_n(0), \partial_t u_n(0))\|_{\mathcal{E}}}{\sqrt{E(u_n(0))}} \geq \frac{1}{\sqrt{C}} > 0. \end{aligned} \tag{62}$$

Thus, we have $\|(w_n(0), \partial_t w_n(0))\|_{\mathcal{E}} \approx 1$ and $(w_n, \partial_t w_n)$ is bounded in $L^\infty([0, T], \mathcal{E})$.

Applying Strichartz estimates to Eq. (61), we get for $C = C(R_0) > 0$

$$\|w_n\|_{L^5([0,T], L^{10})} \leq C(1 + \alpha_n^4 \|w_n\|_{L^5([0,T], L^{10})}^5).$$

Then, using a bootstrap argument, we deduce that $\|w_n\|_{L^5([0,T], L^{10})}$ is bounded and therefore

$$\square w_n + w_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } L^1([0, T], L^2).$$

Proposition 3.2 yields that w_n converges strongly to some w solution of

$$\square w + w = 0; \quad \partial_t w \equiv 0 \quad \text{on } [0, T] \times \omega. \tag{63}$$

We deduce as in J. Rauch and M. Taylor [33] or C. Bardos, G. Lebeau, J. Rauch [3] that the set G_T of solutions in \mathcal{E} fulfilling (63) is finite dimensional. This is also the case for some $G_{T'}$ with $T' < T$. So, for $W = (w_0, w_1) \in G_T$ and ε small enough, $(e^{\varepsilon A} W - W)/\varepsilon$ is also in $G_{T'}$, where e^{tA} is the Klein–Gordon semi-group. By equivalence of the norms in $G_{T'}$, we get $AW \in \mathcal{E}$ (note that we could have proved directly that G_T only contains smooth functions by propagation of regularity and geometric control condition). Then, A and indeed $A^2 = \Delta - I$ maps G_T into itself and so admits an eigenvector W . By unique continuation for second-order elliptic operator, we get $\partial_t w \equiv 0$ for w the associated solution. Multiplying the equation by \bar{w} and integrating, we obtain $w \equiv 0$ (note that, at this stage, the choice of the Klein–Gordon equation instead of the wave equation is crucial to avoid the constant solutions). We conclude that $(w_n(0), \partial_t w_n(0))$ tends to 0 strongly in \mathcal{E} which gives a contradiction to (62). \square

3.2. Controllability

3.2.1. Linear control

In this subsection, we recall some well-known results about linear control theory and HUM method. Let $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$. We solve the system

$$\begin{cases} \square\Phi + \Phi = 0 & \text{on } [0, T] \times M, \\ (\Phi, \partial_t\Phi)|_{t=0} = (\Phi_0, \Phi_1) \end{cases} \tag{64}$$

and

$$\begin{cases} \square v + v = a^2\Phi & \text{on } [0, T] \times M, \\ (v, \partial_t v)|_{t=T} = (0, 0). \end{cases} \tag{65}$$

The HUM operator S from $L^2 \times H^{-1}$ to $L^2 \times H^1$ is defined by

$$S(\Phi_0, \Phi_1) = (-\partial_t v(0), v(0)).$$

Lemma 3.2. *If ω satisfies the geometric control Assumption 0.1, then S is an isomorphism.*

Proof. Multiplying Eq. (65) by $\bar{\Phi}$, integrating over $[0, T] \times M$ and integrating by part, we get the formula

$$\int_0^T \int_M |a\Phi|^2 = - \int_M \partial_t v(0)\bar{\phi}(0) + \int_M v(0)\partial_t \bar{\phi}(0) = \langle S(\Phi_0, \Phi_1), (\Phi_0, \Phi_1) \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^2 \times H^1$ and $L^2 \times H^{-1}$. We get the conclusion thanks to the following observability estimate which can be proved by the same techniques used in the nonlinear problem

$$\|(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_M |a\Phi|^2. \quad \square$$

3.2.2. Controllability for small data

Theorem 3.2. *Let ω satisfy Assumption 0.1 and $T > T_0$. Then, there exists $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq \delta; \quad \|(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{E}} \leq \delta$$

there exists $g \in L^\infty([0, 2T], L^2)$ supported in $[0, 2T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \square u + u + |u|^4 u = g & \text{on } [0, 2T] \times M, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

satisfies $(u(2T), \partial_t u(2T)) = (\tilde{u}_0, \tilde{u}_1)$.

Proof. The proof is very similar to [11] except that the critical exponent do not allow to use compactness argument and we use the classical Picard fixed point instead of Schauder, as done in [9] or [29], [30] for NLS. By a compactness argument, we can select $a \in C_0^\infty(\omega)$ with $a(x) > \eta > 0$ for x in $\tilde{\omega}$ where $\tilde{\omega}$ satisfies Assumption 0.1. Since the equation is reversible, we can assume $(\tilde{u}_0, \tilde{u}_1) \equiv (0, 0)$ and take the time T instead of $2T$. We seek g of the form $a^2(x)\Phi$ where Φ is a solution of the free wave equation as in linear control theory with initial datum $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$. The purpose will be to choose the right $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$ to get the expected data. We consider the solutions of the two systems

$$\begin{cases} \square\Phi + \Phi = 0 & \text{on } [0, T] \times M, \\ (\Phi, \partial_t\Phi)|_{t=0} = (\Phi_0, \Phi_1) \end{cases}$$

and

$$\begin{cases} \square u + u + |u|^4 u = a^2\Phi & \text{on } [0, T] \times M, \\ (u, \partial_t u)|_{t=T} = (0, 0). \end{cases} \tag{66}$$

Let us define the operator

$$\begin{aligned} L : L^2 \times H^{-1} &\rightarrow H^1 \times L^2, \\ (\Phi_0, \Phi_1) &\mapsto L(\Phi_0, \Phi_1) = (u, \partial_t u)|_{t=0}. \end{aligned} \tag{67}$$

We split $u = v + \Psi$ with Ψ solution of

$$\begin{cases} \square\Psi + \Psi = a^2\Phi & \text{on } [0, T] \times M, \\ (\Psi, \partial_t\Psi)|_{t=T} = (0, 0). \end{cases} \tag{68}$$

This corresponds to the linear control, and $(-\partial_t\Psi, \Psi)|_{t=0} = S(\Phi_0, \Phi_1)$. As for function v , it is a solution of

$$\begin{cases} \square v + v = -|u|^4 u & \text{on } [0, T] \times M, \\ (v, \partial_t v)|_{t=T} = (0, 0). \end{cases} \tag{69}$$

Φ belongs to $C([0, T], L^2)$. So, u, v and Ψ belong to $C([0, T], H^1) \cap C^1([0, T], L^2) \cap L^5([0, T], L^{10})$. We can write

$$L(\Phi_0, \Phi_1) = K(\Phi_0, \Phi_1) + S(\Phi_0, \Phi_1)$$

where $K(\Phi_0, \Phi_1) = (-\partial_t v, v)|_{t=0}$. $L(\Phi_0, \Phi_1) = (-u_1, u_0)$ is equivalent to $(\Phi_0, \Phi_1) = -S^{-1}K(\Phi_0, \Phi_1) + S^{-1}(-u_1, u_0)$. Defining the operator $B : L^2 \times H^{-1} \rightarrow L^2 \times H^{-1}$ by

$$B(\Phi_0, \Phi_1) = -S^{-1}K(\Phi_0, \Phi_1) + S^{-1}(-u_1, u_0),$$

the problem $L(\Phi_0, \Phi_1) = (-u_1, u_0)$ is equivalent to finding a fixed point of B . We will prove that if $\|(u_0, u_1)\|_{\mathcal{E}}$ is small enough, B is a contraction and reproduces a small ball B_R of $L^2 \times H^{-1}$.

Since S is an isomorphism, we have

$$\|B(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} \leq C(\|K(\Phi_0, \Phi_1)\|_{L^2 \times H^1} + \|(u_0, u_1)\|_{\mathcal{E}}).$$

So we are led to estimate $\|K(\Phi_0, \Phi_1)\|_{L^2 \times H^1} = \|(v, \partial_t v)|_{t=0}\|_{\mathcal{E}}$. Energy estimates applied to Eq. (69) and the Hölder inequality give

$$\|(v, \partial_t v)|_{t=0}\|_{\mathcal{E}} \leq C\| |u|^4 u \|_{L^1([0, T], L^2)} \leq C\|u\|_{L^5([0, T], L^{10})}^5.$$

But Strichartz estimates applied to Eq. (66) give

$$\begin{aligned} \|u\|_{L^5([0, T], L^{10})} &\leq C(\|a^2 \Phi\|_{L^1([0, T], L^2)} + \|u\|_{L^5([0, T], L^{10})}^5) \\ &\leq C(\|(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} + \|u\|_{L^5([0, T], L^{10})}^5). \end{aligned}$$

Using a bootstrap argument, we get that for $\|(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} \leq R$ small enough, we have

$$\|u\|_{L^5([0, T], L^{10})} \leq C\|(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}}. \tag{70}$$

We finally obtain

$$\|B(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} \leq C(\|(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}}^5 + \|(u_0, u_1)\|_{\mathcal{E}}).$$

Choosing R small enough and $\|(u_0, u_1)\|_{H^1 \times L^2} \leq R/2C$, we obtain $\|B(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} \leq R$ and B reproduces the ball B_R . Let us now prove that B is contracting. We examine the systems

$$\begin{cases} \square(u - \tilde{u}) + (u - \tilde{u}) + |u|^4 u - |\tilde{u}|^4 \tilde{u} = a^2(\Phi - \tilde{\Phi}) & \text{on } [0, T] \times M, \\ (u - \tilde{u}, \partial_t(u - \tilde{u}))|_{t=T} = (0, 0), \end{cases} \tag{71}$$

$$\begin{cases} \square(v - \tilde{v}) + (v - \tilde{v}) + |u|^4 u - |\tilde{u}|^4 \tilde{u} = 0 & \text{on } [0, T] \times M, \\ (v - \tilde{v}, \partial_t(v - \tilde{v}))|_{t=T} = (0, 0). \end{cases} \tag{72}$$

We obtain similarly

$$\begin{aligned} &\|B(\Phi_0, \Phi_1) - B(\tilde{\Phi}_0, \tilde{\Phi}_1)\|_{L^2 \times H^{-1}} \\ &\leq C\|(v - \tilde{v}, \partial_t(v - \tilde{v}))|_{t=0}\|_{\mathcal{E}} \\ &\leq C\| |u|^4 u - |\tilde{u}|^4 \tilde{u} \|_{L^1([0, T], L^2)} \\ &\leq C\|u - \tilde{u}\|_{L^5([0, T], L^{10})} (\|u\|_{L^5([0, T], L^{10})}^4 + \|\tilde{u}\|_{L^5([0, T], L^{10})}^4) \\ &\leq CR^4\|u - \tilde{u}\|_{L^5([0, T], L^{10})} \end{aligned} \tag{73}$$

where we have used estimate (70) for the last inequality. Applying Strichartz estimates to Eq. (71), we get

$$\begin{aligned} \|u - \tilde{u}\|_{L^5([0,T],L^{10})} &\leq C(\| |u|^4 u - |\tilde{u}|^4 \tilde{u} \|_{L^1([0,T],L^2)} + \|a^2(\Phi - \tilde{\Phi})\|_{L^1([0,T],L^2)}) \\ &\leq CR^4 \|u - \tilde{u}\|_{L^5([0,T],L^{10})} + C\|(\Phi_0, \Phi_1) - (\tilde{\Phi}_0, \tilde{\Phi}_1)\|_{L^2 \times H^{-1}}. \end{aligned}$$

If R is taken small enough, it yields

$$\|u - \tilde{u}\|_{L^5([0,T],L^{10})} \leq C\|(\Phi_0, \Phi_1) - (\tilde{\Phi}_0, \tilde{\Phi}_1)\|_{L^2 \times H^{-1}}. \tag{74}$$

Combining (73) and (74), we finally obtain for R small enough

$$\|B(\Phi_0, \Phi_1) - B(\tilde{\Phi}_0, \tilde{\Phi}_1)\|_{L^2 \times H^{-1}} \leq CR^4\|(\Phi_0, \Phi_1) - (\tilde{\Phi}_0, \tilde{\Phi}_1)\|_{L^2 \times H^{-1}}$$

and B is a contraction for R small enough, which completes the proof of Theorem 3.2. \square

3.2.3. Controllability of high frequency data

This subsection is devoted to the proof of the two main theorems of the article: Theorems 0.2 and 0.1.

Proof of Theorem 0.2. First, by decreasing of the energy and Sobolev embedding, there exists some constant $C(R_0)$ such that the assumption $\|(u_0, u_1)\|_{\mathcal{E}} \leq R_0$ implies

$$E(u)(t) \leq C(R_0) \quad \text{and} \quad \|(u, \partial_t u)(t)\|_{\mathcal{E}} \leq C(R_0); \quad \forall t \geq 0. \tag{75}$$

Fix T such that Theorem 3.1 applies. Then, there exists $\varepsilon > 0$ such that for any (u_0, u_1) satisfying

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq C(R_0); \quad \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \varepsilon, \tag{76}$$

we have the strong observability estimate

$$E(u)(0) \leq C \iint_{[0,T] \times M} |a(x)\partial_t u|^2 dt dx,$$

for any solution of the damped equation (3). This means that there exists $0 < C$ such that any solution of the damped equation satisfying (76) fulfills

$$E(u)(T) \leq (1 - C)E(u)(0). \tag{77}$$

Pick $N \in \mathbb{N}$ large enough such that $(1 - C)^N C(R_0) \leq \varepsilon^2/2$.

Corollary 2.4 and (75) allow us to choose δ small enough such that the assumptions

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq R_0; \quad \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta$$

imply

$$\|(u(nT), \partial_t u(nT))\|_{L^2 \times H^{-1}} \leq \varepsilon, \quad 0 \leq n \leq N. \tag{78}$$

So, with that choice, we have $E(u)(NT) \leq (1 - C)^N E(u)(0)$. Then, by the energy decreasing, for any $t \geq NT$, we have

$$\|(u, \partial_t u)(t)\|_{L^2 \times H^{-1}}^2 \leq 2E(u)(t) \leq 2E(u)(NT) \leq \varepsilon^2.$$

Therefore, the decay estimate (77) is true on each interval $[nT, (n + 1)T]$, $n \in \mathbb{N}$ and we have

$$E(u)(nT) \leq (1 - C)^n E(u)(0)$$

which yields the result. \square

Proof of Theorem 0.1. Since the equation is reversible, we can assume $(\tilde{u}_0, \tilde{u}_1) = (0, 0)$. By a compactness argument, we can select $a \in C_0^\infty(\omega)$ with $a(x) > \eta > 0$ for x in $\tilde{\omega}$ where $\tilde{\omega}$ satisfies Assumption 0.2. We will first use the damping term $a(x)^2 \partial_t u$ as a term of control. We apply Theorem 0.2 and Theorem 3.2 once the energy is small enough. \square

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