Abstract

This work continues the study of $F$-manifolds $(M, ◦)$, first defined in [HeMa] (Int. Math. Res. Notices 6 (1999) 277–286, Preprint math.QA/9810132) and investigated in [He] (Frobenius Manifolds and Moduli Spaces for Singularities, Cambridge University Press, Cambridge, 2002). The notion of a compatible flat structure $∇$ is introduced, and it is shown that many constructions known for Frobenius manifolds do not in fact require invariant metrics and can be developed for all such triples $(M, ◦, ∇)$. In particular, we extend and generalize recent Dubrovin’s duality ([Du2], On almost duality for Frobenius manifolds, Preprint math.DG/0307374).

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1. Introduction

The notion of Frobenius (super)manifold $M$ axiomatized and thoroughly studied by B. Dubrovin in the early 1990s, plays a central role in mirror symmetry, theory of...
unfolding spaces of singularities, and quantum cohomology. A full Frobenius structure on $M$ consists of the data $(\circ, e, g, E)$. Here $\circ : T_M \otimes \mathcal{O}_M T_M \to T_M$ is an associative and (super)commutative multiplication on the tangent sheaf, so that $T_M$ becomes a sheaf of (super)commutative $\mathcal{O}_M$-algebras with identity $e \in T_M(M)$; $g$ is a metric on $M$ (nondegenerate quadratic form $S^2(T_M) \to \mathcal{O}_M$), and $E$ is an Euler vector field. These structures are connected by various constraints and compatibility conditions, spelled out in [Du1, Ma], for example, $g$ must be flat and $\circ$-invariant.

Pretty soon it became clear that various weaker versions of the Frobenius structure are interesting in themselves and also appear naturally in different contexts. Here I focus on the core notion of $F$-manifold introduced in [HeMa] and further studied in [He]. This structure consists of an associative and (super)commutative multiplication $\circ$ on the tangent sheaf as above, constrained by the following identity (1.2).

Start with an expression measuring the deviation of the structure $(T_M, \circ, [\cdot, \cdot])$ from that of a (sheaf of) Poisson algebra(s) on $(T_M, \circ)$:

$$PX(Z, W) := [X, Z \circ W] - [X, Z] \circ W - (-1)^{XZ} Z \circ [X, W].$$

Here $X, Y, Z, W$ are arbitrary local vector fields, and a notation like $(-1)^{XZ}$ is a shorthand for $(-1)^{\tilde{X}\tilde{Z}}$ where $\tilde{X}$ is the parity of $X$.

Then we must have

$$P_{X \circ Y}(Z, W) = X \circ P_Y(Z, W) + (-1)^{XY} Y \circ P_X(Z, W).$$

(1.2)

If $\circ$ has an identity $e \in T_M$, we will call $(M, \circ, e)$ an $F$-manifold with identity.

$F$-manifolds keep reappearing in recent research, although they are not always recognized as such. Any Frobenius manifold stripped of $g, E$ and $e$, becomes an $F$-manifold. Solutions of the oriented associativity equations with flat identity defined and studied in [LoMa2, 5.3.1–5.3.2], are exactly $F$-manifolds with compatible flat structure (see Definition 2.2). Quantum $K$-theory produces $F$-manifolds with a flat invariant metric which are not quite Frobenius because $e$ is not flat: see [Lee]. Dubrovin’s almost Frobenius manifolds [Du2, Section 3, Definition 9] are $F$-manifolds with nonflat identity as well.

In a very general context of $dg$-extended deformation theory, S. Merkulov in [Me1, Me2], found out that extended moduli spaces (e.g. deformations of complex or symplectic structure) often carry a natural $F$-structure, and produced a strong homotopy version of Eq. (1.2).

In this paper I introduce and study $F$-manifolds with a compatible flat structure. This notion turns out to share some of the deeper properties of Frobenius manifolds, in particular, Dubrovin’s deformed connections, Dubrovin’s duality, and several versions of an operadic description.

The paper is structured as follows.

The main result of §2 shows that there is an essential equivalence between the notions of an $F$-manifold with a compatible flat structure and that of a pencil of flat torsionless connections. This fact then allows us to borrow various techniques from [LoMa1, LoMa2]. In §3 I generalize to this setup the formalism of extended structure
connections and show, in particular, how Euler fields emerge as classifiers of certain extended connections. Dubrovin’s duality is treated in §4: the approach via external vector bundles with a pencil of flat connections makes Dubrovin’s construction more general and more transparent. Finally, in §5 I reproduce without proofs a representation theoretic description of formal flat manifolds from [LoMa1].

2. \(F\)-manifolds with compatible flat structures

Below I broadly follow the conventions of [Ma], Chapter I. “A manifold” may mean an analytic supermanifold over \(C\) (eventually over an exterior \(C\)-algebra carrying odd constants), or a germ of it, or a formal completion of it along an embedded closed submanifold. The sign rules are determined by the convention that De Rham differentials and connections are odd.

2.1. Compatible flat structures. An (affine) flat structure on a manifold \(M\) can be described in three equivalent ways:

(i) A torsionless flat connection \(\nabla_0 : T_M \to \Omega^1_M \otimes_{O_M} T_M\).

(ii) A local system \(T^f_M \subset T_M\) of flat vector fields, which forms a sheaf of supercommutative Lie algebras of rank \(\dim M\) such that \(T_M = O_M \otimes T^f_M\).

(iii) An atlas whose transition functions are affine linear.

The equivalence is established as follows: given \(\nabla_0\), we put \(T^f_M := \text{Ker} \nabla_0\); given a flat local map \((x^a)\), we define local sections of \(T^f_M\) as constant linear combinations of \(\partial_a = \partial/\partial x^a\). As a notation for a flat structure, we will use indiscriminately \(\nabla_0\) or \(T^f_M\).

Now consider a manifold \(M\) whose tangent sheaf is endowed with an \(O_M\)-bilinear (super)commutative and associative multiplication \(\circ\), and eventually with identity \(e\). In the following definition we do not assume that it satisfies (1.2).

2.2. Definition. (a) A flat structure \(T^f_M\) on \(M\) is called compatible with \(\circ\), if in a neighborhood of any point there exists a vector field \(C\) such that for arbitrary local flat vector fields \(X, Y\) we have

\[
X \circ Y = [X, [Y, C]].
\]

(2.1)

\(C\) is called a local vector potential for \(\circ\).

(b) \(T^f_M\) is called compatible with \((\circ, e)\), if (a) holds and moreover, \(e\) is flat.

2.3. Remarks. (i) If we choose a local flat coordinate system \((x^a)\) and write \(C = \sum C_c \partial_c\), \(X = \partial_a\), \(Y = \partial_b\), then (2.1) becomes

\[
\partial_a \circ \partial_b = \sum C_{ab}^c \partial_c, \quad C_{ab}^c = \partial_a \partial_b C^c.
\]

(2.2)
If moreover \( e \) is flat, we may choose local flat coordinates so that \( e = \partial_0 \), and the conditions \( e \circ \partial_b = \partial_b \) reduce to \( C_{0b}\partial_c = \partial_b \).

(ii) If we choose an arbitrary \( C \) and define a composition \( \circ : T^f_M \otimes T^f_M \to T^f_M \) by formula (2.1), it will be automatically supercommutative in view of the Jacobi formula. Associativity, however, is a quadratic differential constraint on \( C \) which was called “oriented associativity equations” in [LoMa2].

(iii) If \( C \) exists, it is not unique. As one sees from (2.2), locally it is defined modulo the span of vector fields \( \{ \partial_a, x^b \partial_c \} \) which form a subsheaf of Lie algebras in \( T^f_M \) depending on \( \nabla_0 \) and denoted \( T^f_M(1) \).

2.4. Proposition. Assume that a multiplication \( \circ \) on the tangent sheaf of \( M \) admits a compatible flat structure. Then it satisfies (1.2). Thus, \( (M, \circ) \) is an \( F \)-manifold.

Proof. The calculation is essentially the same as in the proof of Theorem 2 of [HeMa]. We reproduce it for completeness and because the context is slightly different.

First of all, identity (1.2) can be rewritten as follows: for any local vector fields \( X, Y, Z, W \) we have

\[
[X \circ Y, Z \circ W] - [X \circ Y, Z] \circ W - (-1)^{(X+Y)Z} Z \circ [X \circ Y, W] \\
- X \circ [Y, Z \circ W] - (-1)^{XY} Y \circ [X, Z \circ W] + X \circ [Y, Z] \circ W \\
+ (-1)^{YZ} X \circ Z \circ [Y, W] + (-1)^{XY} Y \circ [X, Z] \circ W \\
+ (-1)^{X(Y+Z)} Y \circ Z \circ [X, W] = 0. 
\]

(2.3)

This form is convenient because it turns out that the left-hand side of (2.3) is in fact a tensor, that is \( O_M \)-polylinear in \( X, Y, Z, W \): cf. [Me1,Me2] for a discussion and a generalization of this identity.

Therefore to verify (2.3) in our context it suffices to check that the left-hand side vanishes on all quadruples of local flat fields \( (\partial_a, \partial_b, \partial_c, \partial_d) \). Since flat fields (super)commute, the last four summands of (2.3) vanish, and only the first five ones should be taken care of. Let us denote the structure “constants” \( C_{ab}\partial_c \) as in (2.2). Calculating the coefficient of \( \partial_f \) in the left-hand side of (2.3), we represent it as a sum of the respective five summands, for which we introduce a special notation in order to explain the pattern of cancellation:

\[
\sum_e C_{ab}^e \partial_e C_{cd}^f - (-1)^{(a+b)(c+d)} \sum_e C_{cd}^e \partial_e C_{ab}^f = \alpha_1 + \beta_1, \\
(-1)^{(a+b)c} \sum_e \partial_e C_{ab}^e C_{ed}^f = \alpha_2, \quad (-1)^{(a+b+c)d} \sum_e \partial_d C_{ab}^e C_{ec}^f = \gamma_1, \\
-(-1)^{(b+c+d)} \sum_e \partial_b C_{cd}^e C_{ea}^f = \gamma_2, \quad -(-1)^{(c+d)b} \sum_e \partial_d C_{cd}^e C_{eb}^f = \beta_2.
\]

Here we write, say, \((-1)^{(a+b)c}\) as a shorthand for \((-1)^{(\tilde{\partial}_a + \tilde{x}_b)}\tilde{\partial}_c\).
Using the second formula (2.2), we can replace \( \partial_e C_{cd}^f \) in \( \alpha_1 \) by \((-1)^{ec} \partial_e C_{ed}^f \).

After this we see that

\[
\alpha_1 + \alpha_2 = (-1)^{(a+b)c} \partial_c \left( \sum_e C_{ab}^e C_{ed}^f \right).
\]

Similarly, permuting \( a \) and \( e \) in \( \beta_1 \) we find

\[
\beta_1 + \beta_2 = -(-1)^{(c+d)b} \partial_a \left( \sum_e C_{cd}^e C_{eb}^f \right).
\]

Now rewrite \( \gamma_1 \) permuting \( a, d, \) and \( \gamma_2 \) permuting \( b, c \). Calculating finally \( \beta_1 + \beta_2 + \gamma_1 + \gamma_2 \) we see that it cancels with \( \alpha_1 + \alpha_2 \) due to the associativity of \( \circ \) written as in [Ma, I (1.5)].

2.5. Pencils of flat connections. Let \( M \) be a supermanifold endowed with a torsionless flat connection \( \nabla_0 : T_M \to \Omega^1_M \otimes_{\mathcal{O}_M} T_M \) and an odd global section \( A \in \Omega^1_M \otimes_{\mathcal{O}_M} \text{End}(T_M) \) (it is called a Higgs field in other contexts).

We will use this operator in order to define two structures.

First, for any even constant \( \lambda \), consider a connection \( \nabla_\lambda = \nabla_0 + \lambda A \).

\[
\nabla_\lambda := \nabla_0 + \lambda A.
\]

Write the curvature form of \( \nabla_\lambda \) as

\[
\nabla^2_\lambda = \lambda R_1 + \lambda^2 R_2.
\]

Second, define an \( \mathcal{O}_M \)-bilinear composition law \( \circ = \circ^A \) on \( T_M \):

\[
X \circ^A Y := i_X(A)(Y), \quad i_X(df \otimes G) := Xf \cdot G.
\]

2.6. Proposition. (a) \( \circ^A \) is supercommutative if and only if all connections \( \nabla^A_\lambda \) are torsionless.

(b) Assume that (a) holds. Then \( \circ^A \) restricted to \( \nabla_0 \)-flat vector fields can be written everywhere locally in form (2.1) if and only if \( R_1 = 0 \).

(c) Assume that (a) holds. Then \( \circ^A \) is associative if and only if \( R_2 = 0 \).

2.7. Corollary. \( (M, \circ^A, \nabla_0) \) is an \( F \)-manifold with compatible flat structure if and only if \( \nabla^A_\lambda \) is a pencil of torsionless flat connections.

In this case, \( (M, \circ^A, \nabla^A_\lambda) \) is an \( F \)-manifold with compatible flat structure for any \( \lambda \) as well.
Proof of Proposition. (a) Omitting for brevity the superscripts $A$, we can write covariant derivatives as

$$\nabla_{\lambda, X}(Y) = \nabla_{0, X}(Y) + \lambda X \circ Y.$$ 

We will again work with a local basis of $\nabla_0$-flat vector fields $\{\hat{e}_a\}$. Then the vanishing of torsion of $\nabla_{\lambda}$ means that

$$\nabla_{\lambda, \hat{e}_a}(\hat{e}_b) = (-1)^{ab} \nabla_{\lambda, \hat{e}_b}(\hat{e}_a)$$

that is

$$\lambda \hat{e}_a \circ \hat{e}_b = (-1)^{ab} \lambda \hat{e}_b \circ \hat{e}_a$$

which is the supercommutativity of $\circ$.

(b) Define the structure/connection coefficients $A_{abc}^e$ by the formula $\hat{e}_a \circ \hat{e}_b = \sum_c A_{abc}^c \hat{e}_c$. Since we have assumed (a), these coefficients are symmetric in $a, b$. The vanishing of $R_1$ is equivalent to

$$\forall a, b, c, e, \quad \hat{e}_a A_{bc}^e = (-1)^{ab} \hat{e}_b A_{ac}^e.$$ 

This condition means that local 1-forms $\sum_b dx^b A_{bc}^e$ are closed. In view of the De Rham lemma, they are locally exact, which is equivalent to the existence of local functions $B_{ce}^e$ such that

$$A_{bc}^e = \hat{e}_b B_{ce}^e.$$ 

Again, since $A_{bc}^e$ is symmetric in $b, c$, the form $\sum_c dx^c B_{ce}^e$ is closed, and the De Rham lemma implies existence of some local functions $C^e$ such that $B_{ce}^e = \hat{e}_c C^e$. Hence the local vector field $C := \sum C^e \hat{e}_c$ determines our $\circ$ as in formula (2.1). Retracing these arguments in reverse order, one sees that if $\circ$ is locally of form (2.1), then $R_1 = 0$.

(c) Commutativity assumed, the equivalence of associativity with vanishing of $R_2$ is checked exactly as in the proof of Theorem 1.5. (b) in [Ma, p. 21], and we will not repeat it. This completes the proof of the Proposition 2.6. □

The first statement of Corollary 2.7 is thereby proved as well.

The second statement follows from the fact that changing the initial point on the affine line of flat torsionless connections does not affect the vanishing of torsion and curvature.

2.8. Auxiliary formulas. Let $(M, \circ, \nabla)$ be an $F$-manifold with a compatible flat structure. Here and below we write $\nabla$ in place of former $\nabla_0$. We will often use formulas
\( \nabla_X Y - \nabla_Y X = [X, Y] \) (vanishing torsion) and \( \nabla_{[X,Y]} = [\nabla_X, \nabla_Y] \) (vanishing curvature). If \( X \) is flat, \( \nabla_X Y = [X, Y] \). Put

\[
D(X, Y, Z) := \nabla_X (Y \circ Z) - \nabla_X (Y) \circ Z - (-1)^{XY} Y \circ \nabla_X Z \tag{2.6}
\]

(compare this with \( P_X(Y, Z) \) defined by (1.1) and depending only on \( \circ \)).

2.8.1. Lemma. D(X, Y, Z) is a symmetric tensor with values in \( \mathcal{T}_M \).

Proof. One first directly checks that \( D(X, Y, Z) - (-1)^{XY} D(Y, X, Z) \) is a tensor. Then to show that this difference vanishes identically it suffices to prove this for flat \( X, Y, Z \). Only two terms of six survive, and they can be rewritten in view of (2.1) as

\[
\nabla_X [Y, [Z, C]] - (-1)^{XY} \nabla_Y [X, [Z, C]] = [X, [Y, [Z, C]]] - (-1)^{XY} [Y, [X, [Z, C]]]
\]

which vanish thanks to Jacobi. Moreover, \( D(X, Y, Z) \) is obviously symmetric in \( Y, Z \), so symmetric in all three arguments. Finally, it is \( \Omega_M \)-linear in \( X \), which completes the argument. □

2.9. Nonflat identities. In this section, we collect for further use several formulas involving an \( F \)-manifold \( (M, \circ, \nabla, e) \) with a compatible flat structure but nonnecessarily flat identity.

Denote by \( L \) the set of global (or local) vector fields \( \epsilon \) satisfying the following condition:

\((*)\) for any local vector field \( Y \) we have

\[
\nabla_Y \epsilon = Y \circ \nabla_\epsilon \epsilon. \tag{2.7}
\]

Clearly, \( L \) is a (sheaf of) vector space(s). Notice that we could have postulated a formally weaker condition: \( \nabla_Y \epsilon = Y \circ l(\epsilon) \) for some functional \( l \), but putting \( Y = e \) we immediately get the unique possibility \( l(\epsilon) = \nabla_e \epsilon \).

Another useful remark is this: if (2.7) holds for \( \nabla \) (\( \epsilon \) being fixed), then it holds for any \( \nabla_{\lambda} \) in the relevant pencil of flat connections:

\[
\nabla_{\lambda, Y} \epsilon = \nabla_Y \epsilon + \lambda Y \circ \epsilon = Y \circ \nabla_{\lambda, \epsilon} \epsilon.
\]

More generally, if (2.7) holds for any single \( \nabla_{\lambda} \), it holds for all of them.

2.9.1. Proposition. (a) \( L \) contains \( e, \text{ Ker } \nabla_{\lambda} \) for all \( \lambda \), and is closed with respect to \( \nabla_{\epsilon} \) so that it contains the span of vector fields \( \{ \text{ Ker } \nabla_{\lambda}, e, \nabla_\epsilon e, \nabla_\epsilon^2 e, \ldots \} \).

(b) For any \( \epsilon \in L \) we have

\[
ad \epsilon = \nabla_\epsilon - \circ \nabla_\epsilon \epsilon. \tag{2.8}
\]
and

\[ P_\varepsilon(Y, Z) = D(\varepsilon, Y, Z) + Y \circ Z \circ \varepsilon. \quad (2.9) \]

(c) The operator \( \text{ad}_e \) is a ◦-derivation:

\[ [e, Y \circ Z] = [e, Y] \circ Z + Y \circ [e, Z]. \quad (2.10) \]

**Proof.** Clearly, \( \mathcal{L} \) contains \( \text{Ker} \nabla_\lambda \). We have from (2.6)

\[ D(e, e, Y) = -\nabla_{e\circ Y}, \quad D(Y, e, e) = -\nabla_{Y\circ e}. \]

These fields coincide since \( D \) is symmetric, which shows that \( e \in \mathcal{L} \).

Since generally \( [\varepsilon, Y] = \nabla_\varepsilon Y - \nabla_Y \varepsilon \), (2.8) is a consequence of (2.7). Inserting (2.8) in the definition of \( P_\varepsilon \), we get (2.9).

Putting \( X = Y = e \) in (1.2), we obtain (2.10). \( \square \)

Now we will check that \( \mathcal{L} \) is \( \nabla_e \)-stable. In fact, let \( \varepsilon \in \mathcal{L} \). Then

\[ \nabla_Y \nabla_e \varepsilon = \nabla_e \nabla_Y \varepsilon + \nabla_{[Y,e]} \varepsilon = \nabla_e (Y \circ \nabla_e \varepsilon) + [Y, e] \circ \nabla_e \varepsilon. \]

In the last expression, replace \( \nabla_e \) by \( \text{ad}_e + \circ \nabla_e e \) and use the fact that \( \text{ad}_e \) is a ◦-derivation. We get

\[ [e, Y \circ \nabla_e \varepsilon] + Y \circ \nabla_e e \circ \nabla_e e + [Y, e] \circ \nabla_e \varepsilon = Y \circ [e, \nabla_e \varepsilon] + Y \circ \nabla_e \varepsilon \circ \nabla_e e \]

which has the required form (2.7) so that \( \nabla_e \varepsilon \in \mathcal{L} \).

**2.10. Identities with** \( \nabla_e \varepsilon = ce \). Start with an \( F \)-manifold with a compatible flat structure and flat identity \( (M, \circ, e, \nabla_0) \). Replace \( \nabla_0 \) by some shifted connection \( \nabla := \nabla_0 + cA \) in the pencil of flat connections (2.4). In view of Corollary 2.7, it will still be compatible with \( \circ \), but \( e \) will be nonflat: \( \nabla_e e = ce \).

Conversely, if \( \nabla_e e = ce \) for some constant \( c \), then we can choose a new initial point \( \nabla_0 \) in the affine line of flat torsionless connections \( \nabla + \lambda A \) compatible with \( \circ \) in such a way that \( e \) will become \( \nabla_0 \)-flat: simply put \( \lambda = -c \).

Examples below show that an \( F \)-manifold may admit a ◦-invariant flat metric whose Levi–Civita connection belongs to the same affine line of flat torsionless connections compatible with ◦, but does not coincide with the point of flat identity.

**2.10.1. Examples.** (i) Here we will demonstrate that Dubrovin’s “almost Frobenius structure” [Du2, Definition 9] after forgetting the metric but retaining the affine flat structure \( \nabla \) corresponding to its Levi–Civita connection, becomes an \( F \)-manifold with
compatible flat structure and nonflat identity \( e \) satisfying the condition above: \( \nabla e e = c e \) with a constant \( c \).

In fact, the Levi–Civita connection \( \nabla \) is torsionless and flat. Compatibility condition (2.1) follows from Dubrovin’s potentiality formulas (3.20) and (3.21) of [Du2]. Finally, in a system of flat coordinates \( (x^a) \) (Dubrovin’s \( (p^i) \)), the identity \( e \) takes form
\[
\frac{1 - d}{2} \sum_a x^a \partial_a \quad \text{(Dubrovin’s identity/Euler field \( E \), formula (3.23) of [Du2]), where}
\]
d \( \neq 1 \) is a constant.

Hence \( \nabla e e = c e \) where \( c = \frac{(1 - d)^2}{4} \).

(ii) Quantum \( K \)-theory formal \( F \)-manifolds studied in [Gi,Lee] share a similar property: \( \nabla e e = e/2 \). I am thankful to Y.-P. Lee who has shown me a calculation establishing this (it was known to Givental). It follows directly from the form of Christoffel symbols, presented in [Lee, Section 5.2].

2.11. Summary. The “space” of flat affine connections compatible with a given multiplication \( \circ \) is fibered by affine lines. Without an additional structure, there is generally no way to single out a point on such a line. Point of a flat unity and point of the Levi–Civita connection of an invariant flat metric may in general diverge, although they coincide for Frobenius manifolds in the strict sense.

Therefore it makes sense to summarize the results of this section stressing the \( \lambda \)-coordinate free aspect.

Let us start with a structure \((M, \mathcal{P})\) where \( M \) is a (super)manifold, and \( \mathcal{P} \) is an affine line (pencil) of connections on \( T_M \). Such a line determines the following derivative structures on \( M \):

(i) A \( \mathbb{C}^\ast \)-torsor \( A \) of odd tensors \( A \in \Omega^1_M \otimes_{\mathcal{O}_M} \text{End}(T_M) \).

Namely, each \( A \in A \) is a difference \( \nabla_1 - \nabla_2 \) of some \( \nabla_1 \neq \nabla_2 \in \mathcal{P} \), and for any two such differences \( A, B \) we have \( A = \lambda B \) for some \( \lambda \in \mathbb{C}^\ast \).

(ii) A \( \mathbb{C}^\ast \)-torsor \( \mathcal{M} \) of \( \mathcal{O}_M \)-bilinear multiplications \( \circ^A \) on \( T_M \): for each \( A \in A \), we define \( X \circ^A Y := i_X(A)(Y) \).

One (and hence all) \( \circ^A \) is (are) supercommutative, if and only if two (and hence all) \( \nabla \in \mathcal{P} \) are torsionless. Assume that this condition is satisfied.

One (and hence all) \( \circ^A \) is (are) associative and admit local vector potentials, if and only if all \( \nabla \in \mathcal{P} \) are flat.

If one of the multiplications \( \circ^A \) has an identity \( e_A \), then each of them has an identity: \( A = \lambda B \) implies \( X \circ^A Y = \lambda X \circ^B Y \), so that \( e_A = \lambda^{-1} e_B \). Hence this is a property of the whole pencil \( \mathcal{P} \), which we will then call unital. All identities \( e_A \) form a \( \mathbb{C}^\ast \)-torsor as well.

For an unital pencil \( \mathcal{P} \) there may exist at most one \( \nabla_0 \in \mathcal{P} \) such that some (equivalently, any) \( e_A \) is \( \nabla_0 \)-flat. It exists if and only if \( e_A \) belongs to an eigenspace of some (equivalently, any) \( \nabla \in \mathcal{P} \). Such \( \mathcal{P} \) can be called flat unital.

(iii) The sheaf \( \mathcal{L} \) of such local vector fields \( \varepsilon \) that for any local vector field \( Y \), some (equivalently, any) \( \nabla \in \mathcal{P} \), and some (equivalently, any) \( A \in A \) there exists a vector field \( l(\varepsilon) \) such that \( \nabla_Y \varepsilon = Y \circ l(\varepsilon) \).

From Proposition 2.9.1 we know that \( \mathcal{L} \) contains \( \sum_{\nabla \in \mathcal{P}} \text{Ker} \ \nabla \). If \( \mathcal{P} \) is unital, then \( \mathcal{L} \) contains all identities \( e \) as well and is stable with respect to \( \nabla \) for any \( \nabla \in \mathcal{P} \).
3. Extended flat connections and Euler fields

3.1. Notation. We continue considering an $F$-manifold with a compatible flat structure and possibly nonflat identity $(M, \circ, \nabla, e)$. Let $\lambda$ be the coordinate on the affine line of flat connections as in (2.4). We will generally denote by $\hat{M}$ the total space of the constant family of manifolds $M$ over a base which might be the $\lambda$-affine line, or its projectivization, or formal completion, say, at $\lambda = \infty$ etc., to be specified in concrete situations. Let $p r_M : \hat{M} \to M$ be the projection on the base.

We will identify $T_M$ with the subsheaf of $\lambda$-independent vector fields in $T_{\hat{M}}$. We denote by $\partial_{\lambda}$ the vector field on $\hat{M}$ which annihilates $\mathcal{O}_M$ and such that $\partial_{\lambda}\lambda = 1$.

We will now consider $\hat{\nabla}$ defined by covariant derivatives along $\lambda$-independent vector fields $\hat{\nabla}_X := \nabla_X + \lambda X \circ$ (cf. (2.4)) as a part of a connection on $p r_M^*(T_M)$ over $\hat{M}$ (it might be meromorphic or formal). To complete (3.1) to a full connection $\hat{\nabla}$ on $p r_M^*(T_M)$ (not on $T_{\hat{M}}$!), we should choose a covariant derivative along $\partial_{\lambda}$ which on $Y \in p r_M^*(T_M)$ we require to be of the form

$$\hat{\nabla}_{\partial_{\lambda}}(Y) = \partial_{\lambda}Y + H(Y),$$

where $H$ is an even endomorphism of $p r_M^*(T_M))$ which we will have to allow to depend rationally or even formally of $\lambda$. We will write $\hat{\nabla}^H$ for $\hat{\nabla}$ if we need to stress its dependence on $H$.

3.2. Flatness conditions. We want $\hat{\nabla}^H$ to be flat. In this subsection we will spell the implied conditions on $H$, which are equivalent to $[\hat{\nabla}_X, \hat{\nabla}_{\partial_{\lambda}}] = 0$ for all $\lambda$-independent $X$. It suffices to check this identity by applying it to $\lambda$-independent fields $Y$. We have

$$\hat{\nabla}_X \hat{\nabla}_{\partial_{\lambda}}(Y) = \nabla_X H(Y) + \lambda X \circ H(Y),$$

where $\nabla_X$ and $\circ$ are extended to $p r_M^*(T_M)$ in the evident way. Furthermore,

$$\hat{\nabla}_{\partial_{\lambda}} \hat{\nabla}_X(Y) = \hat{\nabla}_{\partial_{\lambda}}(\nabla_X Y + \lambda X \circ Y)$$

$$= X \circ Y + H(\nabla_X Y) + \lambda H(X \circ Y).$$

Combining (3.3) and (3.4), we get the following reformulation of the flatness condition:

$$\forall X, Y, \quad H(X \circ Y) = X \circ H(Y) + \frac{1}{\lambda} (\nabla_X H(Y) - X \circ Y - H(\nabla_X Y)).$$
Putting here $Y = e$ and using $\nabla_X e = X \circ \nabla_e e$ we get a functional equation for $H$ which is in principle only necessary for $\nabla$ to be flat:

$$H(X) = X \circ H(e) + \frac{1}{\lambda} \left( \nabla_X H(e) - X - H(X \circ \nabla e) \right).$$

(3.6)

If (3.6) is satisfied, we can put here $X = e$ and get $\nabla_e H(e) = e + H(\nabla e)$, or better, since $\nabla_e H(e) = [e, H(e)] + \nabla_{H(e)} e = [e, H(e)] + H(e) \circ \nabla e$ (see (2.7) and Proposition 2.12.1),

$$[e, H(e)] + H(e) \circ \nabla e - H(\nabla e) = e.
(3.7)

3.3. Formal solutions to (3.6). Let $T$ be the space of global vector fields on $M$. Put $\mu := \lambda^{-1}$, $e_1 := \nabla_e e$, and

$$(e + \mu e_1)^{-1} := \sum_{i=0}^{\infty} (-1)^i (e_1)^{\circ i} \mu^i \in T[[\mu]].
(3.8)

3.3.1. Theorem. (a) There exists an 1–1 correspondence between the solutions to (3.6) $H \in \text{End} \, T[[\mu]]$ and the solutions $E \in T[[\mu]]$ to the equation

$$(e + \mu e_1) \circ \nabla_{(e + \mu e_1)^{-1}} e - e_1 \circ E = e.
(3.9)

Namely, from $E$ one reconstructs $H$ as follows:

$$H(X) := X \circ E + \mu(\nabla_{(e + \mu e_1)^{-1}} e X - (e + \mu e_1)^{-1} \circ X)
+ ((e + \mu e_1)^{-1} \circ X - X) \circ E,
(3.10)$$

and from $H$ one gets $E$ as $E = H(e)$.

(b) A solution $H = H_e$ as in (a) satisfies the flatness condition (3.5) if and only if for all $\nabla$-flat vector fields $X, Y$ we have

$$P_E(X, Y) - [X, [Y, C - \mu E]] = X \circ H(Y \circ e_1) + \mu[X, H(Y \circ e_1)],
(3.11)$$

where $C$ is defined by $X \circ Y = [X, [Y, C]]$ as in (2.1).

Proof. (a) Assume first that $H$ satisfies (3.6). Put $E := H(e)$. We will prove that (3.10) holds. In fact, (3.6) can be rewritten as

$$H(X) = A(X) + \mu B(X) + \mu H(C(X)),
(3.12)$$
where $A, B, C$ are $O_M[[\mu]]$-linear operators

$$A(X) = X \circ E, \quad B(X) = \nabla X E - X, \quad C(X) = -X \circ e_1.$$  \hfill (3.13)

We can replace in (3.12) $X$ by $C(X)$ and then put the obtained formula for $H(C(X))$ into the right-hand side of (3.12). Infinitely iterating this procedure, we obtain

$$H(X) = A(X) + \sum_{n=1}^{\infty} \mu^n B C^{n-1}(X) + \sum_{n=1}^{\infty} \mu^n A C^n(X)$$

$$= X \circ E + \mu B((e + \mu e_1)^{-1} \circ X) + A((-e + (e + \mu e_1)^{-1}) \circ X)$$

which is equivalent to (3.10).

Now put $X = e$ in (3.10), and multiply the result by $\mu^{-1}(e + \mu e_1)$. We will get (3.9).

Conversely, start with an arbitrary $E \in T[[\mu]]$ and define $H(X)$ by the formula (3.10). Retracing backwards the calculations above, one sees that $H(X)$ satisfies (3.12). This means that a version of (3.6) holds in which $H(e)$ is replaced by $E$. The additional condition $E = H(e)$ is then equivalent to (3.9). This completes the proof.

(b) Now consider the full flatness condition (3.5). Both sides of (3.5) are $O_M$-bilinear in $X, Y$ so that it suffices to check its meaning for $\nabla$-flat $X, Y$, and we will assume this in the following calculations. The term $H(\nabla X Y)$ in (3.5) will vanish. We replace $\nabla X$ by $[X, \star]$ and similarly for $\nabla Y$.

Replace $X$ in (3.6) first by $X \circ Y$ and then by $Y$. Put the resulting expressions for $H(X \circ Y)$ and $H(Y)$ into (3.5). After some cancelations and division by $\mu$ we find

$$\forall \text{ flat } X, Y,$$

$$\nabla_{X \circ Y} E = X \circ [Y, E] + [X, Y \circ E] - X \circ Y + \mu[X, [Y, E]] - X \circ H(Y \circ e_1)$$

$$- \mu[X, H(Y \circ e_1)].$$ \hfill (3.14)

Now rewrite the left-hand side of (3.14) using (2.6) and Lemma 2.8.1:

$$\forall \text{ flat } X, Y,$$

$$\nabla_{X \circ Y} E = [X \circ Y, E] + \nabla E(X \circ Y) = [X \circ Y, E] + D(E, X, Y)$$

$$= [X \circ Y, E] + D(X, Y, E) = [X \circ Y, E] + [X, Y \circ E] - (-1)^{XY} Y \circ [X, E].$$

Putting this into (3.14), after cancelations and regrouping we obtain (3.11). \hfill \Box
3.4. Extended connections and Euler fields. In this subsection we assume that $e_1 = ce$ for a constant $c$ (as in 2.13 above) and moreover, that $E := H(e)$ is independent of $i$. Shifting the base point in the relevant pencil of flat connections we may and will assume that $c = 0$. Equivalently, we should take $\mu/(1 + c\mu)$ for new $\mu$, and of course, adjust the notion of flatness. In this case (3.6) and (3.7) reduce respectively to

$$H(X) = X \circ E + \mu (\nabla_X E - X),$$

(3.15)

$$[e, E] = e.$$  

(3.16)

We now recall the definition from [HeMa].

3.4.1. Definition. (a) A vector field $E$ on an $F$-manifold $(M, \circ)$ is called an Euler field of weight $d_0$ if

$$P_E(X, Y) = d_0 X \circ Y.$$  

(3.17)

(b) The compatibility condition of $E$ with a compatible flat structure $\nabla$ reads

$$[E, \operatorname{Ker} \nabla] \subset \operatorname{Ker} \nabla.$$  

(3.18)

Local Euler fields on $(M, \circ)$ form a sheaf of linear spaces and Lie algebras. Weight is a linear function on this sheaf. Commutator of two Euler fields is an Euler field of weight zero. Identity is an Euler field of weight zero.

These statements hold for Euler fields compatible with a given $\nabla$ as well. Identity as an Euler field is compatible with $\nabla$ if it is $\nabla$-flat or more generally $e \in T^f_M$.

3.4.2. Proposition. The connection $\hat{\nabla}^H$ with $H$ of form (3.15) and $\lambda$-independent $E$ is flat if and only if $E$ is an Euler field of weight 1 compatible with $\nabla$.

Proof. In the general flatness condition (3.11) the right-hand side vanishes. Since $E$ is $\lambda$-independent, the term $[X, [Y, \mu E]]$ in the left-hand side must vanish as well which means that $[E, T^f_M] \subset T^f_M$. After that (3.11) reduces to (3.17) with $d_0 = 1$. This completes the proof. \qed

3.4.3. A further deformation of $\hat{\nabla}$. Assume that we are given an Euler field $E$ of weight 1 compatible with $\nabla$ and that $e \in T^f_M$. Then we can construct a line of Euler fields $E^{(s)} := E + se$ of weight 1 compatible with $\nabla$, and the respective deformation $\hat{\nabla}^{(s)}$ of $\hat{\nabla}$ obtained by using $E^{(s)}$ (or rather $E^{(s\mu)}$) in place of $E$ in (3.15).

See a discussion of this family in the context of Frobenius manifolds on pp. 154–157 of [He].
4. Dubrovin’s duality

4.1. Pencils of flat connections on an external bundle. Consider now a slight variation of the setup described in 2.11: a locally free sheaf $\mathcal{F}$ on a manifold $M$ and a pencil (affine line) $P$ of connections $\nabla : \mathcal{F} \to \Omega^1_M \otimes \mathcal{O}_M$. The difference of any two connections is now an odd global section $\nabla_1 - \nabla_0 = A \in \Omega^1_M \otimes \mathcal{O}_M \text{End} \mathcal{F}$. Any two differences are proportional; we will often choose one arbitrarily. Any $\nabla$ can be extended to an odd derivation (denoted again $\nabla$) of $\Omega^* \otimes \mathcal{O}_M T(\mathcal{F})$ in the standard way, where $T(\mathcal{F})$ is the total tensor algebra of $\mathcal{F}$.

From now on, we will assume that all connections in $P$ are flat. This means that for any $\nabla \in P$, $\nabla A = 0$ and $A \wedge A = 0$. Again, in view of the De Rham lemma, $A$ can be everywhere locally written as $\nabla B$ where $B = B_\nabla$ (it generally depends on $\nabla$) is an even section of $\text{End} \mathcal{F}$. Notice that we cannot assume $P$ to be torsionless: this makes no sense for an external bundle. We will see that this restriction can be replaced by fixing an additional piece of data.

Let now $U$ be a coordinate neighborhood in $M$ over which the linear superspace $F$ of local $\nabla$-flat sections of $\mathcal{F}$ trivializes $\mathcal{F}$. Denote by $\tilde{\mathcal{F}}$ the affine supermanifold associated with $\mathcal{F}$. Let $q : \tilde{\mathcal{F}} \to M$ be the fibration of supermanifolds which is “the total space” of $\mathcal{F}$ as a vector bundle: for example, in algebraic geometry this is the relative affine spectrum of $\text{Symm}_{\mathcal{O}_M}(\mathcal{F}^*)$, $\mathcal{F}^*$ being the dual sheaf. Hence local sections of $\mathcal{F}$ become local sections of $q$.

Clearly, $\nabla$ trivializes $q$ over $U$: we have a well-defined isomorphism $q^{-1}(U) = \tilde{\mathcal{F}} \times U$ turning $q$ into projection.

Let now $u \in F$ be a $\nabla$-flat section of $\mathcal{F}$ over $U$, $\nabla B = A$, $B \in \text{End} \mathcal{F}$. Then $Bu$ is a section of $\mathcal{F}$; we will identify it with a section of $q$ as above. Projecting to $\tilde{\mathcal{F}}$, we finally get a morphism $Bu : U \to \tilde{\mathcal{F}}$. Thus $Bu$ denotes several different although closely related objects; hopefully, this will not lead to a confusion.

In a less fancy language, if we choose a basis of flat sections in $F$ and a system of local coordinates $(x^a)$ on $M$, $B$ becomes a local even matrix function $B(x)$ acting from the left on columns of local functions. Then $Bu$ becomes the map $U \to \tilde{\mathcal{F}} : x \mapsto B(x)u$. Since $B$ is defined up to $\text{Ker} \nabla$, this map is defined up to a constant shift.

4.2. Definition. A section $u$ of $\mathcal{F}$ is called a primitive section with respect to $\nabla \in P$, if it is $\nabla$-flat, and $Bu$ is a local isomorphism of $U$ with a subdomain of $\tilde{\mathcal{F}}$.

Since $F$ is linear, $\tilde{\mathcal{F}}$ has a canonical flat structure $\nabla^F : T_{\tilde{\mathcal{F}}} \to \Omega^1_{\tilde{\mathcal{F}}} \otimes T_{\tilde{\mathcal{F}}}$. If $u$ is primitive, $Bu$ is a local isomorphism allowing us to identify locally $(Bu)^*(T_{\tilde{\mathcal{F}}}) = T_M$. Moreover, we can consider the pullback of $\nabla^F$ with respect to $Bu$:

$$\nabla^* := (Bu)^*(\nabla^F) : T_M \to \Omega^1_M \otimes T_M. \quad (4.1)$$

From the remarks above it is clear that the local flat structures induced by the maps $Bu$ on $M$ do not depend on local choices of $B$ and glue together to a flat structure on all of $M$ determined by $\nabla$ and $u$. 
It follows also that $\nabla^*$ is flat and torsionless.

There is another important isomorphism produced by this embedding. Namely, restricting $(Bu)^*$ to $F \subset T\tilde{F}$ we can identify $(Bu)^*(F) \subset T_F$ with $F \subset \mathcal{F}$ and then by $\mathcal{O}_M$-linearity construct an isomorphism $\beta^*: \mathcal{F} \rightarrow T_M$. The connection $\nabla^*$ identifies with $\nabla$ under this isomorphism. The inverse isomorphism $(\beta^*)^{-1}$ can be described as $X \mapsto i_X(\nabla Bu)$.

Denote $\mathcal{P}^*$ the pencil of flat connections $\beta^*(\mathcal{P})$.

4.2.1. Example. Assume that $(M, \circ, e, \nabla_0)$ is an $F$-manifold endowed with a compatible flat structure and a $\nabla_0$-flat identity. Put $\mathcal{F} := T_M$ and construct $A$ so that $X \circ Y = i_X(A)(Y)$. Then $e$ is a primitive section which induces exactly the initial flat structure $\nabla_0^* = \nabla_0$.

In fact, let $(x^a)$ be a $\nabla_0$-flat local coordinate system on $M$ such that $e = \partial_0$, and $(\partial_0, \ldots, \partial_m)$ the dual basis of flat vector fields. Then an easy argument shows that as a vector field, $Be = \sum_a (x^a + c^a) \partial_a$ where $c^a$ are constants.

The following theorem borrowed from [LoMa2] shows that a converse statement holds as well. We will reproduce the proof here.

4.3. Theorem. Let $(M, \mathcal{F}, \nabla, A, u)$ be (the data for) a pencil of flat connections on an external bundle endowed with a primitive section. Then $\mathcal{P}^*$ is a pencil of torsionless flat connections, and $e := \beta^*(u)$ is an identity for one of the associated $F$-manifold structures $\circ$.

Proof. We know that one point $\nabla^* \in \mathcal{P}^*$ is torsionless. The key remark is that one more point is torsionless, namely $\nabla^* + A^*$ where $A^* = \beta^*(A)$. Once we have established this, everything will follow from the results of §2.

To see this, let us work in local coordinates. Let $\partial_a$ be a basis of $F$ considered simultaneously as sections of $\mathcal{F}$ and of $T_M$, such that $\partial_0 = u$. Let $(x^a)$ be dual local coordinates. Let $A = \nabla^* B^*$ where $B^* := \beta^*(B)$. We represent $B^*$ as an even matrix function $(B^a_c)$ such that its action on sections of $\mathcal{F}$ or $T_M$ is given by

$$B^* \left( \sum_a f^a \partial_a \right) = \sum_a \left( \sum_c B^a_c f^c \right) \partial_a. \quad (4.2)$$

By construction, the map $X \mapsto i_X(\nabla^* B^*)u$ must be identical on $F$. It follows that $B^* u$ as a vector field must be of the form $\sum_a (x^a + c^a) \partial_a$ as in the example above.

We want to prove that

$$i_{\partial_a} (\nabla^* B^*) (\partial_b) = (-1)^{ab} i_{\partial_b} (\nabla^* B^*) (\partial_a) \quad (4.3)$$
or equivalently
\[ \forall c, \; d\omega^c = 0, \quad \omega^c := \sum_a dx^a B^c_a. \tag{4.4} \]

Now, from \( \nabla^* B^* \wedge \nabla^* B^* = 0 \) and \( \nabla^* u = 0 \) it follows that \( \nabla^* B^* \wedge \nabla^* B^* u = 0 \). Let us write this in coordinates:
\[
0 = (dB^c_c) \wedge \left( \sum_a dx^a \partial_a \right) = \sum_a \left( \sum_c d B^c_a dx^c \partial_a \right) \partial_a = \sum_a d\omega^a \partial_a.
\]
This proves (4.3) and (4.4). \( \Box \)

For more details and deeper results, see [LoMa2, §5].

4.4. The tangent bundle considered as an external bundle. Fix now a structure \((M, P)\) of a manifold with a unital pencil of torsionless flat connections as in Section 2.11. Assume moreover that one (hence each) identity \( e_A \) is flat with respect to some \( \nabla_0 \in P \). Choose as origin \( \nabla_0 \) and a coordinate \( u \) on \( P \) so that \((M, P)\) determines an \( F \)-manifold structure \( \circ \) with \( \nabla_0 \)-flat identity \( e \) and multiplication tensor \( A \).

In the following we will study the family of all \( F \)-manifold structures that can be obtained from this one by treating \( TM \) as an external bundle with the pencil \( P \), choosing different \( \nabla \in P \) and different \( \nabla \)-flat primitive sections, and applying Theorem 4.3.

4.4.1. Definition. An even global vector field \( \varepsilon \) is called a virtual identity, if it is invertible with respect to \( \circ \) and its inverse \( u := \varepsilon^{-1} \) belongs to \( \text{Ker} \nabla \) for some \( \nabla \in P \).

4.4.2. Proposition. (a) Inverted virtual identities in \( \text{Ker} \nabla \) are exactly primitive sections of \( (TM, \nabla) \) considered as an external bundle.
(b) The map \((\beta^*)^{-1}\) sends a local vector field \( X \) to \( X \circ u = X \circ \varepsilon^{-1} \).

Proof. As we have already remarked, \((\beta^*)^{-1}\) sends a vector field \( X \) to \( i_X(\nabla B)(u) \). Since \( A = \nabla B \), the last expression equals \( X \circ u \). Thus \( \circ \mapsto B(x)u \) is a local isomorphism if and only if the \( \circ \)-multiplication by \( u \) is an invertible operator, that is, \( \varepsilon \) is a virtual identity. This completes the proof. \( \Box \)

4.5. Dubrovin’s duality. Let now \((M, P, e, \varepsilon)\) be a flat unital pencil on \( TM \) endowed with a \( \nabla_0 \)-flat identity \( e \), and a \( \nabla \)-flat inverse virtual identity \( \varepsilon^{-1} \).

4.5.1. Theorem. (a) Denote by \( * \) the new multiplication on \( TM \) induced by \((\beta^*)^{-1}\) from \( \circ \):
\[
X \ast Y = \varepsilon^{-1} \circ X \circ Y. \tag{4.5}
\]
Then \((M, *, \varepsilon)\) is an \( F \)-manifold with identity \( \varepsilon \) (hence our term “virtual identity”).
(b) With the same notation, assume moreover that $\nabla_0 \neq \nabla$ and that $A$ is normalized as $A = \nabla - \nabla_0$. Then $e$ is an Euler field of weight one for $(M, *, \varepsilon)$.

**Proof.** (a) Clearly, $*$ is a commutative associative $\mathcal{O}_M$-bilinear multiplication. We will now exhibit the associated pencil of torsionless flat connections. We have

$$\nabla_X^*(Y) := \varepsilon \circ \nabla_X(\varepsilon^{-1} \circ Y).$$

More generally, the whole pencil $\mathcal{P}^* = \{ \nabla_{X,*}^* := \nabla_X^* + \lambda X^* \}$ is the pullback of $\mathcal{P}$ with respect to $Y \mapsto Y \circ u$:

$$(\nabla_X^* + \lambda X^*)Y = \varepsilon \circ \nabla_X(\varepsilon^{-1} \circ Y) + \varepsilon \circ (\lambda \varepsilon^{-1} \circ X \circ \varepsilon^{-1} \circ Y). \quad (4.6)$$

Hence the first part follows from the Theorem 4.3.

(b) We must now check the relevant versions of the Euler field relations (3.16) and (3.17):

$$[\varepsilon, e] = \varepsilon, \quad (4.7)$$

$$P_e^*(X, Y) := [e, X * Y] - [e, X] * Y - X * [e, Y] = X * Y. \quad (4.8)$$

In fact, we have $[\varepsilon, e] = -\nabla_0 \varepsilon \varepsilon$ because $\nabla_0 \varepsilon = 0$. Moreover, $\nabla_0 = \nabla - A$ and $\nabla \varepsilon = 0$, hence $\nabla_0 \varepsilon = -e \circ \varepsilon = -\varepsilon$ which shows (4.7).

Furthermore,

$$P_e^*(X, Y) = [e, \varepsilon^{-1} \circ X \circ Y] - \varepsilon^{-1} \circ [e, X] \circ Y - \varepsilon^{-1} \circ X \circ [e, Y].$$

Using the fact that $\text{ad } e$ is a $\circ$-derivation (cf. (2.10)), we can rewrite the first summand on the right-hand side. After cancelations we get

$$P_e^*(X, Y) = [e, \varepsilon^{-1}] \circ X \circ Y.$$ 

Finally,

$$[e, \varepsilon^{-1}] = -\varepsilon^{-2} \circ [e, e] = \varepsilon^{-1}$$

in view of (4.7), which completes the check of (4.8). $\square$

The last property to be checked is $[e, \ker \nabla_0^*] \subset \ker \nabla_0^*$ (notice that $\varepsilon$ is $\nabla_0^*$-flat, because $e$ is $\nabla_0$-flat). But $\ker \nabla_0^* = \varepsilon \circ \ker \nabla_0$, and for a $\nabla_0$-flat $X$ we have $[e, \varepsilon \circ X] = [e, \varepsilon] \circ X = -\varepsilon \circ X.$
4.5.2. Comments. (a) The relationship between \((M, \circ, e)\) and \((M, \ast, \varepsilon)\) is almost symmetric, but not quite. To explain this, we start with a part that admits a straightforward check:

\[
X \circ Y = e^{*-1} \ast X \ast Y \tag{4.9}
\]

which is the same as (4.5) with the roles of \((\circ, e)\) and \((\ast, \varepsilon)\) reversed. Here \(e^{*-1}\) denotes the solution \(v\) to the equation \(v \ast e = \varepsilon\), that is \(\varepsilon^{-1} \circ v \circ e = \varepsilon\), therefore \(v = e^{*-1} = e^{o2}\). After this remark one sees that (4.9) follows from (4.5).

Slightly more generally, one easily checks that the map inverse to \(X \mapsto \varepsilon^{-1} \circ X\) reads \(Y \mapsto e^{*-1} \ast Y\) as expected.

If the symmetry were perfect, we would now expect \(\varepsilon\) to be an Euler field of weight one for \((M, \circ, e)\), however, this contradicts (4.3)!

The reason of this is that the vector field \(e\) is not a virtual identity for \((M, \ast, \varepsilon)\): \(e^{*-1} = \varepsilon^2\) is not flat with respect to any \(\nabla^* \in \mathcal{P}^*\), so that if we start with \((M, \ast, \varepsilon)\) and construct \(\circ\) via (4.9), we cannot apply Theorem 4.5.1 anymore.

This remark suggests two ways of extending our construction.

First, one can iterate it by applying it to \((M, \ast, \varepsilon, \eta)\) where \(\eta\) is an arbitrary virtual identity thus getting new structures of an \(F\)-manifold.

Second, and more interesting, one can try to extend the definition of a virtual identity: say, call an eventual identity for \((M, \circ, e)\) any invertible vector field \(\varepsilon\) such that (4.1) defines a structure of an \(F\)-manifold. We have seen that virtual identities are eventual ones, but not necessarily vice versa. Can one give an independent characterization of eventual identities?

(b) If we take a pair consisting of a Frobenius manifold and the dual almost Frobenius manifold in the sense of [Du2], and retain only their \(F\)-manifold structures with the relevant flat structures, we will get a pair like ours \((M, \circ, e)\) and \((M, \ast, \varepsilon)\).

A reader willing to compare our work with [Du2], should look at this point at Dubrovin’s formulas (3.1) and (3.26) and replace Dubrovin’s \(\ast, \cdot, E, e, u, v\) respectively by our \(\circ, \ast, e, \varepsilon, X, Y\).

Our extension shows that the use of a metric is superfluous, so that the term “duality” is not quite justified in this context, because the construction is more like “twisting”. Moreover, in the realm of \(F\)-manifolds with compatible flat structure the construction becomes somewhat more transparent and natural.

5. Formal flat \(F\)-manifolds and toric compactifications

5.1. Notation. Let \(T\) be a finite-dimensional linear superspace over \(C\) (any characteristic zero ground field or a local Artin superalgebra \(k\) over it will do as well). Consider a basis \((\Delta_a | a \in I)\) of \(T\) and a system of linear coordinates \((x^a)\) on \(T\) and denote by \(M\) the formal completion of \(T\) at 0, that is, the formal spectrum of \(\mathcal{R} := C[[x^a]]\). The space of vector fields \(T_M\) on \(M\) can be canonically identified with \(\mathcal{R} \otimes T\): \(\Delta_a \mapsto \partial_a\).

Let \(\nabla\) be the torsionless free connection on \(T_M\) with kernel \(T\).
According to the results of §2, classification of all formal $F$-manifold structures on $M$ compatible with $\nabla$ is equivalent to the classification of the solutions to the matrix differential equation

$$\nabla C \wedge \nabla C = 0, \quad C \in m \otimes C \text{End } T,$$

(5.1)

where $m$ is the maximal ideal of $\mathcal{R}$.

Namely, given such a solution, we put

$$X \circ Y := i_X(\nabla C)(Y) = (XC)Y,$$

(5.2)

where $X$ acts as derivation on the entries of $C$.

The pencil of torsionless flat connections associated with $(M, \circ, \nabla)$ is $\nabla^+ / \mathfrak{afii9838} \nabla C$.

This remark allows us to apply to this situation a result of [LoMa1] which states essentially that the classification of solutions to (5.1) reduces to a problem in the representation theory of an associative algebra. We will summarize this result below, referring to [LoMa1] for proofs, and for more general statements about connections on an external bundle.

Notice that there exists another description of a formal $F$-manifold with compatible flat structure on $M$: it is the same as the structure of an operadic algebra over the oriented homology operad $(H_*, \text{or})(M_0, n+1)$. Orientation means that the zeroth structure section of each universal curve $C_{0,n+1} \to M_{0,n+1}$ is marked, the underlying combinatorial formalism involves rooted trees with root corresponding to the 0th section, and grafting is allowed only root–to–nonroot. The $n$th component of the operad $H_*, \text{or}(M_{0,n+1})$ is acted upon by the $n$th symmetric group $S_n$, rather than $S_{n+1}$.

For details of this operadic description and its generalizations, see [LoMa2].

5.2. $S$-algebras. Consider a graded associative $C$-algebra $V = \bigoplus_{n=1}^{\infty} V_n$ (without identity) in the category of vector superspaces. I will call it an $S$-algebra, if for each $n$, an action of the symmetric group $S_n$ on $V_n$ is given such that the multiplication map $V_m \otimes V_n \to V_{m+n}$ is compatible with the action of $S_m \times S_n$ embedded into $S_{m+n}$. Example: the tensor algebra (without the rank zero part) of a superspace $T$ is an $S$-algebra.

In any $S$-algebra $V$ the sum of subspaces $J_n$ spanned by $(1-s)v$, $s \in S_n$, $v \in V_n$, is a double-sided ideal in $V$. Hence the sum of the coinvariant spaces $V_S := \oplus_n V_{S_n} = V_n / J_n$ is a graded ring. I will denote it $V_S$.

If $V, W$ are two $S$-algebras, then the diagonal part of their tensor product $\bigoplus_{n=1}^{\infty} V_n \otimes W_n$ is an $S$-algebra as well.

5.3. Algebra $H_*$. We will now define an $S$-algebra whose $n$th component is the homology of the toric variety associated with the $n$th permutohedron.

5.3.1. Permutohedral fans. Let $B$ be a finite set. A partition $\sigma$ of $B$ is defined as a totally ordered set of pairwise disjoint nonempty subsets of $B$ whose union is $B$. If a
partition consists of \( N \) subsets, it is called an \( N \)-partition. If its components are denoted \( \sigma_1, \ldots, \sigma_N \), this means that they are listed in their structure order.

Let \( \tau \) be an \( N + 1 \)-partition of \( B \). If \( N \geq 1 \), it determines a well-ordered family of \( N \) 2-partitions \( \sigma^{(a)} \):

\[
\sigma^{(a)}_1 := \tau_1 \cup \cdots \cup \tau_a, \quad \sigma^{(a)}_2 := \tau_{a+1} \cup \cdots \cup \tau_{N+1}, \quad a = 1, \ldots, N.
\]

Call a sequence of \( N \) 2-partitions \( (\sigma^{(i)}) \) good if it can be obtained by such a construction.

Put \( N_B := \mathbb{Z}^B / \mathbb{Z} \), the latter subgroup being embedded diagonally. Similarly, \( N_B \otimes \mathbb{R} = \mathbb{R}^B / \mathbb{R} \). The vectors in this space (resp. lattice) will be written as functions \( B \to \mathbb{R} \) (resp. \( B \to \mathbb{Z} \)) considered modulo constant functions. For a subset \( \beta \subset B \), let \( \chi_\beta \) be the function equal \( 1 \) on \( \beta \) and \( 0 \) elsewhere.

The fan \( \Phi_B \) in \( N_B \otimes \mathbb{R} \), consists of certain \( l \)-dimensional cones \( C(\tau) \) labeled by \( (l + 1) \)-partitions \( \tau \) of \( B \).

If \( \tau \) is the trivial \( 1 \)-partition, \( C(\tau) = \{0\} \).

If \( \sigma \) is a 2-partition, \( C(\sigma) \) is generated by \( \chi_{\sigma_1} \), or, equivalently, \(-\chi_{\sigma_2}\), modulo constants.

Generally, let \( \tau \) be an \((l + 1)\)-partition, and \( \sigma^{(i)}, i = 1, \ldots, l \), the respective good family of 2-partitions. Then \( C(\tau) \) is defined as a cone generated by \( \{C(\sigma^{(i)})\} \).

### 5.3.2. Permutohedral toric varieties

Denote by \( L_B \) the toric variety which is the compactification of the torus \((\mathbb{G}_m)^B / \mathbb{G}_m\) associated with the fan \( \Phi_B \). One can prove that it is smooth and proper; its dimension is \( \text{card} \ B - 1 \). By construction, the permutation group of \( B \) acts upon it.

### 5.3.3. The \( S \)-algebra \( H_s \)

By definition, the \( n \)th component of \( H_s \) is the homology space \( H_{*n} := H_s(L_n) \) where we write now \( L_n \) for \( L_{\{1, \ldots, n\}} \). The whole homology space is spanned by certain cycles \( \mu(\tau) \) indexed, as well as cones of \( \Phi_B \), by partitions \( \tau \) of \( \{1, \ldots, n\} \): this is a part of the general theory of toric compactifications.

Define now a multiplication \( H_s(L_m) \times H_s(L_n) \to H_s(L_{m+n}) \): if \( \tau^{(1)} \) (resp. \( \tau^{(2)} \)) is a partition of \( \{1, \ldots, m\} \) (resp. of \( \{1, \ldots, n\} \)), then

\[
\mu(\tau^{(1)}) \mu(\tau^{(2)}) := \mu(\tau^{(1)} \cup \tau^{(2)}),
\]

where the concatenated partition of \( \{1, \ldots, m, m + 1, \ldots, m + n\} \) is defined in an obvious way, shifting all the components of \( \tau^{(2)} \) by \( m \). Of course, this map has a geometric origin coming from certain boundary morphisms \( L_m \times L_n \to L_{m+n} \).

### 5.4. Algebra \( H_s T \), its representations and matrix correlators

Let now \( T \) be a linear superspace considered in 5.1. Define \( H_s T \) as the algebra of coinvariants of the diagonal tensor product

\[
H_s T := \left( \bigoplus_{n=1}^\infty H_{*n} \otimes T \otimes n \right)_S.
\]
Consider a linear representation \( \rho : H^*_T \to \text{End} T \) and define the matrix correlators of \( \rho \) as the following family of endomorphisms of \( T \):

\[
\tau^{(n)} (\Delta_{a_1} \ldots \Delta_{a_n}) \rho := \rho(\mu(\tau^{(n)}) \otimes \Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n}).
\]

Here \( \tau^{(n)} \) runs over all partitions of \( \{1, \ldots, n\} \) whereas \( (a_1, \ldots, a_n) \) runs over all maps \( \{1, \ldots, n\} \to I : i \mapsto a_i \).

Top matrix correlators of \( \rho \) constitute the subfamily of correlators corresponding to the identical partitions \( \epsilon^{(n)} \) of \( \{1, \ldots, n\} \):

\[
\langle \Delta_{a_1} \ldots \Delta_{a_n} \rangle \rho := \epsilon^{(n)}(\Delta_{a_1} \ldots \Delta_{a_n}) \rho.
\]

The relevant cycle \( \mu(\epsilon^{(n)}) \) is the fundamental cycle of \( L_n \).

5.5. Main theorem. We can now state the basic correspondence between the solutions to (5.1) and representations \( \rho \).

Given \( \rho \), construct the series

\[
C_\rho = \sum_{n=1}^{\infty} \sum_{(a_1, \ldots, a_n)} x^{a_n} \ldots x^{a_1} \frac{1}{n!} \langle \Delta_{a_1} \ldots \Delta_{a_n} \rangle \rho \in \mathbb{C}[[x]] \otimes \text{End} T. \tag{5.3}
\]

5.5.1. Theorem. (a) We have

\[
\nabla C_\rho \wedge \nabla C_\rho = 0. \tag{5.4}
\]

(b) Conversely, let \( \Delta(a_1, \ldots, a_n) \in \text{End} T \) be a family of linear operators defined for all \( n \geq 1 \) and all maps \( \{1, \ldots, n\} \to I : i \mapsto a_i \). Assume that the parity of \( \Delta(a_1, \ldots, a_n) \) coincides with the sum of the parities of \( \Delta_{a_i} \) and that \( \Delta(a_1, \ldots, a_n) \) is (super)symmetric with respect to permutations of \( a_i \)'s. Finally, assume that the formal series

\[
C = \sum_{n=1}^{\infty} \sum_{(a_1, \ldots, a_n)} x^{a_n} \ldots x^{a_1} \frac{1}{n!} \Delta(a_1, \ldots, a_n) \in \mathbb{C}[[x]] \otimes \text{End} T \tag{5.5}
\]

satisfies Eqs. (5.1). Then there exists a well-defined representation \( \rho : H^*_T \to \text{End} T \) such that \( \Delta(a_1, \ldots, a_n) \) are the top correlators \( \langle \Delta_{a_1} \ldots \Delta_{a_n} \rangle \rho \) of this representation.

References


