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Construction of surfaces by discrete variational splines with parallelism conditions

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Abstract

We present in this paper a discrete problem of constructing some parametric surfaces with parallelism conditions from some given Lagrangean data. We consider the problem of fitting some scattered points with a surface *pseudo-parallel* to a given reference surface in a finite element space. Some convergence results are shown. Finally, we analyze some graphical examples in order to prove the validity and the effectiveness of this method.

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1. Introduction

In some fields such as oil exploration, geology and structural geology the problem of fitting of spatial data points by a surface constrained to be "parallel", in some sense, to a given reference surface is frequently encountered. The reconstruction of a stratigraphic structure when one layer is known and only a few data points in the other layers are available is a typical application. Offset (or parallel) curves and surfaces can provide solutions to the above problem, but only in very restricted situations.

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López de Silanes et al. [6] considered the case of parametric curves. They introduced the notion of *pseudo-parallel* curves, extending, in the plane, that of offset curves, and they proposed a fitting method based in the theory of smoothing D^m-splines (see [1]). Such work is extended in [5] to parametric surfaces (see also the references contained there).

This work completes [5,6] studying parametric surfaces by a discrete method in a finite element space. We proceed as follows. First, we consider the following problem: given a regular patch $\mathbf{x}_0 : \Omega \to \mathbb{R}^3$, where Ω denotes a nonempty open subset of $\mathbb{R}^2$, and two sets $\{\mathbf{u}_1, \ldots, \mathbf{u}_N\} \subset \Omega$ and $\{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \subset \mathbb{R}^3$, construct a C^1-patch $\mathbf{x} : \Omega \to \mathbb{R}^3$, such that $\mathbf{x}_0$ and $\mathbf{x}$ are pseudo-parallel and, for $i = 1, \ldots, N$, $\mathbf{x}(\mathbf{u}_i) = \mathbf{p}_i$. Then, we show how to find an approximate solution to this problem using discrete smoothing variational splines with parallelism conditions (see [4]). These splines minimize certain quadratic functional in a parametric finite element space and converge, under suitable hypotheses, to the solution, if any exists, of the above problem.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some preliminary notations and results. We recall in Section 3 the notion of pseudo-parallel surfaces. In Section 4 we formulate the discrete problem of fitting with pseudo-parallel surfaces and we obtain an approximation of the solution of such a problem by means of a discrete smoothing variational spline with parallelism conditions. In Section 5, we show the convergence of the solution of the discrete problem to the pseudo-parallel surface, if it exists, and the convergence of the tangent space of the surface parameterized by this variational spline to the tangent space of the given reference surface for any point in Ω. In order to test the discrete smoothing method in this paper, we present in the last section some numerical examples.

2. Notations and preliminaries

Let $n, N \in \mathbb{N}^*$ and $m \in \mathbb{N}$. We denote by $\langle \cdot \rangle_{\mathbb{R}^n}$ and $\langle \cdot , \cdot \rangle_{\mathbb{R}^n}$, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$.

Likewise, for any nonempty open set Ω in $\mathbb{R}^2$ we denote by $H^m(\Omega; \mathbb{R}^3)$ the usual Sobolev space of (classes of) functions $\mathbf{x}$ belonging to $L^2(\Omega; \mathbb{R}^3)$, together with all their partial derivatives $\partial^\gamma \mathbf{x}$ in the distribution sense of order $|\gamma| \leqslant m$, where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$ and $|\gamma| = \gamma_1 + \gamma_2$. This space is equipped with the norm

$$\|\mathbf{x}\|_{m,\Omega,\mathbb{R}^3} = \left(\sum_{|\gamma| \leqslant m} \int_\Omega \langle \partial^\gamma \mathbf{x}(\mathbf{u}) \rangle^2_{\mathbb{R}^3} \, \mathrm{d}\mathbf{u} \right)^{1/2},$$

the semi-norms

$$|\mathbf{x}|_{\ell,\Omega,\mathbb{R}^3} = \left(\sum_{|\gamma| = \ell} \int_\Omega \langle \partial^\gamma \mathbf{x}(\mathbf{u}) \rangle^2_{\mathbb{R}^3} \, \mathrm{d}\mathbf{u} \right)^{1/2}, \quad 0 \leqslant \ell \leqslant m,$$

and the corresponding inner semi-products

$$(\mathbf{x}, \mathbf{y})_{\ell,\Omega,\mathbb{R}^3} = \sum_{|\gamma| = \ell} \int_\Omega \langle \partial^\gamma \mathbf{x}(\mathbf{u}), \partial^\gamma \mathbf{y}(\mathbf{u}) \rangle_{\mathbb{R}^3} \, \mathrm{d}\mathbf{u}, \quad 0 \leqslant \ell \leqslant m.$$

Moreover, we denote by $\mathbb{P}_m(\Omega;\mathbb{R}^3)$ the space of all polynomials defined on Ω with values in $\mathbb{R}^3$ of degree $\leqslant m$.

We call parameterized surface to any differentiable mapping $\mathbf{x}:\Omega\to\mathbb{R}^3$, Ω being any open set in $\mathbb{R}^2$. The set $\mathbf{x}(\Omega)$ is the trace $\mathbf{x}$. For any $\mathbf{u}\in\Omega$, we write $T_{\mathbf{x}}(\mathbf{u})=\operatorname{span}\langle D_1\mathbf{x}(\mathbf{u}),D_2\mathbf{x}(\mathbf{u})\rangle$, where $D_1\mathbf{x}$ and $D_2\mathbf{x}$ denote the first partial derivatives of $\mathbf{x}$. $T_{\mathbf{x}}(\mathbf{u})$ is called the *tangent space* of $\mathbf{x}$ at $\mathbf{u}$. Likewise, we say that $\mathbf{x}$ is *regular* if, for all $\mathbf{u}\in\Omega$, the vectors $D_1\mathbf{x}$ and $D_2\mathbf{x}$ are linearly independent. In this case, for any $\mathbf{u}\in\Omega$, the *unit normal vector* $\mathbf{N}(\mathbf{u})$ to the surface $\mathbf{x}(\Omega)$ at the point $\mathbf{x}(\mathbf{u})$ is defined by

$$\mathbf{N}(\mathbf{u})=\frac{D_1\mathbf{x}(\mathbf{u})\times D_2\mathbf{x}(\mathbf{u})}{\langle D_1\mathbf{x}(\mathbf{u}),D_2\mathbf{x}(\mathbf{u})\rangle_{\mathbb{R}^3}},$$

where $\times$ stands for the vector product in $\mathbb{R}^3$. We denote by C a real strictly positive constant.

3. Fitting with pseudo-parallel surfaces

Let Ω be a nonempty open set of $\mathbb{R}^2$ and let $\mathbf{x}$ and $\mathbf{y}$ be two parameterized surfaces defined from Ω to $\mathbb{R}^3$.

Definition 3.1. We say that $\mathbf{x}$ and $\mathbf{y}$ are *pseudo-parallel* if, for any $\mathbf{u}\in\Omega$, $T_{\mathbf{x}}(\mathbf{u})\subset T_{\mathbf{y}}(\mathbf{u})$ or $T_{\mathbf{y}}(\mathbf{u})\subset T_{\mathbf{x}}(\mathbf{u})$.

Remark 3.1. If $\mathbf{x}$ and $\mathbf{y}$ are regular, then the previous definition becomes: $\mathbf{x}$ and $\mathbf{y}$ are *pseudo-parallel* if $T_{\mathbf{x}}(\mathbf{u})=T_{\mathbf{y}}(\mathbf{u})$ for any $\mathbf{u}\in\Omega$.

Suppose we are given:

- a connected, bounded, open subset Ω of $\mathbb{R}^2$;
- a regular parameterized surface $\mathbf{x}_0$ belonging to $C^1(\bar{\Omega};\mathbb{R}^3)$;
- an ordered finite set A of N_1 distinct points of $\bar{\Omega}$;
- for all $\mathbf{a}\in A$, a point $\mathbf{p_a}$ of $\mathbb{R}^3$.

Now, we consider the following problem:

Construct a parameterized surface $\mathbf{x}\in C^1(\bar{\Omega};\mathbb{R}^3)$ such that

$$\mathbf{x}_0 \text{ and } \mathbf{x} \text{ are pseudo-parallel and } \mathbf{x}(\mathbf{a})=\mathbf{p_a} \text{ for all } \mathbf{a}\in A. \tag{3.1}$$

For the discussions of the interpolations and the pseudo-parallelism conditions in (3.1) we suggest the readers to consult [5].

Hence, we can affirm that neither the existence nor the uniqueness of (3.1) is guaranteed. So, let us see how to avoid these situations. We fix a positive number $m\geqslant 3$ and we assume that Ω has a Lipschitz-continuous boundary and that

$$A \text{ contains a } \mathbb{P}_{m-1}(\Omega;\mathbb{R}^3)\text{-unisolvent subset.} \tag{3.2}$$

Now, we consider the following set:

$$\mathscr{P}=\{\mathbf{y}\in H^m(\Omega;\mathbb{R}^3)\,|\,\mathbf{y} \text{ is a solution of problem (3.1)}\}.$$

Theorem 3.1. *Suppose that hypothesis* (3.2) *holds and that the set* $\mathscr{P}$ *is nonempty. Then, there exists in* $\mathscr{P}$ *a unique element of minimal semi-norm* $|\cdot|_{m,\Omega,\mathbb{R}^3}$.

Proof. It is easy to prove that $\mathscr{P}$ is a closed and convex subset of $H^m(\Omega;\mathbb{R}^3)$. Then, the unique element of $\mathscr{P}$ of minimal semi-norm $|\cdot|_{m,\Omega,\mathbb{R}^3}$ turns out to be the projection on $\mathscr{P}$ of the null element of $H^m(\Omega;\mathbb{R}^3)$. □

4. Discrete fitting with pseudo-parallel surfaces

We maintain the hypotheses about $\mathbf{x}_0$, A and $\mathbf{p_a}$ for each $\mathbf{a}\in A$ stated in the previous section. But, now let Ω be a polygonal open subset of $\mathbb{R}^2$. We suppose that an ordered subset B of N_2 distinct points of $\bar{\Omega}$ is given.

Since $\mathbf{x}_0$ is a regular parameterized surface belonging to $C^1(\bar{\Omega};\mathbb{R}^3)$ it makes sense to define the mapping $\mathbf{N}\in C^0(\bar{\Omega};\mathbb{R}^3)$ such that each point $\mathbf{u}\in\bar{\Omega}$ is associated with the unit vector $\mathbf{N}(\mathbf{u})$ to the reference surface $\mathbf{x}_0(\bar{\Omega})$ at the point $\mathbf{x}_0(\mathbf{u})$.

Moreover, let Π be the bilinear form defined from $H^m(\Omega;\mathbb{R}^3)\times H^m(\Omega;\mathbb{R}^3)$ into $\mathbb{R}$ by

$$\Pi(\mathbf{y},\mathbf{z}) = \sum_{\mathbf{b}\in B}\sum_{i=1}^{2} \langle D_i\mathbf{y}(\mathbf{b}),\ \mathbf{N}(\mathbf{b})\rangle_{\mathbb{R}^3}\ \langle D_i\mathbf{z}(\mathbf{b}),\ \mathbf{N}(\mathbf{b})\rangle_{\mathbb{R}^3}.$$

Finally, suppose we are given:

- a subset $\mathscr{H}$ of $(0,+\infty)$ for which 0 is an accumulation point;
- for all $h\in\mathscr{H}$, a partition $\mathscr{T}_h$ of $\bar{\Omega}$ made of rectangles or triangles K of diameter $h_K \leqslant h$;
- for any $h\in\mathscr{H}$, a finite element space X_h constructed on $\mathscr{T}_h$ such that

$$X_h \subset H^m(\Omega)\cap C^k(\bar{\Omega}) \text{ with } k+1 \geqslant m. \tag{4.1}$$

For all $h\in\mathscr{H}$, we then define the parametric finite element space V_h constructed from X_h by $V_h=(X_h)^3$. From (4.1) we deduce

$$V_h \subset H^m(\Omega;\mathbb{R}^3)\cap C^k(\bar{\Omega};\mathbb{R}^3). \tag{4.2}$$

Now, for any $h\in\mathscr{H}$, $\varepsilon>0$ and $\tau>0$, let $J_{\varepsilon\tau}$ be the functional defined in V_h by

$$J_{\varepsilon\tau}(\mathbf{y}) = \sum_{\mathbf{a}\in A}\langle \mathbf{y}(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3} + \tau\Pi(\mathbf{y},\mathbf{y}) + \varepsilon|\mathbf{y}|^2_{m,\Omega,\mathbb{R}^3}. \tag{4.3}$$

Remark 4.1. For each $\mathbf{y}\in V_h$ the functional $J_{\varepsilon\tau}(\mathbf{y})$ can be interpreted as follows:

- The first term, $\sum_{\mathbf{a}\in A}\langle \mathbf{y}(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}$, indicates how well the function $\mathbf{y}$ approaches $\mathbf{p_a}$ in a discrete least-squares sense for each $\mathbf{a}\in A$.
- The second term, $\tau\Pi(\mathbf{y},\mathbf{y})$, represents a measure of the parallelism of $\mathbf{y}$ with respect to the parameterized surface $\mathbf{x}_0$, i.e. a measure of the mean proximity between the tangent spaces to the surfaces $\mathbf{y}$ and $\mathbf{x}_0$, which is weighted by the parameter τ.
- The last term, $\varepsilon|\mathbf{y}|^2_{m,\Omega,\mathbb{R}^3}$, represents a classical smoothness measure.

Then, in this situation we propose the problem of finding a parameterized surface $\mathbf{s}\in V_h$ that approximates the set of points $\{\mathbf{p_a}\}_{\mathbf{a}\in A}$ and is closed to be *pseudo-parallel* to the parameterized surface in the following sense: for each $\mathbf{u}\in\Omega$ the tangent space of $\mathbf{s}$ at $\mathbf{u}$ has to be close to the tangent space of $\mathbf{x}_0$ at $\mathbf{u}$.

Namely, for any $h\in\mathscr{H}$, $\varepsilon>0$ and $\tau>0$ we consider the following problem: find a parameterized surface $\mathbf{s}^h_{\varepsilon\tau}$ such that

$$\begin{cases} \mathbf{s}^h_{\varepsilon\tau}\in V_h, \\ \forall \mathbf{y}\in V_h, \quad J_{\varepsilon\tau}(\mathbf{s}^h_{\varepsilon\tau}) \leqslant J_{\varepsilon\tau}(\mathbf{y}). \end{cases} \tag{4.4}$$

Before proving the existence and the uniqueness of the solution of problem (4.4) we need the following result.

Lemma 4.1. *The mapping* $[[\cdot]]: H^m(\Omega;\mathbb{R}^3)\to\mathbb{R}$, *given by*

$$\mathbf{y}\mapsto[[\mathbf{y}]]=\left(\sum_{\mathbf{a}\in A}\langle\mathbf{y}(\mathbf{a})\rangle^2_{\mathbb{R}^3}+|\mathbf{y}|^2_{m,\Omega,\mathbb{R}^3}\right)^{1/2} \tag{4.5}$$

is a Hilbertian norm in $H^m(\Omega;\mathbb{R}^3)$ *equivalent to the Sobolev norm* $\|\cdot\|_{m,\Omega,\mathbb{R}^3}$.

Proof. This proof is analogous to the proof of Lemma 4.1 of [6] (see also [5, Lemma 3.2]). □

Theorem 4.2. *Problem* (4.4) *has a unique solution called discrete smoothing variational spline with parallelism conditions, which is also the unique solution of the following variational problem: find* $\mathbf{s}^h_{\varepsilon\tau}$ *such that*

$$\begin{cases} \mathbf{s}^h_{\varepsilon\tau}\in V_h, \\ \forall \mathbf{y}\in V_h, \quad \displaystyle\sum_{\mathbf{a}\in A}\langle\mathbf{s}^h_{\varepsilon\tau}(\mathbf{a}),\mathbf{y}(\mathbf{a})\rangle_{\mathbb{R}^3}+\tau\Pi(\mathbf{s}^h_{\varepsilon\tau},\mathbf{y})+\varepsilon(\mathbf{s}^h_{\varepsilon\tau},\mathbf{y})_{m,\Omega,\mathbb{R}^3}=\sum_{\mathbf{a}\in A}\langle\mathbf{p_a},\mathbf{y}(\mathbf{a})\rangle_{\mathbb{R}^3}. \end{cases} \tag{4.6}$$

Proof. Taking into account (3.2), (4.2) and Lemma 4.1, one easily checks that the symmetric bilinear form $\tilde{a}: V_h\times V_h\to\mathbb{R}$ given by

$$\tilde{a}(\mathbf{y},\mathbf{z})=\sum_{\mathbf{a}\in A}\langle\mathbf{y}(\mathbf{a}),\mathbf{z}(\mathbf{a})\rangle_{\mathbb{R}^3}+\tau\Pi(\mathbf{y},\mathbf{z})+\varepsilon(\mathbf{y},\mathbf{z})_{m,\Omega,\mathbb{R}^3}$$

is continuous and V_h-elliptic.

Likewise, the linear form $\mathbf{y}\in V_h\mapsto\sum_{\mathbf{a}\in A}<\mathbf{p_a},\ \mathbf{y}(\mathbf{a})_{\mathbb{R}^3}$ is continuous. The result is then a consequence of the Lax–Milgram Lemma (see [3]). □

Remark 4.2. Of course, if $\tau=0$ then we deal with a discrete smoothing D^m-spline in V_h relative to $A,(\mathbf{p_a})_{\mathbf{a}\in A}$ and ε (see [1]).

In the explicit case and when $\mathbf{x}_0=0$ we deal with the classical formulation of spline under tension, which can be interpreted as a parallelism condition with respect to the horizontal plane.

By construction, $\mathbf{s}^h_{\varepsilon\tau}$ approximates the points $(\mathbf{p_a})_{\mathbf{a}\in A}$ and, for each $\mathbf{b}\in B$, the tangent space of $\mathbf{s}^h_{\varepsilon\tau}$ at $\mathbf{b}$ is close to the tangent space of $\mathbf{x}_0$ at $\mathbf{b}$.

Now, let us see how to obtain in practice any discrete smoothing variational spline with parallelism conditions, but we assume that we know $(\mathbf{p}_\mathbf{a})_{\mathbf{a}\in A}$.

For any $h\in\mathscr{H}$, let I and $\{w_1,\dots,w_I\}$, be respectively, the dimension and a basis of X_h. Let $\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3$. Then, the family $\{\mathbf{v}_1,\dots,\mathbf{v}_Z\}$ is a basis of V_h, with $Z=3I$ and for any $i=I,\dots,I$, $\ell=1,2,3$, $j=3(i-1)+\ell$, $\mathbf{v}_j=w_i e_\ell$.

Thus, $\mathbf{s}^h_{\varepsilon\tau}$ can be written as $\mathbf{s}^h_{\varepsilon\tau}=\sum_{i=1}^{Z}\alpha_i\mathbf{v}_i$, where $\alpha_i\in\mathbb{R}$, for $i=1,\dots,Z$, are unknown coefficients. Applying Theorem 4.2 we obtain the linear system $M\alpha=b$, where

$$M=(m_{ij})_{1\leqslant i,j\leqslant Z},\quad \alpha=(\alpha_1,\dots,\alpha_Z)^{\mathrm{T}},\quad b=(b_1,\dots,b_Z)^{\mathrm{T}}$$

and for $i,j=1,\dots,Z$ we have

$$m_{ij}=\sum_{\mathbf{a}\in A}\langle\mathbf{v}_i(\mathbf{a}),\mathbf{v}_j(\mathbf{a})\rangle_{\mathbb{R}^3}+\tau\Pi(\mathbf{v}_i,\mathbf{v}_j)+\varepsilon(\mathbf{v}_i,\mathbf{v}_j)_{m,\Omega,\mathbb{R}^3},$$

$$b_i=\sum_{\mathbf{a}\in A}\langle\mathbf{p}_\mathbf{a},\mathbf{v}_i(\mathbf{a})\rangle_{\mathbb{R}^3}.$$

Finally, we point out that the matrix M is symmetric, positive definite and of band type.

5. Convergence

We maintain the hypotheses about $\mathbf{x}_0$, A and $\mathbf{p}_\mathbf{a}$, for each $\mathbf{a}\in A$, and suppose we are given:

- a subset $\mathscr{D}$ of $(0,+\infty)$ where 0 is an accumulation point;
- for all $d\in\mathscr{D}$, an ordered subset B^d of $N_2=N_2(d)$ distinct points of $\bar{\Omega}$ that verifies

$$\sup_{\mathbf{u}\in\Omega}\min_{\mathbf{b}\in B^d}\langle\mathbf{u}-\mathbf{b}\rangle_{\mathbb{R}^2}=d; \tag{5.1}$$

- for all $d\in\mathscr{D}$, a linear form $\Pi^d=\Pi(d)$ defined as Π with B^d instead of B.

Likewise, we suppose that the family $(\mathscr{T}_h)_{h\in\mathscr{H}}$ satisfies the inverse assumption (cf. [3]):

$$\exists\nu>0,\quad \forall h\in\mathscr{H},\quad \forall K\in\mathscr{T}_h,\quad \frac{h}{h_K}\leqslant\nu. \tag{5.2}$$

We also assume that the family $(B^d)_{d\in\mathscr{D}}$ satisfies

$$\exists C>0,\quad \forall d\in\mathscr{D},\quad \mathrm{card}(B^d)\leqslant\frac{C}{d^2}, \tag{5.3}$$

and that there exists a constant $C>0$ and, for any $h\in\mathscr{H}$, a linear operator $\rho_h:L^2(\Omega;\mathbb{R}^3)\to V_h$ verifying

(i) $\forall l=0,\dots,m,\quad \forall\mathbf{y}\in H^m(\Omega;\mathbb{R}^3)$,

$$|\mathbf{y}-\rho_h\mathbf{y}|_{l,\Omega,\mathbb{R}^3}\leqslant Ch^{m-l}|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}; \tag{5.4}$$

(ii) $\forall\mathbf{y}\in H^m(\Omega;\mathbb{R}^3)$, $\lim_{h\to 0}|\mathbf{y}-\rho_h\mathbf{y}|_{m,\Omega,\mathbb{R}^3}=0$.

Before proving some convergence results we need the following lemma.

Lemma 5.1. *We suppose that the hypotheses (5.2)–(5.4) hold. Then, there exists some constant $C>0$ such that for any $\mathbf{y}\in H^m(\Omega;\mathbb{R}^3)$, $d\in\mathscr{D}$ and $h\in\mathscr{H}$, one has*

$$\sum_{\mathbf{a}\in A}\langle(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{a})\rangle_{\mathbb{R}^3}^2\leqslant Ch^{2m-2}|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}^2 \tag{5.5}$$

and

$$\sum_{\mathbf{b}\in B^d}\langle D_i(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{b})\rangle_{\mathbb{R}^3}^2\leqslant C\,\frac{h^{2m-4}}{d^2}|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}^2,\quad i=1,2. \tag{5.6}$$

Proof. Reasoning as in the proof of Lemma 6.1 of [2], we deduce that there exists a constant $C>0$ such that for any $\mathbf{y}\in H^m(\Omega;\mathbb{R}^3)$, $d\in\mathscr{D}$ and $h\in\mathscr{H}$, and for any $K\in\mathscr{T}_h$, one has

$$\max_{\mathbf{u}\in K}\langle\mathbf{y}(\mathbf{u})\rangle_{\mathbb{R}^3}^2\leqslant Ch^{-2}\sum_{\ell=0}^{m}h^{2\ell}|\mathbf{y}|_{\ell,K,\mathbb{R}^3}^2, \tag{5.7}$$

and

$$\max_{\mathbf{u}\in K}\langle D_i\mathbf{y}(\mathbf{u})\rangle_{\mathbb{R}^3}^2\leqslant Ch^{-2}\sum_{\ell=0}^{m-1}h^{2\ell}|\mathbf{y}|_{\ell+1,K,\mathbb{R}^3}^2,\quad i=1,2. \tag{5.8}$$

Thus, taking $\rho_h\mathbf{y}-\mathbf{y}$ instead of $\mathbf{y}$ in (5.7), we deduce that

$$\begin{aligned}\sum_{\mathbf{a}\in A}\langle(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{a})\rangle_{\mathbb{R}^3}^2&\leqslant\sum_{K\in\mathscr{T}_h}\sum_{\mathbf{a}\in A\cap K}\langle(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{a})\rangle_{\mathbb{R}^3}^2\\&\leqslant Ch^{-2}N_1\sum_{K\in\mathscr{T}_h}\sum_{\ell=0}^{m}h^{2\ell}|\rho_h\mathbf{y}-\mathbf{y}|_{\ell,K,\mathbb{R}^3}^2\\&\leqslant Ch^{-2}N_1\sum_{\ell=0}^{m}h^{2\ell}|\rho_h\mathbf{y}-\mathbf{y}|_{\ell,\Omega,\mathbb{R}^3}^2.\end{aligned}$$

Hence, from (5.4), we have

$$\begin{aligned}\sum_{\mathbf{a}\in A}\langle(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{a})\rangle_{\mathbb{R}^3}^2&\leqslant Ch^{-2}N_1\sum_{\ell=0}^{m}h^{2\ell}h^{2m-2\ell}\,|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}^2\\&\leqslant Ch^{-2}N_1(m+1)h^{2m}|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}^2,\end{aligned}$$

and we conclude that (5.5) holds.

Analogously, taking $\rho_h\mathbf{y}-\mathbf{y}$ instead of $\mathbf{y}$ in (5.8) we deduce, for $i=1,2$, that

$$\sum_{\mathbf{b}\in B^d}\langle D_i(\rho_h\mathbf{y}-\mathbf{y})(\mathbf{b})\rangle_{\mathbb{R}^3}^2\leqslant Ch^{-2}N_2\sum_{\ell=0}^{m-1}h^{2\ell}|\rho_h\mathbf{y}-\mathbf{y}|_{\ell+1,\Omega,\mathbb{R}^3}^2.$$

Hence, from (5.4), we have

$$\sum_{\mathbf{b}\in B^d}\langle D_i(\rho_h\mathbf{y}-\mathbf{y})\rangle_{\mathbb{R}^3}^2\leqslant Ch^{-2}N_2mh^{2m-2}|\mathbf{y}|_{m,\Omega,\mathbb{R}^3}^2,$$

and, from (5.3), we conclude that (5.6) holds. □

Let $\varepsilon:\mathscr{D}\to(0,+\infty)$, $\tau:\mathscr{D}\to(0,+\infty)$ and $h:\mathscr{H}\to(0,+\infty)$ tree functions such that

$$\varepsilon(d)=\mathrm{o}(1),\quad d\to 0, \tag{5.9}$$

$$\lim_{d\to 0}\tau(d)=+\infty \tag{5.10}$$

and

$$\frac{\tau h^{2m-4}}{d^2\varepsilon}=\mathrm{o}(1),\quad d\to 0. \tag{5.11}$$

For simplicity, we write ε,τ and h instead of $\varepsilon(d)$, $\tau(d)$ and $h(d)$. Now, for any $d\in\mathscr{D}$, let $\mathbf{s}^{hd}_{\varepsilon\tau}$ be the unique solution of problem: find $\mathbf{s}^{dh}_{\varepsilon\tau}$ such that

$$\mathbf{s}^{dh}_{\varepsilon\tau}\in V_h,\quad J^d_{\varepsilon\tau}=\inf_{\mathbf{y}\in V_h}J^d_{\varepsilon\tau}(\mathbf{y}),$$

where $J^d_{\varepsilon\tau}$ is defined as $J_{\varepsilon\tau}$ in (4.3) with Π^d instead of Π, $\varepsilon=\varepsilon(d)$ and $\tau=\tau(d)$.

Finally, we suppose that the set $\mathscr{P}$ is nonempty and let $\mathbf{s}$ the unique element of minimal semi-norm $|\cdot|_{m,\Omega,\mathbb{R}^3}$ in $\mathscr{P}$.

Theorem 5.2. *Suppose that the hypotheses* (3.2), (5.1)–(5.4) *and* (5.9)–(5.11) *hold. Then*, *one has*

$$\lim_{d\to 0}\|\mathbf{s}^{dh}_{\varepsilon\tau}-\mathbf{s}\|_{m,\Omega,\mathbb{R}^3}=0.$$

Proof. (1) For all $d\in\mathscr{D}$, we have

$$\begin{aligned}J^d_{\varepsilon\tau}(\mathbf{s}^{dh}_{\varepsilon\tau})&=\sum_{\mathbf{a}\in A}\langle\mathbf{s}^{dh}_{\varepsilon\tau}(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}+\tau\Pi^d(\mathbf{s}^{dh}_{\varepsilon\tau},\mathbf{s}^{dh}_{\varepsilon\tau})+\varepsilon|\mathbf{s}^{dh}_{\varepsilon\tau}|^2_{m,\Omega,\mathbb{R}^3}\\&\leqslant J^d_{\varepsilon\tau}(\mathbf{s}^h),\end{aligned} \tag{5.12}$$

where $\mathbf{s}^h=\rho_h\mathbf{s},\rho_h$ being the operator given in (5.4). This means that

$$\mathbf{s}^{dh}_{\varepsilon\tau}|^2_{m,\Omega,\mathbb{R}^3}\leqslant\frac{1}{\varepsilon}\sum_{\mathbf{a}\in A}\langle\mathbf{s}^h(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}+\frac{\tau}{\varepsilon}\Pi^d(\mathbf{s}^h,\mathbf{s}^h)+|\mathbf{s}^h|^2_{m,\Omega,\mathbb{R}^3}. \tag{5.13}$$

From Lemma 5.1 there exists $C_1>0$ such that

$$\sum_{\mathbf{a}\in A}\langle\mathbf{s}^h(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}=\sum_{\mathbf{a}\in A}\langle(\rho_h\mathbf{s}-\mathbf{s})(\mathbf{a})\rangle^2_{\mathbb{R}^3}\leqslant C_1h^{2m-2}|\mathbf{s}|^2_{m,\Omega,\mathbb{R}^3}. \tag{5.14}$$

Furthermore, we have

$$\begin{aligned}\Pi^d(\mathbf{s}^h,\mathbf{s}^h)&=\sum_{\mathbf{b}\in B^d}\sum_{i=1,2}\langle D_i\mathbf{s}^h(\mathbf{b}),\mathbf{N}(\mathbf{b})\rangle^2_{\mathbb{R}^3}\\&=\sum_{\mathbf{b}\in B^d}\sum_{i=1,2}\langle D_i(\rho_h\mathbf{s}-\mathbf{s})(\mathbf{b}),\mathbf{N}(\mathbf{b})\rangle^2_{\mathbb{R}^3},\end{aligned}$$

which implies, by using Lemma 5.1, that there exists $C_2 > 0$ such that

$$\Pi^d(\mathbf{s}^h, \mathbf{s}^h) \leqslant C_2 \frac{h^{2m-4}}{d^2} |\mathbf{s}|^2_{m,\Omega,\mathbb{R}^3}. \tag{5.15}$$

From (5.4) it is easy to see that

$$|\mathbf{s}^h|^2_{m,\Omega,\mathbb{R}^3} = \mathrm{o}(1) + |\mathbf{s}|^2_{m,\Omega,\mathbb{R}^3}, \quad d \to 0. \tag{5.16}$$

Finally, using (5.14)–(5.16) on (5.13) we deduce that

$$|\mathbf{s}^{dh}_{\varepsilon\tau}|^2_{m,\Omega,\mathbb{R}^3} \leqslant \left(C_1 \frac{h^{2m-2}}{\varepsilon} + C_2 \frac{h^{2m-4}}{\varepsilon d^2} + \mathrm{o}(1) + 1 \right) |\mathbf{s}|^2_{m,\Omega,\mathbb{R}^3}, \quad d \to 0. \tag{5.17}$$

From (5.9)–(5.11) we can obtain that $\lim_{d\to 0} h(d) = \lim_{d\to 0} (h^{2m-2}/\varepsilon) = 0$ and, taking into account Lemma 4.1 and (5.17), we deduce that there exists a constant $C > 0$ such that, for any $d \in \mathscr{D}$, $\|\mathbf{s}^{dh}_{\varepsilon\tau}\|_{m,\Omega,\mathbb{R}^3} \leqslant C$.

Hence, the family $(\mathbf{s}^{dh}_{\varepsilon\tau})_{d\in\mathscr{D}}$ is bounded in $H^m(\Omega;\mathbb{R}^3)$. Then, there exists a sequence $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell})_{\ell\in\mathbb{N}}$ extracted from such a family with

$$\lim_{\ell\to+\infty} d_\ell = \lim_{\ell\to+\infty} h_\ell = \lim_{\ell\to 0} \varepsilon_\ell = 0, \quad \lim_{\ell\to+\infty} \tau_\ell = +\infty \quad \text{and} \quad \lim_{\ell\to+\infty} (\tau_\ell h_\ell^{2m-4}/d_\ell^2 \varepsilon_\ell) = 0$$

and there also exists an element $\mathbf{s}^* \in H^m(\Omega;\mathbb{R}^3)$ such that

$$\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell} \text{ converges weakly to } \mathbf{s}^* \text{ in } H^m(\Omega;\mathbb{R}^3) \text{ as } \ell = \to +\infty. \tag{5.18}$$

Therefore, using (5.9)–(5.11) and (5.17), we have

$$|\mathbf{s}^*|_{m,\Omega,\mathbb{R}^3} \leqslant \liminf_{\ell\to+\infty} |\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}|_{m,\Omega,\mathbb{R}^3} \leqslant \limsup_{\ell\to+\infty} |\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}|_{m,\Omega,\mathbb{R}^3} \leqslant |\mathbf{s}|_{m,\Omega,\mathbb{R}^3}. \tag{5.19}$$

Likewise, we obtain

$$\forall i = 1,2, \quad \forall \mathbf{u} \in \Omega, \quad |\langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{u}) - D_i \mathbf{s}^*(\mathbf{u}), \mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}|$$
$$\leqslant \langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{u}) - D_i \mathbf{s}^*(\mathbf{u})\rangle_{\mathbb{R}^3} \leqslant \|\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell} - \mathbf{s}^*\|_{C^1(\Omega,\mathbb{R}^3)},$$

which implies, together with (5.18), that

$$\forall i = 1,2, \quad \lim_{\ell\to+\infty} \sup_{\mathbf{u}\in\Omega} |\langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{u}) - D_i \mathbf{s}^*(\mathbf{u}), \mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}| = 0. \tag{5.20}$$

(2) Let us see that $\mathbf{s}^* = \mathbf{s}$. For this, we will prove that $\mathbf{s}^*$ is pseudo-parallel to $\mathbf{x}_0$ and also that $\mathbf{s}^*(\mathbf{a}) = \mathbf{p_a}$, for all $\mathbf{a} \in A$.

We argue by contradiction. Suppose that there exists $\mathbf{u}_0 \in \Omega$ and $i \in \{1,2\}$ such that $\langle D_i \mathbf{s}^*(\mathbf{u}_0), \mathbf{N}(\mathbf{u}_0)\rangle_{\mathbb{R}^3} \neq 0$. Then, by the continuity of the mapping $\mathbf{u} \mapsto \langle D_i \mathbf{s}^*(\mathbf{u}), \mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}$, for any $\mathbf{u} \in \Omega$, there exists a constant $\gamma > 0$ and an open neighbourhood ω of $\mathbf{u}_0$ such that

$$\forall \mathbf{u} \in \omega, \quad |\langle D_i \mathbf{s}^*(\mathbf{u}), \mathbf{N}(\mathbf{s})\rangle_{\mathbb{R}^3}| > \gamma. \tag{5.21}$$

From (5.20) and (5.21), we have

$$\exists \ell_0 \in \mathbb{N}, \quad \forall \ell \geqslant \ell_0, \quad \forall \mathbf{u} \in \omega, \quad |\langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{u}), \mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}| > \frac{\gamma}{2}. \tag{5.22}$$

Now, from (3.2), there follows the existence of a sequence $(\mathbf{b}_\ell)_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \to +\infty} \mathbf{b}_\ell = \mathbf{u}_0$ and, for all $\ell \in \mathbb{N}$, $\mathbf{b}_\ell \in B^{d_\ell}$. Thus, there exists $\ell_1 \in \mathbb{N}$ such that, for all $\ell \geqslant \ell_1$, $\mathbf{b}_\ell \in \omega \cap B^{d_\ell}$. This implies, together with (5.22), that

$$\forall \ell \geqslant \max\{\ell_0, \ell_1\}, \quad |\langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{b}_\ell), \mathbf{N}(\mathbf{b}_\ell)\rangle_{\mathbb{R}^3}| > \frac{\gamma}{2}. \tag{5.23}$$

But, from (5.9)–(5.16), we have

$$\lim_{\ell \to +\infty} |\langle D_i \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{b}_\ell), \mathbf{N}(\mathbf{b}_\ell)\rangle_{\mathbb{R}^3}| = 0,$$

leading to a contradiction with (5.23). Therefore $\mathbf{s}^*$ is pseudo-parallel to $\mathbf{x}_0$.

Moreover, from (5.14) we have

$$\forall \mathbf{a} \in A, \quad \lim_{\ell \to +\infty} \mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}(\mathbf{a}) = \mathbf{p_a}, \tag{5.24}$$

which implies, together with (5.18), that for all $\mathbf{a} \in A$, $\mathbf{s}^*(\mathbf{a}) = \mathbf{p_a}$.

Thus, taking into account that $\mathbf{s}^* \in H^m(\Omega, \mathbb{R}^3)$, we derive that $\mathbf{s}^*$ belongs to $\mathscr{P}$. Since $\mathbf{s}$ is the unique element of minimal semi-norm $|\cdot|_{m,\Omega,\mathbb{R}^3}$ in the set $\mathscr{P}$, we conclude from (5.17) and (5.18) that $\mathbf{s}^* = \mathbf{s}$.

(3) From (5.19), we have that $\lim_{\ell \to +\infty} |\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}|_{m,\Omega,\mathbb{R}^3} = |\mathbf{s}|_{m,\Omega,\mathbb{R}^3}$. Using (5.24), we then deduce that $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell})_{\ell \in \mathbb{N}}$ is a sequence weakly convergent to $\mathbf{s}$ in $H^m(\Omega; \mathbb{R}^3)$ such that $\lim_{\ell \to +\infty} [[\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell}]] = [[\mathbf{s}]]$. Therefore, from Lemma 4.1, $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell})_{\ell \in \mathbb{N}}$ is strongly convergent to $\mathbf{s}$ in $H^m(\Omega, \mathbb{R}^3)$, i.e.,

$$\lim_{\ell \to +\infty} \|\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell} - \mathbf{s}\|_{m,\Omega,\mathbb{R}^3} = 0.$$

(4) To complete the proof we argue by contradiction.

Suppose that $\lim_{d \to 0} \|\mathbf{s}^{dh}_{\varepsilon\tau} - \mathbf{s}\|_{m,\Omega,\mathbb{R}^3} \neq 0$. This means that there exists a real number $\gamma > 0$ and a sequence $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell})_{\ell \in \mathbb{N}}$ with

$$\lim_{\ell \to +\infty} d_\ell = \lim_{\ell \to +\infty} h_\ell = \lim_{\ell \to 0} \varepsilon_\ell = 0, \quad \lim_{\ell \to +\infty} \tau_\ell = +\infty \quad \text{and} \quad \lim_{\ell \to +\infty} (\tau_\ell h_\ell^{2m-4}/d_\ell^2 \varepsilon_\ell) = 0$$

such that

$$\forall \ell \in \mathbb{N}, \quad \|\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell} - \mathbf{s}\|_{m,\Omega,\mathbb{R}^3} > \gamma. \tag{5.25}$$

By reasoning for the family $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell \tau_\ell})_{\ell \in \mathbb{N}}$ as for the family $(\mathbf{s}^{dh}_{\varepsilon\tau})_{d \in \mathscr{D}}$ we arrive at a contradiction with (5.25).

In consequence, $\lim_{d \to 0} \|\mathbf{s}^{dh}_{\varepsilon\tau} - \mathbf{s}\|_{m,\Omega,\mathbb{R}^3} = 0$. □

We observe that an essential hypothesis of Theorem 5.2 is that $\mathscr{P}$ is nonempty. But, in practice, due to the lack of regularity of the existing solution or, simply, to the contradiction between pseudo-parallelism and interpolation conditions, usually $\mathscr{P} = \emptyset$. In such cases, if the quality of the fit of the points is not crucial, the following result show that "almost pseudo-parallel" surfaces can be still obtained. For this, (5.9)–(5.11) should be replaced by the following hypothesis:

$$\inf_{d \in \mathscr{D}} \varepsilon(d) > 0, \quad \lim_{d \to 0} \tau(d) = +\infty. \tag{5.26}$$

Theorem 5.3. *Suppose that the hypotheses* (3.2), (5.1)–(5.4) *and* (5.26) *hold. Then, for each* $\mathbf{u}\in\Omega$ *and for* $i=1,2$, *one has*

$$\limsup_{d\to 0}\langle D_i \mathbf{s}^{dh}_{\varepsilon\tau}, \mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}=0.$$

Proof. For all $d\in\mathscr{D}$, we have

$$\begin{aligned} J^d_{\varepsilon\tau}(\mathbf{s}^{dh}_{\varepsilon\tau}) &= \sum_{\mathbf{a}\in A}\langle \mathbf{s}^{dh}_{\varepsilon\tau}(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}+\tau\Pi^d(\mathbf{s}^{dh}_{\varepsilon\tau},\mathbf{s}^{dh}_{\varepsilon\tau})+\varepsilon|\mathbf{s}^{dh}_{\varepsilon\tau}|^2_{m,\Omega,\mathbb{R}^3}\\ &\leqslant J^d_{\varepsilon\tau}(\mathbf{0})=\sum_{\mathbf{a}\in A}\langle \mathbf{p_a}\rangle^2_{\mathbb{R}^3}.\end{aligned}$$

Hence, the family $(\mathbf{s}^{dh}_{\varepsilon\tau})_{d\in\mathscr{D}}$ satisfies the relations

$$\forall \mathbf{s}\in A,\quad \langle \mathbf{s}^{dh}_{\varepsilon\tau}(\mathbf{a})-\mathbf{p_a}\rangle^2_{\mathbb{R}^3}\leqslant\sum_{\mathbf{a}\in A}\langle \mathbf{p_a}\rangle^2_{\mathbb{R}^3}$$

and

$$|\mathbf{s}^{dh}_{\varepsilon\tau}|^2_{m,\Omega,\mathbb{R}^3}\leqslant\frac{1}{\inf_{d\in\mathscr{D}}\varepsilon(d)}\sum_{\mathbf{a}\in A}\langle \mathbf{p_a}\rangle^2_{\mathbb{R}^3}.$$

Taking into account Lemma 4.1 we deduce that the family $(\mathbf{s}^{dh}_{\varepsilon\tau})_{d\in\mathscr{D}}$ is bounded in $H^m(\Omega;\mathbb{R}^3)$. Thus, there exists a sequence $(\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell\tau_\ell})_{\ell\in\mathbb{N}}$ extracted from such a family with

$$\lim_{\ell\to+\infty} d_\ell=0,\quad \liminf_{\ell\to+\infty}\varepsilon_\ell>0\quad\text{and}\quad\lim_{\ell\to+\infty}\tau_\ell=+\infty,$$

and there also exists an element $\mathbf{s}^*\in H^m(\Omega;\mathbb{R}^3)$ such that

$$\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell\tau_\ell}\text{ converges weakly to }\mathbf{s}^*\text{ in }H^m(\Omega;\mathbb{R}^3)\text{ as }\ell\to+\infty.$$

By reasoning as the proof of Theorem 5.2, we can prove that

$$\forall i=1,2,\quad\forall\mathbf{u}\in\Omega,\quad\langle D_i\mathbf{s}^*(\mathbf{u}),\mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}=0$$

and

$$\forall i=1,2,\quad\forall\mathbf{u}\in\Omega,\quad\limsup_{\ell\to+\infty}|\langle D_i\mathbf{s}^{d_\ell h_\ell}_{\varepsilon_\ell\tau_\ell},\mathbf{N}(\mathbf{u})\rangle_{\mathbb{R}^3}=0.$$

Then, to complete the proof, it suffices to argue by contradiction as in point (4) of the proof of Theorem 5.2. □

Observe that the previous result is not depending of h.

6. Numerical and graphical results

To check the fitting method presented in Section 4, we have considered problem (4.4) with two different data sets. For the first example, we have taken

- $\Omega = (-2, 2) \times (-2, 2)$,
- $\mathbf{x}_0(u, v) = (u + (u^2 + v^2 + 1)^{-1}, v + (u^2 + v^2 + 1)^{-1}, 2(u + v)(u^2 + v^2 + 1)^{-1})$,
- $A = \{(-2 + 2i/3, -2 + 2j/3) \mid i = 0, \ldots, 6,\ j = 0, \ldots, 6\}$,
- for all $\mathbf{a} = (a_1, a_2) \in A$, $\mathbf{p_a} = \mathbf{x}_0(\mathbf{a}) + 4a_1(1 - a_1)a_2(1 - a_2)\mathbf{N}(\mathbf{a}) + (0, 0, 2)$,

while for the second example we have set

- $\Omega = (0, 1) \times (0, 1)$,
- $\mathbf{x}_0(u, v) = (u, v, \mathrm{e}^{-(u-0.5)^2-(v-0.5)^2})$,
- $A = \{(i/2, j/2) \mid i = 0, 1, 2,\ j = 0, 1, 2\}$,
- for all $\mathbf{a} = (a_1, a_2) \in A$, $\mathbf{p_a} = (-0.1 + 0.5a_1, -0.1 + 0.5a_2, 0.9065 + 0.686(a_1 - a_1^2 + a_2 - a_2^2) + 1.608(a_1a_2 - a_1^2a_2 - a_1a_2^2 + a_1^2a_2^2))$.

For both examples, for some values of d and h we have applied the fitting method with $m = 3$, $\varepsilon = 10^{-7}$, $\tau = 10$ and a set B^d consisting of 2500 points uniformly distributed in $\bar{\Omega}$. Problem (4.6) has been numerically solved using a tessellation of $\bar{\Omega}$ into 6×6 equal squares from the generic finite element BFS of class C^2. After solving a linear system of dimension $\dim V_h = 1323$, we have obtained a discrete smoothing variational spline $\mathbf{s}_{\varepsilon\tau}^{dh}$, which is the approximate solution to our problem in each case.

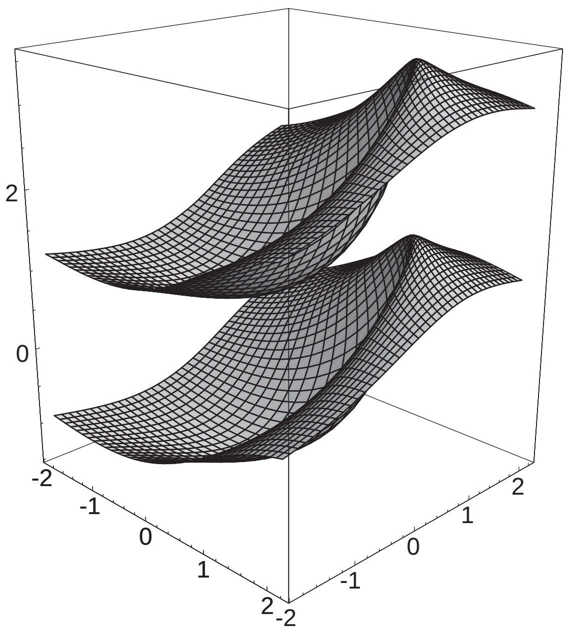

Fig. 1. Example 1. Traces of $\mathbf{x}_0$ (bottom) $\mathbf{s}_{\varepsilon\tau}^{dh}$ (top).

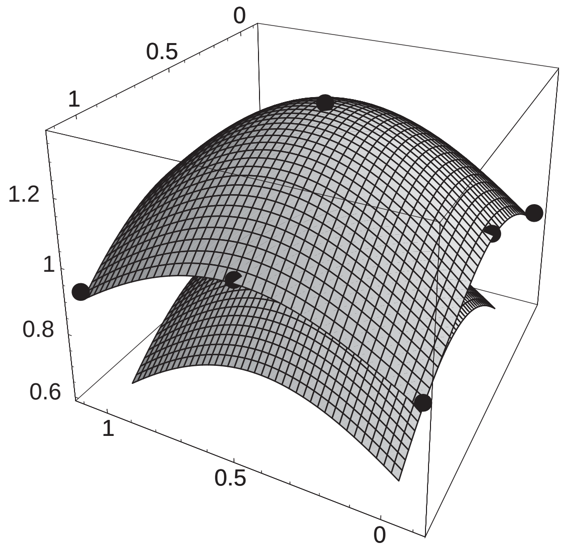

Fig. 2. Example 2. Traces of $\mathbf{x}_0$ (bottom) $\mathbf{s}^{dh}_{\varepsilon\tau}$ (top) and data points $\{\mathbf{p_a}\}_{\mathbf{a}\in A}$.

Figs. 1 and 2 show, for the first and second example, respectively, the trace of the reference patch $\mathbf{x}_0$ (bottom) and the trace of the approximating patch $\mathbf{s}^{dh}_{\varepsilon\tau}$ (top). Fig. 2 also shows the points $\{\mathbf{p_a}\}_{\mathbf{a}\in A}$.

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