THE DISTRIBUTION OF VALUES OF TRIGONOMETRIC SUMS WITH LINEARLY INDEPENDENT FREQUENCIES:
KAC'S PROBLEM REVISITED

Richard BARAKAT
Division of Applied Sciences, Harvard University, Cambridge, MA 02138, U.S.A.
Bolt Beranek and Newman Inc., Cambridge, MA 02138, U.S.A.

Received 20 September 1977
Revised 9 March 1978

The average frequency with which a trigonometric sum having linearly independent frequencies achieves a given value in a specified time interval was solved by Kac in the form of a double integral. In the present paper, Kac's solution is generalized by allowing the amplitudes of the sinusoidal terms to be statistically independent random variables possessing compact probability density functions. The average frequency is expressed as a Fourier cosine series. Representative numerical calculations are shown. The asymptotic form of the average frequency is also discussed.

### 1. Introduction

It was Kac's considerable achievement [1] to derive an explicit expression for the average frequency with which the trigonometric sum

\[ x(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \alpha_n) \]  

(1.1)

achieves a given value \( x_0 \) in a specified time interval. The amplitude \( a_n > 0 \), initial phases \( \alpha_n \), and linearly independent frequencies \( \omega_n \) are real; the time \( t \) ranges over the infinite interval. Kac obtained the average frequency in the form of a double integral with a very complicated, oscillatory integrand. Although he evidently made no attempt to evaluate this integral, he produced a valuable asymptotic approximation valid for large \( N \).

The present communication is devoted to an extension of Kac's analysis by allowing the amplitudes \( a_n \) to be statistically independent random variables. Each \( a_n \) is governed by a probability density function \( f_{a_n}(a_n) \) that is required to vanish identically for \( a_n \geq a_n^{(\text{max})} \) (\( = \) some specified finite constant). The average frequency
is again obtained in the form of a double integral with an even more complicated, oscillatory integrand. The fact that the probability density functions of the $a_n$ are compact permits the use of cardinal function (sampling) expansions of the bivariate characteristic functions associated with $x(t)$ and its time derivative. The final result is that the average frequency is expressed as a Fourier cosine series over a fundamental interval determined by the $a_n^{\text{max}}$ with Fourier coefficients that are simple to evaluate numerically. Representative numerical calculations are carried out when all the probability density functions are identical for two cases: $f(a) = \delta(a - 1)$; $f(a) = 1$ for $a \leq 1$, $\equiv 0$ for $a > 1$. The asymptotic form of the average frequency is also investigated. The numerical computations indicate a very rapid approach to the limiting asymptotic form.

2. $E(x_0)$ as an integral

Our problem is to determine the mean frequency of $x_0$ conditioned on the $a$’s, $E(x_0 \mid \{a_n\})$, which is given by

$$E(x_0 \mid \{a_n\}) = \lim_{T \to \infty} \frac{1}{2T} N(x_0 \mid T, \{a_n\})$$

(2.1)

where $N(x \mid T, \{a_n\})$ is the number of zeros of $g(t) = x(t) - x_0$ in the interval $|t| \leq T$.

Ka. [1], see also Mazur and Montroll [2], has shown that when the $\{a_n\}$ are numerical constants then

$$E(x_0 \mid \{a_n\}) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \phi_{xx}(\alpha, \beta \mid \{a_n\}) - \phi_{x\beta}(\alpha, \beta \mid \{a_n\}) \right] \beta^{-2} \cos(x_0\alpha) \, d\alpha \, d\beta$$

(2.2)

where $\phi_{xx}(\alpha, \beta \mid \{a_n\})$ is given by

$$\phi_{xx}(\alpha, \beta \mid \{a_n\}) = \prod_{n=1}^{N} J_0[a_n(\alpha^2 + \omega_n^2\beta)]$$

(2.3)

We now take the $a_n$ to be independent random variables and assign a probability density function $f_{a_n}(a_n)$ to each. Consequently,

$$\phi_{xx}(\alpha, \beta) = \prod_{n=1}^{N} \int_{0}^{\infty} f_{a_n}(a_n) J_0[a_n(\alpha^2 + \omega_n^2\beta)] \, da_n$$

(2.4)

so that $E(x_0)$ is given by eq. 2.2 with $\phi_{xx}(\alpha, \beta \mid \{a_n\})$ replaced by $\phi_{xx}(\alpha, \beta)$. The function $\phi_{xx'}$ is the joint characteristic function of $x(t)$ and its derivative, $x'(t)$. 
3. Evaluation of $E(x_0)$

The evaluation of $E(x_0)$ will now be carried out under the assumption that

$$f_{a_n}(a_n) = 0 \quad \text{for} \quad a_n > a_n^{(\text{max})}$$

(3.1)

where $a_n^{(\text{max})} < \infty$. Note that

$$|x(t)_{\text{max}}| = \sum_{n=1}^{N} a_n^{(\text{max})} = a,$$

(3.2)

$$|x'(t)_{\text{max}}| = \sum_{n=1}^{N} \omega_n a_n^{(\text{max})} = a'.$$

(3.3)

Thus $f_\omega(x, x')$, the joint PDF of $x$ and $x'$, is identically zero outside the rectangle $(|x| \leq a, |x'| \leq a')$.

We can express $f_{xx}(x, x')$ as a two-dimensional Fourier series in this basic rectangle

$$f_{xx}(x, x') = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f_r \cos \left( \frac{\pi r x}{a} \right) \cos \left( \frac{\pi s x'}{a'} \right)$$

(3.4)

with the Fourier coefficients given by

$$f_r = \frac{1}{aa'} \int_{-a}^{a} \int_{-a'}^{a'} f_{xx}(x, x') \cos \left( \frac{\pi r x}{a} \right) \cos \left( \frac{\pi s x'}{a'} \right) dx \, dx'$$

$$= \frac{1}{aa'} \phi_{xx} \left( \frac{\pi r}{a}, \frac{\pi s}{a'} \right),$$

(3.5)

so that the Fourier coefficients are given by sampled values of $\phi_{xx}$. Upon taking the inverse Fourier transform of eq. 3.5, we obtain the cardinal function (sampling expansion) of $\phi_{xx}(\alpha, \beta)$

$$\phi_{xx}(\alpha, \beta) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \phi_{xx} \left( \frac{\pi r}{a}, \frac{\pi s}{a'} \right) \sin(a \alpha - r \pi) \sin(a' \beta - s \pi) \frac{\sin(a' \beta - s \pi)}{(aa' - r \pi)}.$$

(3.6)

Substitution of eq. 3.6 into eq. 2.2 and termwise integration of the double series yields

$$E(x_0) = \frac{1}{2\pi^2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left[ \phi_{xx} \left( \frac{\pi r}{a}, 0 \right) - \phi_{xx} \left( \frac{\pi r}{a}, \frac{\pi s}{a'} \right) \right] B(s) A(x_0)$$

(3.7)

where

$$B(s) = \int_{\infty}^{\infty} \sin(a' \beta - s \pi) d\beta,$$

(3.8)

$$A(x_0) = \int_{-\infty}^{\infty} \frac{\sin(a \alpha - r \pi)}{(a \alpha - r \pi)} \cos x_0 \alpha \, d\alpha.$$
To evaluate $B(s)$ set $a' \beta - s \pi = x$, then

$$B(s) = a' \int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x+s\pi)}. \tag{3.10}$$

Next consider the Hilbert transform

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x+y)} = \pi \left( \frac{1 - \cos y}{y} \right). \tag{3.11}$$

Differentiate both sides with respect to $y$ and set $y = \pi s$ where $s$ is an integer, the final result is

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x+s\pi)} = \frac{\pi(1 - \cos \pi s)}{(\pi s)^2}. \tag{3.12}$$

Consequently,

$$B(r) = \frac{2a'}{\pi s^2}, \quad s = \pm 1, \pm 3, \pm 5, \ldots,$$

$$= 0, \quad s = 0, \pm 2, \pm 4, \ldots \tag{3.13}$$

The integral $A(x_0)$ is simply a convolution integral and immediately yields

$$A(x_0) = \frac{\pi}{a} \cos \left( \frac{\pi r_0 x_0}{a} \right), \quad |x_0| \leq a,$$

$$= 0, \quad |x_0| > a. \tag{3.14}$$

Upon substituting these expressions in eq. 3.7, we have

$$\mathbb{E}(x_0) = \frac{1}{\pi} \left( \frac{a'}{a} \right) \left[ E_0 + 2 \sum_{r=1}^{\infty} E_r \cos \left( \frac{\pi r x_0}{a} \right) \right], \quad |x_0| \leq a$$

$$= 0, \quad |x_0| > a \tag{3.15}$$

where

$$E_r = E_{-r} = \sum_{s=0}^{\infty} (2s+1)^{-2} \left\{ \phi_{xx} \left( \frac{r\pi}{a}, 0 \right) - \phi_{xx} \left[ \frac{r\pi}{a}, \frac{(2s+1)\pi}{a'} \right] \right\}$$

$$= \frac{\pi^2}{8} \phi_{xx} \left( \frac{r\pi}{a}, 0 \right) - \sum_{s=0}^{\infty} (2s+1)^{-2} \phi_{xx} \left[ \frac{r\pi}{a}, \frac{(2s+1)\pi}{a'} \right]. \tag{3.16}$$

This sampling expansion is easily computed. The rate of convergence of the series will, of course, depend on the "smoothness" of $\mathbb{E}(x_0)$.

There is an internal check on the numerical calculations because $\mathbb{E}(x_0 = \pm a) = 0$ for $N \geq 2$; this implies that

$$E_0 = -2 \sum_{r=1}^{\infty} (-1)^r E_r \tag{3.17}$$
A reviewer has pointed out the important (but relatively unknown) work of Slater [3] on this problem in the context of molecular reaction kinetics. Slater does not evaluate $E[x_0]$, but by an argument that I am unable to follow asserts that $E[x_0]$ is equal to the quantity $E[N_{x_0}]$, the rate of upcrossings at level $x_0$ of a stationary stochastic process $X(t)$. Subject to some mild restrictions, $E[N_{x_0}]$ is given by [4, 5]

$$E[N_{x_0}] = \int_0^\infty |x'| f_{xx}(x_0, x') \, dx'$$  (3.18)

where $f_{xx}(x, x')$ is the joint PDF of $X(t)$ and its time derivative. Slater evaluates this integral and finally arrives at eq. 3.16, in the special case where all the amplitudes are fixed, but not necessarily equal. It is straightforward to show that one also gets the same result when the amplitudes are random. Consequently, it appears that $E[x_0]$ and $E[N_{x_0}]$ are numerically equal as asserted by Slater. Nevertheless, $E[x_0]$ and $E[N_{x_0}]$ are really different. The quantity $E[N_{x_0}]$ is defined for a stationary stochastic process, while $E[x_0]$ is defined for a deterministic situation. The function $x(t)$, as given in eq. 1.1, is strictly deterministic. The statistical features arise from the linear independence of the $\omega_n$ through the Kronecker–Weyl theorem [6, 7] allowing the replacement of time averages, eq. 2.1, by ensemble averages.

Before discussing the numerical calculations, let us consider the asymptotic form of $E[x_0]$ as $N$ is increased.

### 4. Asymptotics of $E(x_0)$

At this point, we will make the assumption that the random variables $\omega_n$ all have the same probability density function, $f_\omega(\omega)$.

In order to derive an asymptotic expression for $E(x_0)$, we note that the main contribution to the integral form of $E(x_0)$, eq. 2.2, is from the integrand in the vicinity of the origin $\alpha, \beta \approx 0$. Consequently, we have

$$\phi_{xx}(\alpha, \beta) = \prod_{n=1}^N \int_0^\infty f_\omega(\omega) J_0[a(\alpha^2 + \omega_n^2 \beta^2)^{1/2}] \, da$$

$$\approx \prod_{n=1}^N \left\{ 1 - \frac{1}{2}(\alpha^2 + \omega_n^2 \beta^2) + \frac{1}{4}\langle \omega_n^4 \rangle (\alpha^2 + \omega_n^2 \beta^2)^2 + \cdots \right\}$$

$$\approx \prod_{n=1}^N \exp[-(\alpha^2 + \omega_n^2 \beta^2)/4] \times$$

$$\times \left\{ 1 + \frac{1}{\langle \omega^4 \rangle} \left( \frac{\langle \omega^4 \rangle}{\langle \omega^2 \rangle^2} - 2 \right) (\alpha^2 + \omega_n^2 \beta^2)^2 + \cdots \right\}$$  (4.1)

where

$$\langle a^{2k} \rangle = \int_0^\infty a^{2k} f_\omega(a) \, da, \quad k = 1, 2, \ldots$$  (4.2)
Substitution of eq. 4.1 into eq. 2.2 yields

\[ 2\pi^2 E(x_0) = \int \int \left\{ \prod_{n=1}^{N} \exp[-(a^2)\alpha^2/4](1+q\alpha^4) - \right. \]
\[ \int_{-\infty}^{\infty} \prod_{n=1}^{N} \exp[-(a^4)(\alpha^2 + \omega_n^2\beta^2)/4](1+q\alpha^4)] \beta^{-2} \cos x_0\alpha \, d\alpha \, d\beta \]
\[ = \int_{-\infty}^{\infty} \exp[-N(a^2)\alpha^2/4](1+Nq\alpha^4) \cos x_0\alpha \, d\alpha \]
\[ \times \int_{-\infty}^{\infty} \beta^{-2}(1-\exp[-(a^2)\Omega^2\beta^2/4]) \, d\beta \quad (4.3) \]

where

\[ q \equiv \frac{1}{64} \left( \frac{a^4}{a^2} - 2 \right), \quad \Omega^2 \equiv \sum_{n=1}^{N} \omega_n^2 \quad (4.4) \]

The \( \beta \)-integral is

\[ \int_{-\infty}^{\infty} \beta^{-2}(1-\exp[-(a^2)\Omega^2\beta^2/4]) \, d\beta = (\pi(a^2)\Omega^2)^{1/2} \quad (4.5) \]

while the \( \alpha \)-integral is a well-known Fourier cosine transform. Upon carrying out the integrations, we finally arrive at

\[ E(x_0) \sim \frac{1}{\pi} \left[ N^{-1} \sum_{n=1}^{N} \omega_n^2 \right]^{1/2} \exp[-x_0^2/N(a^2)] \times \]
\[ \left\{ 1 + \frac{1}{N} \cdot \frac{a^4}{1024(a^2)^2} \cdot \text{He}_2 \left[ \left( \frac{2}{N(a^2)^2} \right)^{1/2} x_0 \right] + \ldots \right\} \quad (4.6) \]

The first term is a Gaussian, while the second term is a correction term that decays as \( N^{-1} \) and depends upon the shape of \( f_\alpha(a) \) through \( \langle a^2 \rangle \) and \( \langle \alpha^4 \rangle \). Eq. 4.6 represents a central limit like behavior and cannot be expected to be useful except for \( x_0^2 \sim 0(N(a^2)) \). Kac [1] obtained the leading Gaussian term, when all the amplitudes are fixed and equal.

Some numerical calculations were performed in order to determine how rapidly \( E(x_0) \) approaches its limiting Gaussian form as a function of \( f_\alpha(a) \) and \( N \). Linearly independent frequencies \( \omega_n \) were taken from a table of random numbers, see Table 1. The probability density function was taken to be of the form

(a) \[ f_\alpha(a) = \delta(a-1) \quad (4.7) \]

(b) \[ f_\alpha(a) = 1, \quad 0 \leq a \leq 1, \]
\[ = 0, \quad \text{elsewhere.} \quad (4.8) \]
Table 1. Linearly independent frequencies $\omega_n$ employed in calculations of Figs. 1–4

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\omega_n$ (Case I)</th>
<th>$\omega_n$ (Case II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.534798</td>
<td>0.980361</td>
</tr>
<tr>
<td>2</td>
<td>0.973447</td>
<td>0.973447</td>
</tr>
<tr>
<td>3</td>
<td>0.660233</td>
<td>0.745236</td>
</tr>
<tr>
<td>4</td>
<td>0.997767</td>
<td>0.997763</td>
</tr>
</tbody>
</table>

Case a, of course, corresponds to Kac's original situation. The results of the calculations, using eqs. 3.15 and 3.16, are displayed in Figs. 1–4. The most obvious
Fig. 3. $E(x_0)$ for rectangular distribution of amplitudes and case I distribution of frequencies: \(N=1\), \(N=2\), \(N=3\), \(N=4\).

Fig. 4. $E(x_0)$ for rectangular distribution of amplitudes and case II distribution of frequencies: \(N=1\), \(N=2\), \(N=3\), \(N=4\).

feature is the extreme rapidity of the approach to the Gaussian behavior, an $N$ as small as 4 is sufficient to assure the limiting behavior. The rectangular density function, eq. 4.8, obviously reaches its limiting behavior faster than the Dirac delta density function.

Numerical calculations were also carried out for linearly independent frequencies taken to be logarithms of prime numbers. The results are qualitatively the same as for the displayed curves and are omitted.
References


