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Module Homomorphisms and Topological Centres Associated with Weakly Sequentially Complete Banach Algebras*

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This paper is a contribution to the theory of weakly sequentially complete Banach algebras A. We require them to have bounded approximate identities and, for the most part, to be ideals in their second duals, so that examples are the group algebras $L^1(G)$ for compact groups G or the Fourier algebras $A(G)$ for discrete amenable groups G. In Section 2 we present our main result, that the topological centre (or set of weak* bicontinuous elements) of A^{**} is identifiable with A. As a corollary, we deduce in Section 3 that each left A-module homomorphism from A^* to A^*A can be realized as right translation by an element of A. These conclusions generalise recent advances in the subject. In Section 4 we take a special algebra, $l^1(S)$ for a commutative discrete semigroup S, and show that if its second dual has an identity then that identity must lie in $l^1(S)$. © 1998 Academic Press

INTRODUCTION

This paper had its genesis in a beautiful and brief argument which can be found in its clearest form in Lemma 2.1 of Ülger's paper [27]. Let us recall it by proving informally a simple result: let A be a Banach algebra

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which is weakly sequentially complete as a Banach space, whose second dual A^{**} has an identity ε, and suppose that the bounded approximate identity (bai) (e_n) (which must then exist in A) is sequential; then $\varepsilon \in A$ (regarded as a subset of A^{**}). The proof is this. Any weak* cluster point in A^{**} of (e_n) is quite easily seen to be a right identity for A^{**} and therefore must be ε. This means that $e_n \to \varepsilon$ in the weak* topology. Hence (e_n) is weakly Cauchy in A, and so convergent in A by weak sequential completeness. Uniqueness of limits in A^{**} shows that $\varepsilon \in A$.

The simplicity of the argument prompts two questions. The first is whether it can be applied in other situations. In fact it has already been used to establish variations on the result given above. In [19] (Theorem 2.1) it was shown that a weakly sequentially complete Banach algebra A with a sequential bai cannot be Arens regular, and later in the same paper (Theorem 3.4(i)) that, with the additional hypothesis that A is a right ideal in A^{**} (actually a weaker assumption suffices), the topological centre of A^{**} (definition below) is A itself.

The second question is whether the restriction to *countable* bai's is necessary. Ülger himself has already made progress in this direction. In [27] he removed the hypothesis that the bai should be sequential from the Arens regularity result already mentioned. In [28] he characterised the elements of the topological centre of weakly sequentially complete algebras without bai's, although he needed to impose other strong conditions (we give more information in Section 2), and this generalises the result quoted above. In Theorem 2.1(iii) we give a version of the latter result which requires only a (non-sequential) bai but which again needs a further hypothesis, that the algebra should be an ideal in its second dual. Our approach to this problem requires a way of passing from countable situations to general ones. We find that our needs are met by the Eberlein–Šmulian Theorem which says that a weakly sequentially compact subset of a Banach space is weakly compact.

In Section 3 we show that the theory of the topological centre is of other than intrinsic interest by applying Theorem 2.2 to an apparently unconnected situation. In Theorem 3.1 we show that if A is a weakly sequentially complete Banach algebra with a bai which is an ideal in its second dual then any left A-module homomorphism $T: A^* \to A^*A$ is of the form $Tf = fz$ $(f \in A^*)$ for some $z \in A$. This applies in particular to $L^1(G)$-homomorphisms $T: L^\infty(G) \to C(G)$ when G is a compact group, and to $A(G)$-homomorphisms $T: VN(G) \to UCB(\hat{G})$ when $A(G)$ is the Fourier algebra of a discrete amenable group G.

One aspect of our second question concerns the result with which we began. If A is a weakly sequentially complete Banach algebra whose second dual has an identity, must A itself have an identity? An obvious place to begin looking for a counter-example is amongst semigroup algebras $l^1(S)$.

Section 4 is devoted to a proof that such a quest must fail if S is commutative, for in that case if $l^1(S)^{**}$ has an identity, so does $l^1(S)$ (although there need not be an identity in S itself). Of course, in this situation, $l^1(S)$ may not be an ideal in $l^1(S)^{**}$. The actual result proved (Theorem 4.2) is much stronger, for it determines the exact (infinite) vector space dimension of the set of right identities in $l^1(S)^{**}$ if $l^1(S)$ has no identity.

An essential ingredient in our development is a general structure theory for second duals due to M. Grosser [7]. That this work has had less impact than it deserves may be due to its generality (we shall present only special cases below) and perhaps also to the fact that it is written in the German language. We shall include in Section 1 the proofs of the results from [7] which we use.

We should like to thank the referee for some significant contributions to the paper, and in particular for telling us how to obtain the results of Section 3 in the present quick manner.

1. GENERAL BANACH ALGEBRAS

Let A be a Banach algebra which for definiteness we take as over $\mathbb{C}$, though our results are equally valid for real algebras.

For $\mu, \nu \in A^{**}$ there are nets (x_i), (y_j) in A with w*-$\lim_i x_i = \mu$ (the weak* limit in A^{**} with A considered as embedded in A^{**}) and w*-$\lim_j y_j = \nu$. Then the first (resp. second) Arens product in A^{**} is determined by

$$\mu\nu = \text{w*-}\lim_i \,(\text{w*-}\lim_j \, x_i y_j) \qquad (\text{resp. } \mu \circ \nu = \text{w*-}\lim_j \,(\text{w*-}\lim_i \, x_i y_j)).$$

This description makes the distinction between the two multiplications clear, though it does not show why the limits exist; that comes from an alternative approach which defines successively $fx \in A^*$, $\nu f \in A^*$ and $\mu\nu \in A^{**}$ for $x, y \in A$, $f \in A^*$ and $\mu, \nu \in A^{**}$ by

$$\langle fx, y \rangle = \langle f, xy \rangle, \qquad \langle \nu f, x \rangle = \langle \nu, fx \rangle, \qquad \langle \mu\nu, f \rangle = \langle \mu, \nu f \rangle$$

and similarly

$$\langle yf, x \rangle = \langle f, xy \rangle, \qquad \langle f\mu, y \rangle = \langle \mu, yf \rangle, \qquad \langle \mu \circ \nu, f \rangle = \langle \nu, f\mu \rangle.$$

These formulas show that $\mu \circ y = \mu y$ and $x \circ \nu = x\nu$ for $x, y \in A$, $\mu, \nu \in A^{**}$. The product $\mu\nu$ is weak* continuous in μ for fixed ν, and weak* continuous in ν when μ is fixed in A. Thus $A \subseteq \Lambda(A^{**})$ where

$$\Lambda(A^{**}) = \{\mu \in A^{**} : \nu \mapsto \mu\nu \text{ is continuous in the weak* topology}\};$$

$\Lambda(A^{**})$ is the *topological centre* of A^{**}. When $\Lambda(A^{**})=A^{**}$, A is called *Arens regular*, and in that case $\mu\circ\nu=\mu\nu$ for all $\mu, \nu\in A^{**}$. In this paper when A^{**} is referred to as an algebra it will be with respect to the first Arens product unless the second is mentioned explicitly.

When A has a bounded right approximate identity (*brai*) (e_i), any weak* cluster point ε of (e_i) in A^{**} satisfies firstly $x\varepsilon=\text{w*-lim}_i\, xe_i=x$ (since $xe_i\to x$ even in norm) and then (on taking weak* limits in x) $\mu\varepsilon=\mu$ for $\mu\in A^{**}$. Thus ε is a right identity for A^{**}. If (e_i) is a (two-sided) bounded approximate identity (*bai*) for A, then any weak* cluster point ε is a right identity for A^{**} and a left identity for $(A^{**}, \circ)$, but not usually an identity for A^{**}; however for $x\in A$ it is true that $x\varepsilon=x=\varepsilon x$.

The closed subspace

$$\begin{aligned} WAP(A) &= \{f\in A^*\colon \langle \mu\nu, f\rangle=\langle \mu\circ\nu, f\rangle \text{ for all } \mu, \nu\in A^{**}\}\\ &= \{f\in A^*\colon x\mapsto xf\,(A\to A^*) \text{ is weakly compact}\}\end{aligned}$$

(for the equivalence of these two descriptions see, for example, Theorem 1 of [2] and the remarks which follow it) is called the space of *weakly almost periodic functionals* on A. The dual $WAP(A)^*$ has two multiplications defined on it in the same manner as A^{**} does, but these two coincide. Restriction to $WAP(A)$ is a homomorphism of both A^{**} and $(A^{**}, \circ)$ onto $WAP(A)^*$.

We write A^*A for the closed linear span in A^* of $\{fx : f\in A^*, x\in A\}$. When A has a brai the Cohen-Hewitt factorisation theorem (Theorem 32.22 of [10]) shows that in fact $A^*A=\{fx : f\in A^*, x\in A\}$. In our first theorem and its corollary we show that, when A has a brai, A^{**} is isomorphic, in the category of Banach spaces and continuous linear maps, to the direct sum $A^{**}\cong(A^*A)^*\oplus(A^*A)^\perp$ and that this decomposition carries with it a great deal of the algebraic structure. The results are, in all their essentials, due to M. Grosser ([7], especially Satz 4.14 (i) and pages 181, 182). We give the simple proof.

THEOREM 1.1. *Let A have a brai. Then $(A^*A)^\perp$ is the ideal of right annihilators in A^{**}. $(A^*A)^*$ has a natural Banach algebra structure under which it is isomorphic (as a Banach algebra, but not necessarily isometric) with the Banach algebra* $\mathrm{Hom}_A(A^*, A^*)$ *of continuous homomorphisms of the right A-module A^* to itself (so that $T\in\mathrm{Hom}_A(A^*, A^*)$ means $T(fx)=(Tf)x$ for $f\in A^*$, $x\in A$). In particular, $(A^*A)^*$ has a two-sided identity.*

Proof. If $\nu\in(A^*A)^\perp$ then $\langle \nu f, x\rangle=\langle \nu, fx\rangle=0$ $(f\in A^*, x\in A)$, so that $\langle \mu\nu, f\rangle=\langle \mu, \nu f\rangle=0$ $(\mu\in A^{**}, f\in A^*)$ and ν is a right annihilator. Conversely if ν is a right annihilator then in particular $x\nu=0$ for all $x\in A$, whence $\langle \nu, fx\rangle=\langle x\nu, f\rangle=0$ for all $f\in A^*$, $x\in A$, so that $\nu\in(A^*A)^\perp$.

$(A^*A)^*$ is, as a Banach space, isomorphic with the quotient $A^{**}/(A^*A)^\perp$; since this is the quotient of a Banach algebra by a closed ideal, $(A^*A)^*$ is a Banach algebra.

For each $\nu \in A^{**}$, $T_\nu f = \nu f$ $(f \in A^*)$ obviously defines a continuous right A-module homomorphism of A^*. The kernel of the continuous linear map $\nu \mapsto T_\nu$ consists of $\{\nu : \nu f = 0 \text{ for all } f \in A^*\}$, so ν is in the kernel if and only if $\langle \nu, fx \rangle = \langle \nu f, x \rangle = 0$ $(f \in A^*, x \in A)$, that is $\nu \in (A^*A)^\perp$. Thus $A^{**}/(A^*A)^\perp$ is continuously and injectively mapped onto a subalgebra of $\mathrm{Hom}_A(A^*, A^*)$. Conversely let ε be any right identity for A^{**}. We associate with $T \in \mathrm{Hom}_A(A^*, A^*)$ the element ν_T of A^{**} obtained by writing $\langle \nu_T, f \rangle = \langle \varepsilon, Tf \rangle$ $(f \in A^*)$. Obviously $T \mapsto \nu_T$ is continuous. Also $\langle \nu_T f, x \rangle = \langle \nu_T, fx \rangle = \langle \varepsilon, T(fx) \rangle = \langle \varepsilon, (Tf)\, x \rangle = \langle x\varepsilon, Tf \rangle = \langle x, Tf \rangle = \langle Tf, x \rangle$ $(f \in A^*, x \in A)$, so that $Tf = \nu_T f$. Since $T \mapsto \nu_T \mapsto T_{\nu_T}$ is the identity map the isomorphism between $\mathrm{Hom}_A(A^*, A^*)$ and $(A^*A)^*$ is established.

To conclude we observe that $\mathrm{Hom}_A(A^*, A^*)$ obviously has an identity. ∎

The defining relation for ν_T in the above proof shows that $\varepsilon \nu_T = \nu_T$. From this observation we easily deduce

COROLLARY 1.2. *Let ε be any right identity of A^{**}. Then εA^{**} is isomorphic* (*as a Banach algebra, but not necessarily isometric*) *with* $(A^*A)^*$, *and* $(A^*A)^\perp = \{\nu - \varepsilon\nu : \nu \in A^{**}\}$. *Thus, as Banach spaces,* $A^{**} \cong (A^*A)^* \oplus (A^*A)^\perp$.

Together 1.1 and 1.2 provide a general form for much of the structure theorem for the second dual of $L^1(G)$ where G is a compact group (Theorem 3.3 of [12]).

Although the algebras εA^{**} are Banach algebra isomorphic with $(A^*A)^*$, they are not usually isomorphic with $(A^*A)^*$ when εA^{**} has the topology induced by the weak* topology of A^{**} and $(A^*A)^*$ has its weak* topology. For example, whereas $(A^*A)^*$ always has a compact unit ball, when A has a two-sided bai εA^{**} is not closed unless $\varepsilon A^{**} = A^{**}$ (because εA^{**} contains $\varepsilon A = A$ and so is dense in A^{**}).

The reason why $(A^*A)^*$ plays such an important part in the theory of A^{**} was recognised long ago by M. Grosser ([9], Lemma 1). He introduced the action $\bigstar$ of our next theorem.

THEOREM 1.3. *The projection π: $A^{**} \to (A^*A)^*$ adjoint to the inclusion $A^*A \to A^*$ is a weak* continuous surjective homomorphism.*

There is a natural action $\mu \mapsto \mu \bigstar \xi$ $(\mu \in A^{**}, \xi \in (A^*A)^*)$ of $(A^*A)^*$ on A^{**} with the property that $\mu\nu = \mu \bigstar \pi(\nu)$ $(\mu, \nu \in A^{**})$. Moreover,

(i) *if A has a left approximate identity, π is injective on A (regarded as a subalgebra of A^{**})*;

(ii) *if (v_i) is a bounded net in A^{**}, $\pi(v_i) \to \pi(v)$ in the weak* topology of $(A^*A)^*$ and $\lambda \in \Lambda(A^{**})$, then $\lambda v_i \to \lambda v$ in the weak* topology of A^{**}*;

(iii) *if $\mu \in A^{**}$ is such that $A\mu \subseteq A$, then $\eta \mapsto \pi(\mu)\eta$ is weak* continuous on $(A^*A)^*$.*

Proof. That π is a weak* continuous surjective homomorphism is immediate from the observation that $(A^*A)^*$ is the quotient of A^{**} by its ideal $(A^*A)^{\perp}$, and that from this viewpoint π is just the quotient mapping.

Now we define an action $f \mapsto \xi \star f$ of $(A^*A)^*$ on A^* by writing for $\xi \in (A^*A)^*$ and $f \in A^*$, $\langle \xi \star f, x\rangle = \langle \xi, fx\rangle$ $(x \in A)$. We define $\mu \mapsto \mu \star \xi$ to be the adjoint of $f \mapsto \xi \star f$.

For $v \in A^{**}$, $f \in A^*$ and $x \in A$ we have $\langle vf, x\rangle = \langle v, fx\rangle = \langle \pi(v), fx\rangle = \langle \pi(v) \star f, x\rangle$. therefore, for $\mu \in A^{**}$, $\langle \mu v, f\rangle = \langle \mu, vf\rangle = \langle \mu, \pi(v) \star f\rangle = \langle \mu \star \pi(v), f\rangle$.

To prove (i), let (e_i) be the left approximate identity for A. If $\pi(x) = 0$ for some $x \in A$, then $e_i x = e_i \star \pi(x) = 0$ for all i, whence $x = 0$. Thus π is injective on A.

To prove (ii) we use the following lemma. We state it precisely because we need it frequently.

LEMMA 1.4. *Let K be a compact set, let (a_i) be a net in K and let $a \in K$. If each convergent subnet of (a_i) has limit a then $a_i \to a$.*

(The proof is simple. If $a_i \not\to a$, there exist an open neighbourhood V of a and a subnet (a_{i_j}) of (a_i) with $a_{i_j} \notin V$ for all j. Since K is compact, (a_{i_j}) has a convergent subnet whose limit cannot be in the open set V, contrary to the hypothesis that it is a.)

To finish the proof of (ii), take any weak* convergent subnet of (v_i), say $v_{i_j} \to \tau$. Then $\pi(v_{i_j}) \to \pi(\tau)$. By hypothesis, $\pi(v_{i_j}) \to \pi(v)$, so that $\pi(\tau) = \pi(v)$. Since $\lambda \in \Lambda(A^{**})$ we find $\lambda v_{i_j} \to \lambda\tau = \lambda \star \pi(\tau) = \lambda \star \pi(v) = \lambda v$. The conclusion now follows from the Lemma.

Finally we come to (iii). Take $f \in A^*$, $x \in A$. Then $x\mu \in A$, so also $fx\mu \in A^*A$. For $\eta \in (A^*A)^*$ we have $\eta = \pi(v)$ for some $v \in A^{**}$ and since π is a homomorphism we find

$$\langle \pi(\mu)\eta, fx\rangle = \langle \pi(\mu)\pi(v), fx\rangle = \langle \mu v, fx\rangle = \langle x\mu v, f\rangle$$

$$= \langle v, fx\mu\rangle = \langle \pi(v), fx\mu\rangle = \langle \eta, fx\mu\rangle$$

which shows that $\pi(\mu)\eta$ is weak* continuous in η. ∎

Item (i) of the above theorem suggests that π need not always be injective on A, or in other words that the natural map from A to $(A^*A)^*$ need not be injective. Here is a simple example which confirms that this is the case. Let X be a Banach space made into an algebra by taking a fixed $f_0 \neq 0$ in X^* and writing $xy = \langle f_0, y \rangle x$. Then for $f \in X^*$, $x, y \in X$ we have $\langle fx, y \rangle = \langle f, xy \rangle = \langle f_0, y \rangle \langle f, x \rangle$, whence $fx = \langle f, x \rangle f_0$, so that X^*X is one-dimensional.

Most of the Banach algebras A we consider are at least one-sided ideals in A^{**}. Under such conditions, conclusions like those of Theorem 1.1 or Corollary 1.2 can often be strengthened.

THEOREM 1.5. *Let A have a brai and suppose A is a right ideal in A^{**}. Then π: $A^{**} \to (A^*A)^*$ is a weak* continuous surjective homomorphism of both A^{**} and $(A^{**}, \circ)$ onto $(A^*A)^*$. The multiplication in $(A^*A)^*$ is weak* separately continuous. If A has a two-sided bai, then $WAP(A) = A^*A$.*

Proof. We first prove separate continuity in $(A^*A)^*$. Since π is surjective, $\pi(\mu)\,\pi(\nu)$ represents an arbitrary product in $(A^*A)^*$ for $\mu, \nu \in A^{**}$. That $\pi(\mu)\,\pi(\nu) = \pi(\mu\nu)$ is weak* continuous in $\pi(\mu)$ follows from the weak* continuity of $\mu\nu$ in μ and the weak* continuity of π. Continuity in $\pi(\nu)$ follows immediately from Theorem 1.3(iii) since A is a right ideal in A^{**}.

Since π is a homomorphism on A, taking weak* limits shows that it is a homomorphism for $(A^{**}, \circ)$.

(Readers familiar with second duals will recognize that when A is a right ideal in A^{**}, the subspace A^*A of A^* is right—as well as left—introverted, so that π is a homomorphism of the second Arens multiplications on A^{**} and $(A^*A)^*$. Separate continuity in $(A^*A)^*$ says that the two multiplications in this space coincide. Notice that we are *not* saying that, for a right identity ε of A^{**}, multiplication in εA^{**} is separately continuous in the topology induced by the weak* topology of A^{**}.)

Weak* separate continuity in $(A^*A)^*$ shows that $A^*A \subseteq WAP(A)$. Now take ε to be a weak* cluster point of a bai, so that it is a right identity for A^{**} and a left identity for $(A^{**}, \circ)$. Let $f \in WAP(A)$ so that $\langle \mu\nu, f \rangle = \langle \mu \circ \nu, f \rangle$ for all $\mu, \nu \in A^{**}$. Put $\mu = \varepsilon$ to find that for all $\nu \in A^{**}$ we have

$$\langle \nu - \varepsilon\nu, f \rangle = \langle \varepsilon \circ \nu - \varepsilon\nu, f \rangle = 0.$$

Thus, using Corollary 1.2, $f \in (A^*A)^{\perp\perp} = A^*A$. ∎

The inclusion $WAP(A) \subseteq A^*A$ is true more generally (Proposition 3.3 of [16]). The equivalence (i) $\Leftrightarrow$ (iii) of the next theorem is in [19] (Theorem 2.6). One of the significant problems concerning A^* as a right module over A is to determine when it factorizes, that is, when $A^* = A^*A$. We give some simple equivalences in terms of A^{**}.

THEOREM 1.6. *The statements* (i)–(iv) *are equivalent for a Banach algebra with a brai.*

(i) A^* *factorizes on the right* (*that is*, $A^* = A^*A$).

(ii) A^{**} *has no non-zero right annihilator.*

(iii) A^{**} *has an identity.*

(iv) A^{**} *has a unique right identity.*

Moreover if A *has a two-sided bai and is a right ideal in* A^{**}, *each of* (i)–(iv) *is equivalent to*

(v) A *is Arens regular.*

Proof. The equivalence of (i), (ii) and (iii) follows from Theorem 1.1 since $A^* = A^*A \Leftrightarrow (A^*A)^{\perp} = 0 \Leftrightarrow A^{**} = (A^*A)^*$. To see the equivalence with (iv), simply observe that, given a right identity ε of A^{**}, ε' is a right identity if and only if $\varepsilon' = \varepsilon + \rho$ where ρ is a right annihilator. The equivalence with (v) follows from Theorems 1.1 and 1.5. ∎

Of course any algebra A with identity satisfies (i)–(iv) of the last theorem, so it is obvious that (v) is not equivalent to the other conditions in general. The real point of the theorem is to suggest that the size of the ideal of right annihilators in A^{**} is a measure of the non-factorizability of A^*.

Many of our arguments use sequential approximate identities. We next give some variants of a lemma used by Ülger ([27], Lemma 2.2) to reduce the general to the sequential case.

LEMMA 1.7. *Let* (e_i) *be a bai* (*resp. brai*) *for a Banach algebra* A. *Let* $X = \{x_1, x_2, \ldots\}$ *be any countable subset of* A. *Then there exist a closed subalgebra* $A(X)$ *of* A *and a sequence* (i_n) *such that* $X \subseteq A(X)$, $e_{i_n} \in A(X)$ *for all* n *and* (e_{i_n}) *is a bai* (*resp. brai*) *for* $A(X)$.

In each case, the closed left ideal $L(X)$ *generated by* $A(X)$ *has* (e_{i_n}) *as a brai.*

The algebras $A(X)$ *are not uniquely determined, but for any two countable sets* X_1, X_2 *there is a third* X_3 *such that* $A(X_1) + A(X_2) \subseteq A(X_3)$.

Proof. We sketch the proof for the case of the bai. The construction is inductive. Enumerate the elements of the subalgebra of A over the countable field $\mathbb{Q} + i\mathbb{Q}$ generated by x_1 as $u_{11}, u_{12}, \ldots$. Choose i_1 such that $\|u_{11} - u_{11}e_{i_1}\| < 1$ and $\|u_{11} - e_{i_1}u_{11}\| < 1$. When $(u_{rm})_{m=1}^{\infty}$ and i_r have been chosen for $r \leqslant n-1$, enumerate the elements of the subalgebra over $\mathbb{Q} + i\mathbb{Q}$ generated by $\{x_1, \ldots x_n\} \cup \{e_{i_1}, \ldots e_{i_{n-1}}\}$ as $u_{n1}, u_{n2}, \ldots$ and choose i_n such that $\|u_{rm} - u_{rm}e_{i_n}\| < 1/n$ and $\|u_{rm} - e_{i_n}u_{rm}\| < 1/n$ for $1 \leqslant r,\ m \leqslant n$. Then for any element u in the subalgebra over $\mathbb{Q}+i\mathbb{Q}$ generated by $\{x_1, x_2, ...\} \cup \{e_{i_1}, e_{i_2}, ...\}$ we have $\|u-ue_{i_n}\| \to 0$ and $\|u-e_{i_n}u\| \to 0$, and these relations hold for u in the closure of this subalgebra too. We take this closure to be $A(X)$, so that $A(X)$ is just the Banach subalgebra of A (over $\mathbb{C}$) generated by $\{x_1, x_2, ...\} \cup \{e_{i_1}, e_{i_2}, ...\}$.

For $u \in A(X)$ we have $ue_{i_n} \to u$ and so for each $x \in A$ also $xue_{i_n} \to xu$. Therefore (e_{i_n}) is a brai for the closed left ideal $L(X) = AA(X)$ generated by $A(X)$. (We can also see that $L(X) = \{x \in A : xe_{i_n} \to x\}$.)

If X_1 and X_2 are two countable sets, $A(X_1)$ has bai (e_{i_m}) and $A(X_2)$ has bai (e_{j_n}), take $X_3 = X_1 \cup X_2 \cup \{e_{i_1}, e_{i_2}, ...\} \cup \{e_{j_1}, e_{j_2}, ...\}$ to get the required result. (The subalgebra $A(X)$ is easily seen not to be uniquely determined because of the arbitrary choices of i_n in its construction.) ▮

COROLLARY 1.8. *Let (e_i) be a brai for A and let $X \subset A$ be countable. If A is a left ideal in A^{**}, then $L(X)$ is a left ideal in A^{**} (and so in $L(X)^{**}$). Let (e_{i_n}) be the brai for $A(X)$ produced in Lemma 1.7. Let ε_X be any weak* cluster point of (e_{i_n}) in A^{**}. Then ε_X is a right identity for the weak* closure of $A(X)$ in A^{**} (which can be identified with $A(X)^{**}$). Moreover $L(X)^{**} = A^{**}\varepsilon_X$. If in addition A is a right ideal in A^{**}, then $L(X)$ is a right ideal in $L(X)^{**}$ and $L(X) = A\varepsilon_X$.*

When (e_i) is a two-sided bai, then also $\varepsilon_X x = x$ for $x \in A(X)$. Moreover if A is a right ideal in A^{**} the idempotent $\pi(\varepsilon_X)$ is an identity for the algebra $\pi(A(X)^{**})$. Thus $\pi(\varepsilon_X)$ is uniquely determined by $A(X)$, and moreover $\pi(e_{i_n}) \xrightarrow{w^*} \pi(\varepsilon_X)$. Further if $A(X) \subseteq A(Y)$ then $\pi(\varepsilon_X)\,\pi(\varepsilon_Y) = \pi(\varepsilon_X) = \pi(\varepsilon_Y)\,\pi(\varepsilon_X)$.

Proof. Since $L(X) = AA(X)$ we have $A^{**}L(X) \subseteq A^{**}AA(X) \subseteq AA(X) = L(X)$. It is elementary Banach space theory that $A(X)^{**}$ is the weak* closure of $A(X)$ and we know that any weak* cluster point of a brai for $A(X)$ is a right identity for $A(X)^{**}$. From $AL(X) \subseteq L(X)$ we get by weak* continuity first that $AL(X)^{**} \subseteq L(X)^{**}$ and then that $A^{**}L(X)^{**} \subseteq L(X)^{**}$. Since $\varepsilon_X \in L(X)^{**}$ we deduce that $A^{**}\varepsilon_X \subseteq L(X)^{**}$. On the other hand, since ε_X is a right identity for $L(X)^{**}$ we have $L(X)^{**} = L(X)^{**}\,\varepsilon_X \subseteq A^{**}\varepsilon_X$, so that $L(X)^{**} = A^{**}\varepsilon_X$. In particular, $A\varepsilon_X \subseteq L(X)^{**}$. If A is a right ideal in A^{**}, then $A\varepsilon_X \subseteq A$, so $A\varepsilon_X \subseteq L(X)^{**} \cap A = L(X)$. Since $L(X) = L(X)\,\varepsilon_X \subseteq A\varepsilon_X$, we conclude $L(X) = A\varepsilon_X$. Then $L(X)\,L(X)^{**} \subseteq AA^{**}\varepsilon_X \subseteq A\varepsilon_X = L(X)$.

When (e_i) is two-sided, Lemma 1.7 gives $e_{i_n}x \to x$ in norm for $x \in A(X)$, and taking the weak* limit gives $\varepsilon_X x = x$. Moreover $(\pi(e_{i_n}))$ is a two-sided bai for $\pi(A(X))$, and since multiplication in $(A^*A)^*$ is separately continuous in the weak* topology when A is a right ideal in A^{**} (Theorem 1.5) we deduce that $\pi(\varepsilon_X)$ is a two-sided identity for the weak* closure of $\pi(A(X))$ in $(A^*A)^*$, that is, for $\pi(A(X)^{**})$. Thus $\pi(\varepsilon_X)$ is unique and so does not depend on the cluster point ε_X of (e_{i_n}) chosen, and we deduce that $\pi(e_{i_n}) \xrightarrow{w^*} \pi(\varepsilon_X)$ in $(A^*A)^*$. The remaining conclusions follow easily. ∎

The meaning of Corollary 1.8 is illuminated by the special case in which $A = L^1(G)$ where G is a compact group. Here A is a two-sided ideal in A^{**} and A^*A is the space $LUC(G)$ of left uniformly continuous functions on G which, since G is compact, is just the space $C(G)$ of continuous functions. Thus $(A^*A)^*$ is the convolution algebra $M(G)$ of bounded measures on G. We take the bai (e_i) to consist of positive functions supported by neighbourhoods of the identity in G. Each idempotent ε_X is a weak* cluster point of a sequence (e_{i_n}) taken from (e_i). Because (e_{i_n}) is a bai for a subalgebra of A its weak* cluster points are idempotents in A^{**}. Also $\pi(\varepsilon_X)$ is a weak* cluster point in $M(G)$ of $(\pi(e_{i_n}))$, and so is a positive idempotent measure. Therefore $\pi(\varepsilon_X)$ is the Haar measure m_H of a compact subgroup H of G [29] and since (e_{i_n}) is countable H must be a G_δ, so that G/H is metrizable. $A\varepsilon_X$ (or $L^1(G) * m_H$) is not only a subalgebra of A but is also a quotient space via the map $x \mapsto x\varepsilon_X$; as such it is identifiable with $L^1(G/H)$ when G/H is equipped with its unique bounded left G-invariant measure. Since H need not be normal, $L^1(G/H)$ need not have a natural structure as a convolution algebra. Thus studying the algebra A by means of the subalgebras $A\varepsilon_X$ is an abstract version of studying $L^1(G)$ by considering the spaces $L^1(G/H)$ for metrizable quotients G/H.

Since G is compact, its group algebra $L^1(G)$ has a two-sided bai which is central (i.e., $e_i x = xe_i$ for all i and all $x \in A = L^1(G)$) (see [21]). For such a bai, taking weak* limits gives $\varepsilon_X x = x\varepsilon_X$ for all $x \in A$, which means that the corresponding subgroup H is normal. The map $x \mapsto x\varepsilon_X$ is then a Banach algebra homomorphism of A onto $A\varepsilon_X = L(X)$, and $L(X)$ is both a left ideal and a quotient algebra of A.

2. ELEMENTS IN THE TOPOLOGICAL CENTRE

The main aim of this section is to produce a variant of a theorem of Lau and Ülger [19] which is valid for all weakly sequentially complete algebras with bai's and not just those in which the bai is sequential. Whether this is possible without making additional hypotheses we do not know. Ülger himself [28] has given a result which removes the requirement for a bai altogether, but he has some strong extra assumptions. We state for comparison, as one theorem, three results which are (i) Theorem 3.4 of [19], (ii) Theorem 2.2 of [28], and (iii) a consequence of our Theorem 2.2 below.

THEOREM 2.1. *Let A be a Banach algebra which is weakly sequentially complete as a Banach space.*

(i) *Let A have a sequential bai and suppose $A\Lambda(A^{**})\subseteq A$. Then $\Lambda(A^{**})=A$.*

(ii) *Let A be commutative, semisimple and completely continuous (definition below). Then $\mu\in\Lambda(A^{**})$ if and only if $\mu A^{**}\subseteq A$ and $A^{**}\mu\subseteq A$. In particular if A has a bai, then $\Lambda(A^{**})=A$.*

(iii) *If A has a bai, is a right ideal in A^{**} and $\Lambda(A^{**})A\subseteq A$, then $\Lambda(A^{**})=A$.*

A *completely continuous* Banach algebra A is one for which the maps $x\mapsto xy$ and $x\mapsto yx$ are compact for each $y\in A$. Since A is a right ideal in A^{**} if and only if $x\mapsto yx$ is weakly compact for all $y\in A$ (Lemma 4.3 of [2] for example), completely continuous algebras A are left (and similarly right) ideals in A^{**}. On Banach spaces isomorphic with any L^1-space, the composition of two weakly compact operators is compact ([4], VI.8.13). If A has a bai, then its elements factorize (Cohen's Theorem, 32.26 in [10]), with the consequence that if multiplication by every element is weakly compact, then it is compact. We deduce that in this case, being completely continuous is the same as being an ideal in A^{**}.

The reader might notice that where (i) has the hypothesis $A\Lambda(A^{**})\subseteq A$, (iii) has $\Lambda(A^{**})\,A\subseteq A$. This is of little significance, and is a consequence of how the proof is arranged. However, hypotheses of the kind $\Lambda(A^{**})\,A\subseteq A$ are unsatisfactory. To see why, consider the case when $A=L^1(G)$ where G is a locally compact, non-compact group. Here A does satisfy the hypothesis but at present there is no way of verifying this apart from proving that $\Lambda(A^{**})=A$ by other means (see [17]). The most natural condition which implies this hypothesis is that A should be a left ideal in A^{**}, so that Theorem 2.1(iii) really requires that A should be an ideal in A^{**}. Of course, if A is commutative and is a one-sided ideal in A^{**} it is automatically two-sided (because A is then in the algebraic centre of A^{**}).

Theorem 2.1(iii) is an obvious consequence of the following more general result. We take a brai (e_i) in A and let E be a closed linear subspace containing all the e_i. Then E^{**} is the weak* closure of E in A^{**}.

THEOREM 2.2. *Let A be weakly sequentially complete. Let $\lambda\in\Lambda(A^{**})$ satisfy $\lambda A\subseteq A$.*

(i) *Suppose A has a sequential brai. Then $\lambda\in A$.*

(ii) *Suppose (e_i) is a two-sided bai and that $AE^{**}\subseteq A$ where E is some closed subspace containing (e_i). Then $\lambda\in A$.*

Proof. (i) Let (e_n) be the sequential brai. Each weak* cluster point ε of (e_n) is a right identity for A^{**}. Since $\lambda\in\Lambda(A^{**})$, each weak* cluster point of (λe_n) is of the form $\lambda\varepsilon$ for some ε, and so equal to λ. Therefore by Lemma 1.4, $\lambda e_n \xrightarrow{w^*} \lambda$. Thus (λe_n) is weakly Cauchy and so convergent in A to a limit which can only be λ, whence $\lambda \in A$.

(ii) We shall show that $\{\lambda e_i\}$ is weakly relatively compact in A. Then A must contain all weak* cluster points of (λe_i), and these are of the form $\lambda\varepsilon = \lambda$ because any weak* cluster point ε of (e_i) is a right identity.

We take K such that $\|e_i\| \leqslant K$ for all i, and we write E_K^{**} for the closed ball in E^{**} with radius K. We shall show $\{\lambda\mu : \mu \in E_K^{**}\}$ is weakly sequentially compact and so (by Eberlein–Šmulian) weakly compact. We take a sequence $X = (x_m)$ in the ball E_K in E and form a left ideal $L(X)$ with a sequential brai (e_{i_n}) such that $e_{i_n} x_m \to x_m$ for each m (Lemma 1.7).

Now let ε_X be weak* cluster point of (e_{i_n}). We prove $\lambda\varepsilon_X \in L(X)^{**}$. Indeed, for any m, n we have $(\lambda e_{i_m})\, e_{i_n} \in A e_{i_n} \subseteq L(X)$. Since (e_{i_n}) is a brai for $L(X)$ and $e_{i_m} \in L(X)$, taking the limit over n gives $\lambda e_{i_m} \in L(X)$. Since $\lambda \in \Lambda(A^{**})$ we deduce that $\lambda\varepsilon_X$ is in the weak* closure (in A^{**}) of $L(X)$, that is, in $L(X)^{**}$.

We now show that in fact $\lambda\varepsilon_X \in \Lambda(L(X)^{**})$. From the Krein–Šmulian Theorem (in particular Theorem V.5.6 of [4]) it is enough to show that $\nu \mapsto \lambda\varepsilon_X \nu$ is weak*–weak* continuous on bounded sets in $L(X)^{**}$. Further, $L(X)^{**}$ can be regarded as a subalgebra and a normed subspace of A^{**} (using the second adjoint of the inclusion $L(X) \to A$) and its weak* topology $\sigma(L(X)^{**}, L(X)^*)$ is just the topology induced on $L(X)^{**}$ by $\sigma(A^{**}, A^*)$ (this fact is easy to see directly because $A^* \to L(X)^*$ is surjective, and also follows from Theorem 16.11(i) of [14]). Now let (ν_j) be a bounded net with $\nu_j \xrightarrow{w^*} \nu$ in $L(X)^{**}$, and so also in A^{**}. Since π: $A^{**} \to (A^*A)^*$ is a weak* continuous homomorphism and $A\varepsilon_X \subset A$ by hypothesis (because $\varepsilon_X \in E^{**}$), we can apply Theorem 1.3(iii) to find $\pi(\varepsilon_X \nu_j) = \pi(\varepsilon_X)\, \pi(\nu_j) \xrightarrow{w^*} \pi(\varepsilon_X)\, \pi(\nu) = \pi(\varepsilon_X \nu)$. Then from Theorem 1.3(ii) we deduce that $\lambda\varepsilon_X \nu_j \xrightarrow{w^*} \lambda\varepsilon_X \nu$, in A^{**} and so also in $L(X)^{**}$, as required.

Since $L(X)$ has a sequential brai, we can apply the first part of the present theorem to get $\lambda\varepsilon_X \in L(X) \subseteq A$. Therefore from our hypotheses $\lambda\varepsilon_X E_K^{**} \subseteq A$. Since $\nu \mapsto (\lambda\varepsilon_X)\, \nu$ is now seen to be continuous from E^{**} with its weak* topology to A with its weak topology, we find that $\lambda\varepsilon_X E_K^{**}$ is weakly compact. But since $\varepsilon_X x_m = x_m$ for each m (Corollary 1.8), $\{\lambda x_m\} \subseteq \lambda\varepsilon_X E_K^{**}$, and our conclusion follows from this. ∎

Corollary 2.3. *Let A be weakly sequentially complete with a brai and suppose that A^{**} has a unique right identity ε.*

(i) *If A has a sequential brai then $\varepsilon \in A$.*

(ii) *If the brai is two-sided and there is E as above for which $AE^{**} \subseteq A$, then $\varepsilon \in A$.*

Proof. Theorem 1.6 tells us that ε is actually the identity of A^{**}. It therefore trivially satisfies $\varepsilon \in \Lambda(A^{**})$ and $\varepsilon A \subseteq A$. So (i) (resp. (ii)) follows from Theorem 2.2(i) (resp. 2.2(ii)). ∎

EXAMPLES 2.4. (a) When G is a compact group, $L^1(G)$ is a two-sided ideal in $L^1(G)^{**}$ (as in Examples 3.2(a)) so that $\Lambda(L^1(G)^{**}) = L^1(G)$. (This was proved in [12], Theorem 3.3(vi).)

(b) For a locally compact group G which is not compact, $L^1(G)$ is not an ideal in its second dual ([8]). The best our theory can achieve here is to take a compact neighbourhood V of the identity in G and write $L^1(V)$ for the set of functions in $L^1(G)$ with supports in V. We take E of Theorem 2.2(ii) to be $L^1(V)$; this certainly contains a two-sided bai. Then $\pi(L^1(V)^{**})$ consists of those elements of $LUC(G)^*$ carried by V, that is, the bounded measures $M(V)$ on V. Thus $AE^{**} = L^1(G)\, L^1(V)^{**} = L^1(G) \bigstar \pi(L^1(V)^{**}) = L^1(G) * M(V) \subseteq L^1(G)$. So Theorem 2.2(ii) holds. Of course we again meet the problem of knowing which elements λ belong to $\Lambda(L^1(G)^{**})$. Here we can use symmetry to deduce $L^1(V)^{**}\, L^1(G) = L^1(V)^{**} \circ L^1(G) \subseteq L^1(G)$, and Theorem 2.2(ii) then gives $L^1(V)^{**} \cap \Lambda(L^1(G)^{**}) \subseteq L^1(G)$. This is not the best possible result: it is shown in [17] that $\Lambda(L^1(G)^{**}) = L^1(G)$.

(c) The weighted convolution algebra $L^1(\mathbb{R}^+, w)$ (where w is a continuous weight with $w(0) = 1$ which is regulated, that is, $\lim_{s \to \infty} w(s+t)/w(s) = 0$ for all $t > 0$) and the Volterra algebra $V = L^1(0, 1)$ are both ideals in their second duals, so that Theorem 2.1 applies. The conclusion about their topological centres is due to Ghahramani and McLure ([5], Theorem 2.2(a)), but since these two algebras have *sequential* bai's, this result follows from Theorem 2.1(i) as was observed in [19]. However, the Volterra-like algebras of [25], which generalize the Volterra algebra, need not be commutative and they have bai's which need not be sequential, so that Theorem 2.1(ii) is needed to obtain a new proof of Theorem 3.7 of [25].

(d) When G is discrete and amenable, the Fourier algebra $A(G)$ has a bai and is an ideal in its second dual (see Example 3.2(d)). We conclude from Theorem 2.1(iii) that $\Lambda(A(G)^{**}) = A(G)$; this was proved in [18] (Theorem 6.5(i)) and also follows from Corollary 2.4 of [28].

(e) In [13], Kamyabi-Gol discusses the second duals of hypergroups. A hypergroup A is a Banach algebra of measures on a locally compact space with certain properties which include having a bai. Being a Banach space of measures, A is weakly sequentially complete. Also, Kamyabi-Gol shows that A is an ideal in its second dual if and only if the underlying space is compact. Thus we recover his conclusion that the centre of A^{**} is A in this case. (Medghalchi ([20], Theorem 14(a)) has a similar result—also a consequence of Theorem 2.1(iii) when the underlying space is compact—using a different definition of hypergroup.)

(f) The conclusion that $\Lambda(A^{**}) = A$ may not hold when A is not an ideal in A^{**}. An illustration of this fact is easy to provide. Let X be any Banach space and turn it into a Banach algebra by writing $xy = 0$ for $x, y \in X$. Let A be X with an identity adjoined (so that A trivially has a bai). Then A^{**} is just X^{**} with an identity adjoined, and X^{**} also has the multiplication in which every product is zero. Therefore even when A is weakly sequentially complete , $\Lambda(A^{**}) = A^{**}$.

A more satisfying example is supplied by Saghafi [26]. She presents some commutative convolution L^1-algebras on semigroups, with bai's, for which the centres of the second duals are much larger than the original algebra; these algebras are, of course, not ideals in their second duals. A specific example is provided by $L^1(\mathbb{R})$ with the mutiplication *max* in $\mathbb{R}$. (See [26], Theorem 4.12).

(g) The existence in A of a bai cannot simply be omitted if the conclusion $\Lambda(A^{**}) = A$ is required. An easy example is provided by the algebra $A = l^1(\mathbb{N})$ with the *pointwise* multiplication. Then A^{**} can be identified with the space $M(\beta\mathbb{N})$ of bounded measures on the Stone–Čech compactification of the discrete space $\mathbb{N}$; we can write $M(\beta\mathbb{N})$ as a direct sum $l^1(\mathbb{N}) \oplus M(\beta\mathbb{N} \setminus \mathbb{N})$ and the multiplication is just the original one in l^1 together with the zero multiplication in $M(\beta\mathbb{N} \setminus \mathbb{N})$ (that is, $\mu\nu = 0$ for all μ, ν). It is easy to see that A has no bai, is an ideal in its second dual, and that $\Lambda(A^{**}) = A^{**}$. This also follows from Corollary 2.4 of [28].

3. MODULE HOMOMORPHISMS

In Theorem 1.1. we saw that the algebra of all *right* A-module homomorphisms from A^* to itself can be identified with the algebra $(A^*A)^*$ when A has a brai. In the present section we shall see that the algebra of *left* A-module homomorphisms from A^* to A^*A is naturally isomorphic with A itself when A has a two-sided bai, is a right ideal in its second dual, and is weakly sequentially complete as a Banach space. This conclusion was proved for $A = L^1(G)$ where G is a compact group by Hewitt and Ross ([10], Theorem 35.13) and also applies when $A = A(G)$, the Fourier algebra of the discrete amenable group G—see Example 3.2(d). Because it depends on the general structure of algebras our proof is simpler than the original one for $L^1(G)$. It is possibly even a little shorter (even allowing for the need to establish 2.2(ii)), and when the approximate identity is sequential—for example when the G in $L^1(G)$ is metrizable—so that only 2.2(i) is required, it is much shorter.

We wish to thank the referee for pointing out to us that Theorem 3.1 was a consequence of Theorem 2.2. This insight has considerably shortened the proof.

We begin by stating our result precisely. As for Theorem 2.2, there is a stronger result for the case in which the brai is sequential. Although the proof involves second duals in an essential way, the statement of the conclusion only involves first duals. We therefore prefer to state our hypotheses in terms of A and A^*. To relate these to our other results, we recall ([2], Section 4, Lemma 3) that A is a right ideal in A^{**} if and only if for each $x \in A$ the map $y \mapsto xy$ is weakly compact.

THEOREM 3.1. *Let A be a Banach algebra for which all maps $y \mapsto xy$ $(x \in A)$ are weakly compact, which has a brai and which is weakly sequentially complete as a Banach space. Let $T: A^* \to A^*A$ be continuous, linear and satisfy $T(xf) = xTf$ for all $x \in A$, $f \in A^*$. Then under each of the additional hypotheses*

(i) *A has a sequential brai,*

(ii) *A has a two-sided bai,*

there is $z \in A$ with $Tf = fz$ $(f \in A^*)$.

Proof. There is a single proof for the two cases; the difference is that appeal is made to Theorem 2.2(i) or (ii) as appropriate.

We first extend the defining property of T by showing that $T(vf) = vTf$ for $v \in A^{**}$, $f \in A^*$. Let (e_i) be a brai for A. Then for any $x \in A$,

$$\begin{aligned}\langle T(vf), x\rangle &= \lim_i \langle T(vf), xe_i\rangle = \lim_i \langle e_i T(vf), x\rangle = \lim_i \langle T(e_i vf), x\rangle \\ &= \lim_i \langle (e_i v)(Tf), x\rangle = \lim_i \langle vTf, xe_i\rangle = \langle vTf, x\rangle\end{aligned}$$

which establishes our assertion.

Now let ε be a right identity of A^{**}, so that $\pi(\varepsilon)$ is the identity of $(A^*A)^*$ (Theorems 1.1 and 1.3). We find, for any $v \in A^{**}$, $f \in A^*$,

$$\begin{aligned}\langle T^*\pi(v), f\rangle &= \langle T^*(\pi(\varepsilon)\,\pi(v)), f\rangle = \langle \pi(\varepsilon v), Tf\rangle = \langle \varepsilon v, Tf\rangle = \langle \varepsilon, vTf\rangle \\ &= \langle \varepsilon, T(vf)\rangle = \langle \pi(\varepsilon), T(vf)\rangle \\ &= \langle T^*\pi(\varepsilon), vf\rangle = \langle (T^*\pi(\varepsilon))\, v, f\rangle.\end{aligned}$$

Therefore if we write $\xi = T^*\pi(\varepsilon) \in A^{**}$, we see that $T^*\pi(v) = \xi v$. (This formula actually determines T^* since π is surjective. Although the right hand side appears to involve v and not merely $\pi(v)$, Theorem 1.3 tells us that

$\xi v = \xi \star \pi(v)$.) We next prove that $\xi \in \Lambda(A^{**})$. Let $v_j \xrightarrow{\mathrm{w}^*} v$ in A^{**}. Because T^* is an adjoint and therefore weak* continuous, we find

$$\xi v_j = T^*(\pi(v_j)) \xrightarrow{\mathrm{w}^*} T^*(\pi(v)) = \xi v,$$

as required. From Theorem 2.2 (either (i) or (ii) for the corresponding cases of the present theorem) we deduce that $\xi \in \Lambda(A^{**}) = A$, so that we can write $\xi = z \in A$. Finally for any $v \in A^{**}$, $f \in A^*$ we have

$$\langle v, Tf \rangle = \langle \pi(v), Tf \rangle = \langle T^*(\pi(v)), f \rangle = \langle zv, f \rangle = \langle v, fz \rangle.$$

Thus $Tf = fz$. ∎

We now comment on Theorem 3.1 by means of examples and applications. Although the conclusion of the theorem is asymmetric (it involves *left* module homomorphisms, A^*A rather than AA^*, and so on) the applications we give are all in fact symmetric since $A^*A = AA^*$ when, for example, A is $L^1(G)$ or $A(G)$. We do not know whether there are significant asymmetric applications.

EXAMPLES 3.2. (a) For a compact group G the convolution algebra $L^1(G)$ has a bai and is an ideal in its second dual ([8]). Theorem 2.1 therefore applies to show that left module homomorphisms T from $L^\infty(G)$ to $L^\infty(G) * L^1(G) = C(G)$ are all given by convolution on the right by elements of $L^1(G)$. This is Theorem 35.13 of Hewitt and Ross [10].

(b) The conclusion of (a) does not hold for non-compact groups, and this shows that Theorem 3.1 may not be true if A is not an ideal in A^{**}. To illustrate using a simple case, let $A = l^1(\mathbb{Z})$ (where $\mathbb{Z}$ is the usual additive group). Since A has an identity, $A^*A = A^* = l^\infty(\mathbb{Z})$. Let μ be an invariant mean on $l^\infty(\mathbb{Z})$. Then $Tf = \langle \mu, f \rangle\, 1$ (where 1 is the constant function) defines an A-module homomorphism from A^* to itself which, as its range consists of the constant functions, is obviously not convolution by any element of A.

A conclusion for commutative locally compact groups which is parallel to Example (a) was presented by Pigno [24]. His homomorphisms are from $L^1(G) \cap L^\infty(G)$, and his result does not appear to be related to our theorem.

(c) The Volterra algebra $A = L^1(0, 1)$ consists of all functions integrable on $(0, 1)$ with respect to Lebesgue measure with multiplication given by the convolution product $x * y(t) = \int_0^t x(t-s)\, y(s)\, ds$ a.e. $(0 < t < 1)$. It is a commutative radical Banach algebra which has a sequential bai and is an ideal in its second dual (see for example [5] or [25]). For $f \in A^* = L^\infty(0, 1)$ and $x \in A$ we find easily that $fx(t) = \int_{s=0}^{1-t} f(s+t)\, x(s)\, ds$ a.e., so that fx, being essentially the convolution on $\mathbb{R}$ of an L^∞ function with an L^1 function, is continuous and bounded, and it is easy to see that $\lim_{t \nearrow 1} fx(t) = 0$. Thus A^*A is contained in the space $C_0[0, 1)$ of continuous functions on $[0, 1)$ which vanish at 1. Since A^*A is closed, $L^1(0, 1)$ has a bai carried by arbitrarily small neighbourhoods of 0, and functions in $C_0[0, 1)$ are uniformly continuous, it is not hard to show that $A^*A = C_0[0, 1)$. We deduce from Theorem 3.1 that the space of operators from $L^\infty(0, 1)$ to $C_0[0, 1)$ commuting with the action of $L^1(0, 1)$ can be identified with A.

The Volterra algebra has a sequential bai and so the full generality of Theorem 3.1 is not needed to deal with it. The more general Volterra-like algebras of [25] do have two-sided bai's and are ideals in their second duals, so the conclusion also holds for them.

(d) When G is discrete and amenable, the Fourier algebra $A(G)$ is an ideal in its second dual ([15], Theorem 3.7) and has a bai ([23], Section 4.34). The dual of $A(G)$ is identifiable with the von Neumann algebra $VN(G)$ defined by the left regular representation of G. The space $VN(G)\,A(G)$ $(= A(G)\,VN(G)$ because $A(G)$ is commutative) is the subalgebra $UCB(\hat{G})$ generated by the operators in $VN(G)$ of compact support (see [6], p. 65). Theorem 3.1 applies to show that the algebra of left $A(G)$-homomorphisms from $VN(G)$ to $UCB(\hat{G})$ can be identified with $A(G)$.

4. RIGHT IDENTITIES IN $l^1(S)^{**}$

In Corollary 2.3 we saw that certain types of Banach algebras can only have identities in their second duals if they themselves have identities. As usual in this paper, there was a contrast there between the situations in which A the bai was sequential and when it was not, with the latter case requiring an additional hypothesis. Ülger has another such result in which an additional hypothesis is needed ([27], Theorem 2.4): a weakly sequentially complete Banach algebra with a bai which is also Arens regular must have an identity (and, of course, a regular algebra with a bai has an identity in its second dual). The motivation for the work in this section was the hope of finding an example which would indicate that some extra hypothesis was essential; in fact, we shall show that for the convolution algebra $l^1(S)$ of a discrete commutative semigroup S it is not:

THEOREM 4.1. *If S is commutative and $l^1(S)^{**}$ has an identity then $l^1(S)$ has an identity.*

We shall establish a result more general than Theorem 4.1. From the proof of Theorem 1.6 we can see that the set of right identities of $l^1(S)^{**}$

is a translate of the subspace consisting of the right annihilators. It therefore makes sense to speak of its vector space dimension. We shall show that if $l^1(S)^{**}$ has a right identity but $l^1(S)$ does not, then the dimension of the set of right identities in $l^1(S)^{**}$ is infinite (we give the precise number in Theorem 4.2). In this form it recalls the result of [1]: if βS has a right identity but S does not, then βS has infinitely many right identities. In fact we shall need the latter result to establish our conclusion. However, Theorem 4.1 is not a straightforward consequence of [1], for $l^1(S)^{**}$ can have right identities even though its subsemigroup βS has none.

The restriction to commutative S follows the tradition set by Hewitt and Zuckermann [11]. For commutative S they were able to find necessary and sufficient conditions for $l^1(S)$ to have an identity (notice that it is not necessary for S to have an identity—this is apparent from Lemma 4.12). We shall need to use what is essentially their construction for an identity. The only work on the non-commutative case we are aware of is for inverse semigroups in [3]; we shall consider their situation in Corollary 4.14 below.

We begin by describing our cardinal number. If ε is a right identity in $l^1(S)^{**}$ we write

$$\kappa(\varepsilon) = \min\{\mathrm{card}(U) : U \subseteq l^1(S) \text{ and } \varepsilon \in \text{w*-cl}(U)\}$$

and then

$$\kappa(S) = \min\{\kappa(\varepsilon) : \varepsilon \text{ is a right identity for } l^1(S)^{**}\}.$$

Observe that if $\varepsilon \notin l^1(S)$ then $\kappa(\varepsilon)$ is infinite, while if $\varepsilon \in l^1(S)$ then $\kappa(\varepsilon) = 1$.

THEOREM 4.2. *Let S be a commutative discrete semigroup, and suppose $l^1(S)$ has no identity. If $l^1(S)^{**}$ has a right identity then the vector space dimension of the set of right identities is at least $2^{2^{\kappa(S)}}$.*

In this section, we shall denote the unit mass at a point s by $\bar{s}$.

We present the proof through a sequence of lemmas. The first is trivial, but we state it formally as we use it so often.

LEMMA 4.3. *If $s, t, u \in S$, $st = s$ and $tu = t$ then $su = s$.*

Proof. $su = (st)u = s(tu) = st = s$. ∎

Lemma 4.4. *If $l^1(S)^{**}$ has a right identity ε, then for each $s \in S$ there is $t \in S$ with $st = s$.*

Proof. Because $l^1(S)^{**}$ has a right identity, $l^1(S)$ has a brai (e_i) ([22], Theorem 5.1.8) and so, being commutative, a bai. In particular we can find

i such that $\|\bar{s}-\bar{s}e_i\|<1$. If $e_i=\sum_{t\in S} a_t\bar{t}$ where $a_t\neq 0$ for at most countably many t, this means that $\|\bar{s}-\sum_t a_t\overline{st}\|<1$. Since $\|\bar{s}\|=1$, this can only happen if $\bar{s}=\overline{st}$ for at least one t, that is $s=st$ for some t. ▮

DEFINITION 4.5. Let $A\subseteq S$. A set B is called an *identity set for A* if for each $a\in A$ there is $b\in B$ with $ab=a$. $R(A)$ denotes the set of all identity sets for A. We write $\rho(A)=\min\{\operatorname{card}(B) : B\in R(A)\}$.

From Lemmas 4.3 and 4.4 we get immediately

LEMMA 4.6. *If $B\in R(A)$ and $C\in R(B)$ then $C\in R(A)$. If A is finite, then $\rho(A)\leqslant\operatorname{card}(A)$.*

Notation 4.7. Fix a finite subset A of S. Take $B\in R(A)$ with $\operatorname{card}(B)=\rho(A)=m$ (say), so that $B=\{b_1, ..., b_m\}$. Write for $1\leqslant i\leqslant m$,

$$S_i=\{s\in S : \text{there is } t\in S \text{ with } b_i st=b_i\}.$$

Lemma 4.8. *Let $l^1(S)^{**}$ have a right identity. Then $S_1, ..., S_m$ are disjoint prime subsemigroups of S (that is, $st\in S_i$ implies both $s\in S_i$ and $t\in S_i$). Any identity for b_i or for any $u\in S_i$ is in S_i.*

Proof. That each S_i is a semigroup is easy from associativity and commutativity. If $st\in S_i$ then there is $u\in S$ with $b_i s(tu)=b_i(st)\,u=b_i$ so that $s\in S_i$. Since S is commutative, $t\in S_i$ also follows, so that S is prime.

To show the S_i are disjoint, suppose there is $s\in S_i\cap S_j$ with $i\neq j$. Then there exist t_i, t_j with $b_i st_i=b_i$, $b_j st_j=b_j$. Let u be an identity for s (Lemma 4.4). By Lemma 4.3, $b_i u=b_i$ and $b_j u=b_j$. Then, using Lemma 4.6, we see that $B'=(\{b_1, ..., b_m\}\backslash\{b_i, b_j\})\cup\{u\}\in R(A)$. But $\operatorname{card}(B')\leqslant m-1$, contradicting $\rho(A)=m$.

Now take any identity s for b_i. Let t be an identity for s. Then $b_i st=b_i s=b_i$, whence $s\in S_i$. Again, given $u\in S_i$, let t be an identity for u, so that $ut=u$. Then for some s we have $b_i us=b_i$ and $b_i t(us)=b_i(ut)\,s=b_i us=b_i$, so that $t\in S_i$. ▮

LEMMA 4.9. *For any right identity ε of $l^1(S)^{**}$, $m\leqslant\|\varepsilon\|$.*

Proof. Let E be the finite semigroup with elements $e_1, ..., e_m, \infty$ in which each element is idempotent and any product of distinct elements is ∞. It is easy to check that $l^1(E)$ has identity $e=\overline{e_1}+\cdots+\overline{e_m}-(m-1)\,\overline{\infty}$; this has norm $2m-1$. Define φ: $S\to E$ by $\varphi(s)=e_i$ if and only if $s\in S_i$ and $\varphi(s)=\infty$ if $s\notin\bigcup_{i=1}^m S_i$. Then φ is a surjective homomorphism since the S_i are prime subsemigroups. Now φ extends to a surjective algebra homomorphism $\bar{\varphi}$: $l^1(S)\to l^1(E)$, and so the second adjoint $\bar{\varphi}^{**}$ is a surjective homomorphism $\bar{\varphi}^{**}$: $l^1(S)^{**}\to l^1(E)^{**}=l^1(E)$ (because $l^1(E)$ is finite dimensional). Therefore $\bar{\varphi}^{**}(\varepsilon)$ is a right identity for $l^1(E)$; but this can only be e. Since $\|\bar{\varphi}\| \leqslant 1$ we find $2m-1 = \|e\| = \|\bar{\varphi}^{**}(\varepsilon)\| \leqslant \|\varepsilon\|$. Since $m \geqslant 1$ is an integer, $m \leqslant \|\varepsilon\|$ follows. ∎

The point of Lemma 4.9 is that the set of integers $\rho(A)$ is bounded as A runs through the finite subsets of S.

Notation 4.10. Let $M = \max\{\rho(A) : A \subseteq S \text{ is finite}\}$. Fix A_M with $\rho(A_M) = M$ and take $B_M = \{b_1, ..., b_M\} \in R(A_M)$. Form $S_1, ..., S_M$ as in 4.7.

LEMMA 4.11. (i) *For each $t \in S$ there exist j with $1 \leqslant j \leqslant M$ and $u \in S_j$ with $tu = t$.*

(ii) *Take j with $1 \leqslant j \leqslant M$. Let $F \subseteq S_j$ be finite. Then there is $u \in S_j$ with $tu = t$ for all $t \in F$.*

Proof. The proofs of the two parts of the lemma use the same technique.

(i) The set $\{b_1, ..., b_M, t\}$ is finite. There is therefore a finite set $\{s_1, ..., s_k\}$ of minimal cardinal k, with $k \leqslant M$ from 4.10, such that for $1 \leqslant i \leqslant M$ there is r_i with $b_i s_{r_i} = b_i$ and also there is j with $ts_j = t$. But this means that $\{s_{r_1}, ..., s_{r_M}\} \in R(B_M)$, and therefore it is also in $R(A_M)$ and consequently has at least M elements. Thus $s_{r_1}, ..., s_{r_M}$ must be distinct, and so by permuting them we may assume that $k = M$ and $b_i s_i = b_i$ for all i. Then $b_j s_j = b_j$ tells us that $s_j \in S_j$ (Lemma 4.8) and $ts_j = t$ gives the conclusion of (i) with $u = s_j$.

(ii) For simplicity in notation take $j = M$. We find a finite set of minimal cardinal $\{s_1, ..., s_k\}$ of identities for the finite set $\{b_1, ..., b_{M-1}\} \cup F$. As in (i) we find $k \leqslant M$ and we may suppose $b_i s_i = b_i$ for $1 \leqslant i \leqslant M-1$. Since $t \in F$ implies $t \in S_M$, $ts_i = t$ for some i with $1 \leqslant i \leqslant M-1$ is impossible from Lemma 4.8. Therefore $k = M$ and $ts_M = t$ for all $t \in F$. ∎

We can now construct right identities for $l^1(S)^{**}$.

LEMMA 4.12. *Let $S_1, ..., S_M$ be as in* 4.10. *Then each $l^1(S_i)^{**}$ has a right identity ε_i and*

$$\varepsilon = \left(\sum_i \varepsilon_i\right) - \left(\sum_{i<j} \varepsilon_i \varepsilon_j\right) + \left(\sum_{i<j<k} \varepsilon_i \varepsilon_j \varepsilon_k\right) - \cdots + (-1)^{M+1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_M$$

is a right identity for $l^1(S)^{**}$.

Proof. From Lemma 4.11(ii) for each finite set $F \subseteq S_i$ there is an identity e_F for F in S_i. Direct the net $(\overline{e_F})_F$ by inclusion. Then $(\overline{e_F})_F$, which consists of elements of norm 1 in $l^1(S_i) \subseteq l^1(S)$, must have a weak* cluster point ε_i in $l^1(S_i)^{**} \subseteq l^1(S)^{**}$, and this is a right identity for $l^1(S_i)^{**}$.

Now take $t \in S$. From Lemma 4.11(i) there exist j and $u \in S_j$ with $tu = t$. Thus $\bar{t}\varepsilon_j = \bar{t}\bar{u}\varepsilon_j = \bar{t}$. Since $\bar{t} \in l^1(S)$ commutes with all elements of $l^1(S)^{**}$, we find (for example) for any i, k,

$$\bar{t}\varepsilon_i\varepsilon_j\varepsilon_k = \varepsilon_i\bar{t}\varepsilon_j\varepsilon_k = \varepsilon_i\bar{t}\varepsilon_k = \bar{t}\varepsilon_i\varepsilon_k.$$

From such relationships it is easy to calculate that $\bar{t}\varepsilon = \bar{t}$. That ε is a right identity for $l^1(S)^{**}$ follows. ∎

LEMMA 4.13. *Suppose $l^1(S)$ has no identity. Then there are at least $2^{2^{\kappa(S)}}$ linearly independent right identities ε for $l^1(S)^{**}$ of the form given in Lemma* 4.12.

Proof. Fix any j. Make $S_j \cup \{\infty\}$ a semigroup by taking $\{\infty\}$ to be an adjoined semigroup zero. Then $\psi_j : S \to S_j \cup \{\infty\}$, defined to be the identity on S_j and to map $S \backslash S_j$ to ∞, is a surjective semigroup homomorphism because S_j is prime (Lemma 4.8). The element ε_j of $l^1(S_j)^{**}$ is a right identity for $l^1(S_j \cup \{\infty\})^{**}$. ψ_j extends to a surjective homomorphism $\overline{\psi_j}^{**} : l^1(S)^{**} \to l^1(S_j \cup \{\infty\})^{**}$ and $\overline{\psi_j}^{**}(\varepsilon_i) = \bar{\infty}$ if $i \neq j$, but $\overline{\psi_j}^{**}(\varepsilon_j) = \varepsilon_j$. Therefore $\overline{\psi_j}^{**}(\varepsilon) = \varepsilon_j$ for each ε as in Lemma 4.12. We conclude that $\overline{\psi_j}^{**}$ maps the set of right identities of $l^1(S)^{**}$ onto the set of right identities of $l^1(S_j)^{**}$ (regarded as a subalgebra of $l^1(S_j \cup \{\infty\})^{**}$).

If the semigroup S_j has an identity e_j then the point mass $\overline{e_j}$ is an identity ε_j for $l^1(S_i)$, and so also for $l^1(S_j)^{**}$. If every S_j has an identity, Lemma 4.12 shows that $l^1(S)$ has an identity. We deduce that under our hypotheses, at least one S_j has no identity. We take such a j. According to [1], the Stone–Čech compactification βS_j of the discrete semigroup S_j has $2^{2^{\kappa(j)}}$ right identities, where $\kappa(j)$ is the smallest cardinal of a set $U_j \subseteq S_j$ whose closure in βS_j contains a right identity for βS_j. We may regard βS_j, which can be described as the set of complex homomorphisms of $l^\infty(S_j) = l^1(S_j)^*$, as a subset of $l^1(S_j)^{**}$, and as such all its elements are linearly independent. Thus the dimension of the set of right identities in $l^1(S_j)^{**}$ is $2^{2^{\kappa(j)}}$. We conclude (using $\overline{\psi_j}^{**}$ above) that the set of right identities in $l^1(S)^{**}$ has dimension at least $2^{2^{\kappa(j)}}$.

We finish by relating $\kappa(j)$ to $\kappa(S)$. For each j we can find a subset U_j of S_j with $\operatorname{card}(U_j) = \kappa(j)$ and $\varepsilon_j \in \operatorname{cl} U_j$ for some right identity ε_j of $\beta(S_j)$ (recall that if $\varepsilon_j \in S_j$ then $\kappa(j) = 1$). Take $u_j \in U_j$ for each j. Consider

$$u = \left(\sum_i \overline{u_i}\right) - \left(\sum_{i<j} \overline{u_i u_j}\right) + \left(\sum_{i<j<k} \overline{u_i u_j u_k}\right) - \cdots + (-1)^{M+1}\, \overline{u_1 u_2} \cdots \overline{u_M}.$$

The iterated limit of u when we let first $\overline{u_M} \xrightarrow{w^*} \varepsilon_M$, then $\overline{u_{M-1}} \xrightarrow{w^*} \varepsilon_{M-1}, ...,$ and finally $\overline{u_1} \xrightarrow{w^*} \varepsilon_1$ is ε. Thus the right identity ε is in the closure of the set of all such u's, and this has cardinal $\max\{\kappa(1), ..., \kappa(M)\}$ since at least one of the $\kappa(j)$'s is infinite. Therefore $\kappa(S) \leqslant \kappa(j)$ for at least one j, and our proof is complete. ▮

Theorem 4.2 can be applied to prove a result for type of non-commutative semigroup. S is called an *inverse semigroup* if for each $s \in S$ there exists a unique s^* in S with $ss^*s = s$, $s^*ss^* = s^*$ (see [3] for more information).

COROLLARY 4.14. *If S is an inverse semigroup and $l^1(S)$ has a brai but no right identity, then $l^1(S)^{**}$ has at least $2^{2^{\kappa(S)}}$ right identities.*

Proof. The set $I(S)$ of idempotents in S is a commutative semigroup. According to the proof of Lemma 13 of [3], $l^1(S)$ has a brai if and only if $l^1(I(S))$ does, and the brai for $l^1(I(S))$ will serve as a brai for $l^1(S)$. As $l^1(I(S))$ is commutative, its brai is actually a bai. We need only apply Theorem 4.2 to $l^1(I(S))$. ▮

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