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Boundary Value Problems of Fractional Differential Equations at Resonance

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Abstract

By using the coincidence degree theory due to Mawhin and constructing the suitable operators, the existence of solutions for boundary value problems of fractional differential equations at resonance is obtained. An example is given to illustrate our result.

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1 Introduction

Fractional differential equations arise in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, etc. (see [1]-[2] and references cited therein). Recently, boundary value problems of fractional differential equations have been studied by many authors (see [3]-[5] and references cited therein).

Motivated by the results of [3-8], in this paper, we investigate the existence of solutions for the fractional differential equation at resonance:

\[ D_0^\alpha u(t) + f(t, u(t)) = 0, \quad a.e. \quad t \in [0, 1], \]

\[ u(0) = 0, \quad u(1) = \beta u(\xi), \]

where \( \alpha \leq 2 \), \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies Carathéodory condition.
0 < \xi < 1, \beta \xi^{\alpha-1} = 1.

In this paper, we will always suppose the following conditions hold.

\((H_1)\) \(1 < \alpha \leq 2, 0 < \xi < 1, \beta \xi^{\alpha-1} = 1.\)

\((H_2)\) \(f : [0, 1] \times R \rightarrow R\) satisfies Carathéodory condition, that is \(f(.,u)\) is measurable for each fixed \(u \in R\), \(f(t,.)\) is continuous for a.e. \(t \in [0, 1]\), and for each \(r > 0\), there exists \(\Phi_r \in L^\infty[0, 1]\) such that \(|f(t,u)| \leq \Phi_r(t)\) for all \(u \in [0, r]\), a.e. \(t \in [0, 1]\).

2 Preliminaries

For the convenience, we introduce some notations and a theorem. For more detail see [9].

Let \(X\) and \(Y\) be real Banach spaces and let \(L : domL \subset X \rightarrow Y\) be a Fredholm operator with index zero, \(P : X \rightarrow X, Q : Y \rightarrow Y\) be projectors such that

\[ImP = KerL, \quad KerQ = ImL,\]
\[X = KerL \oplus KerP, \quad Y = ImL \oplus ImQ.\]

It follows that

\[L|_{domL \cap KerP} : domL \cap KerP \rightarrow ImL\]

is invertible. We denote the inverse by \(K_P\).

If \(\Omega\) is an open bounded subset of \(X\), \(domL \cap \Omega \neq \emptyset\) the map \(N : X \rightarrow Y\) will be called \(L\)-compact on \(\Omega\) if \(QN((\Omega))\) is bounded and \(K_P(I-Q)N : \Omega \rightarrow X\) is compact.

**Theorem 2.1** [9]. Let \(L : domL \subset X \rightarrow Y\) be a Fredholm operator of index zero and \(N : X \rightarrow Y\). Assume that the following conditions are satisfied:

1. for every \((x, \lambda) \in ((domL \setminus KerL) \cap \partial \Omega) \times (0, 1)\),
2. \(N x \notin ImL\) for every \(x \in KerL \cap \partial \Omega\),
3. \(\text{deg}(QN|_{KerL}, \Omega \cap KerL, 0) \neq 0\), where \(Q : Y \rightarrow Y\) is a projection such that \(ImL = KerQ\).

Then the equation \(Lx = Nx\) has at least one solution in \(domL \cap \Omega\).

The following definitions and lemma can be found in [1], [2].

**Definition 2.1.** The fractional integral of order \(\alpha > 0\) of a function \(y : (0, \infty) \rightarrow R\) is given by

\[I_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad (2.1)\]

provided the right-hand side is pointwise defined on \((0, \infty)\).

**Definition 2.2.** The fractional derivative of order \(\alpha > 0\) of a function \(y : (0, \infty) \rightarrow R\) is given by

\[D_0^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \quad (2.2)\]

provided the right-hand side is pointwise defined on \((0, \infty)\), where \(n = [\alpha] + 1\).

**Lemma 2.1.** Assume \(\alpha > 0\):

1. \(D_0^\alpha I_0^\alpha u(t) = u(t)\) for a.e. \(t \in [0, 1]\), where \(u \in L^1[0, 1]\).
2. \(D_0^\alpha u(t) = 0\) if and only if \(u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}\) for some
$c_i \in \mathbb{R}, i = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

Take $X = C[0, 1]$ with norm $\|x\| = \max_{t \in [0, 1]} |x(t)|$. $Y = L^1[0, 1]$ with norm $\|x\|_1 = \int_0^1 |x(t)| dt$, $L = D_0^\alpha$.

$\text{dom} L = \{ u \in C[0, 1] \ | \ u(0) = 0, \ u(1) = \beta u(\xi) \}$

with $1 < \alpha \leq 2$. Let $N : X \to Y$ be defined as $Nu(t) = f(t, u(t)), t \in [0, 1]$.

Then problem (1.1), (1.2) is $Lu = Nu$.

3. Main results

In order to obtain our main result, we firstly present and prove the following lemmas.

**Lemma 3.1.** Suppose (H1) holds, then $L : \text{dom} L \subset X \to Y$ is a Fredholm operator of index zero and the linear continuous projector $Q : Y \to Y$ can be defined as

$$Qy = \frac{\alpha}{1 - \xi} \left[ \int_0^1 (1 - s)^{\alpha - 1} y(s) ds - \beta \int_0^\xi (\xi - s)^{\alpha - 1} y(s) ds \right]$$

and the linear operator $K_P : \text{Im} L \cap \text{Ker} P$ can be written by

$$K_P y = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds - \frac{t^{\alpha - 1}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} y(s) ds.$$  

**Proof:** It is easy to get that $\text{Ker} L = \{ ct^{\alpha - 1} | c \in \mathbb{R} \}$.

Define linear projector $P : X \to X$ by $Pu(t) = u(1)t^{\alpha - 1}$.

It is clear that

$$P^2 u = Pu, \quad \text{Im} P = \text{Ker} L, \quad X = \text{Ker} L \oplus \text{Ker} P.$$  

We will show that

$$\text{Im} L = \{ y \in Y | \int_0^1 (1 - s)^{\alpha - 1} y(s) ds - \beta \int_0^\xi (\xi - s)^{\alpha - 1} y(s) ds = 0 \}.$$  

In fact, if $y \in \text{Im} L$, there exists $u \in \text{dom} L$ such that $y = D_0^\alpha u \in Y$. So, we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds + c_1 t^{\alpha - 1} + c_2 e^{\alpha t}.$$  

By $y(0) = 0, \ u(1) = \beta u(\xi)$, we get that $y$ satisfies

$$\int_0^1 (1 - s)^{\alpha - 1} y(s) ds - \beta \int_0^\xi (\xi - s)^{\alpha - 1} y(s) ds = 0.$$  

On the other hand, if $y \in Y$ satisfies (3.5), take
\[ u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} y(s) ds \]

It follows from \( D_0^\alpha I_0^\alpha y = y \) for \( y \in Y \) and \( D_0^\alpha t^{\alpha-1} = 0 \) that \( D_0^\alpha u(t) = y(t) \). Obviously, \( u(0) = 0 \), \( u(1) = \beta u(\xi) \) So, \( u \in domL \).

Define the operator \( Q : Y \to Y \) as follows:
\[ Qy = \frac{\alpha}{1-\xi} \left[ \int_0^1 (1-s)^{\alpha-1} y(s) ds - \beta \int_0^\xi (\xi-s)^{\alpha-1} y(s) ds \right] . \]

It is clear that \( \text{Ker}Q = \text{Im}L \). For \( y \in Y \), we can easily get that \( Q^2 y = Qy \), i.e. \( Q : Y \to Y \) is a projector. Set \( y = (y - Qy) + Qy \). Then, \( y - Qy \in \text{Ker}Q = \text{Im}L \), \( Qy \in \text{Im}Q = R \). It is easy to get that \( \text{Im}Q \cap \text{Im}L = \{0\} \). So, we have \( Y = \text{Im}L \oplus \text{Im}Q \). This, together with (3.3), means that \( L \) is a Fredholm operator of index zero.

Define operator \( K_P : \text{Im}L \to domL \cap \text{Ker}P \) as follows
\[ K_P y = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \]

We will prove that \( K_P \) is the inverse of \( L \big|_{domL \cap \text{Ker}P} \).

It is clear that \( LK_P y = y \), for \( y \in \text{Im}L \). On the other hand, if \( u \in \text{dom}L \cap \text{Ker}P \), then then \( u(0) = u(1) = 0 \). By Lemma 2.1, we have
\[ K_P Lu(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} Lu(s) ds \]

It follows from \( u(0) - u(1) = K_P Lu(0) - K_P Lu(1) = 0 \) that \( K_P Lu = u \). So, \( K_P = (L \big|_{domL \cap \text{Ker}P})^{-1} \). The proof is completed.

Lemma 3.2. Assume \( \Omega \subset \bar{X} \) is an open bounded subset and \( domL \cap \overline{\Omega} \neq \emptyset \), then \( N \) is L-compact on \( \overline{\Omega} \).

Proof. By (H2), we can get that \( QN(\overline{\Omega}) \) and \( K_P(I-Q)N(\overline{\Omega}) \) are bounded. Now we will show that \( K_P(I-Q)N : \overline{\Omega} \to \bar{X} \) is compact.

It follows from (H2) and Lebesgue Dominated Convergence theorem that \( K_P(I-Q)N : \overline{\Omega} \to \bar{X} \) is continuous. By (H2), there exists a constant \( M > 0 \) such that \( |(I-Q)Nu| \leq M, u \in \Omega, \) a.e. \( t \in [0,1] \). For \( 0 \leq t_1 < t_2 \leq 1 \), \( u \in \overline{\Omega} \), we have
Since \( t_1^\alpha \) and \( t_1^{\alpha-1} \) are uniformly continuous on \([0,1]\), we can get that \( K_P(I - Q)Nu(t_1) \) is equicontinuous. It follows from Ascoli-Arzela theorem that \( K_P(I - Q)N : \overline{\Omega} \to X \) is compact. The proof is completed.

To obtain our main result, we need the following conditions.

\( (H_3) \) \(|f(t, u)| \leq \psi(t) + g(t)|u| \) where \( \psi(t) \) and \( g(t) \) are nondecreasing on \([0,1]\).

\( \psi, \ g \in L^1[0, 1] \) and \( \|g\|_1 < \frac{\Gamma(\alpha)}{1+\beta} \).

\( (H_4) \) For \( u \in dom L \setminus Ker L \) there exists a constant \( r > 0 \) such that if \( |u(t)| > r, t \in [\xi, 1] \) then \[ \int_0^1 (1 - s)^{\alpha-1}N u(s)ds - \beta \int_0^\xi (\xi - s)^{\alpha-1}N u(s)ds \neq 0. \]

\( (H_5) \) There exists a constant \( c_0 > 0 \) such that for either

1. \( e[\int_0^1 (1 - s)^{\alpha-1}N(cs^{\alpha-1})ds - \beta \int_0^\xi (\xi - s)^{\alpha-1}N(cs^{\alpha-1})ds] > 0 \)

or

2. \( e[\int_0^1 (1 - s)^{\alpha-1}N(cs^{\alpha-1})ds - \beta \int_0^\xi (\xi - s)^{\alpha-1}N(cs^{\alpha-1})ds] > 0 \)

Lemma 3.3. Suppose \((H_1)-(H_4)\) hold, then the set

\[ \Omega_1 = \{ u \in dom L \setminus Ker L \mid Lu = \lambda Nu, \ \lambda \in (0, 1) \} \]

is bounded.

**Proof.** Take \( u \in \Omega_1 \), then \( u \in dom L \setminus Ker L \) and \( Nu \in Im L \) By (3.4), we have

\[ \int_0^1 (1 - s)^{\alpha-1}N u(s)ds - \beta \int_0^\xi (\xi - s)^{\alpha-1}N u(s)ds = 0. \]

It follows from \((H_5)\) that there exists a constant \( t_0 \in [\xi, 1] \) such that \( |u(t_0)| \leq r \). Since \( Lu = \lambda Nu, u(0) = 0 \), we have

\[ u(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}N u(s)ds + c t^{\alpha-1} \]

Considering \( |u(t_0)| \leq r \), we get \( |ct^{\alpha-1}| \leq r + \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1}|Nu(s)|ds \), so, we have
\[ |u(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Nu(s)| ds + \frac{t^{\alpha-1}}{t_0^{\alpha-1}} |c t_0^{\alpha-1}| \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Nu(s)| ds \]
\[ + \frac{1}{\xi^{\alpha-1}} r + \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0-s)^{\alpha-1} |Nu(s)| ds \]
\[ \leq \beta r + \frac{1+\beta}{\Gamma(\alpha)} (\|\psi\|_1 + \|\eta\|_1 \|u\|). \]

By (H3), we get that \( \Omega_1 \) is bounded.

**Lemma 3.4.** Suppose (H1), (H2) and (H5) hold, then the set
\[
\Omega_2 = \{ u \mid u \in \text{Ker} L, \ Nu \in \text{Im} L \}
\]
is bounded.

**Proof.** For \( u \in \Omega_2 \), there exists a constant \( c \in \mathbb{R} \) such that \( u = ct^{\alpha-1} \) and \( N(ct^{\alpha-1}) \in \text{Im} L \). By (3.4), we have
\[
\int_0^1 (1-s)^{\alpha-1} N(cs^{\alpha-1}) ds - \beta \int_0^\xi (\xi-s)^{\alpha-1} N(cs^{\alpha-1}) ds = 0.
\]
By (H5), we get that \( |c| \leq c_0 \). So, \( \Omega_2 \) is bounded.

**Lemma 3.5.** Suppose (H1), (H2) and the first part of (H5) hold. The set
\[
\Omega_3 = \{ u \in \text{Ker} L \mid \lambda Ju + (1-\lambda) QNu = 0, \ \lambda \in [0,1] \}
\]
is bounded, where \( J : \text{Ker} L \to \text{Im} Q \) is a linear isomorphism given by \( J(ct^{\alpha-1}) = c, \ c \in \mathbb{R} \).

**Proof.** For \( u = ct^{\alpha-1} \in \Omega_3 \), we have
\[
\lambda c = -(1-\lambda) \frac{\alpha}{1-\xi} \int_0^1 (1-s)^{\alpha-1} N(cs^{\alpha-1}) ds - \beta \int_0^\xi (\xi-s)^{\alpha-1} N(cs^{\alpha-1}) ds.
\]
If \( \lambda = 1 \), then \( c = 0 \). If \( \lambda = 0 \), by (H5), we get that \( |c| \leq c_0 \) for \( 0 < \lambda < 1 \), if \( |c| > c_0 \) by the first part of (H5), we get that
\[
\lambda c^2 = -(1-\lambda) \frac{\alpha}{1-\xi} c \int_0^1 (1-s)^{\alpha-1} N(cs^{\alpha-1}) ds - \beta \int_0^\xi (\xi-s)^{\alpha-1} N(cs^{\alpha-1}) ds < 0.
\]
A contradiction. So, \( \Omega_3 \) is bounded.

**Remark 3.1.** If (H1), (H2) and the second part of (H5) hold. Take
\[
\Omega_3 = \{ u \in \text{Ker} L \mid \lambda Ju + (1-\lambda) QNu = 0, \ \lambda \in [0,1] \}.
\]

Similarly, we can prove that \( \Omega_3 \) is bounded.

The following theorem is our main result.

**Theorem 3.1.** Suppose (H1) \(- (H5) \) hold. Then the problem (1:1) \(- (1:2) \) has at least one solution in
Proof. Let $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i} \cup \{0\}$ be a bounded open subset of $X$. It follows from Lemma 3.2 that $N$ is $L$-compact on $\overline{\Omega}$. By Lemma 3.3 and Lemma 3.4, we have

1. $Lu \neq \lambda Nu$ for every $(u, \lambda) \in \left[ (\text{dom} L \setminus \text{Ker} L) \cap \partial \Omega \right] \times (0, 1)$
2. $Nu \notin \text{Im} L$ for every $u \in \text{Ker} L \cap \partial \Omega$.

We need only to prove:

3. $\text{deg}(QN|_{\text{Ker} L}, \Omega \cap \text{Ker} L, 0) \neq 0$.

Take $H(u, \lambda) = \pm \lambda Ju + (1 - \lambda)QN u$.

According to Lemma 3.5, we know $H(u, \lambda) \neq 0$ for $u \in \partial \Omega \cap \text{Ker} L$. By the homotopy of degree, we get that

\[
\text{deg}(QN|_{\text{Ker} L}, \Omega \cap \text{Ker} L, 0) = \text{deg}(H(\cdot, 0), \Omega \cap \text{Ker} L, 0) = \text{deg}(H(\cdot, 1), \Omega \cap \text{Ker} L, 0) = \text{deg}(\pm J, \Omega \cap \text{Ker} L, 0) \neq 0.
\]

By Theorem 2.1, we can get that $Lu = Nu$ has at least one solution in $\text{dom} L \cap \overline{\Omega}$, i.e. (1.1) - (1.2) has at least one solution in $X$. The prove is completed.

4 Example

Let's consider the following boundary value problem

\[
D^{3/2}_{0+} u(t) = f(t, u(t)), \quad 0 < t < 1, \quad u(0) = 0, \quad u(1) = \sqrt{2} u(1/2).
\]

(4.1)

where

\[
f(t, u) = \begin{cases}
    t^2, & t \in [0, 1/2], \\
    t^2 + \Gamma(3/2) t^2 u^{3/2}, & |u| > 1, \ t \in (1/2, 1], \\
    t^2 + \Gamma(3/2) t^2 u^{3}, & |u| \leq 1, \ t \in (1/2, 1].
\end{cases}
\]

Corresponding to the problem (1.1)-(1.2), we have that $\alpha = 3/2$, $\beta = \sqrt{2}$, $\psi(t) = t^2$, and

\[
g(t) = \begin{cases}
    0, & t \in [0, 1/2], \\
    \Gamma(3/2) t^2, & t \in (1/2, 1].
\end{cases}
\]

By simple calculation, we can get that (H1)-(H4) and the first part of (H5) hold.

By Theorem 3.1, we obtain that problem (4.1)-(4.2) has at least one solution.
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