Periodic solutions for Rayleigh type $p$-Laplacian equation with deviating arguments

Minggang Zong$^{a,b,*}$, Hongzhen Liang$^a$

$^a$Faculty of Science, Jiangsu University, Zhenjiang 212013, Jiangsu, PR China
$^b$School of Statistics, Renmin University of China, Beijing, 100872, PR China

Received 18 June 2005; received in revised form 15 February 2006; accepted 22 February 2006

Abstract

By using topological degree theory and some analysis skill, we obtain some sufficient conditions for the existence of periodic solutions for Rayleigh type $p$-Laplacian differential equation with deviating arguments.

Keywords: $p$-Laplacian; Periodic solutions; Rayleigh equation; Deviating argument; Topological degree

In the last few years, the existence of periodic solutions for delay differential equations has received a lot of attention. For example, in [1–4], the following second order differentials equations with delay:

\begin{align*}
    x'(t) + f(x'(t)) + g(x(t - \tau)) &= p(t), \\
    x''(t) + m^2 x(t) + g(x(t - \tau)) &= p(t), \\
    x''(t) + f(t, x'(t - \tau)) + g(x(t - \sigma)) &= p(t),
\end{align*}

were studied. However, so far as we are aware, fewer papers discuss the Rayleigh type equation with deviating arguments. In [5] Lu and Ge studied the following Rayleigh equation with a deviating argument:

\begin{equation}
    x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t),
\end{equation}

and by using coincidence degree theory, the authors obtained sufficient conditions for the existence of periodic solutions for Eq. (1).

In this work we deal with the existence of periodic solutions for the Rayleigh type $p$-Laplacian differential equation with deviating arguments of the form

\begin{equation}
    (\varphi_p(x'(t)))' + f(t, x'(t - \sigma(t))) + g(t, x(t - \tau(t))) = e(t),
\end{equation}

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $f, g$ are continuous functions defined on $R^2$ and are periodic about $t$ with $f(t, \cdot) = f(t + 2\pi,)$, $g(t, \cdot) = g(t + 2\pi,)$, $f(t, 0) = 0$ for

$^*$Corresponding author at: Faculty of Science, Jiangsu University, Zhenjiang 212013, Jiangsu, PR China.
E-mail address: zong-mg@163.com (M. Zong).

0893-9659/S - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2006.02.021
\( t \in R, e, \sigma \) and \( \tau \) are continuous periodic functions defined on \( R \) with period \( 2\pi \) and \( \int_0^{2\pi} e(t) \, dt = 0. \) Obviously, as \( p = 2, \varphi_2(x) = x \) is a linear function and when \( \sigma(t) = \tau(t) = 0, f(t,x) = f(x), g(t,x) = g(x), \) Eq. (2) reduces to Eq. (1). So the results in [4] are contained in our work. However, because the \( p \)-Laplacian operator \( \varphi_p : \varphi_p(x) = |x|^{p-2}x \) is nonlinear, the continuation theorem of Mawhin [6] is not applicable, which leads to difficulty for solving the problem (2). By using topological degree theory and some analysis skill, we obtain the sufficient conditions for the existence of periodic solutions for Eq. (2).

For convenience, let us define

\[
C_T^1 := C_T^1[0, T] = \{x \in C^1[0, T], x(0) = x(T), x'(0) = x'(T), T > 0\}.
\]

For the periodic boundary value problem

\[
(\varphi_p(x'(t)))' = \tilde{f}(t, x, x'), \quad x(0) = x(T), x'(0) = x'(T) \tag{3}
\]

where \( \tilde{f} \in C(R^3, R) \), we have the following lemma:

**Lemma ([7]).** Let \( \Omega \) be an open bounded set in \( C_T^1 \); suppose that the following conditions hold:

(i) For each \( \lambda \in (0, 1) \) the problem

\[
(\varphi_p(x'(t)))' = \lambda \tilde{f}(t, x, x'), \quad x(0) = x(T), x'(0) = x'(T)
\]

has no solution on \( \partial \Omega \).

(ii) The equation

\[
F(a) := \frac{1}{T} \int_0^T \tilde{f}(t, a, 0) \, dt = 0,
\]

has no solution on \( \partial \Omega \bigcap \bar{R} \).

(iii) The Brouwer degree of \( F \)

\[
\deg \left(F, \quad \Omega \bigcap \bar{R}, \quad 0\right) \neq 0.
\]

Then the periodic boundary value problem (3) has at least one periodic solution on \( \bar{\Omega} \).

By using this lemma, we obtain our main results:

**Theorem 1.** Suppose there exist positive constants \( K, D, M \) such that:

(H1) \( |f(t, x)| \leq K \) for \( (t, x) \in R \times R \).

(H2) \( xg(t, x) > 0 \) and \( |g(t, x)| > K \) for \( |x| > D \) and \( t \in R \).

(H3) \( g(t, x) \geq -M, \) for \( x \leq -D \) and \( t \in R \).

Then Eq. (2) has at least one solution with period \( 2\pi \).

**Proof.** Consider the homotopic equation of Eq. (2) as follows:

\[
(\varphi_p(x'(t)))' + \lambda f(t, x'(t - \sigma(t))) + \lambda g(t, x(t - \tau(t))) = \lambda e(t), \quad \lambda \in (0, 1). \tag{4}
\]

Let \( x(t) \in C_{2\pi}^1 \) be an arbitrary solution of Eq. (4) with period \( 2\pi \). Firstly, we give \( x'(t) \) a priori bound which is independent of \( \lambda \). By integrating both sides of Eq. (4) over \([0, 2\pi]\), and noticing that \( x'(0) = x'(2\pi) \), we have

\[
\int_0^{2\pi} [f(t, x'(t - \sigma(t))) + g(t, x(t - \tau(t)))] \, dt = 0.
\]

As \( x(0) = x(2\pi) \), there exists \( t_0 \in [0, 2\pi] \) such that \( x'(t_0) = 0 \), while \( \varphi_p(0) = 0 \); we see

\[
|\varphi_p(x'(t))| = \left| \int_{t_0}^t (\varphi_p(x'(s)))' \, ds \right| \leq \lambda \int_0^{2\pi} |f(t, x'(t - \sigma(t)))| \, dt + \lambda \int_0^{2\pi} |g(t, x(t - \tau(t))| \, dt + \lambda \int_0^{2\pi} |e(t)| \, dt \leq 2\pi K + \int_0^{2\pi} |g(t, x(t - \tau(t))| \, dt + 2\pi \max_{t \in [0, 2\pi]} |e(t)|. \tag{6}
\]
and then we can assert that there exists some positive constant $D_1$ such that

$$\int_0^{2\pi} |g(t, x(t - \tau(t)))| \, dt \leq 4\pi D_1 + 2\pi K. \quad (7)$$

In fact, in view of condition (H1) and Eq. (5) we have

$$\int_0^{2\pi} \{g(t, x(t - \tau(t))) - K\} \, dt \leq \int_0^{2\pi} \{g(t, x(t - \tau(t))) - |f(t, x'(t - \sigma(t)))|\} \, dt$$

$$\leq \int_0^{2\pi} \{g(t, x(t - \tau(t))) + f(t, x'(t - \sigma(t)))\} \, dt$$

$$= 0. \quad (8)$$

Defining

$$E_1 = \{t : t \in [0, 2\pi], x(t - \tau(t)) > D\}$$

$$E_2 = \{t : t \in [0, 2\pi], |x(t - \tau(t))| \leq D\} \cup \{t : t \in [0, 2\pi], x(t - \tau(t)) < -D\}$$

we can get

$$\int_{E_1} |g(t, x(t - \tau(t)))| \, dt \leq 2\pi \max\{M, \sup_{t \in [0, 2\pi], |x| \leq D} |g(t, x)|\}$$

$$\int_{E_1} \{|g(t, x(t - \tau(t)))| - K\} \, dt = \int_{E_1} \{|g(t, x(t - \tau(t)))| - K\} \, dt$$

$$\leq -\int_{E_2} \{|g(t, x(t - \tau(t)))| - K\} \, dt$$

$$\leq \int_{E_2} \{|g(t, x(t - \tau(t)))| + K\} \, dt$$

which yields

$$\int_{E_1} |g(t, x(t - \tau(t)))| \, dt \leq \int_{E_2} |g(t, x(t - \tau(t)))| \, dt + \int_{E_1 + E_2} K \, dt$$

$$= \int_{E_2} |g(t, x(t - \tau(t)))| \, dt + 2\pi K,$$

that is

$$\int_0^{2\pi} |g(t, x(t - \tau(t)))| \, dt = \int_{E_1} |g(t, x(t - \tau(t)))| \, dt + \int_{E_2} |g(t, x(t - \tau(t)))| \, dt$$

$$\leq 2\int_{E_2} |g(t, x(t - \tau(t)))| \, dt + 2\pi K$$

$$\leq 4\pi \max\{M, \sup_{t \in [0, 2\pi], |x| \leq D} |g(t, x)|\} + 2\pi K$$

$$= 4\pi D_1 + 2\pi K$$

where $D_1 = \max\{M, \sup_{t \in [0, 2\pi], |x| \leq D} |g(t, x)|\}$. Now substituting inequality (7) into (6) we obtain

$$|\varphi_p(x'(t))| \leq 4\pi D_1 + 4\pi K + 2\pi \max_{t \in [0, 2\pi]} |e(t)| \coloneqq M_1.$$ 

By using the monotonicity of $\varphi_p$, we can get some positive constant $M_2$ such that for all $t \in R$,

$$|x'(t)| \leq M_2.$$ 

Since (8) is satisfied, there must exist $t_1 \in (0, 2\pi)$ such that

$$f(t_1, x'(t_1 - \sigma(t_1))) + g(t_1, x(t_1 - \tau(t_1))) = 0.$$
By applying condition \((H_1)\) we have
\[
|g(t_1, x(t_1 - \tau(t_1)))| = |f(t_1, x'(t_1 - \sigma(t_1)))| \leq K,
\]
and from condition \((H_2)\) we can get that \(|x(t_1 - \tau(t_1))| \leq D\). Since \(x(t)\) is periodic with period \(2\pi\), so \(t_1 - \tau(t_1) = 2n\pi + t_2\), \(t_2 \in (0, 2\pi)\) where \(n\) is some integer; therefore \(|x(t_2)| \leq D\). So for all \(t \in R\),
\[
|x(t)| = |x(t_2) + \int_{t_2}^t x'(s) \, ds| \leq D + 2\pi M_2 := M_3. \quad (9)
\]

Setting
\[
\Omega = \{x \in C^1_{2\pi}(R, R) \| x \| \leq M_3 + 1, |x'| \leq M_2 + 1\}
\]
we know that Eq. \((4)\) has no solution on \(\partial \Omega\) as \(\lambda \in (0, 1)\) and when \(x(t) \in \partial \Omega \cap R\), \(x(t) = M_3 + 1\) or \(x(t) = -M_3 - 1\), from \((9)\) we know that \(D + 2\pi M_2 = M_3\), so \(M_3 + 1 > D\); we see that
\[
\frac{1}{2\pi} \int_0^{2\pi} g(t, M_3 + 1) + e(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} g(t, -M_3 - 1) + e(t) \, dt > 0
\]
so condition \((ii)\) is also satisfied. Setting
\[
H(x, \mu) = \mu x + (1 - \mu) \frac{1}{2\pi} \int_0^{2\pi} g(t, x) \, dt,
\]
when \(x \in \partial \Omega \cap R\), \(\mu \in [0, 1]\) we have
\[
xH(x, \mu) = \mu x^2 + (1 - \mu)x \frac{1}{2\pi} \int_0^{2\pi} g(t, x) \, dt > 0
\]
and thus \(H(x, \mu)\) is a homotopic transformation and
\[
\deg \left\{ f, \Omega \cap R, 0 \right\} = \deg \left\{ \frac{1}{2\pi} \int_0^{2\pi} g(t, x) \, dt, \Omega \cap R, 0 \right\} = \deg \left\{ x, \Omega \cap R, 0 \right\} \neq 0
\]
so condition \((iii)\) is satisfied. In view of the previous lemma, there exists at least one solution with period \(2\pi\). This completes the proof. \(\square\)

By using a similar argument, we can obtain the following theorem:

**Theorem 2.** Suppose there exist positive constants \(K, D, M\) such that:

\(H_1)\) \(|f(t, x)| \leq K\) for \((t, x) \in R \times R\).

\(H_2)\) \(xg(t, x) > 0\) and \(|g(t, x)| > K\) for \(|x| > D\) and \(t \in R\).

\(H_3)\) \(g(t, x) \leq M, \text{for} t \in R \text{ and } x \geq D\).

Then Eq. \((2)\) has at least one solution with period \(2\pi\).

As an application, let us consider the following equation:
\[
(\varphi p x'(t))' + \cos t \sin x'(t - \sin t) + \arctan \left( \frac{x(t - \sin t)}{1 + \cos^2 t} \right) = \cos t \quad (10)
\]
where we have \(f(t, x) = \cos t \sin x\), \(g(t, x) = \arctan \frac{x}{1 + \cos^2 t}\), \(\sigma(t) = \tau(t) = \sin t\) and \(e(t) = \cos t\). We can easily choose \(K = 1, D > \frac{\pi}{2}\) and \(M = \frac{\pi}{2}\) such that \((H_1)-(H_3)\) hold. By Theorem 1, Eq. \((10)\) has at least one solution.
Remark 1. If \( \int_{0}^{2\pi} e(t) \, dt \neq 0 \) and \( f(t, 0) \neq 0 \), the problem of existence of \( 2\pi \)-periodic solutions to Eq. (2) can be converted to the existence of \( 2\pi \)-periodic solutions to the equation

\[
(\varphi_p(x'(t)))' + f_1(t, x'(t - \sigma(t))) + g_1(t, x(t - \tau(t))) = e_1(t)
\]  

(11)

where \( f_1(t, x) = f(t, x) - f(t, 0) \), \( g_1(t, x) = g(t, x) - \int_{0}^{2\pi} e(t) \, dt + f(t, 0) \) and \( e_1(t) = e(t) - \int_{0}^{2\pi} e(t) \, dt \). Clearly, \( \int_{0}^{2\pi} e_1(t) \, dt = 0 \) and \( f_1(t, 0) = 0 \). Eq. (11) can be discussed by using Theorem 1 (or Theorem 2).

Remark 2. When \( f(t, x) \equiv 0 \), Eq. (2) reduces to a Duffing type equation; the results also hold.

Acknowledgement

The authors are grateful to the referee for his or her suggestions on the first draft of the work.

References

[1] X. Huang, Z. Xiang, On the existence of \( 2\pi \)-periodic solution for delay Duffing equation \( x''(t) + g(t, x(t - \tau)) = p(t) \), Chinese Sci. Bull. 39 (3) (1994) 201–203.