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On two problems in extension theory

A.V. Karasev

*Computer Science and Mathematics Department, Nipissing University,
100 College Drive, Box 5002, North Bay, Ont., P1B 8L7, Canada*

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Abstract

In this note we introduce the concept of a quasi-finite complex. Next, we show that for a given countable simplicial complex L the following conditions are equivalent:

- L is quasi-finite.
- There exists a $[L]$-invertible mapping of a metrizable compactum X with $\text{e-dim}\, X \leqslant [L]$ onto the Hilbert cube.

  Finally, we construct an example of a quasi-finite complex L such that its extension type $[L]$ does not contain a finitely dominated complex.

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1. Introduction

One of the most interesting open problems in extension theory is to characterize all countable CW complexes L for which there exists a universal metric compactum of a given extension dimension $[L]$. In the case of L being the Eilenberg–MacLane complex $K(\mathbb{Z}, n)$ the above problem can be restated as follows: does there exist a universal metric

E-mail address: alexandk@nipissingu.ca (A.V. Karasev).

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compactum of a given integral cohomological dimension? If in the previous question "universal metric compactum" is replaced with "universal separable metric space" the question can be answered positively. Namely, Dydak and Mogilski [11] proved that for a given n there exists a Polish space X of integral cohomological dimension n which contains a topological copy of any separable metric space Y with $\dim_{\mathbb{Z}} Y \leqslant n$. This result was generalized by Olszewski [12], who proved the existence of universal separable metric space of given extension dimension $[L]$ for each countable complex L. As a corollary, this implies the existence of universal separable metrizable space of a given cohomological dimension with respect to any countable Abelian coefficient group.

It was shown in [10] that a universal compactum of extension dimension $[L]$ exists if L is a finitely dominated complex. It should be emphasized that all such universal compacta were obtained as preimages of the Hilbert cube with respect to certain $[L]$-invertible mappings.

In [2] Chigogidze stated the following two problems and showed that they are equivalent:

Problem 1. Characterize connected locally compact simplicial complexes P such that $P \in AE(X)$ iff $P \in AE(\beta X)$ for any space X.

Problem 2. Characterize connected locally compact simplicial complexes P such that there exists a P-invertible map $f : X \to I^{\omega}$ where X is a metrizable compactum with $P \in AE(X)$.

As noted above, the following problem is closely related to the Problems 1 and 2 [4,10]:

Problem 3. Let L be a countable CW complex such that the class of metrizable compacta $\{X\colon L \in AE(X)\}$ has a universal space. Is it true that the extension type $[L]$ of this complex contains a finitely dominated complex?

In this note we introduce the notion of quasi-finite complexes and show that the class of quasi-finite complexes yields the characterization required in Problems 1 and 2. Next, we construct an example of a quasi-finite complex L such that its extension type $[L]$ does not contain a finitely dominated complex. This provides a negative solution for Problem 3.

2. Preliminaries

For spaces X and L, the notation $L \in AE(X)$ means that every map $f : A \to L$, defined on a closed subspace A of X, admits an extension $\bar{f}$ over X. Let L and K be countable and locally finite CW complexes. Following Dranishnikov [6], we say that $L \leqslant K$ if for each Polish space X the condition $L \in AE(X)$ implies the condition $K \in AE(X)$. This definition leads to a preorder relation $\leqslant$ on the class of countable and locally finite CW complexes. This preorder relation generates the equivalence relation. The equivalence class of complex L is called the extension type of L and is denoted by $[L]$. By e-dim X we denote extension dimension of space X [6,7]. Inequality e-dim $X \leqslant [\mathrm{L}]$ means that $L \in AE(X)$. More information about extension dimension and extension types can be found in [3].

Following Chigogidze [3] we say that a map $f: X \to Y$ is $[L]$-*invertible* if for each Polish space Z with e-dim $Z \leqslant [L]$ and for any map $g: Z \to Y$ there exists a map $h: Z \to X$ satisfying the conditions $f \circ h = g$. The following theorem [2, Theorem 2.1] shows that Problems 1 and 2, stated in the introduction, are equivalent.

Theorem 2.1. (A. Chigogidze) *Let P be a Polish ANR-space. Then the following statements are equivalent*:

(a) *$P \in AE(\beta X)$ whenever X is a space with $P \in AE(X)$.*
(b) *$P \in AE(\beta X)$ whenever X is a normal space with $P \in AE(X)$.*
(c) *$P \in AE(\beta(\bigoplus\{X_t\colon t \in T\}))$ whenever T is an arbitrary indexing set and X_t, $t \in T$, is a separable metrizable space with $P \in AE(X_t)$.*
(d) *$P \in AE(\beta(\bigoplus\{X_t\colon t \in T\}))$ whenever T, is an arbitrary indexing set and X_t, $t \in T$, is a Polish space with $P \in AE(X_t)$.*
(e) *There exists a P-invertible map $f: X \to I^\omega$ where X is a metrizable compactum with $P \in AE(X)$.*

3. Quasi-finite complexes

A pair of spaces $V \subset U$ is called $[L]$-*connected for Polish spaces* [1] if for every Polish space X with e-dim $X \leqslant [L]$ and for every closed subspace $A \subset X$ any mapping of A to V can be extended to a mapping of X into U.

Definition 3.1. We say that a CW complex L is *quasi-finite* if for every finite subcomplex P of L there exists a finite subcomplex P' of L containing P such that the pair $P \subset P'$ is $[L]$-connected for Polish spaces.

Theorem 3.2. *Let L be a countable and locally finite* CW *complex. Then the following conditions are equivalent*:

(i) *L is quasi-finite.*
(ii) *Any of the conditions* (a)–(e) *of Theorem* 2.1 *is satisfied.*

Proof. It is enough to show that condition (i) is equivalent to the condition (d) of Theorem 2.1.

Suppose that L is quasi-finite. Let $\{X_t\colon t \in T\}$ be an arbitrary family of Polish spaces with $L \in AE(X_t)$ for all $t \in T$. Consider a closed subspace $A \subset \beta(\bigoplus\{X_t\colon t \in T\})$ and a mapping $f: A \to L$. Let P be a finite subcomplex of L containing $f(A)$. Since L is quasi-finite there exists a finite subcomplex P' of L containing P such that the pair $P \subset P'$ is $[L]$-connected for Polish spaces. Let $\tilde{f}: \tilde{A} \to P$ be an extension of f over some closed neighborhood $\tilde{A}$ of A in $\beta(\bigoplus\{X_t\colon t \in T\})$. For any $t \in T$, let $f_t: X_t \to P'$ be an extension of $\tilde{f}|_{\tilde{A} \cap X_t}$. Consider a mapping $f' = \bigoplus\{f_t\colon t \in T\}: \bigoplus\{X_t\colon t \in T\} \to P'$. Since P' is compact, the mapping f' can be extended to a mapping $\bar{f}:\beta(\bigoplus\{X_t\colon t\in T\})\to P'\subset L$. Clearly $\bar{f}$ provides a necessary extension of f. Thus $\text{e-dim}\,\beta(\bigoplus\{X_t\colon t\in T\})\leqslant[L]$.

Now suppose that condition (d) of the Theorem 2.1 is satisfied. Consider a finite subcomplex $P\subset L$. Let $\{X_t, A_t, f_t\colon t\in T\}$ be the set of all triples such that X_t is a Polish space with $\text{e-dim}\,X_t\leqslant[L]$, A_t is a closed subspace of X_t and $f_t:A_t\to P$ is a mapping. Put $f=\bigoplus\{f_t\colon t\in T\}:\bigoplus\{A_t\colon t\in T\}\to P$. Since P is compact there exists a mapping $\tilde{f}:\overline{\bigoplus\{A_t\colon t\in T\}}^{\beta(\bigoplus\{X_t\colon t\in T\})}=\beta(\bigoplus\{A_t\colon t\in T\})\to P$ extending f. By assumption $\text{e-dim}\,\beta(\bigoplus\{X_t\colon t\in T\})\leqslant[L]$ and we can extend the mapping $\tilde{f}$ to a mapping $\bar{f}:\beta(\bigoplus\{X_t\colon t\in T\})\to L$. Let P' be a finite subcomplex of L containing $\bar{f}(\beta(\bigoplus\{X_t\colon t\in T\}))$. It is easy to see that the pair $P\subset P'$ is $[L]$-connected for Polish spaces. □

For each $n\geqslant 4$ there exists a space X with integral cohomological dimension $\text{c-dim}\,X\leqslant n$ and $\text{c-dim}\,\beta X\geqslant n+1$ [5]. Therefore the above theorem implies that the Eilenberg–MacLane complex $K(\mathbb{Z},n)$ is not quasi-finite for all $n\geqslant 4$.

4. Example

In this section we show that there exists a quasi-finite complex M such that its extension type $[M]$ does not contain a finitely dominated complex. By Theorem 3.1 for such complex M there exists $[M]$-invertible mapping of a metrizable compactum X with $\text{e-dim}\,X\leqslant[M]$ onto the Hilbert cube. This implies that X is universal for the class of metrizable compacta $\{X\colon M\in AE(X)\}$. Thus the solution of Problem 3 is negative.

Let M be a countable and locally finite CW complex homotopically equivalent to the bouquet $\bigvee\{M_p\colon p\text{-prime}\}\bigvee S^3$ where $M_p=M(\mathbb{Z}_P,2)$ is a Moore space of the type $(\mathbb{Z}_p,2)$. Clearly M is quasi-finite and $[S^2]\leqslant[M]<[S^3]$ (further consideration will imply that $[S^2]<[M]$). We shall show that the extension type $[M]$ of M does not contain a finitely dominated complex.

Suppose the opposite. Let L be a countable finitely dominated CW complex such that $[M]=[L]$. Then $[S^2]\leqslant[L]<[S^3]$. In particular, L is simply connected and therefore $H_2(L)\cong\pi_2(L)$. Since L is finitely dominated, groups $H_2(L)\cong\pi_2(L)$ are finitely generated. Since $[L]<[S^3]$ it follows that $\pi_2(L)$ is non-trivial. Consider two cases.

Case 1. $H_2(L)\cong\pi_2(L)\cong\mathbb{Z}\oplus H$ where H is finitely generated.

Note that groups $H^1(M;\mathbb{Q})$ and $H^2(M;\mathbb{Q})$ are trivial. On the other hand, suspension isomorphism and Hurewicz theorem imply that $H_3(\Sigma L)\cong\mathbb{Z}\oplus H$ and hence the group $H^3(\Sigma L;\mathbb{Q})$ is non-trivial. Therefore we can apply construction of Dranishnikov and Repovš ([8, Theorem 2.4] and [9, Theorem 1.3]) and use the idea from the proof of Theorem 1.4 from [9, p. 351] to obtain a metrizable compactum X with $\text{e-dim}\,X\leqslant[M]$ and a mapping $f:X\to\Sigma L$ which is not null-homotopic. Let $A=f^{-1}(L)$ where L is considered as the equator of ΣL. Then $f|_A$ does not have an extension $\bar{f}:X\to L$. Indeed, such an extension $\bar{f}$ would be null-homotopic. On the other hand it would be homotopic to f. This shows that $L\notin AE(X)$ which leads to a contradiction.

Case 2. $H_2(L)\cong\bigoplus\{\mathbb{Z}_{n_i}\colon i=1,\dots,m\}$.

Choose a prime $p > \max\{n_i\colon\ i = 1, \ldots, m\}$. Then $H^1(L; \mathbb{Z}_p)$ and $H^2(L; \mathbb{Z}_p)$ are trivial. Note that $H^3(\Sigma M_p; \mathbb{Z}_p)$ is non-trivial. As before we can find a metrizable compactum X with $\text{e-dim}\, X \leqslant [L]$ and a mapping $f : X \to \Sigma M_p$ which is not null-homotopic. By the same arguments as above we conclude that $M_p \notin AE(X)$. Since $[M] \leqslant [M_p]$ we obtain a contradiction with $[M] = [L]$.

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