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Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators, Cauchy Problems

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1. INTRODUCTION

In this paper, we shall study the existence and regularity of local and global (in time t) strong solutions for the following abstract Cauchy problem in a real separable Hilbert space H:

$$\left.\begin{aligned} &\frac{du}{dt}(t) + \partial\varphi^t(u(t)) + B(t, u(t)) \ni f(t), \qquad 0 < t < T, && (1.1)\\ &u(0) = u_0, && (1.2)\end{aligned}\right\} \text{(C.P.)}$$

where $\partial\varphi^t$ is the subdifferential of a time-dependent lower semicontinuous convex function φ^t from H into $[0, +\infty]$ with $\varphi^t \not\equiv +\infty$, and $B(t, \cdot)$ is a possibly nonmonotone multivalued nonlinear operator from $D(B(t, \cdot)) \subset H$ into H such that $D(\partial\varphi^t) \subset D(B(t, \cdot))$ for all $t \in [0, T]$. When $B(t, \cdot)$ is a monotone-type operator, many results on the existence, uniqueness and regularity of the (global) strong solution for (C.P.) have been established. In particular, we refer to Brézis [8], Watanabe [32], Maruo [23], Attouch and Damlamian [1], Kenmochi [18] and Yamada [33].

On the other hand, the study for the case that $B(t, \cdot)$ is not monotone has been developed recently by several authors under some compactness assumptions on $D(\varphi^t) := \{u \in H; \varphi^t(u) < +\infty\}$ similar to one other. For example, Attouch and Damlamian [2] and Biroli [3] dealt with the case where $\partial\varphi^t \equiv \partial\varphi$ ($\partial\varphi^t$ is independent of t) and $B(t, \cdot)$ belongs to a class of (time-dependent) upper semicontinuous operators. The case that $\partial\varphi^t \equiv \partial\varphi$ and $B(t, \cdot) \equiv -\partial\psi$ was studied by Koi and Watanabe [19] and the author [25].

The main purpose of the present paper is to extend these results in the following three directions:

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(I) Allow the domains $D(\partial\varphi^t)$ and $D(B(t, \cdot))$ of $\partial\varphi^t$ and $B(t, \cdot)$ to move as time t varies.

(II) Treat a much wider class of perturbations $B(t, \cdot)$ aiming at an abstract theory applicable to the Navier–Stokes equation.

(III) Study the case that the initial data u_0 belong to intermediate classes between $D(\varphi^0)$ and $\overline{D(\varphi^0)}$ (the closure of $D(\varphi^0)$ in the H-norm).

As for the t-dependence of $D(\partial\varphi^t)$, we shall employ a condition similar to those of Yamada [33, 34] and Ôtani [26] (see $(A.\varphi^t)$ below). Concerning $B(t, \cdot)$, we shall assume only certain demiclosedness and boundedness conditions relative to $\partial\varphi^t$ and a measurability condition with respect to time t (see (A.2)). In order to pursue the last plan, we shall make use of the nonlinear interpolation theory developed by D. Brézis [5], which makes it possible to classify intermediate classes between $D(\varphi^0)$ and $\overline{D(\varphi^0)}$ by measuring how fast $(I+\lambda\,\partial\varphi^0)^{-1}u$ converges to u in H as $\lambda \downarrow 0$. As will be exemplified in Section 5, generalization in these directions gives a unified abstract treatment for initial-boundary value problems of some nonlinear heat equations with a difference term of monotone operators and the Navier–Stokes-type equations in bounded regions with moving boundaries.

This paper is composed of five sections. Section 2 contains some notions and known results on the nonlinear interpolation theory and a Schauder–Tychonoff-type fixed point theorem for multivalued mappings which will be used later. The existence results for local strong solutions of (C.P.) are given in Section 3, where method of proofs of them is based on a Schauder–Tychonoff-type fixed point theorem as in [2] and [3]. In Section 4, in order to study the global existence, we shall discuss continuation of the local strong solutions constructed in Section 3. The last section is devoted to applications of the preceding abstract results to some initial-boundary value problems for the Navier–Stokes-type equations in a bounded noncylindrical domain. In particular, concerning the 3-dimensional nonstationary Navier–Stokes equation, it is shown that the results of Fujita and Kato [12] still hold in a form reflecting the noncylindrical nature of the domain.

2. Preliminaries

2.1. *Notations and Subdifferential Operators*

Let H be a real Hilbert space with the inner product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$, which are often denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. We denote by $C([a, b]; H)$ the set of all H-valued continuous functions on $[a, b]$. $L^p(a, b; H)$, $1 \leqslant p \leqslant \infty$, denotes the set of all strongly measurable functions on $[a, b]$ such that

$$|v|_{L^p(a,b;H)} = \left(\int_a^b |v(t)|_H^p\, dt\right)^{1/p} < +\infty \qquad (1 \leqslant p < +\infty),$$

$$|v|_{L^\infty(a,b;H)} = \operatorname*{ess\,sup}_{a<t<b} |v(t)|_H < +\infty \qquad (p = \infty).$$

We also denote by $L^p_{\text{loc}}((0, S]; H)$ the set of all functions v on $(0, S)$ such that $v \in L^p(\delta, S; H)$ for all $\delta \in (0, S)$. In the case of $H = \mathbb{R}^1$, we simply write $L^p(a, b)$ or $L^p_{\text{loc}}(0, S]$ as usual.

Let φ be a *proper lower semicontinuous convex function* from H into $(-\infty, +\infty]$, where "*proper*" means $\varphi \not\equiv +\infty$. Define the *effective domain* $D(\varphi)$ of φ by $D(\varphi) = \{u \in H; \varphi(u) < +\infty\}$ and the subdifferential $\partial\varphi$ of φ by

$$\partial\varphi(u) = \{u \in H; \varphi(v) - \varphi(u) \geqslant (f, v - u) \text{ for all } v \in H\}$$

with domain $D(\partial\varphi) = \{u \in H; \partial\varphi(u) \neq \varnothing\}$. Then, as is well known, $\partial\varphi$ is a (possibly multivalued) maximal monotone operator in H and $\overline{D(\varphi)}$ (the closure of $D(\varphi)$ in the H-norm) coincides with $\overline{D(\partial\varphi)}$. We designate by $\partial^0\varphi$ the *minimal section* of $\partial\varphi$, i.e., $\partial^0\varphi(u)$ is the unique element of least norm in $\partial\varphi(u)$.

2.2. *Nonlinear Interpolation Classes*

Let A be a maximal monotone operator in H with domain $D(A)$, and let $J_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$. For each $\alpha \in (0, 1)$ and $p \in [1, \infty]$, we define the intermediate class $\mathscr{B}_{\alpha,p}(A)$ between $D(A)$ and $\overline{D(A)}$ by

$$\mathscr{B}_{\alpha,p}(A) = \{u \in \overline{D(A)}; t^{-\alpha} |u - J_t u|_H \in L^p_*(0, 1)\},$$

where $L^p_* = L^p(dt/t)$, i.e., $|f|_{L^p_*(0,1)} = (\int_0^1 |f(t)|^p (dt/t))^{1/p}$ if $1 \leqslant p < \infty$ and $L^\infty_*(0, 1) = L^\infty(0, 1)$.

In what follows, we frequently use the notation

$$|u|_{\mathscr{B}_{\alpha,p}} = |t^{-\alpha} |u - J_t u|_H|_{L^p_*(0,1)}.$$

Since the function $|t^{-1}(I - J_t)u|_H$ is a monotone decreasing function of $t \in (0, 1]$ for each $u \in H$, the following proposition is derived (see Brézis [5–7]).

PROPOSITION 2.1. *The following inclusion holds*:

$$|u|_{\mathscr{B}_{\alpha,q}} < 2\, |u|_{\mathscr{B}_{\alpha,p}} \qquad \textit{for all} \quad \alpha \in (0, 1) \textit{ if } 1 \leqslant p < q \leqslant \infty, \tag{2.1}$$

$$|u|_{\mathscr{B}_{\beta,q}} < \frac{2}{\alpha - \beta} |u|_{\mathscr{B}_{\alpha,p}} \qquad \textit{for all} \quad 1 \leqslant p, q \leqslant \infty \textit{ if } 0 < \beta < \alpha < 1. \tag{2.2}$$

In the case of $A = \partial\varphi$, $\mathscr{B}_{\alpha,p}(\partial\varphi)$ can be characterized by the behavior (near $t=0$) of the semigroup $S(t)=e^{-t\partial\varphi}$ generated by $-\partial\varphi$ as follows (see [5]).

THEOREM 2.2. *Let* $\alpha \in (0, \frac{1}{2})$ *and* $u \in \overline{D(\partial\varphi)}$. *Then the following properties are equivalent to each other*:

(i) $u \in \mathscr{B}_{\alpha,p}(\partial\varphi)$,

(ii) $t^{-\alpha}\,|S(t)u-u|_H \in L^p_*(0,1)$,

(iii) $t^{1/2-\alpha}\,|\varphi(S(t)u)|^{1/2} \in L^p_*(0,1)$.

More precisely, we have

$$|t^{-\alpha}\,|S(t)u-u|_H\,|_{L^p_*(0,1)} \leqslant 3\,|u|_{\mathscr{B}_{\alpha,p}}, \tag{2.3}$$

$$\begin{aligned} &|t^{1/2-\alpha}\,|\varphi(S(t)u)|^{1/2}\,|_{L^p_*(0,1)} \\ &\quad < \frac{2^{1/2-\alpha}}{2^{1/2-\alpha}-1}\,(3\,|u|_{\mathscr{B}_{\alpha,p}} + |u-w|_H + |\varphi(w)|^{1/2}) \qquad \textit{for all} \quad w \in D(\varphi). \end{aligned} \tag{2.4}$$

This theorem is proved by the standard argument on the subdifferential operator theory with the aid of the following two lemmas due to Brézis.

LEMMA 2.3. *Let $F(t)$ and $G(t)$ be positive functions on $(0, S]$ such that $F(t) \in L^p_{\mathrm{loc}}(0, S]$ $(1 \leqslant p \leqslant \infty)$, $F(t) \to 0$ as $t \to 0$, and that*

$$F(t) \leqslant F(t/2) + G(t) \qquad \textit{for all} \quad t \in (0, S]. \tag{2.5}$$

Let $t^{-\gamma}G(t) \in L^p_*(0, S)$ with $\gamma > 0$. Then we have

$$|t^{-\gamma}F(t)|_{L^p_*(0,S)} \leqslant \frac{2^\gamma}{2^\gamma-1}\,|t^{-\gamma}G(t)|_{L^p_*(0,S)}. \tag{2.6}$$

LEMMA 2.4. *Let $F(t)$ and $G(t)$ be positive functions on $(0, S]$ such that $F(t) \in L^p_{\mathrm{loc}}(0, S]$ $(1 \leqslant p \leqslant \infty)$, and that*

$$F(t) \leqslant F(2t) + G(t) \qquad \textit{for all} \quad t \in (0, S/2]. \tag{2.7}$$

Let $t^\gamma G(t) \in L^p_*(0, S/2)$ with $\gamma > 0$. Then we have

$$|t^\gamma F(t)|_{L^p_*(0,S)} \leqslant \frac{2^\gamma}{2^\gamma-1}\,(|t^\gamma G(t)|_{L^p_*(0,S/2)} + |t^\gamma F(t)|_{L^p_*(S/2,S)}). \tag{2.8}$$

2.3. *A Fixed Point Theorem for Multivalued Mappings*

In this subsection, we mention a couple of results on upper semicontinuous multivalued mappings without their proofs. (For details, we refer to the text of Browder [9].) Let us begin with the definition of *upper semicontinuous mappings.*

DEFINITION 2.5. Let X and Y be two topological spaces and T be a multivalued mapping from X into Y. Then T is said to be an upper semicontinuous multivalued mapping from X into Y, if for each point x of X and each open neighborhood V of $T(x)$ in Y, there exists a neighborhood U of x in X (with U depending upon V) such that $T(U) \subset V$, i.e., $y \in U$ implies $T(y) \subset V$.

We have the following criterion for the upper semicontinuity of multivalued mappings (see [9, Proposition 6.2]).

PROPOSITION 2.6. *Let K and K_1 be two compact topological spaces and T be a multivalued mapping from K into K_1 with $T(x)$ closed for each x in K. Then T is upper semicontinuous if and only if the graph $G(T) := \{[u, w] \in K \times K_1;\ w \in T(u)\}$ of T is a closed subset of $K \times K_1$.*

Now we mention an important Schauder–Tychonoff-type fixed point theorem (see [9, Corollary 2 to Theorem 6.3]).

THEOREM 2.7. *Let K be a compact convex subset of a locally convex topological vector space X. Let T be an upper semicontinuous multivalued mapping from K into X such that for each x in K, $T(x)$ is a closed convex subset of X whose intersection with K is nonempty. Then T has a fixed point in K, i.e., there exists an element x_0 in K such that $x_0 \in T(x_0)$.*

3. LOCAL EXISTENCE

In this section, we study the local (in time) existence of strong solutions of (C.P.) when the initial data u_0 belongs to $\mathscr{B}_{\alpha,p}(\partial\varphi^0)$ $(0 < \alpha < \frac{1}{2})$, $D(\varphi^0) = \mathscr{B}_{1/2,2}(\partial\varphi^0)$ and $\overline{D(\varphi^0)} = \overline{D(\partial\varphi^0)}$. Here and henceforth, we are concerned with *strong solutions* of (C.P.) in the following sense.

DEFINITION 3.1. A function $u(t) \in C([0, S]; H)$ is said to be a *strong solution* of (C.P.) in $[0, S]$, if the following (i) and (ii) are satisfied.

(i) $u(t)$ is an H-valued absolutely continuous function on $[\delta, S]$ for all $\delta > 0$ and $u(t) \to u_0$ as $t \downarrow 0$.

(ii) $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in (0, S)$ and there exist two functions $g(t)$, $b(t) \in L^2_{\text{loc}}((0, S]; H)$ such that $g(t) \in \partial\varphi^t(u(t))$, $b(t) \in B(t, u(t))$ and

$$du(t)/dt + g(t) + b(t) = f(t) \tag{3.1}$$

hold for a.e. $t \in (0, S)$. In addition, if $du(t)/dt$, $g(t)$ and $b(t)$ belong to $L^2(0, S; H)$, then $u(t)$ is said to be *s-strong solution* of (C.P.) in $[0, S]$.

3.1. *Results*

As was noted before, the operator $B(t, \cdot)$ is not assumed to be monotone. Instead of it, roughly speaking, we assume the same type of compactness condition on $D(\varphi^t)$ just as in the previous paper [25], i.e., (A.1) below. Besides, boundedness conditions on $B(t, \cdot)$ are quite similar to those of [25]. In the present paper, however, $-B(t, \cdot)$ need not be of the form $\partial\psi^t$, i.e., another (t-dependent) subdifferential operator, nor maximal monotone, but a certain kind of demiclosedness, together with a measurability condition with respect to time t, is imposed on $B(t, \cdot)$ (see (A.2) below).

Throughout this paper, we assume that Hilbert space H is separable and denote by $\mathscr{M}$ the *family of all positive monotone increasing functions* on $[0, +\infty)$ and by $M(\cdot)$ a general element of $\mathscr{M}$ which will have different forms in different places. In order to formulate our results, we gradually introduce several conditions for $\partial\varphi^t$ and $B(t, \cdot)$. We first introduce the following four conditions.

(A.φ^t) For each $t \in [0, T]$, φ^t is a proper lower semicontinuous convex function from H into $[0, +\infty]$. Furthermore, for each $t_0 \in [0, T]$ and $x_0 \in D(\varphi^{t_0})$, there exist a positive constant δ_0 (independent of t_0 and x_0) and an H-valued function $x(t)$ on $I(t_0) := [\max(0, t_0 - \delta_0), \min(t_0 + \delta_0, T)]$ such that

$$|x(t) - x_0|_H \leqslant m_1(|x_0|_H)\,|t - t_0|\,(\varphi^{t_0}(x_0) + m_2(|x_0|_H))^{\beta} \tag{3.2}$$

and

$$\varphi^t(x(t)) \leqslant \varphi^{t_0}(x_0) + m_1(|x_0|_H)\,|t - t_0|\,(\varphi^{t_0}(x_0) + m_2(|x_0|_H)) \tag{3.3}$$

hold for all $t \in I(t_0)$, where β is a constant in $[0, 1]$ and $m_i(\cdot)$ are continuous functions belonging to $\mathscr{M}$ $(i = 1, 2)$.

(A.1) For each $t \in [0, T]$ and $L \in (0, +\infty)$, the set $\{u \in H;\ \varphi^t(u) + |u|_H^2 \leqslant L\}$ is compact in H.

(A.2) For each interval $[a, b]$ in $[0, T]$, the following (i)–(iii) hold.

(i) $B(t, u)$ is a convex subset of H for all $t \in [a, b]$ and $u \in D(\partial\varphi^t)$.

(ii) $B(t, \cdot)$ is *measurable* in the following sense: For each function $u(t) \in C([a, b]; H)$ such that $du(t)/dt \in L^2(a, b; H)$ and there exists a function $g(t) \in L^2(a, b; H)$ with $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [a, b]$, there exists an H-valued measurable function $b(t)$ such that $b(t) \in B(t, u(t))$ for a.e. $t \in [a, b]$.

(iii) $B(t, \cdot)$ is *demiclosed* in the following sense: If $u_n \to u$ in $C([a, b]; H)$ $g_n \to g$ weakly in $L^2(a, b; H)$ with $g_n(t) \in \partial\varphi^t(u_n(t))$, $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [a, b]$, and if $b_n \to b$ weakly in $L^2(a, b; H)$ with $b_n(t) \in B(t, u_n(t))$ for a.e. $t \in [a, b]$, then $b(t) \in B(t, u(t))$ holds for a.e. $t \in [a, b]$.

Remark 3.2. If $B(t, \cdot)$ (or $-B(t, \cdot)$) is a t-independent maximal monotone operator in H, then (A.2) is always satisfied.

$(\mathrm{A}.3)_\alpha$ For an exponent $\alpha \in (0, \frac{1}{2})$, there exist functions $\ell_0(\cdot)$, $\ell(\cdot) \in \mathscr{M}$ and $a(\cdot) \in L^2(0, T)$ such that

$$|||B(t, u)|||_H \leqslant \ell(|u|_H)\{\varepsilon\, |\partial^0 \varphi^t(u)|_H + \ell_0(1/\varepsilon)\, |\varphi^t(u)|^{(1-\alpha)/(1-2\alpha)} + |a(t)|\} \quad (3.4)$$

for all $\varepsilon > 0$, $t \in [0, T]$ and $u \in D(\partial\varphi^t)$, where $|||B(t, u)|||_H = \sup\{|b|_H;\ b \in B(t, u)\}$ and $\partial^0\varphi^t$ denotes the minimal section of $\partial\varphi^t$.

Then, in the case of $u_0 \in \mathscr{B}^0_{\alpha,p} := \mathscr{B}_{\alpha,p}(\partial\varphi^0)$, $0 < \alpha < \frac{1}{2}$, we have

THEOREM I. *Let* $(\mathrm{A}.\varphi^t)$, (A.1), (A.2) *and* $(\mathrm{A}.3)_\alpha$ *be satisfied. Let* $u_0 \in \mathscr{B}^0_{\alpha,p}$ *with* $p \in [1, 2]$ *and* $f(t) \in L^2(0, T; H)$. *Then there exists a positive number* $T_0 \in (0, T]$ *depending on* $|u_0|_H$ *and* $|u_0|_{\mathscr{B}^0_{\alpha,p}}$ *such that* (C.P.) *has a strong solution* $u(t)$ *in* $[0, T_0]$ *satisfying*

$$t^{1/2-\alpha}\, du(t)/dt, \quad t^{1/2-\alpha} g(t), \quad t^{1/2-\alpha} b(t) \in L^2(0, T_0; H), \quad (3.5)$$

$$t^{-\alpha}\, |u(t) - u_0|_H,\ t^{1/2-\alpha}\, |\varphi^t(u(t))|^{1/2} \in L^q_*(0, T_0) \quad \textit{for all } q \in [2, \infty], \quad (3.6)$$

where $g(t)$ *and* $b(t)$ *are the sections of* $\partial\varphi^t(u(t))$ *and* $B(t, u(t))$ *satisfying* (3.1) *in Definition* 3.1.

COROLLARY I. *Let* $u_0 \in \mathscr{B}^0_{\alpha,p}$ *with* $\alpha \in (0, \frac{1}{2})$, $p \in (2, \infty]$ *and* $f(t) \in L^2(0, T; H)$. *Let* $(\mathrm{A}.\varphi^t)$, (A.1) (A.2) *and* $(\mathrm{A}.3)_{\alpha'}$ *be satisfied for some* $\alpha' \in (0, \alpha)$. *Then, as in Theorem* I, *there exists a local strong solution* $u(t)$ *of* (C.P.) *in* $[0, T_0]$ *satisfying* (3.5) *and* (3.6) *with* α *replaced by* α'.

As for the cases $u_0 \in D(\varphi^0)$ and $\overline{D(\varphi^0)}$, we assume:

(A.4) There exist functions $\ell(\cdot) \in \mathscr{M}$, $c(\cdot) \in L^1(0, T)$ and a constant $k \in (0, 1)$ such that

$$|||B(t, u)|||_H^2 \leqslant k\, |\partial^0\varphi^t(u)|_H^2 + \ell(\varphi^t(u) + |u|_H)\, |c(t)|$$
$$\text{for all} \quad t \in [0, T] \text{ and } u \in D(\partial\varphi^t). \quad (3.7)$$

(A.5) There exist functions $\ell(\cdot) \in \mathscr{M}$, $a(\cdot) \in L^2(0, T)$ and a constant $\gamma \in (0, 1)$ such that

$$|||B(t, u)|||_H \leqslant \ell(|u|_H)(|\partial^0\varphi^t(u)|_H^{1-\gamma} + |\varphi^t(u)|^{1-\gamma} + |a(t)|)$$
$$\text{for all} \quad t \in [0, T] \text{ and } u \in D(\partial\varphi^t). \quad (3.8)$$

Then our results are stated as follows.

THEOREM II. *Let* $(\mathrm{A}.\varphi^t)$, (A.1), (A.2) *and* (A.4) *be satisfied. Let* $u_0 \in$

$D(\varphi^0)$ and $f(t) \in L^2(0, T; H)$. Then there exists a positive number $T_0 \in (0, T]$ depending on $|u_0|_H$ and $\varphi^0(u_0)$ such that (C.P.) *has an s-strong solution $u(t)$ in $[0, T_0]$.*

THEOREM III. *Let* (A.φ^t), (A.1), (A.2) *and* (A.5) *be satisfied. Let $u_0 \in \overline{D(\varphi^0)}$ and $f(t) \in L^2(0, T; H)$. Then there exists a positive number $T_0 \in (0, T]$ depending on $|u_0|_H$ such that* (C.P.) *has a strong solution $u(t)$ in $[0, T_0]$ satisfying* (3.5) *with $\alpha = 0$ and*

$$\varphi^t(u(t)) \in L^1(0, T_0), \qquad t \cdot \varphi^t(u(t)) \in L^\infty(0, T_0). \tag{3.9}$$

Remark 3.3. Under assumption (A.φ^t), if $u(t)$ is a strong (resp. s-strong) solution of (C.P.) in $[0, S]$, then $\varphi^t(u(t))$ is absolutely continuous on $(0, S]$ (resp. $[0, S]$). (See Proposition 3.5 of [26].)

3.2. *Some Lemmas*

To prove our theorems, we need some *a priori* estimates. First of all we investigate the behavior (near $t = 0$) of strong solutions of the equations

$$\left.\begin{aligned} &d\hat{u}(t)/dt + \partial\varphi^t(\hat{u}(t)) \ni 0, \qquad 0 < t < S, &&(3.10)\\ &\hat{u}(0) = u_0. &&(3.11)\end{aligned}\right\} \text{(C.P.)}_0$$

(Note that under (A.φ^t), for all $u_0 \in \overline{D(\varphi^0)}$, there exists a unique strong solution of (C.P.)$_0$; see, e.g., [33] and [26].) To this end, we here prepare the following proposition, which often plays an important role in the following argument. (For a proof, see [26] and [28].)

PROPOSITION 3.4. *Let* (A.φ^t) *be satisfied and $u(t)$ be an H-valued absolutely continuous function on $[0, S]$ with $\varphi^t(u(t))$ absolutely continuous on $[0, S]$. Put $\mathscr{L} = \{t \in [0, S];\ du(t)/dt$ and $d\varphi^t(u(t))/dt$ exist, $u(t) \in D(\partial\varphi^t)\}$. Then*

$$\begin{aligned}\left|\frac{d}{dt}\varphi^t(u(t)) - \left(g, \frac{du}{dt}(t)\right)_H\right| &\\ \leqslant m_1(|u(t)|_H)\,|g|_H\{\varphi^t(u(t)) &+ m_2(|u(t)|_H)\}^\beta \\ + m_1(|u(t)|_H)\{\varphi^t(u(t)) &+ m_2(|u(t)|_H)\}\end{aligned}$$

holds for all $t \in \mathscr{L}$ and $g \in \partial\varphi^t(u(t))$.

Remark 3.5. When assumption (A.φ^t) is applied in the following argument of this section, we restrict our consideration to the special case that $m_1(\cdot) \equiv m$ (a positive constant), $m_2(\cdot) \equiv 1$ and $I(t_0) \equiv [0, T]$ for the sake of simplicity; nevertheless, all the results of this section can be verified for the general case with simple modification.

Concerning the behavior near $t=0$ of solutions of $(\mathrm{C.P.})_0$, we have

LEMMA 3.6. *Let* $(\mathrm{A}.\varphi^t)$ *be satisfied and* $u_0 \in \mathscr{B}^0_{\alpha,p}$ *with* $0<\alpha<\frac{1}{2}$. *Then there exists a function* $M_1(\cdot) \in \mathscr{M}$ *(depending on* α*) such that the unique strong solution* $\hat{u}(t)$ *of* $(\mathrm{C.P.})_0$ *in* $[0,S]$, $0<S\leqslant 1$, *satisfies*

$$|t^{-\alpha}\,|\hat{u}(t)-u_0|_H|_{L^p_*(0,S)} \leqslant M_1(|u_0|_H)\cdot G_{\alpha,p}(u_0,w,S), \tag{3.12}$$

$$|t^{1/2-\alpha}\,|\varphi^t(\hat{u}(t))|^{1/2}|_{L^p_*(0,S)} \leqslant M_1(|u_0|_H)\cdot G_{\alpha,p}(u_0,w,S), \tag{3.13}$$

$$\left| t^{-\alpha}\left(\int_0^t s\left|\frac{d\hat{u}}{ds}(s)\right|_H^2 ds\right)^{1/2}\right|_{L^p_*(0,S)} \leqslant M_1(|u_0|_H)\cdot G_{\alpha,p}(u_0,w,S), \tag{3.14}$$

$$\left| t^{-\alpha}\left(\int_0^t \varphi^s(\hat{u}(s))\,ds\right)^{1/2}\right|_{L^p_*(0,S)} \leqslant M_1(|u_0|_H)\cdot G_{\alpha,p}(u_0,w,S), \tag{3.15}$$

where $G_{\alpha,p}(u_0,w,S)=|u_0|_{\mathscr{B}^0_{\alpha,p}}+|u_0-w|_H+|\varphi^0(w)|^{1/2}+S^{1-\alpha}$ *and* w *is an arbitrary element in* $D(\varphi^0)$.

Proof. By virtue of $(\mathrm{A}.\varphi^t)$ (see Remark 3.5), for each $v\in D(\varphi^0)$, there exists an H-valued function $x(t)$ such that

$$|x(s)-v| \leqslant ms(\varphi^0(v)+1) \qquad \text{for all} \quad s\in[0,T],$$
$$\varphi^s(x(s)) \leqslant \varphi^0(v)+ms(\varphi^0(v)+1) \qquad \text{for all} \quad s\in[0,T],$$

which, together with the definition of $\partial\varphi^s$, imply

$$\begin{aligned}\varphi^s(\hat{u}(s)) &\leqslant \varphi^s(x(s)) + (-d\hat{u}(s)/ds,\ \hat{u}(s)-v+v-x(s))\\ &\leqslant \varphi^0(v)+ms(\varphi^0(v)+1)-\frac{1}{2}\frac{d}{ds}|\hat{u}(s)-v|^2\\ &\quad + \left|\frac{d\hat{u}}{ds}(s)\right| ms(\varphi^0(v)+1).\end{aligned}$$

Then integration of this inequality on $[0,t]$ gives

$$\begin{aligned}\int_0^t \varphi^s(\hat{u}(s))\,ds &\leqslant t\varphi^0(v)+mt^2(\varphi^0(v)+1)/2+|u_0-v|^2/2\\ &\quad +\varepsilon\int_0^t s\left|\frac{d\hat{u}}{ds}(s)\right|^2 ds + m^2t^2(\varphi^0(v)+1)^2/8\varepsilon\\ &\qquad \text{for all} \quad \varepsilon>0 \text{ and } t\in[0,S].\end{aligned} \tag{3.16}$$

On the other hand, multiplying (3.10) by $d\hat{u}(t)/dt$, we obtain, by Proposition 3.4, that

$$\left|\frac{d\hat{u}}{dt}(t)\right|^2+\frac{d}{dt}\varphi^t(\hat{u}(t))$$
$$\leqslant\left(\left|\frac{d\hat{u}}{dt}(t)\right|+1\right)m(\varphi^t(\hat{u}(t))+1)\qquad\text{for}\quad\text{a.e. } t\in[0,S].\tag{3.17}$$

Since the standard argument (as in the proof of Lemma 3.10) shows that $\varphi^s(\hat{u}(s))\in L^1(0,t)$ and $t\varphi^t(\hat{u}(t))\leqslant M(|u_0|)$, $M(\cdot)\in\mathscr{M}$, holds for a.e. $t\in[0,S]$, we deduce, by (3.17),

$$\frac{1}{2}\int_0^t s\left|\frac{d\hat{u}}{ds}(s)\right|^2 ds+t\varphi^t(\hat{u}(t))\leqslant(1+m^2M(|u_0|)+mt)\int_0^t\varphi^s(\hat{u}(s))\,ds$$
$$+(m+m^2)t^2/2\qquad\text{for all}\quad t\in(0,S].\tag{3.18}$$

Thus, combining (3.16) with (3.18), we easily find that there exists another function $M(\cdot)\in\mathscr{M}$ such that

$$\frac{1}{4}\int_0^t s\left|\frac{d\hat{u}}{ds}(s)\right|^2 ds+t\varphi^t(\hat{u}(t))\tag{3.19}$$
$$\leqslant M(|u_0|)\{t\varphi^0(v)+|u_0-v|^2+(t\varphi^0(v))^2+t^2\}\qquad\text{for all}\quad t\in(0,S].$$

Put $v=v(t):=e^{-t\partial\varphi^0}u_0$ in (3.19), then, sice $t\varphi^0(v(t))\leqslant M(|u_0|)$ for a.e. $t\in[0,S]$, by virtue of (2.3) and (2.4) of Theorem 2.2, we deduce (3.13) and (3.14). Hence (3.16) with $v=v(t)$ also assures (3.15). In order to verify (3.12), we note by (3.17) that

$$|\hat{u}(t)-u_0|-\left|\hat{u}\left(\frac{t}{2}\right)-u_0\right|$$
$$\leqslant\left|\frac{d\hat{u}}{ds}(s)\right|_{L^1(t/2,t;H)}\leqslant\sqrt{t/2}\left|\frac{d\hat{u}}{ds}(s)\right|_{L^2(t/2,t;H)}$$
$$\leqslant\{t\varphi^{t/2}(\hat{u}(t/2))\}^{1/2}+\left\{(m^2+m)t\int_{t/2}^t(\varphi^s(\hat{u}(s))+1)^2\,ds\right\}^{1/2}.\tag{3.20}$$

Here, making use of (3.13), (3.15) and the fact that $t\varphi^t(\hat{u}(t))\leqslant M(|u_0|)$ for a.e. $t\in[0,S]$, we easily find that the $L^p_*(0,S)$-norms of $t^{-\alpha}\{t\varphi^{t/}(\hat{u}(t/2))\}^{1/2}$ and $t^{-\alpha}\{t|\varphi^s(\hat{u}(s))+1|^2_{L^2(t/2,t)}\}^{1/2}$ are dominated by $M(|u_0|)\cdot G_{\alpha,p}(u_0,w,S)$. Thus (3.12) follows from Lemma 2.3. Q.E.D.

We now consider the following auxiliary equation:

$$\left.\begin{aligned} &du_h(t)/dt + \partial\varphi^t(u_h(t)) \ni -h(t) + f(t), \qquad 0 < t < S, &&(3.21)\\ &u_h(0) = u_0. &&(3.22)\end{aligned}\right\} \text{(C.P.)}^*$$

Let X_S^α be a Banach space with the norm

$$|u|_{\alpha,S} = \left(\int_0^S t^{1-2\alpha}\,|u(t)|_H^2\,dt\right)^{1/2}, \qquad 0 < \alpha < \tfrac{1}{2}.$$

Let u_0 and $f(t)$ be given elements in $\mathscr{B}^0_{\alpha,p}$ and $L^2(0, T; H)$, respectively. Then, under assumption (A.φ^t), for each $h(t) \in X_S^\alpha$, (C.P.)* has a unique strong solution $u_h(t)$ in $[0, S]$ (see [33] and [26]). In parallel with Lemma 3.6, we have the following *a priori* estimates.

LEMMA 3.7. *Let* (A.φ^t) *be satisfied. Let* $u_0 \in \mathscr{B}^0_{\alpha,p}$ *and* $h(t), f(t) \in X_S^\beta$ *with* $0 < \alpha < \beta < \frac{1}{2}$, *and put* $R = |h|_{\beta,S} + |f|_{\beta,S}$. *Then there exists a function* $M_2(\cdot) \in \mathscr{M}$ *such that the unique strong solution* $u_h(t)$ *of* (C.P.)* *in* $[0, S]$ *satisfies*

$$\begin{aligned} &|t^{-\alpha}\,|u_h(t) - u_0|_H|_{L^p_*(0,S)} \\ &\qquad \leqslant M_2(|u_0|_H) \cdot (G_{\alpha,p}(u_0, w, S) + R), \qquad (3.23)\\ &|t^{1/2-\alpha}\,|\varphi^t(u_h(t))|^{1/2}|_{L^p_*(0,S)} \\ &\qquad \leqslant M_2(|u_0|_H + R) \cdot (G_{\alpha,p}(u_0, w, S) + R), \qquad (3.24)\end{aligned}$$

where $G_{\alpha,p}(u_0, w, S)$ *is the function appearing in Lemma* 3.6.

LEMMA 3.8. *Let* (A.φ^t) *be satisfied. Let* $u_0 \in \mathscr{B}^0_{\alpha,\infty}$ *and* $h(t), f(t) \in X_S^\alpha$, *and put* $R = |h|_{\alpha,S} + |f|_{\alpha,S}$. *Then estimates* (3.23) *and* (3.24) *hold true with* $p = \infty$.

LEMMA 3.9. *Let* (A.φ^t) *be satisfied. Let* $u_0 \in \mathscr{B}^0_{\alpha,2}$ *and* $h(t), f(t) \in X_S^\alpha$, *and put* $R = |h|_{\alpha,S} + |f|_{\alpha,S}$. *Then estimates* (3.23) *and* (3.24) *hold true with* $p = 2$. *Moreover, the following* (3.25) *and* (3.26) *also hold.*

$$t^{1-2\alpha}\varphi^t(u_h(t)) \to 0 \qquad \text{as} \quad t \downarrow 0, \qquad (3.25)$$

$$\left(\int_0^S t^{1-2\alpha}\,|g_h(t)|_H^2\,dt\right)^{1/2} \leqslant M_2(|u_0|_H + R) \cdot (G_{\alpha,2}(u_0, w, S) + R), \qquad (3.26)$$

where $g_h(t) = -du_h(t)/dt - h(t) + f(t) \in \partial\varphi^t(u_h(t))$.

Proof of Lemma 3.7. Let $\hat{u}(t)$ be the strong solution of $(\text{C.P.})_0$ in $[0, S]$, then $u_h(t) - \hat{u}(t)$ satisfies

$$\frac{d}{dt}(u_h(t) - \hat{u}(t)) + \partial\varphi^t(u_h(t)) - \partial\varphi^t(\hat{u}(t))$$
$$\ni -h(t) + f(t) \qquad \text{for} \quad \text{a.e. } t \in [0, S]. \tag{3.27}$$

Multiply (3.27) by $u_h(t) - \hat{u}(t)$, then we have, by the monotonicity of $\partial\varphi^t$,

$$|u_h(t) - \hat{u}(t)| \leqslant |h - f|_{L^1(0,t;H)}$$
$$\leqslant Rt^\beta/\sqrt{2\beta} \qquad \text{for all} \quad t \in [0, S]. \tag{3.28}$$

Hence (3.23) follows from (3.12) at once. On the other hand, by the definition of $\partial\varphi^s$,

$$\varphi^s(u_h(s)) \leqslant \varphi^s(\hat{u}(s)) + (-du_h(s)/ds - h(s) + f(s), u_h(s) - \hat{u}(s))$$
$$\leqslant \varphi^s(\hat{u}(s)) - \frac{1}{2}\frac{d}{ds}|u_h(s) - \hat{u}(s)|^2 + \left|\frac{d\hat{u}}{ds}(s)\right| |u_h(s) - \hat{u}(s)|$$
$$+ (|h(s)| + |f(s)|)\,|u_h(s) - \hat{u}(s)|. \tag{3.29}$$

Integration of (3.29) on $[0, t]$ and (3.28) give

$$\int_0^t \varphi^s(u_h(s))\,ds \leqslant \int_0^t \varphi^s(\hat{u}(s))\,ds + \frac{1}{2}\int_0^t s\left|\frac{d\hat{u}}{ds}(s)\right|^2 ds$$
$$+ R^2t^{2\beta}\left(1 + \frac{1}{4\beta}\right)\Big/2\beta, \qquad \text{for all} \quad t \in [0, S], \tag{3.30}$$

whence we deduce, by (3.14) and (3.15),

$$\left| t^{-\alpha}\left\{\int_0^t \varphi^s(u_h(s))\,ds\right\}^{1/2}\right|_{L^p_*(0,S)} \leqslant M(|u_0|)\cdot(G_{\alpha,p}(u_0, w, S) + R). \tag{3.31}$$

Since $\varphi^s(u_h(s)) \in L^1(0, S)$, multiplying (3.21) by $tg_h(t) = t\{-du_h(t)/dt - h(t) + f(t)\} \in t\partial\varphi^t(u_h(t))$ and integrating on $[0, t]$, we deduce, by Proposition 3.4 (c.f. (3.17)),

$$\tfrac{1}{2}\int_0^t s\,|g_h(s)|^2\,ds + t\varphi^t(u_h(t))$$
$$\leqslant \int_0^t \varphi^s(u_h(s))\,ds + (m^2 + m)\int_0^t s(\varphi^s(u_h(s)) + 1)^2\,ds$$
$$+ R^2t^{2\beta} \qquad \text{for all} \quad t \in (0, S], \tag{3.32}$$

where we used the fact that $|\sqrt{s}(h(s)-f(s))|^2_{L^2(0,t;H)} \leqslant R^2 t^{2\beta}$. Moreover, in view of (3.30), applying Gronwall's inequality to (3.32), we find that $t\varphi^t(u_h(t)) \leqslant M(|u_0|+R)$ for all $t \in (0, S]$. Then (3.24) follows from (3.31) and (3.32). Q.E.D.

Proof of Lemma 3.8. It is clear that if $p=\infty$, then the very same argument as in the above proof is also valid with $\alpha=\beta$.

Proof of Lemma 3.9. To verify (3.23) and (3.24) with $p=2$, it suffices to repeat the same procedure as in the proof of Lemma 3.7 with the aid of the inequality of Hardy:

$$\left| t^{-\alpha} \int_0^t |\phi(s)| \frac{ds}{s} \right|_{L^p_*(0,S)} \leqslant \frac{1}{\alpha} |t^{-\alpha}\phi(t)|_{L^p_*(0,S)} \qquad (\alpha > 0,\ p \geqslant 1).$$

For example, we have, by (3.28),

$$\begin{aligned} |t^{-\alpha}|u_h(t)-\hat{u}(t)||_{L^2_*(0,S)} &\leqslant \left| t^{-\alpha} \int_0^t s\,|h(s)-f(s)| \frac{ds}{s} \right|_{L^2_*(0,S)} \\ &\leqslant \frac{1}{\alpha} |h-f|_{\alpha,S}. \end{aligned}$$

Then (3.23) follows from (3.12). Furthermore, since $t^{-2\alpha}\varphi^t(u_h(t)) \in L^1(0,S)$ by (3.24), multiplying (3.21) by $t^{1-2\alpha} g_h(t)$, we can easily deduce, by Proposition 3.4 (cf. (3.32)),

$$\begin{aligned} &\tfrac{1}{2} \int_0^t s^{1-2\alpha} |g_h(s)|_H^2\, ds + t^{1-2\alpha}\varphi^t(u_h(t)) \\ &\qquad \leqslant \int_0^t (1-2\alpha)\, s^{-2\alpha}\varphi^s(u_h(s))\, ds + (m^2+m) \int_0^t s^{1-2\alpha}(\varphi^s(u_h(s))+1)^2\, ds \\ &\qquad\quad + \int_0^t s^{1-2\alpha} |h(s)-f(s)|^2\, ds \qquad \text{for all} \quad t \in (0, S]. \end{aligned} \tag{3.33}$$

Hence, since $t\varphi^t(u_h(t)) \leqslant M(|u_0|+R)$, (3.24) implies (3.25) and (3.26). Q.E.D.

For each $u_0 \in \overline{D(\partial\varphi^0)}$, $f(t) \in L^2(0,T;H)$, $S \in (0,T]$ and $h \in X^\alpha_S$, we denote by $\mathbb{E}_{u_0,f,S}(h)$ the unique strong solution of (C.P.)* in $[0,S]$. We put

$$K^\alpha_{S,R} := \text{the set } \{u \in X^\alpha_S;\, |u|_{\alpha,S} \leqslant R\} \quad \text{furnished with the weak topology of } X^\alpha_S. \tag{3.34}$$

Then $K^\alpha_{S,R}$ is metrizable, since H is assumed to be separable (see Dunford and Schwarz [10, p. 426]). We further introduce a possibly multivalued operator $\mathbb{B}^{\alpha}_{u_0,f,S,R}$ from $K^{\alpha}_{S,R}$ into itself by

$$\begin{aligned} \mathbb{B}^{\alpha}_{u_0,f,S,R}(h) &= \{b \in K^{\alpha}_{S,R};\, b(t) \in B(t, \mathbb{E}_{u_0,f,S}(h)(t)) \\ &\qquad \text{for a.e. } t \in (0, S)\}, \qquad (3.35) \\ D(\mathbb{B}^{\alpha}_{u_0,f,S,R}) &= \{h \in K^{\alpha}_{S,R};\, \mathbb{B}^{\alpha}_{u_0,f,S,R}(h) \neq \varnothing\}. \end{aligned}$$

In what follows, we shall simply write $\mathbb{E}$ and $\mathbb{B}$ (or $\mathbb{B}^{\alpha}_{S,R}$) instead of $\mathbb{E}_{u_0,f,S}$ and $\mathbb{B}^{\alpha}_{u_0,f,S,R}$, respectively, if no confusion arises. On the continuity of $\mathbb{E}$, we prepare the following lemma.

LEMMA 3.10. *Let* (A.φ^t) *and* (A.1) *be satisfied. Let* h^n, $h \in X^{\alpha}_S$ *and* h^n *converge to* h *weakly in* X^{α}_S *as* $n \to +\infty$. *Then* $\mathbb{E}(h^n)$ *also converges to* $\mathbb{E}(h)$ *strongly in* $C([0, S]; H)$ *as* $n \to +\infty$.

Proof. Note that there exists an H-valued function $\bar{v}(t)$ on $[0, S]$ such that (see [33] and [26])

$$r_0 := \max\{|\bar{v}(t)|_H + \varphi^t(\bar{v}(t));\, t \in [0, S]\} + |d\bar{v}(t)/dt|_{L^2(0,S;H)}\} < +\infty.$$

Put $u^n = \mathbb{E}_{u_0,f,S}(h^n)$ and multiply (3.21) by $u^n(t) - \bar{v}(t)$. Then, from the definition of $\partial\varphi^t$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u^n(t) - \bar{v}(t)|^2 + \varphi^t(u^n(t)) - \varphi^t(\bar{v}(t)) \\ &\qquad \leqslant |u^n(t) - \bar{v}(t)|\{|d\bar{v}(t)/dt| + |h^n(t)| + |f(t)|\} \qquad \text{for a.e. } t \in [0, S]. \end{aligned}$$

Hence there exists a constant C_0 depending only on $|u_0|_H$, r_0, $|f|_{L^1(0,T;H)}$ and $\sup_n |h^n|_{L^1(0,S;H)}$ (but not on S explicitly) such that

$$\max_{0 \leqslant t \leqslant S} |u^n(t)| + \int_0^S \varphi^t(u^n(t))\, dt \leqslant C_0. \qquad (3.36)$$

Moreover, since u^n also satisfies the same estimate as (3.32) (with $R^2 t^{2\beta}$ replaced by $|\sqrt{s}(h^n(s) - f(s))|_{L^2(0,t;H)}$), Gronwall's inequality, together with (3.36), yields

$$\sup_{0 < t < S} t\varphi^t(u^n(t)) + \int_0^S t\, |g^n(t)|^2\, dt \leqslant C_1, \qquad (3.37)$$

where $g^n(t) = -du^n(t)/dt - h^n(t) + f(t) \in \partial\varphi^t(u^n(t))$ and C_1 is a constant depending only on C_0, $|\sqrt{t} f(t)|_{L^2(0,T;H)}$ and $\sup_n |\sqrt{t}\, h^n(t)|_{L^2(0,S;H)}$. Hence $\sqrt{t}\, du^n(t)/dt$ is also bounded in $L^2(0, S; H)$.

Let ε be an arbitrary (small) positive number. Then, since $du^n(t)/dt$ is bounded in $L^2(\varepsilon, S; H)$, $\{u^n(t)\}_n$ is equicontinuous on $[\varepsilon, S]$. In addition, by (3.36), (3.37) and (A.1), $\{u^n(t)\}_n$ forms a compact set in H for all $t \in [\varepsilon, S]$. Thus, by Ascoli's theorem, there exist a subsequence $\{n'\}$ of $\{n\}$ and two functions u_ε and g_ε on $[\varepsilon, S]$ such that

$$\begin{aligned} u^{n'} &\to u_\varepsilon && \text{strongly in } C([\varepsilon, S]; H),\\ du^{n'}(t)/dt &\to du_\varepsilon(t)/dt && \text{weakly in } L^2(\varepsilon, S; H),\\ g^{n'} &\to g_\varepsilon && \text{weakly in } L^2(\varepsilon, S; H). \end{aligned}$$

Here, by using Proposition 1.1 of Kenmochi [18], it is known that $g_\varepsilon(t) \in \partial\varphi^t(u_\varepsilon(t))$ for a.e. $t \in [\varepsilon, S]$ (see also [26]). Hence, by the usual diagonal process, we can construct two functions $u(t)$ and $g(t)$ on $(0, S]$ such that $\sqrt{t}\, du(t)/dt$, $\sqrt{t}\, g(t) \in L^2(0, S; H)$ and $g(t) = -du(t)/dt - h(t) + f(t) \in \partial\varphi^t(u(t))$ holds for a.e. $t \in (0, S]$. Then, in order to see $u = \mathbb{E}_{u_0,f,S}(h)$, it suffices to verify that $u(t) \to u_0$ as $t \downarrow 0$. To this end, set $\bar{u}(t) = \mathbb{E}_{u_0,f,S}(0)$, then, in parallel with (3.28), we find

$$|u^n(t) - \bar{u}(t)| \leqslant |h^n|_{L^1(0,t,H)} \leqslant C_2 t^\alpha \qquad \text{for all } n, \tag{3.38}$$

where $C_2 = \sup_n |h^n|_{\alpha,S}/\sqrt{2\alpha} < +\infty$, whence follows

$$|u(t) - \bar{u}(t)| \leqslant C_2\, t^\alpha \qquad \text{for all} \quad t \in (0, S]. \tag{3.39}$$

Therefore, since $|\bar{u}(t) - u_0| \to 0$ s $t \downarrow 0$, we obtain that $|u(t) - u_0| \to 0$ as $t \downarrow 0$, i.e., $u = \mathbb{E}_{u_0,f,S}(h)$. Furthermore, by (3.38) and (3.39),

$$|u^n - u|_{C([0,S];H)} \leqslant 2C_2\varepsilon^\alpha + |u^n - u|_{C([\varepsilon,S];H)} \qquad \text{for all} \quad \varepsilon > 0 \text{ and } n.$$

Hence, for a suitable choice of $\{n'\}$, we find that $\mathbb{E}(h^{n'})$ converges to $\mathbb{E}(h)$ strongly in $C([0, S]; H)$ as $n' \to +\infty$. However, the above argument does not depend on the choice of subsequences, then the original sequence $\mathbb{E}(h^n)$ converges to $\mathbb{E}(h)$ as $n \to +\infty$. Q.E.D.

As for the operator $\mathbb{B}$, we have

LEMMA 3.11. *Let* (A.φ^t), (A.1) *and* (A.2) *be satisfied. Then* $G(\mathbb{B}^\alpha_{S,R})$, *the graph of* $\mathbb{B}^\alpha_{S,R}$, *is closed in* $K^\alpha_{S,R} \times K^\alpha_{S,R}$. *Moreover, for each* $h \in D(\mathbb{B}^\alpha_{S,R})$, $\mathbb{B}^\alpha_{S,R}(h)$ *is a closed convex subset of* $K^\alpha_{S,R}$.

Proof. Since $K^\alpha_{S,R}$ is metrizable, it is enough to show the closedness of $G(\mathbb{B})$ on sequences. Let $h^n \to h$ in $K^\alpha_{S,R}$, $b^n \in \mathbb{B}(h^n)$, and $b^n \to b$ in $K^\alpha_{S,R}$. Then, by Lemma 3.10, $u^n := \mathbb{E}(h^n)$ converges to $u := \mathbb{E}(h)$ in $C([0, S]; H)$. Let δ be an arbitrary number in $(0, S)$, then $b^n \to b$ weakly in $L^2(\delta, S; H)$. Furthermore, as in the proof of Lemma 3.10, there exist functions g^n, $g \in L^2(\delta, S; H)$ such that $g^n(t) \in \partial\varphi^t(u^n(t))$, $g(t) \in \partial\varphi^t(u(t))$ for a.e.

$t \in (\delta, S)$ and $g^n \to g$ weakly in $L^2(\delta, S; H)$. Hence it follows from (iii) of (A.2) that $b(t) \in B(t, u(t))$ for a.e. $t \in (0, S)$, i.e., $h \in D(\mathbb{B})$ and $b \in \mathbb{B}(h)$. The above reasoning also assures that $\mathbb{B}(h)$ is closed in $K^\alpha_{S,R}$ for each $h \in D(\mathbb{B})$. Then, by (i) of (A.2), $\mathbb{B}(h)$ is a closed convex subset of $K^\alpha_{S,R}$ for all $h \in D(\mathbb{B})$. Q.E.D.

3.3. *Proofs of Theorems*

Proof of Theorem I. For an arbitrary element w in $D(\varphi^0)$, put

$$R = |u_0|_{\mathscr{B}^0_{\alpha,2}} + |u_0 - w|_H + \{\varphi^0(w)\}^{1/2} + |f(t)|_{\alpha,T}. \tag{3.40}$$

Let $h \in K^\alpha_{T_0,R}$ and $u = \mathbb{E}_{u_0,f,T_0}(h)$. Here T_0 is a positive number in $(0, \min(1, T, R^{1/(1-\alpha)}))$ which will be determined later more precisely. Then, by (3.36), there exists a constant C_0 depending on R but not on T_0 such that

$$\max_{0 \leqslant t \leqslant T_0} |u(t)| + \int_0^{T_0} \varphi^t(u(t))\, dt \leqslant C_0. \tag{3.41}$$

Moreover, by virtue of (3.24), (3.26) (in Lemma 3.9) and (3.40), there exists a function $M_3(\cdot) \in \mathscr{M}$ such that

$$|t^{-2\alpha}\varphi^t(u(t))|_{L^1(0,T_0)} \leqslant \{M_3(R) \cdot R\}^2, \tag{3.42}$$

$$|g(t)|_{\alpha,T_0} \leqslant M_3(R) \cdot R, \tag{3.43}$$

where $g(t) = -du(t)/dt - h(t) + f(t) \in \partial\varphi^t(u(t))$.

Let $b(t)$ be a measurable function such that $b(t) \in B(t, u(t))$ for a.e. $t \in (0, T_0)$. Then, by $(\mathrm{A.3})_\alpha$ and (3.41), we have

$$\begin{aligned} |b(t)|_{\alpha,T_0} \leqslant \ell(C_0)\{\varepsilon\, |g(t)|_{\alpha,T_0} &+ \ell_0(1/\varepsilon)\, |t^{1-2\alpha}(\varphi^t(u(t)))^{(2-2\alpha)/(1-2\alpha)}|^{1/2}_{L^1(0,T_0)} \\ &+ |a(t)|_{\alpha,T_0}\} \qquad \text{for all} \quad \varepsilon > 0. \end{aligned} \tag{3.44}$$

Put $\varepsilon = (3\ell(C_0)\, M_3(R))^{-1}$, then $\ell(C_0)\varepsilon\, |g(t)|_{\alpha,T_0} \leqslant R/3$ by (3.43). Furthermore, since, by (3.42),

$$\begin{aligned} &|t^{1-2\alpha}(\varphi^t(u(t)))^{(2-2\alpha)/(1-2\alpha)}|_{L^1(0,T_0)} \\ &\quad \leqslant \sup_{0<t<T_0} \{t^{1-2\alpha}\varphi^t(u(t))\}^{1/(1-2\alpha)} \{M_3(R)R\}^2, \end{aligned} \tag{3.45}$$

recalling (3.25), we can choose a (sufficiently small) positive number T_0 such that

$$\ell(C_0)\, \ell_0(1/\varepsilon)\, |t^{1-2\alpha}(\varphi^t(u(t)))^{(2-2\alpha)/(1-2\alpha)}|^{1/2}_{L^1(0,T_0)} \leqslant R/3, \tag{3.46}$$

$$\ell(C_0)\, |a(t)|_{\alpha,T_0} \leqslant R/3. \tag{3.47}$$

Thus, for a suitable T_0, $b(t)$ belongs to $K^{\alpha}_{T_0,R}$. Therefore it follows from (ii) of (A.2) and Lemma 3.11 that $D(\mathbb{B}^{\alpha}_{T_0,R})=K^{\alpha}_{T_0,R}$; $G(\mathbb{B}^{\alpha}_{T_0,R})$ is closed in $K^{\alpha}_{T_0,R}\times K^{\alpha}_{T_0,R}$; and that $\mathbb{B}^{\alpha}_{T_0,R}(h)$ is a nonempty closed convex subset of $K^{\alpha}_{T_0,R}$ for all $h\in K^{\alpha}_{T_0,R}$. Then, by Proposition 2.6, $\mathbb{B}^{\alpha}_{T_0,R}$ is an upper semicontinuous mapping from $K^{\alpha}_{T_0,R}$ into itself.

Hence Theorem 2.7 says that there exists an element $b\in K^{\alpha}_{T_0,R}$ such that $b\in \mathbb{B}^{\alpha}_{T_0,R}(b)$, i.e., $u=\mathbb{E}_{u_0,f,T_0}(b)$ satisfies

$$\begin{aligned} &du(t)/dt+\partial\varphi^t(u(t))+b(t)\ni f(t) && \text{for a.e. } t\in(0,T_0),\\ &b(t)\in B(t,u(t)) && \text{for a.e. } t\in(0,T_0),\\ &u(0)=u_0. \end{aligned}$$

Thus $u(t)$ is the desired strong solution of (C.P.) in $[0,T_0]$. Since $\mathscr{B}^0_{\alpha,2}\subset\mathscr{B}^0_{\alpha,\infty}$ (see (2.1)), Lemmas 3.8 and 3.9 give estimates (3.26) and (3.23)–(3.24) with $p=2$ and ∞, whence follows (3.5) and (3.6). Q.E.D.

Proof of Corollary I. Since $u_0\in\mathscr{B}^0_{\alpha,p}\subset\mathscr{B}^0_{\alpha',2}$ by (2.2), we can apply Theorem I with α replaced by α'. Q.E.D.

In proving Theorems II and III, we apply the same idea as in the proof of Theorem I. However, topological spaces where the operator $\mathbb{B}$ works have to be reset.

Let X^0_S be a Banach space with the norm

$$|u|_{0,S}=\left(\int_0^S t\,|u(t)|_H^2\,dt\right)^{1/2}+\left(\int_0^S |u(t)|_H^{(2+\gamma)/2}\,dt\right)^{2/(2+\gamma)},$$

where γ is the number appearing in (A.5). By X_S and $|\cdot|_S$, we mean $L^2(0,S;H)$ and $|\cdot|_{L^2(0,S;H)}$, respectively. Define $K_{S,R}$ (or $K^0_{S,R}$) by (3.34) with X^{α}_S and $|\cdot|_{\alpha,S}$ replaced by X_S and $|\cdot|_S$ (or X^0_S and $|\cdot|_{0,S}$), respectively. We also introduce $\mathbb{B}_{u_0,f,S,R}$ (or $\mathbb{B}^0_{u_0,f,S,R}$) by (3.35) with $K^{\alpha}_{S,R}$ replaced by $K_{S,R}$ (or $K^0_{S,R}$). Then it is clear that Lemmas 3.10 and 3.11 remain true with X^{α}_S, $\mathbb{B}^{\alpha}_{S,R}$ and $K^{\alpha}_{S,R}$ replaced by X_S, $\mathbb{B}_{S,R}$ and $K_{S,R}$ (or X^0_S, $\mathbb{B}^0_{S,R}$ and $K^0_{S,R}$), respectively.

Proof of Theorem II. Define a positive number R by

$$R^2=\max\left(1,\ \frac{2(1+3k)}{1-k}\,\varphi^0(u_0)+\frac{2(1+3k)^2}{(1-k)^2}\,|f(t)|_T^2\right), \tag{3.48}$$

where k is the given number in (A.3). Let $h\in K_{T_0,R}$ and $u=\mathbb{E}_{u_0,f,T_0}(h)$. Here T_0 is a positive number which will be determined later. Multiply (3.21) by $g(t)=-d(u(t))/dt-h(t)+f(t)\in\partial\varphi^t(u(t))$, then Proposition 3.4 yields (c.f. (3.33))

$$|g(t)|^2 + \frac{d}{dt}\varphi^t(u(t)) \leqslant \left(\frac{1}{2} + \varepsilon_k\right)|g(t)|^2 + \frac{1}{2}|h(t)|^2$$
$$+ \frac{1}{2\varepsilon_k}\{|f(t)|^2 + m^2(\varphi^t(u(t)) + 1)^2\}$$
$$+ m(\varphi^t(u(t)) + 1), \tag{3.49}$$

where $\varepsilon_k = (1-k)/2(1+3k)$. Hence by (3.41) and Gronwall's inequality, there exists a constant C_2 depending on C_0 and R but not on T_0 such that $\max\{\varphi^t(u(t)); t \in [0, T_0]\} \leqslant C_2$, whence integration of (3.49) on $[0, T_0]$ gives

$$|g(t)|_{T_0}^2 \leqslant (1-2\varepsilon_k)^{-1}\{R^2 + 2\varphi^0(u_0) + \varepsilon_k^{-1}|f(t)|_T^2$$
$$+ \varepsilon_k^{-1}(m^2+m)(C_2+1)^2 T_0\}. \tag{3.50}$$

We now take T_0 in $(0, T]$ such that

$$\varepsilon_k^{-1}(m^2+m)(C_2+1)^2 T_0 \leqslant (1-k)R^2/4, \tag{3.51}$$

$$\ell(C_2 + C_0)|c(t)|_{L^1(0,T_0)} \leqslant (1-k)R^2/4. \tag{3.52}$$

Then, since $k(1-2\varepsilon_k)^{-1}(2\varphi^0(u_0) + \varepsilon_k^{-1}|f(t)|_T^2) \leqslant (1-k)R^2/4$ by (3.48), it follows from (A.4) and (3.50)–(3.52) that $|b(t)|_{T_0} \leqslant R$ for all measurable sections $b(t)$ of $B(t, u(t))$. Hence, by Lemmas 3.10 and 3.11, $\mathbb{B}_{u_0,f,T_0,R}$ has a fixed point in $K_{S,R}$. Q.E.D.

Proof of Theorem III. For $w \in D(\varphi^0)$, put

$$R = |u_0 - w|_H + \{\varphi^0(w)\}^{1/2} + |f(t)|_{0,T}.$$

Let $h \in K^0_{T_0,R}$ and $u = \mathbb{E}_{u_0,f,T_0}(h)$, where $T_0 \in (0, \min(1, T, R))$. Making use of (3.16), (3.19) with $v = v(t) := e^{-t\partial\varphi^0}u_0$; (3.28), (3.30) and (3.32) with $\beta = 0$; and the fact that $v(t)$ satisfies (see [8, p. 77] or repeat the verification for (3.19) with $\partial\varphi^t \equiv \partial\varphi^0$)

$$\int_0^t s\left|\frac{dv}{ds}(s)\right|^2 ds + t\varphi^0(v(t)) + \tfrac{1}{2}|v(t) - w|^2 \leqslant t\varphi^0(w) + \tfrac{1}{2}|u_0 - w|^2,$$

we can deduce

$$|\sqrt{t}\, g(t)|_{T_0}^2 + \sup_{0<t<T_0} t\varphi^t(u(t)) \leqslant (M_3(R)R)^2. \tag{3.53}$$

Here we take a positive number T_0 such that

$$\max\{(1/2)^{\gamma/2}(M_3(R)R)^{1-\gamma}, (1/2\gamma)^{1/2}(M_3(R)R)^{2-2\gamma}\}\, \ell(C_0)\, T_0^{\gamma} \leqslant R/6, \tag{3.54}$$

$$\max\left\{\left(\frac{\gamma^2+\gamma+2}{2\gamma(1+\gamma)}\right)^{(\gamma^2+\gamma+2)/2(2+\gamma)}(M_3(R)R)^{1-\gamma},\right.$$
$$\left.\left(\frac{2}{\gamma(1+\gamma)}\right)^{2/(2+\gamma)}(M_3(R)R)^{2-2\gamma}\right\}\cdot \ell(C_0)\,T_0^{\gamma(1+\gamma)/(2+\gamma)}\leqslant R/6, \quad (3.55)$$

$$\ell(C_0)\,|a(t)|_{0,T_0}\leqslant R/3. \quad (3.56)$$

Then (3.41), (3.53) and (A.5) with simple calculation show that $b(t)\in K^0_{T_0,R}$ for all measurable sections $b(t)$ of $B(t, u(t))$. Q.E.D.

Remark 3.12. Obviously we have only to assume that $f(t)$, $a(t)\in X^\alpha_T$ (resp. X^0_T) in Theorem I (resp. III).

4. Global Existence

In this section, in order to establish the existence of global (in time) strong solutions of (C.P.), we shall study the extension of the local strong solutions constructed in the previous section.

4.1. *Extension of Local Solutions with Arbitrary Data*

In this subsection, we give a sufficient condition which guarantees that every local strong solution can be continued globally to the given interval $[0, T]$. Let us assume

(A.6) There exist a positive constant α and a positive function $d(t)\in L^1(0, T)$ such that

$$\langle -\partial\varphi^t(u)-B(t,u),u\rangle_H+\alpha\varphi^t(u)$$
$$\leqq d(t)(|u|_H^2+1) \qquad \text{for a.e. } t\in[0,T] \text{ and all } u\in D(\partial\varphi^t), \quad (4.1)$$

where $\langle -\partial\varphi^t(u)-B(t,u),u\rangle_H=\sup\{(-g-b,u)_H;\, g\in\partial\varphi^t(u),\, b\in B(t,u)\}$.
Then we have

THEOREM IV. *Assume that* (A.φ^t), (A.1), (A.2), (A.6) *and* (A.4) *be satisfied with* (3.7) *replaced by*

$$|||B(t,u)|||_H^2\leqslant k\,|\partial^0\varphi^t(u)|_H^2+\ell(|u|_H)(\{\varphi^t(u)\}^2+|c(t)|)$$
$$\textit{for all} \quad t\in[0,T] \textit{ and } u\in D(\partial\varphi^t). \quad (3.7)'$$

Let $f(t)\in L^2_{\text{loc}}((0,T];H)$ (*resp.* $L^2(0,T;H)$), *then every local strong* (*resp. s-strong*) *solution of* (C.P.) *can be continued globally as a strong* (*resp. s-strong*) *solution of* (C.P.) *in* $[0, T]$. *In particular, the assertion of Theorem* II *holds true with* $T_0=T$.

COROLLARY IV. *Let all assumptions in Theorems* IV *and* I (*resp.* III) *be satisfied. Then the assertion of Theorem* I (*resp.* III) *holds true with* $T_0 = T$.

Proof of Theorem IV. Let $u(t)$ be an s-strong solution of (C.P.) in $[0, S]$ $(0 < S \leqslant T)$. Then multiplication of (3.1) by $u(t)$ and (A.6) yield

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \alpha\varphi^t(u(t)) \leqslant d(t)(|u(t)|^2 + 1) + |f(t)|\,|u(t)| \qquad \text{for a.e. } t \in (0, S).$$

Hence there exists a constant C_0 depending on $|u_0|$ and $|f(t)|_{L^1(0,T)}$ but not on S such that

$$\max_{0 \leqslant t \leqslant S} |u(t)|_H + \int_0^S \varphi^t(u(t))\,dt \leqslant C_0. \tag{4.2}$$

Moreover, from (3.49), (4.2) and (3.7)$'$, there exists a function $M(\cdot) \in \mathscr{M}$ such that

$$\frac{3k(1-k)}{2(1+3k)}|g(t)|^2 + \frac{d}{dt}\varphi^t(u(t)) \leqslant M(C_0)(\{\varphi^t(u(t))\}^2 + 1 + |c(t)| + |f(t)|^2) \qquad \text{for a.e. } t \in (0, S). \tag{4.3}$$

Whence, by Gronwall's inequality and (4.2), we obtain the *a priori* bound for $\varphi^t(u(t))$ independent of S. Then Theorem II assures that $u(t)$ can be continued globally to $[0, T]$ as an s-strong solution of (C.P.). As for the extension of strong solution $u(t)$ in $[0, S]$, it suffices to recall that $u(S) \in D(\varphi^S)$, i.e., $\varphi^S(u(S)) < +\infty$ (see Remark 3.3). Q.E.D.

4.2. *Extension of Local Solutions with Small Data*

When condition (A.6) is absent, there is a case that if u_0 and $f(t)$ satisfy certain conditions, then the corresponding (local) strong solution $u(t)$ of (C.P.) blows up in a finite time T_1, i.e., $|u(t)|_H \to +\infty$ as $t \uparrow T_1$ (see, e.g., Fujita [11], Tsutsumi [31], Ishii [17] and the author [27]). In such a case, at the same time, it is often possible to construct the so-called *stable set* W such that if $u_0 \in W$, then $u(t)$ stays in W for all $t > 0$, so $u(t)$ can be continued globally. For example, see [17], [25] and [27], where the case that $\partial\varphi^t \equiv \partial\varphi$ and $B(t, \cdot) \equiv -\partial\psi$ is treated. Nevertheless, for the more general perturbing operator $B(t, \cdot)$ and the time-dependent function φ^t, it seems hardly possible to construct the same kind of stable set as in [17], [25] and [27]. From another point of view, however, the stable set is, roughly speaking, often composed of small elements in a sense (see [25, Proposition 4.2]). So, under this observation, we here intend to investigate the global existence of strong solutions of (C.P.) for sufficiently small initial data u_0 and external forces $f(t)$ in some sense. To this end, set

$$D^t_\varepsilon = \{u \in D(\partial\varphi^t);\ \varphi^t(u) < \varepsilon\}, \qquad \varepsilon > 0,$$

and introduce the following condition (A.7).

(A.7) The following (i) and (ii) hold.

(i) $\varphi^0(0) = 0$.

(ii) There exist positive constants ε_0, k, α, K_0, p and a function $\ell(\cdot) \in \mathscr{M}$ such that

$$|||B(t, u)|||^2_H \leqslant k\,|\partial^0\varphi^t(u)|^2_H + \ell(\varphi^t(u)), \qquad 0 < k < 1, \tag{4.4}$$

$$\langle -\partial\varphi^t(u) - B(t, u), u\rangle_H + \alpha\varphi^t(u) \leqslant 0, \tag{4.5}$$

$$K_0\,|u|^p_H \leqslant \varphi^t(u), \qquad 2 \leqslant p < +\infty, \tag{4.6}$$

hold for a.e. $t \in (0, T)$ and all $u \in D^t_{\varepsilon_0}$.

Remark 4.1. Assume that there exist functions $\ell(\cdot) \in \mathscr{M}$ and $\ell_1(\cdot)$, $\ell_2(\cdot) \in \mathscr{M}_0 := \{\ell(\cdot) \in \mathscr{M};\ \lim_{r\to 0} \ell(r) = 0\}$ such that

$$|||B(t, u)|||^2_H \leqslant \{k + \ell_1(\varphi^t(u))\}\,|\partial^0\varphi^t(u)|^2_H + \ell(\varphi^t(u)), \qquad 0 < k < 1, \tag{4.4$'$}$$

$$\langle -\partial\varphi^t(u) - B(t, u), u\rangle_H + \alpha\varphi^t(u) \leqslant \ell_2(\varphi^t(u)) \cdot \varphi^t(u) \tag{4.5$'$}$$

hold for a.e. $t \in (0, T)$ and all $u \in D(\partial\varphi^t)$. Then (4.4) and (4.5) are fulfilled for a sufficiently small ε_0.

Remark 4.2. Relation (4.6) with $p \in (0, 2)$ for all $u \in D(\partial\varphi^t)$ implies (4.6) with $p = 2$ for all $u \in D^t_{\varepsilon_0}$ if $\varepsilon_0 \in (0, K_0]$.

Assuming $T \geqslant 1$ without loss of generality, we use the notation

$$\|f(t)\|_{q,T} = \sup\{|f(t)|_{L^q(s-1,s;H)};\ 1 \leqslant s \leqslant T\}.$$

Then our extension results are stated as follows.

THEOREM V. *Let* (A.φ^t), (A.1), (A.2) *and* (A.7) *be satisfied. Then there exists a positive number r independent of T such that if* $|u_0|_H + \varphi^0(u_0) \leqslant r$ *and* $\|f(t)\|_{2,T} \leqslant r^{p-1}$, *then every (local) s-strong solution of* (C.P.) *can be continued globally as an s-strong solution of* (C.P.) *in* $[0, T]$, *in particular,* (C.P.) *has a global s-strong solution.*

THEOREM VI. *Let all assumptions in Theorem* V *and* $(A.3)_\alpha$ *(resp.* (A.5)) *be satisfied with* $a(t) \equiv 1$. *Then there exists a positive number r independent of T such that if* $|u_0|_H + |u_0|_{\mathscr{D}^0_{\alpha,2}} \leqslant r$ *(resp.* $|u_0|_H \leqslant r$) *and* $\|f(t)\|_{2,T} \leqslant r^{p-1}$, *then every (local) strong solution of* (C.P.) *satisfying* (3.5) *(resp.* (3.5) *with* $\alpha = 0$) *can be continued globally as a strong solution of* (C.P.) *in* $[0, T]$, *in particular, the assertion of Theorem* I (*resp.* III) *holds true with* $T_0 = T$.

Before we proceed to he proofs of Theorems V and VI, we prepare the following Theorem II′, a variant of Theorem II, and Lemma 4.3.

THEOREM II′. *Let* (A.φ^t), (A.1), (A.2) *and* (A.7) *be satisfied, and let* $u_0 \in D(\varphi^0)$ *and* $f(t) \in L^2(0, T; H)$. *Then there exists a number* $\varepsilon_1 \in (0, \varepsilon_0)$ *independent of* $f(t)$ *such that if* $\varphi^0(u_0) \leqslant \varepsilon_1$, *then there exists a positive number* T_0 *depending on* $|u_0|_H$ *and* $f(t)$ *such that* (C.P.) *has an s-strong solution in* $[0, T_0]$.

Proof. Let $R^2 = \varepsilon_0$ and h, u be as in the proof of Theorem II. Then it easily follows from (3.41) and (3.49) that there exist positive numbers $\varepsilon_0' \in (0, \varepsilon_0)$ (independent of $f(t)$) and S (depending on $|u_0|_H$ and $f(t)$) such that if $\varphi^0(u_0) \leqslant \varepsilon_0'$, then $\max\{\varphi^t(u(t));\ 0 \leqslant t \leqslant S\} \leqslant \varepsilon_0$. Hence, since (4.4) is applicable for $u(t)$ in $[0, S]$, taking $\varphi^0(u_0) \leqslant \varepsilon_1 := \min(\varepsilon_0', (1-k)\varepsilon_0/4(1+3k))$, we can repeat the same procedure as in the proof of Theorem II with T replaced by S. Q.E.D.

LEMMA 4.3. *Let* $f(t) \in L^1(0, T)$ *and* $j(t)$ *be an absolutely continuous function on* $[0, T]$ *such that*

$$\frac{d}{dt}|j(t)| + \alpha\,|j(t)|^{p-1} \leqslant K\,|f(t)| \qquad \textit{for} \quad \text{a.e. } t \in (0, T), \tag{4.7}$$

where $\alpha > 0$, $K > 0$ *and* $p \geqslant 2$. *Suppose that* $|j(0)| \leqslant r$ *and* $\|f(t)\|_{1,T} \leqslant r^{p-1}$ $(r > 0)$, *then there exists a function* $M_{\alpha,p,K}(\cdot) \in \mathscr{M}$ *depending on* α, p *and* K *such that* $|j(t)| \leqslant M_{\alpha,p,K}(r)r$ *for all* $t \in [0, T]$.

Proof. In the case of $p = 2$, by (4.7),

$$|j(t)| \leqslant |j(0)| \exp(-\alpha t) + \int_0^t K\,|f(s)| \exp\{-\alpha(t-s)\}\, ds,$$

whence we can take $M_{\alpha,2,K}(r) = 1 + K/(1-\exp)(-\alpha))$. As for the case $p > 2$, put

$$\phi_a(t) = (|j(a)|^{2-p} + \alpha(p-2)(t-a))^{-1/(p-2)} + K\int_a^t |f(s)|\, ds, \qquad a \in [0, T].$$

Then $\phi_a(t)$ satisfies

$$d\phi_a(t)/dt + \alpha\phi_a(t)^{p-1} \geqslant K\,|f(t)| \qquad \text{for a.e. } t \in (a, T),$$
$$\phi_a(a) = |j(a)|.$$

Therefore we obtain, by (4.7),

$$|j(t)| \leqslant \phi_a(t) \leqslant \min(|j(a)|, \{\alpha(p-2)(t-a)\}^{-1/(p-2)}) \\ + K\int_a^t |f(s)|\, ds \qquad \text{for all} \quad a \in [0, T] \text{ and } t \in [a, T].$$

Hence, putting $r_0 = r^{-(p-2)}$, we deduce

(i) $|j(t)| \leqslant \phi_0(t) \leqslant r + K(1+r_0)\, r^{p-1}$ for all $t \in [0, r_0]$,

(ii) $|j(t)| \leqslant \phi_{t-r_0}(t) \leqslant \{\alpha(p-2)\}^{-1/(p-2)}\, r + K(1+r_0)\, r^{p-1}$ for all $t \in [r_0, T]$ (if $r_0 \leqslant T$).

Thus we can take $M_{\alpha,p,K}(r) = 1 + \{\alpha(p-2)\}^{-1/(p-2)} + K(1 + r^{p-2})$.

Q.E.D.

Proof of Theorem V. First of all, we set

$$N_0 = \ell(\varepsilon_0)/2 + m^2(\varepsilon_0+1)^2\, C_k + m(\varepsilon_0+1),$$
$$m = m_1(\varepsilon_0), \qquad C_k = (1+3k)/(1-k),$$
$$N = M_4(\varepsilon_0) + \{M_4(\varepsilon_0)^2/2 + (\varepsilon_0+1)\, M_4(\varepsilon_0)\, \varepsilon_0^{p-2}\}/\alpha \\ + C_k(\varepsilon_0+1)\, \varepsilon_0^{2p-3} + N_0 + 1,$$
$$M_4(\cdot) = M_{\alpha K_0, p, 1}(\cdot) \quad \text{(the function in Lemma 4.3)},$$

which will appear in the following calculation.

For an arbitrary positive number r satisfying $Nr < \varepsilon_0$, let $|u_0|_H + \varphi^0(u_0) \leqslant r$, $\|f(t)\|_{2,T} \leqslant r^{p-1}$ and $u(t)$ be an s-strong solution of (C.P.) in $[0, S]$. Then we here claim that $\max\{E(t) := |u(t)|_H + \varphi^t(u(t));\ t \in [0, S]\} \leqslant Nr$. Suppose that $E(t) > Nr$ for some $t \in [0, S]$, then, since $E(t)$ is continuous in $t \in [0, S]$ and $E(0) \leqslant r < Nr$, there exists a number $t_1 \in (0, S]$ such that $E(t)$ attains Nr at $t = t_1$ for the first time. Since $\varphi^t(u(t)) \leqslant Nr < \varepsilon_0$ for all $t \in [0, t_1]$, multiplying (3.1) by $u(t)$ and using (4.5), we have

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \alpha\varphi^t(u(t)) \leqslant |f(t)| \cdot |u(t)| \qquad \text{for a.e. } t \in [0, t_1]. \tag{4.8}$$

Then, by (4.6),

$$\frac{d}{dt}|u(t)| + \alpha K_0\, |u(t)|^{p-1} \leqslant |f(t)| \qquad \text{for a.e. } t \in [0, t_1]. \tag{4.9}$$

Hence, by Lemma 4.3,

$$|u(t)| \leqslant M_4(r)r \qquad \text{for all} \quad t \in [0, t_1]. \tag{4.10}$$

Thus integration of (4.8) on $[t_2, t_1]$, $0 \leqslant t_2 < t_1$, gives

$$|\varphi^t(u(t))|_{L^1(t_2,t_1)} \leqslant \{M_4(r)^2\, r^2/2 + (|t_1 - t_2| + 1)\, M_4(r)\, r^p\}/\alpha. \tag{4.11}$$

On the other hand, we obtain by (3.49), (4.4) and (4.10) that

$$d\varphi^t(u(t))/dt \leqslant N_0 + C_k\, |f(t)|^2 \qquad \text{for a.e. } t \in [0, t_1],$$

whence follows

$$\varphi^{t_1}(u(t_1)) \leqslant \varphi^s(u(s)) + |t_1 - s|\, N_0 \\ + C_k(1 + |t_1 - s|)\, r^{2(p-1)} \qquad (0 < s < t_1). \tag{4.12}$$

If $t_1 < r$, then (4.12) with $s = 0$ contradicts the definition of N. So, integrating (4.12) with respect to s on $[t_1 - r, t_1]$, we find by (4.11) with $t_2 = t_1 - r$ that

$$\varphi^{t_1}(u(t_1)) \leqslant [\{M_4(\varepsilon_0)^2/2 + (\varepsilon_0 + 1)\, M_4(\varepsilon_0)\, \varepsilon_0^{p-2}\}/\alpha \\ + N_0 + C_k(\varepsilon_0 + 1)\, \varepsilon_0^{2p-3}]r. \tag{4.13}$$

Then (4.10) and (4.13) contradict the definition of N. Thus we obtain the *a priori* bound $\max\{E(t); 0 \leqslant t \leqslant S\} \leqslant Nr$, which is independent of S. Therefore Theorem II$'$ assures that if $Nr < \varepsilon_1$, then $u(t)$ can be continued globally to $[0, T]$ as an s-strong solution of (C.P.). Q.E.D.

Proof of Theorem VI. Since $\varphi^0(0) = 0$, as a special case of (3.40), we reset

$$R = |u_0|_{\mathscr{B}^0_{\alpha,2}} + |u_0|_H + |f(t)|_{\alpha,1}. \tag{4.14}$$

Let $u(t)$ be an arbitrary strong solution of (C.P.) in $[0, S]$ satisfying (3.5). Then we claim that for sufficiently small R, $u(t)$ can be continued to $[0, T_1]$ with $T_1 = \{2(1-\alpha)R^2/(3\ell(C_0) + 1)^2\}^{1/2(1-\alpha)}$. To see this, we note that $u(t)$ can be continued to the right of $[0, S]$ as long as $|b(t)|_{\alpha,S} < +\infty$. Indeed, by (3.24) with $p = \infty$ of Lemma 3.8, $|b(t)|_{\alpha,S} < +\infty$ implies $\varphi^S(u(S)) < +\infty$, then Theorem II is applied with u_0 replaced by $u(S)$. Suppose here that $|b(t)|_{\alpha,S_1} = R$ for some $S_1 \in (0, T_1)$. Then (3.24) with $p = +\infty$ gives

$$\varphi^t(u(t)) \leqslant M_2(R)(R^2 t^{2\alpha-1} + t) \qquad \text{for all} \quad t \in (0, S_1). \tag{4.15}$$

On the other hand, since $S_1 < T_1$ and $a(t) \equiv 1$, (3.47) is easily verified with $T_0 = S_1$. Moreover, by (3.45) and (4.15), we find that (3.46) is also satisfied with $T_0 = S_1$ for a sufficiently small R. Hence, as in the proof of Theorem I, we deduce $|b(t)|_{\alpha,S_1} < R$, which is a contradiction. Thus we find that (4.15) holds true for all $t \in (0, T_1]$, whence follows

$$\varphi^{T_1}(u(T_1)) \leqslant M(R)\, R^{1/(1-\alpha)}, \qquad M(\cdot) \in \mathscr{M}.$$

Then, taking R small enough, we can apply Theorem V with u_0 replaced by $u(T_1)$.

As for the case $u_0 \in \overline{D(\varphi^0)}$, repeating the same argument as above (with $\alpha = 0$) for (3.54)–(3.56), we can take

$$T_1 = R^{(2+\gamma)/(1+\gamma)}/(M(R)+1) \qquad \text{and} \qquad \varphi^{T_1}(u(T_1)) \leqslant M(R)\, R^{\gamma/(1+\gamma)}.$$

Q.E.D.

Remark 4.4. In Theorem VI, condition $\|f(t)\|_{2,T} \leqslant r^{p-1}$ can be replaced by $|f(t)|_{\alpha,1} + \sup_{2 \leqslant s \leqslant T} |f(t)|_{L^2(s-1,s;H)} \leqslant r^{p-1}$, $\alpha \in [0, \frac{1}{2})$ (see Remark 3.12).

5. Applications

In this section we shall exemplify the applicability of our abstract theorems to some initial-boundary value problems for the Navier–Stokes equations and their variants in bounded regions with moving boundaries. Let T be a given number in $[1, +\infty)$ and $Q(t)$ be a bounded domain in $\mathbb{R}^n_x$ with smooth boundary $\Gamma(t)$ for each $t \in [0, T]$. When t moves over $(0, T)$, $Q(t)$ generates a (x, t)-domain $Q = \bigcup_{0<t<T} (Q(t) \times \{t\})$ and $\Gamma(t)$ generates a (x, t)-hypersurface $\Gamma = \bigcup_{0<t<T} (\Gamma(t) \times \{t\})$. Throughout this section, we always make the following assumptions on Q as in Yamada [34] and Ôtani and Yamada [28].

(Q.1) For each $t \in [0, T]$, the boundary $\Gamma(t)$ of $Q(t)$ is a $(n-1)$-dimensional sufficiently smooth manifold (say, of class C^3).

(Q.2) Q is covered by m slices $Q(s_i, t_i) = \bigcup_{s_i<t<t_i} (Q(t)\, x\{t\})$ ($i = 1, 2, \ldots, m$) such that for each slice $Q(s_i, t_i)$ is mapped onto the cylindrical domain $Q(s_i) \times (s_i, t_i)$ by a diffeomorphism Ψ_i which is of class C^3 up to the boundary and preserves the time coordinate t.

In stating our results, we need an auxiliary open ball $\mathcal{O}$ in $\mathbb{R}^n_x$ such that the closure of Q is contained in $\mathcal{O} \times [0, T]$. We mean by $C(I; X(Q(t)))$ the set of all functions v on Q such that $v(\cdot, t)$ belongs to $X(Q(t))$ for all $t \in I$ and that the zero extension $\hat{v}$ of v to $\mathcal{O} \times I$ is an $X(\mathcal{O})$-valued continuous function in $t \in I$, where I is an interval in $[0, T]$ and $X(\Omega)$ ($\Omega = Q(t)$ or $\mathcal{O}$) is a function space defined on Ω such as $L^2(\Omega)$, $W^{1,p}_0(\Omega)$, etc.

Example I. *Navier–Stokes Equations*

We first consider the following initial-boundary value problem for the Navier–Stokes equation in Q:

$$\left.\begin{array}{lll} \dfrac{\partial \mathbf{u}}{\partial t}(x,t) - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p_* & \text{in } Q, & (5.1) \\ \operatorname{div} \mathbf{u}(x,t) = 0 & \text{in } Q, & (5.2) \\ \mathbf{u}(x,t) = 0 & \text{on } \Gamma, & (5.3) \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{in } Q(0), & (5.4) \end{array}\right\} \text{(Pr.NS)}$$

where the unknown $\mathbf{u}$ and given $\mathbf{f}$, $\mathbf{u}_0$ are real n-dimensional vector functions, while the unknown p_* is a real scalar function.

In recent years, this kind of problem has been studied by several authors. Fujita and Sauer [14] established the existence of a Hopf-type weak solution by the so-called penalty method. In Bock [4], the existence of a unique weak solution of the Kiselev–Ladyzhenskaya type is proved. We also refer to Inoue and Wakimoto [16] and Ôtani and Yamada [28]. Our result here is a natural extension to the noncylindrical case of Fujita and Kato [12], where (Pr.NS) is solved in a class of strong solutions for the cylindrical case $Q(t) \equiv Q(0)$. In particular, our method can give an explicit estimate of the time derivative of the solution (near the lateral boundary).

Let Ω be a bounded domain in $\mathbb{R}^n_x$ with smooth boundary and put

$$\mathbb{C}^\infty_\sigma(\Omega) = \{\mathbf{u} = (u^1, u^2, \ldots, u^n);\ u^i \in C^\infty_0(\Omega)\ (i = 1, 2, \ldots, n),\ \operatorname{div} \mathbf{u} = 0\},$$

$$\mathbb{H}(\Omega) = (L^2(\Omega))^n := \{\mathbf{u} = (u^1, u^2, \ldots, u^n);\ u^i \in L^2(\Omega)\ i = 1, 2, \ldots, n\},$$

$$\mathbb{H}_\sigma(\Omega) = \text{the completion of } \mathbb{C}^\infty_\sigma(\Omega) \text{ under the } \mathbb{H}(\Omega)\text{-norm},$$

$$P_\Omega = \text{the orthogonal projection from } \mathbb{H}(\Omega) \text{ onto } \mathbb{H}_\sigma(\Omega),$$

$$\mathbb{W}^{1,p}_\sigma(\Omega) = (W^{1,p}_0(\Omega))^n \cap \mathbb{H}_\sigma(\Omega),$$

$$\mathbb{H}^s_{\gamma_0}(\Omega) = \{\mathbf{u} = (u^1, {}^2, \ldots, u^n);\ u^i \in H^s(\Omega),\ u^i|_{\partial\Omega} = 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{H}^s_\sigma(\Omega) = \mathbb{H}^s_{\gamma_0}(\Omega) \cap \mathbb{H}_\sigma(\Omega), \qquad \tfrac{1}{2} < s,$$

where $W^{1,p}_0(\Omega)$ and $H^s(\Omega)$ are the usual Sobolev spaces of order 1 and s, respectively. We denote by A_Ω the *Stokes operator* $-P_\Omega \Delta$, which works on $D(A_\Omega) = \mathbb{H}^2_{\gamma_0}(\Omega) \cap \mathbb{H}_\sigma(\Omega) = \mathbb{H}^2_\sigma(\Omega)$. Let A^α_Ω be the fractional power of A_Ω of order α, then Fujita and Morimoto [13], together with Fujiwara [15], give the concrete characterization of the domain $D(A^\alpha_\Omega)$ of A^α_Ω. For example, $D(A^\alpha_\Omega) = \mathbb{H}^{2\alpha}_\sigma(\Omega)$ for $\frac{1}{4} < \alpha \leqslant \frac{1}{2}$.

Now our results are stated as follows.

THEOREM 5.1. *Let* $n=3$, $\mathbf{f}\in L^2(0,T;\mathbb{H}(Q(t)))$ *and* $\mathbf{u}_0\in D(A^{\alpha}_{Q(0)})$ *with* $\frac{1}{4}\leqslant\alpha\leqslant\frac{1}{2}$. *Then there exists a positive number* T_0 ($T_0\leqslant T$) *depending on* $|\mathbf{u}_0|_{A^\alpha}$, *the* $A^{\alpha}_{Q(0)}$-*norm of* $\mathbf{u}_0$, *such that* (Pr.NS) *has a unique strong solution* $\mathbf{u}(\cdot,t)$ *in* $[0,T_0]$ *satisfying*

$$\begin{aligned}
&\mathbf{u}(\cdot,t)\in C([0,T_0];\mathbb{H}(Q(t)))\cap C((0,T_0];\mathbb{H}^1_\sigma(Q(t)))\\
&\qquad\qquad\qquad\textit{for}\quad \alpha\in[\tfrac{1}{4},\tfrac{1}{2});\ \mathbf{u}(\cdot,t)\in C([0,T_0];\mathbb{H}^1_\sigma(Q(t)))\\
&\qquad\qquad\qquad\textit{for}\quad \alpha=\tfrac{1}{2},
\end{aligned}\tag{5.5}$$

$$t^{1/2-\alpha}\,\partial\mathbf{u}(\cdot,t)/\partial t,\quad t^{1/2-\alpha}\,\Delta\mathbf{u}(\cdot,t),$$
$$t^{1/2-\alpha}(\mathbf{u}\cdot\nabla)\,\mathbf{u}(\cdot,t)\in L^2(0,T_0;\mathbb{H}(Q(t))),\tag{5.6}$$

$$t^{1/2-\alpha}\,|\mathbf{u}(\cdot,t)|_{\mathbb{H}^1_\sigma(Q(t))}\in L^p_*(0,T_0)\qquad\textit{for all}\quad p\in[2,+\infty],\tag{5.7}$$

$$t^{-\alpha}\,|\mathbf{u}(\cdot,t)-\mathbf{u}_0(\cdot)|_{\mathbb{H}_\sigma(Q(t))}\in L^p_*(0,T_0)\qquad\textit{for all}\quad p\in[2,+\infty].\tag{5.8}$$

Moreover, there exists a (*suffciently small*) *positive number* r *independent of* T *such that if*

$$|\mathbf{u}_0|_{A^\alpha}+\sup_{1\leqslant t\leqslant T}\left(\int_{t-1}^{t}|\mathbf{f}(s)|^2_{\mathbb{H}(Q(s))}\,ds\right)^{1/2}\leqslant r,\tag{5.9}$$

then (Pr.NS) *has a unique global strong solution* $\mathbf{u}(\cdot,t)$ *in* $[0,T]$ *satisfying* (5.5)–(5.8) *with* $T_0=T$.

THEOREM 5.2. *Let* $n=2$, $\mathbf{f}\in L^2(0,T;\mathbb{H}(Q(t)))$ *and* $\mathbf{u}_0\in D(A^{\alpha}_{Q(0)})$ *with* $0<\alpha\leqslant\frac{1}{2}$. *Then* (Pr.NS) *has a unique global strong solution* $\mathbf{u}(\cdot,t)$ *in* $[0,T]$ *satisfying* (5.5)–(5.8) *with* $T_0=T$.

Proof of Theorem 5.1. Set

$$\begin{aligned}
\varphi_{\mathcal{O}}(\mathbf{u})&=\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathcal{O}}\left|\frac{\partial u^j}{\partial x_i}(x)\right|^2dx\qquad\text{if}\quad \mathbf{u}\in\mathbb{H}^1_\sigma(\mathcal{O}),\\
&=+\infty\qquad\qquad\qquad\qquad\qquad\quad\ \text{if}\quad \mathbf{u}\in\mathbb{H}_\sigma(\mathcal{O})\backslash\mathbb{H}^1_\sigma(\mathcal{O}).
\end{aligned}\tag{5.10}$$

Then $\varphi_{\mathcal{O}}$ is a proper lower semicontinuous convex function on $\mathbb{H}_\sigma(\mathcal{O})$ and $\partial\varphi_{\mathcal{O}}$ coincides with the Stokes operator $A_{\mathcal{O}}$. We next put $K(t)=\{\mathbf{u}\in\mathbb{H}_\sigma(\mathcal{O});\ \mathbf{u}=0$ a.e. $x\in\mathcal{O}\backslash Q(t)\}$ and denote by $I_{K(t)}$ the indicator function of $K(t)$, i.e., $I_{K(t)}(\mathbf{u})=0$ if $\mathbf{u}\in K(t)$, and $I_{K(t)}(\mathbf{u})=+\infty$ if $\mathbf{u}\in\mathbb{H}_\sigma(\mathcal{O})\backslash K(t)$. Define another proper lower semicontinuous convex function φ^t by

$$\varphi^t(\mathbf{u})=\varphi_{\mathcal{O}}(\mathbf{u})+I_{K(t)}(\mathbf{u})\qquad\text{for all}\quad \mathbf{u}\in\mathbb{H}_\sigma(\mathcal{O}).\tag{5.11}$$

Then we have

$$\begin{aligned} D(\varphi^t) &= \{\mathbf{u} \in \mathbb{H}_\sigma(\mathscr{O}); \mathbf{u}|_{Q(t)} \in \mathbb{H}^1_\sigma(Q(t)), \mathbf{u}|_{\mathscr{O}\setminus Q(t)} = 0\}, \\ D(\partial\varphi^t) &= \{\mathbf{u} \in D(\varphi^t); \mathbf{u}|_{Q(t)} \in \mathbb{H}^2_\sigma(Q(t))\}, \\ \partial\varphi^t(u) &= \{\mathbf{f} \in \mathbb{H}_\sigma(\mathscr{O}); P_{Q(t)}\mathbf{f}|_{Q(t)} = A_{Q(t)}\mathbf{u}|_{Q(t)}\}. \end{aligned}$$

We further introduce another nonlinear operator $B(t, \cdot)$ by

$$\begin{aligned} B(t, \mathbf{u}) &= P_{\mathscr{O}}(\mathbf{u} \cdot \nabla)\mathbf{u} \qquad \text{for} \quad \mathbf{u} \in D(\partial\varphi^t), \\ D(B(t, \cdot)) &= D(\partial\varphi^t). \end{aligned} \tag{5.12}$$

Then there exists a constant C such that

$$|B(t, \mathbf{u})|_{\mathbb{H}_\sigma(\mathscr{O})} \leqslant C\{\varphi^t(\mathbf{u})\}^{3/4} |\partial^0\varphi^t(\mathbf{u})|^{1/2}_{\mathbb{H}_\sigma(\mathscr{O})} \qquad \text{for all} \quad \mathbf{u} \in D(\partial\varphi^t), \tag{5.13}$$

(see Fujita and Kato [12], Ladyzhenskaya [20], and Temam [30]), which implies $(\text{A}.3)_\alpha$ with $\alpha = \frac{1}{4}$ and $a(t) \equiv 0$.

We here consider the following abstract Cauchy problem in $\mathbb{H}_\sigma(\mathscr{O})$:

$$(\text{C.P.})\hat{}_1 \quad \begin{cases} d\hat{\mathbf{u}}(t)/dt + \partial\varphi^t(\hat{\mathbf{u}}(t)) + B(t, \hat{\mathbf{u}}(t)) \ni P_{\mathscr{O}}\hat{\mathbf{f}}(t), & 0 < t < T, \\ \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}_0, \end{cases}$$

where $\hat{\mathbf{f}}(t)$ and $\hat{\mathbf{u}}_0$ are the zero extensions of $\mathbf{f}(t)$ and $\mathbf{u}_0$ to $\mathscr{O}$, respectively. Since $(B(t, \mathbf{u}), \mathbf{u})_{\mathbb{H}_\sigma(\mathscr{O})} = 0$, $(\mathbf{g}, \mathbf{u})_{\mathbb{H}_\sigma(\mathscr{O})} = 2\varphi^t(\mathbf{u})$ for all $\mathbf{u} \in D(\partial\varphi^t)$ and $\mathbf{g} \in \partial\varphi^t(\mathbf{u})$, and since there exists a constant C such that $|\mathbf{u}|^2_{\mathbb{H}_\sigma(\mathscr{O})} \leqslant C|\mathbf{u}|^2_{\mathbb{H}^1_\sigma(\mathscr{O})} \leqslant \varphi^t(\mathbf{u})$ for all $\mathbf{u} \in D(\varphi^t)$ (see [20] and [30]), conditions (A.1) and (A.7) are fulfilled. To verify (A.2), it suffices to note that $((\mathbf{u} \cdot \nabla)\mathbf{u}, \phi)_{\mathbb{H}_\sigma(\mathscr{O})} = -((\mathbf{u} \cdot \nabla)\phi, \mathbf{u})_{\mathbb{H}_\sigma(\mathscr{O})}$ for all $\mathbf{u} \in D(\partial\varphi^t)$ and $\phi \in \mathbb{C}^\infty_\sigma(\mathscr{O})$. Moreover, under assumptions (Q.1) and (Q.2), we can verify $(\text{A}.\varphi^t)$ by the same reasoning as in the proof of Lemma 3.1 of [28] (see also Yamada [33, 34]). Thus, noting that $\mathbf{u}_0 \in D(A^\alpha_{Q(0)})$ if and only if $\hat{\mathbf{u}}_0 \in \mathscr{B}_{\alpha,2}(\partial\varphi^0)$ for $0 < \alpha < \frac{1}{2}$ (see [13] and [5]), we can apply Theorems I, II, V and VI to $(\text{C.P.})\hat{}_1$. Then the desired strong solution $\mathbf{u}(\cdot, t)$ is given by the restriction of the strong solution $\hat{\mathbf{u}}(\cdot, t)$ of $(\text{C.P.})\hat{}_1$ to $Q(t)$, i.e., $\mathbf{u}(\cdot, t) = \hat{\mathbf{u}}(\cdot, t)|_{Q(t)}$, since $\partial\hat{\mathbf{u}}(\cdot, t)/\partial t|_{Q(t)} = \partial\mathbf{u}(\cdot, t)/dt$ and $P_{Q(t)}(P_{\mathscr{O}}\mathbf{h})|_{Q(t)} = P_{Q(t)}\mathbf{h}|_{Q(t)}$ for all $\mathbf{h} \in \mathbb{H}_\sigma(\mathscr{O})$. Estimates (5.6)–(5.8) are derived from (3.5), (3.6) and the inequality $|\mathbf{u}|_{\mathbb{H}^2(Q(t))} \leqslant \text{Const.}\, |\partial^0\varphi^t(\mathbf{u})|$ for all $\mathbf{u} \in D(\partial\varphi^t)$ (see [20] and [30]). Furthermore, by Remark 3.3, $|\hat{\mathbf{u}}(\cdot, t)|_{\mathbb{H}^1_\sigma(\mathscr{O})}$ is absolutely continuous on any compact interval of $(0, T_0]$ (or on $[0, T]$). Hence, since $\mathbf{u}(t)$ is also continuous in $(0, T_0]$ (or $[0, T_0]$) in the weak topology of $\mathbb{H}^1_\sigma(\mathscr{O})$, $\mathbf{u}(t)$ turns out to be continuous in the strong topology of $\mathbb{H}^1_\sigma(\mathscr{O})$. The uniqueness part is proved by much the same argument as in Serrin [29]. Q.E.D.

Proof of Theorem 5.2. If $n=2$, then instead of (5.13) we now have (see [30])

$$|B(t, \mathbf{u})|^2_{\mathbb{H}_\sigma(\mathcal{O})} \leqslant C\,|\mathbf{u}|_{\mathbb{H}_\sigma(\mathcal{O})}\{\varphi^t(\mathbf{u})\} \times |\partial^0\varphi^t(\mathbf{u})|_{\mathbb{H}_\sigma(\mathcal{O})} \quad \text{for all} \quad \mathbf{u}\in D(\partial\varphi^t), \tag{5.14}$$

which implies $(\mathrm{A.3})_\alpha$ for all $\alpha\in(0,\frac{1}{2}]$ and (3.7)′ in Theorem IV. Then we can apply Theorems I, II, IV and Corollary IV for $(\mathrm{C.P.})\hat{}_1$. Q.E.D.

Remark 5.3. As for the case $n=4$, we can obtain a result similar to (but somewhat weaker than) that of Theorem 5.2; i.e., there exists a (sufficiently small) positive number r independent of T such that if $|\mathbf{u}_0|_{\mathbb{H}^1_\sigma(Q(0))} + \sup_{1\leqslant t\leqslant T}(\int_{t-1}^t |\mathbf{f}(s)|^2_{\mathbb{H}_\sigma(Q(s))}\,ds)^{1/2}\leqslant r$, then (Pr.NS) has a unique strong solution $u(\cdot,t)$ satisfying (5.5)–(5.8) with $\alpha=\frac{1}{2}$ and $T_0=T$. Indeed, since $|B(t,\mathbf{u})|_{\mathbb{H}_\sigma(\mathcal{O})}\leqslant \mathrm{Const.}\{\varphi^t(\mathbf{u})\}^{1/2}\,|\partial^0\varphi^t(\mathbf{u})|_{\mathbb{H}_\sigma(\mathcal{O})}$ holds for all $\mathbf{u}\in D(\partial\varphi^t)$ (see [16]), we can apply Theorem V.

Example II. *Modified Navier–Stokes Equations*

We now consider the initial-boundary value problem for the Modified Navier–Stokes equation of the form

$$(\mathrm{Pr.MNS})\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t}(x,t) + A_p\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p_* & \text{in } Q,\\ \text{with the same conditions (5.2)–(5.4) as in (Pr.NS),} \end{cases}$$

where $A_p\mathbf{u} = -\sum_{i=1}^n(\partial/\partial x_i)(|\nabla\mathbf{u}|^{p-2}(\partial\mathbf{u}/\partial x_i))$, $p\geqslant 2$, $|\nabla\mathbf{u}|^2=\sum_{i,j=1}^n |(\partial u^j/\partial x_i)|^2$. This problem is posed in the book of Lions [22] for the case that Q is a cylindrical domain $\Omega\times[0,T]$, where the Galerkin's method is employed to construct weak solutions belonging to $L^p(0,T;W^{1,p}_\sigma(\Omega))\cap L^\infty(0,T;\mathbb{H}_\sigma(\Omega))$ (cf. [24]).

THEOREM 5.4. *Suppose that* $p>2$ *and* $p\geqslant 4n/(n+2)$. *Let* $\mathbf{u}_0\in\mathbb{H}_\sigma(Q(0))$ *and* $\mathbf{f}\in L^2(0,T;\mathbb{H}(Q(t)))$, *then* (Pr.MNS) *has a global solution* $\mathbf{u}(\cdot,t)$ *in* $[0,T]$ *satsifying*

$$\mathbf{u}(\cdot,t)\in C([0,T];\mathbb{H}_\sigma(Q(t)))\cap C((0,T];\mathbb{W}^{1,p}_\sigma(Q(t))) \cap L^p(0,T;\mathbb{W}^{1,p}_\sigma(Q(t))), \tag{5.15}$$

$$\sqrt{t}\,\partial\mathbf{u}(\cdot,t)/\partial t\in L^2(0,T;\mathbb{H}(Q(t))), \tag{5.16}$$

$$t^{2/p}(\mathbf{u}\cdot\nabla)\,\mathbf{u}(\cdot,t)\in L^\infty(0,T;\mathbb{H}(Q(t))). \tag{5.17}$$

Furthermore, if $\mathbf{u}_0 \in \mathbb{W}_\sigma^{1,p}(Q(0))$, *then* (Pr.MNS) *has a solution* $\mathbf{u}(\cdot, t)$ *satisfying*

$$\mathbf{u}(\cdot, t) \in C([0, T]; \mathbb{W}_\sigma^{1,p}(Q(t))), \tag{5.18}$$

$$\partial \mathbf{u}(\cdot, t)/\partial t \in L^2(0, T; \mathbb{H}(Q(t))), \tag{5.19}$$

$$(\mathbf{u} \cdot \nabla)\, \mathbf{u}(\cdot, t) \in L^\infty(0, T; \mathbb{H}(Q(t))). \tag{5.20}$$

Proof. We can carry out the proof as in the proof of Theorem 5.1. Instead of (5.10), put $\varphi_{\mathscr{O}}(\mathbf{u}) = |\nabla \mathbf{u}|^p_{L^p(\mathscr{O})}/p$ if $\mathbf{u} \in \mathbb{W}_\sigma^{1,p}(\mathscr{O})$ and $\varphi_{\mathscr{O}}(\mathbf{u}) = +\infty$ if $\mathbf{u} \in \mathbb{H}_\sigma(\mathscr{O}) \backslash \mathbb{W}_\sigma^{1,p}(\mathscr{O})$, and define φ^t by (5.11).

Then we find that $D(\varphi^t) = \{\mathbf{u} \in \mathbb{H}_\sigma(\mathscr{O}); \mathbf{u}|_{Q(t)} \in \mathbb{W}_\sigma^{1,p}(Q(t)), \mathbf{u}|_{\mathscr{O}\backslash Q(t)} = 0\}$; and that $\mathbf{u} \in D(\partial\varphi^t)$ and $\mathbf{f} \in \partial\varphi^t(\mathbf{u})$ if and only if $\mathbf{u} \in D(\varphi^t)$ and

$$\sum_{i=1}^n \left(|\nabla \mathbf{u}|^{p-2} \frac{\partial \mathbf{u}}{\partial x_i}, \frac{\partial \mathbf{w}}{\partial x_i}\right)_{\mathbb{H}(\mathscr{O})} = (\mathbf{f}, \mathbf{w})_{\mathbb{H}(\mathscr{O})} \qquad \text{for all} \quad \mathbf{w} \in D(\varphi^t)$$

(formally, $P_{Q(t)}\mathbf{f}|_{Q(t)} = P_{Q(t)} A_p \mathbf{u}|_{Q(t)}$). Define $B(t, \cdot)$ again by (5.12), then, since $\mathbb{W}_\sigma^{1,p}(\mathscr{O})$ is embedded in $(L^{2p/(p-2)}(\mathscr{O}))^n$, we easily have

$$|B(t, \mathbf{u})|_{\mathbb{H}_\sigma(\mathscr{O})} \leqslant \text{Const.}\{\varphi^t(\mathbf{u})\}^{2/p} \qquad \text{for all} \quad \mathbf{u} \in D(\varphi^t),$$

which implies (A.5) with $a(t) \equiv 0$ and (3.7)′ in Theorem IV. Thus we can apply Theorems II, III and IV to $(\mathrm{C.P.})_1^\frown$. Q.E.D.

Remark 5.5. Another type of modified Navier–Stokes equation,

$$\partial \mathbf{u}/\partial t - \left(1 + \nu \left(\int_{Q(t)} |\nabla \mathbf{u}|^2\, dx\right)^\alpha\right) \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} - \nabla p_* \qquad (\nu, \alpha \geqslant 0),$$

can be treated within our abstract framework. Indeed, repeating the routine with (5.10) replaced by $\varphi_0(\mathbf{u}) = |\nabla \mathbf{u}|^2_{L^2(\mathscr{O})}/2 + \nu |\nabla \mathbf{u}|^{(1+\alpha)\cdot 2}_{L^2(\mathscr{O})}/2(1+\alpha)$ if $\mathbf{u} \in \mathbb{H}_\sigma^1(\mathscr{O})$, and $+\infty$ if $\mathbf{u} \in \mathbb{H}_\sigma(\mathscr{O}) \backslash \mathbb{H}_\sigma^1(\mathscr{O})$, we can obtain some existence results similar to those of Theorems 5.1 and 5.2. In particular, if $n = 3$ and $\alpha > 1$, then there exists a global strong solution for any $\mathbf{u}_0$ and $\mathbf{f}$ as in Theorem 5.4 (cf. [20, 22, 24]).

Remark 5.6. Our abstract framework can also deal with some nonlinear heat equations such as

$$(\text{PR.NH}) \begin{cases} \dfrac{\partial u}{\partial t}(x, t) - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - |u|^\alpha u = f(x, t) & \text{in } Q, \\ u(x, t) = 0 \qquad \text{on } Q, & \\ u(x, 0) = u_0(x) \qquad \text{in } Q(0). & \end{cases}$$

For example, for the case $p=2$, assume $0<\alpha<+\infty$ if $n \leqq 2$, $0<\alpha<4/(n-2)$ if $n \geqq 3$ (resp. $0<\alpha<4/n$), then (Pr.NH) has a local strong solution for any $u_0(x) \in H_0^1(Q(0))$ (resp. $u_0 \in L^2(Q(0))$). Moreover, if $|u_0|_{H^1}$ (resp. $|u_0|_{L^2}$) is sufficiently small, then the solution can be continued globally.

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