The quasi-orthogonality of the derivatives of semi-classical polynomials

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ABSTRACT

The distributional equation of a semi-classical functional allows an efficient study of other characterizations and properties of the semi-classical OPS/functionals. In [6] an extensive survey of this approach has been presented. A particular case of semi-classical OPS/functionals are the classical ones. For the distributional equation of a classical functional a regularity condition holds. We have give a family counterexamples to show that the regularity condition does not hold in general for semi-classical functionals [9]. Here we investigate the consequences of the failure of the regularity condition in the quasi-orthogonality of the derivatives of the semi-classical OPS and, in general, the behaviour of the derivatives of order k. This study leads us to another condition that holds for the distributional equation of classical functional, the coprimality condition.

INTRODUCTION

Semi-classical functionals and the corresponding orthogonal polynomial sequence, in short OPS, are a generalization of classical functionals/OPS. J.A. Shohat [10] introduced this generalization looking at linear functionals associated with weight functions satisfying a Pearson equation

\[ \frac{Dw}{w} = \frac{\psi - D\phi}{\phi}, \quad \text{deg} \phi \geq 0, \quad \text{deg} \psi \geq 1. \]

He proved that the related OPS, \((P_n)\), satisfies the following differential-difference equation

\[ \phi DP_{n+1} = \sum_{k=n-s}^{n} a_{nk} P_k, \quad s = \max\{\text{deg} \phi - 2, \text{deg} \psi - 1\}, \]
the so-called structure relation. G. Szegő and S. Karlin proposed the problem to determine all the OPS that satisfy this structure relation [3]. E. Hendriksen and H. Van Rossum proved the more general statement: for the Shohat's functionals the sequence of derivatives of the corresponding OPS [2] constitutes a quasi-orthogonal family and they called them semi-classical OPS. Later on, P. Maroni gave the complete answer of the problem [5] with the help of the distributional equation

\[ D(\phi u) = \psi u, \quad \deg \phi \geq 0, \quad \deg \psi \geq 1, \]

that characterizes the semi-classical functionals, where \( \phi \) and \( \psi \) are the same polynomials of (1), and \( u \) is a linear functional.

The distributional equation is a powerful tool but some questions arise in view of the classical case. Actually, the distributional equation read as difference equation for the moments of the functional. The polynomial coefficients of such a classical case satisfy the following condition (regularity condition):

\[ na_2 + b_1 \neq 0, \quad n \geq 0, \]

where: \( \phi(x) = a_2x^2 + a_1x + a_0 \), and \( \psi(x) = b_1x + b_0 \). This has an important consequence in the difference equation of the moments: \((na_2 + b_1)u_{n+1} + (na_1 + b_0)u_n + na_0u_{n-1} = 0, \quad n \geq 0\). The mentioned condition means that this equation is not anomalous and two consecutive moments yield the next moment, i.e.,

\[ u_{n+1} = \frac{na_1 + b_0}{na_2 + b_1} u_n + \frac{na_0}{na_2 + b_1} u_{n-1}, \quad n \geq 0. \]

The deal with a singular difference equation is quite complicated. In the semi-classical case they are excluded if

\[ na_{s+2} + b_{s+1} \neq 0, \quad n \geq 0, \quad \phi(x) = \sum_{i=0}^{s+2} a_ix^i, \quad \psi(x) = \sum_{i=0}^{s+1} b_ix^i. \]

In the classical case that is right: there does not exist a linear quasi-definite functional satisfying a singular distributional equation.

The aforementioned difference equation for the classical functionals (4) yields another consequence: a unique non-zero functional solves the equation if the functional is normalized, i.e., \( u_0 = 1 \). We can characterize the polynomials \( \phi \) and \( \psi \) in such a way that the unique non-zero functional satisfying the differential/difference equation of polynomials \( \phi \) and \( \psi \) is quasi-definite. The regularity condition (3) is one condition, the other is the coprimality of \( \phi \) and the polynomials \( \psi + n\phi' \), for \( n \geq 0 \), i.e.

\[ \gcd(\phi, \psi + nD\phi') = c, \quad c \neq 0, \quad n \geq 0. \]

Both are necessary and sufficient conditions to determine the quasi-definite character of the non-zero functional that satisfies the equation (2).

A semi-classical functional \( u \) satisfies multiple distributional equations, and we can associate to each of them a non-negative entire number, the order of the equation. The set of all equations of \( u \) has a lattice structure related to the order.
with an equation of minimum order of polynomials \((\phi, \psi)\). This minimum order is said to be the class of the functional \(u\). The order of the equations is related in the following way to the quasi-orthogonality: If \(D(\phi x) = \psi x\) is one of these equations of order \(s\), then the sequence of derivatives of the corresponding OPS is quasi-orthogonal related to the functional \(\phi u\) and the quasi-orthogonality is also of order \(s\). If the functional is regular, i.e., (5) holds for \(\phi\) and \(\psi\), then the quasi-orthogonality is strict. Moreover, the sequence of second order derivatives is strict quasi-orthogonal of order \(2s\) with respect to \(\phi^2 u\), and so on. This is the background presented in Section 1.

In Section 2 we study what happens with the strict quasi-orthogonality of the sequence of \(k\)-order derivatives related to \(\phi^k u\) if (5) does not hold for \(\phi\) and \(\psi\). The result is that the sequence of the derivatives of order \(k\) can have at most \(k\) consecutive singularities in the strict quasi-orthogonality with \(\phi^k u\). A particular case is presented. These \(\phi^k\)-modifications of \(u\) are the standard modifications of the functional. The question is if all functionals with respect to which the sequence of derivatives of order \(k\) is quasi-orthogonal, say \(w\), are standard modifications of \(u\). In Section 3 we give nearly a complete answer to this question. The discussion of this result in Section 4 leads to the coprimality condition (6). If this condition holds then we can assure that \(w\) is a polynomial modification of the standard modification of \(u\) with \(\phi^k\): \(w = \pi \phi^k u, \pi \in \mathbb{P}\). Otherwise, a divisor of \(\phi\) plays the role of \(\phi\). We have not found such a semi-classical functional, but our assumption is that such a functional exists.

1. PRELIMINARIES

In this section we shall give some basic definitions and propositions which will be useful for the rest of the paper.

1.1. Orthogonal polynomials and quasi-definite functionals

Let \(\mathbb{P}\) be the linear space of polynomials with complex coefficients, and \(\mathbb{P}^*\) be its algebraic dual space, i.e., \(\mathbb{P}^*\) is the linear space of all linear applications \(u: \mathbb{P} \rightarrow \mathbb{C}\). Let \((B_n), n \geq 0\) be a sequence of polynomials, such that \(\deg B_n = n\) for all \(n \geq 0\). Such a sequence is said to be a basis sequence of \(\mathbb{P}\). Since the elements of \(\mathbb{P}^*\) are linear functionals, it is possible to determine them from their actions on a given basis \((B_n)\) of \(\mathbb{P}\). We shall use here, without loss of generality, the canonical basis \((x^n), n \geq 0\) of \(\mathbb{P}\). In general, we shall represent the action of a functional over a polynomial by: \(\langle u, \pi \rangle, u \in \mathbb{P}^*, \pi \in \mathbb{P}\). Therefore a functional is completely determined by a sequence of complex numbers \(\langle u, x^n \rangle = u_n, n \geq 0\), the so-called moments of the functional.

We shall use the following definitions and theorems for an orthogonal polynomial sequence and the corresponding quasi-definite functional:

**Definition 1.1.** Let \((P_n)\) be a basis sequence of \(\mathbb{P}\). \((P_n)\) is said to be an orthogonal polynomial sequence (OPS in short), in widespread sense, if and only if there exists a linear functional \(u \in \mathbb{P}^*\) such that \(\langle u, P_m P_n \rangle = k_n \delta_{mn}, k_n \neq 0, n \geq 0\), where \(\delta_{mn}\)
is the Kronecker delta. \((P_n)\) is said to be positive definite if and only if the coefficients of \(P_n, n \geq 0\), are real numbers and \(k_n > 0\) for all \(n \geq 0\).

**Definition 1.2.** Let \(u\) be a linear functional. We say that \(u\) is a quasi-definite functional if and only if there exists a polynomial sequence \((P_n)\), which is orthogonal with respect to \(u\). We say that \(u\) is positive definite if and only if the corresponding OPS is positive definite.

**Remark 1.3.** Given two polynomial sequences, \((P_n)\) and \((R_n)\), orthogonal with respect to the same linear functional \(u\), then there exists \(c_n \in \mathbb{C} \setminus \{0\}, n \geq 0\) such that \(P_n = c_nR_n, n \geq 0\). This means that, if we normalize the OPS in any way, then we have a unique polynomial sequence orthogonal with respect to a given functional. In this paper we shall consider the monic polynomials. Moreover, if \((P_n)\) is orthogonal with respect to the functionals \(u\) and \(v\), then there exists \(c \in \mathbb{C} \setminus \{0\}\), such that \(v_n = cu_n, n \geq 0\), where \(v_n\) and \(u_n\) are the moments corresponding to the functionals \(v\) and \(u\), respectively. If we normalize the functionals, \(u_0 = 1\), then we have a one-to-one correspondence between the monic OPS \((P_n)\) and the quasi-definite normalized functional \(u\) and we shall represent it \((P_n) = \text{mops } u\).

**Theorem 1.4. (Favard's Theorem)** Let \((P_n)\) be a monic polynomial basis sequence; then, \((P_n)\) is a monic OPS if and only if there exist two sequences of complex numbers \((d_n)_{n\geq 0}\) and \((g_n)_{n\geq 1}, g_n \neq 0, n \geq 1\) such that \((P_n)\) satisfies the following three-term recurrence relation, in short \(TTNR\),

\[
xP_n(x) = P_{n+1}(x) + d_nP_n(x) + g_nP_{n-1}(x), \quad P_{-1}(x) = 0,
\]

\[P_0(x) \equiv 1, \quad n \geq 0.\]  

Moreover, \((P_n)\) is positive definite if and only if the coefficients of \(P_n, n \geq 0\), are real numbers, \((d_n)_{n\geq 0}\) is also a sequence of real numbers and \(g_n > 0, n \geq 1\).

**Corollary 1.5.** Let \((P_n)\) be a monic polynomial basis sequence; if \((P_n)\) satisfies \((7)\), then for all \(n \geq 0\), \(P_n\) and \(P_{n+1}\) are coprime: \(\gcd\{P_n, P_{n+1}\} = 1\).

**Proof.** Suppose \(a \in \mathbb{C}\) and

\[
P_n(a) = 0 = P_{n+1}(a) \implies g_nP_{n-1}(a) = 0 \implies P_{-1}(a)
\]

\[= 0 \ldots \implies g_1P_0(a) = 0, \text{ contradiction.} \]

**Remark 1.6.** We use the following definitions and notations related to the divisibility: for \(\phi, \psi \in \mathbb{P}\)

\[
\phi \nmid \psi \equiv \phi \text{ and } \psi \text{ are coprime} \iff \gcd(\phi, \psi) = c, c \in \mathbb{C} \setminus \{0\}; \not\nmid \text{ for the negation.} \]

\[
\psi \mid \phi \equiv \psi \text{ divides } \phi \iff \gcd(\phi, \psi) = \psi; \not\mid \text{ for the negation.} \]

In the vector space \(\mathbb{P}^*\), we define the following operators: the derivative of a functional and the polynomial or \(n\)-modification of a functional \(u, \pi u\).
Definition 1.7. The functionals $\pi u$ with $\pi \in \mathbb{P}$, and $D u$ act in the following way over all $\rho \in \mathbb{P}$

\[(8) \quad \langle \pi u, \rho \rangle = \langle u, \pi \cdot \rho \rangle, \quad \langle D u, \rho \rangle = -\langle u, D \rho \rangle,\]

where $D$ is the derivative operator.

1.2. The lattice of the distributional equations of a functional

Here is the definition of semi-classical functional/OPS done by Maroni [6]

Definition 1.8. Let $u \in \mathbb{P}'$ be a quasi-definite functional. We say that $u$ and the corresponding OPS are semi-classical if and only if $u$ satisfies the equation

\[(9) \quad D(\pi b x) = \pi b x, \quad \deg \phi \geq 0, \quad \deg \psi \geq 1,\]

where $x$ is a functional variable and $\phi$ and $\psi$ polynomials.

Remark 1.9. The classical functionals/OPS—Hermite, Laguerre, Jacobi, Bessel—can be characterized by the polynomials $\phi$ and $\psi$ of the Definition 1.8: $\deg \phi \leq 2$ and $\deg \psi = 1$. Moreover the classical OPS satisfies the Sturm-Liouville equation $\phi y'' + \psi y' = \lambda_n y$, with $\lambda_n \neq 0$, $n > 0$. In [4, 7], the classical functionals/OPS are studied starting from the distributional equation.

The next definitions and propositions are related with the equation and are independent of the quasi-definite character of the functional.

Remark 1.10. The following functional equations are equivalent: $D(\phi x) = \psi x \iff \phi Dx = \xi x$, with $\xi = \psi + D\phi \iff \psi = \xi - D\phi$. Notice that $\langle D(\phi u), \rho \rangle = -\langle u, (\phi D\rho) - \rho D\phi \rangle$.

Definition 1.11. Let $D(\phi x) = \psi x$, $\phi, \psi \in \mathbb{P}$, be an equation of a variable $x \in \mathbb{P}'$; we say that the order of the polynomial coefficients—and also the order of the equation with these coefficients—is: $\operatorname{ord}(\phi, \psi) = \max\{\deg \phi - 2, \deg \psi - 1\}$.

Remark 1.12. We can also define the order of the equation $\phi Dx = \xi x$ in the same way, $\operatorname{ord}(\phi, \xi) = \max\{\deg \phi - 2, \deg \xi - 1\}$, and so $\operatorname{ord}(\phi, \psi) = \operatorname{ord}(\phi, \xi)$.

Proposition 1.13. Let $u \in \mathbb{P}'$. If $u$ satisfies $D(\phi x) = \psi x$, $\operatorname{ord}(\phi, \psi) = s$, then $u$ also satisfies $D(\pi \phi \cdot x) = (\pi \psi + \phi D\pi)x$, $\forall \pi \in \mathbb{P}$, $\operatorname{ord}(\pi \phi, \pi \psi + \phi D\pi) = s + \deg \pi$.

Proof. $\phi D u = (\psi + D \phi)u \iff \pi \phi Du = (\pi \psi + \pi D\phi)u \iff D(\pi \phi u) = (\pi \psi + \pi D\phi - D(\pi \phi)) \iff D(\pi \phi u) = (\pi \psi - \phi D\pi)u$. Concerning the order: $\deg \phi = s + 2$ yields $\deg \pi \phi = \deg \pi + s + 2$ and $\deg(\pi \psi + \phi D\pi) \leq \deg \pi + s + 1$; and $\deg \phi < s + 2$ yields $\deg \pi \phi < \deg \pi + s + 2$ and $\deg(\pi \psi + \phi D\pi) = \deg \pi + s + 1$. In both cases the order of the new equation is $s + \deg \pi$. \[\square\]
Definition 1.14. Let \( u \in \mathbb{P} \) be a functional that satisfies a functional equation (1.8). We represent the set of all polynomial pairs corresponding to equations that \( u \) satisfies:

\[
\mathbb{P}^2_u = \{ (\phi, \psi) \in \mathbb{P}^2 : D(\phi u) = \psi u \},
\]

and we say that \( \mathbb{P}^2_u \) is the set of all compatible polynomial pairs with \( u \). We assume an equivalence relation in \( \mathbb{P}^2_u \) so that \( (\phi, \psi) \sim (c\phi, c\psi), \forall c \in \mathbb{C} \setminus \{0\} \). We define in \( \mathbb{P}^2_u \) the partial order

\[
(\phi, \psi) \prec (\tilde{\phi}, \tilde{\psi}) \iff \exists \pi \in \mathbb{P}, \deg \pi > 0, \quad \tilde{\phi} = \pi \phi, \quad \tilde{\psi} = \pi \psi - \phi D\pi,
\]

and we call the pair \((\tilde{\phi}, \tilde{\psi})\) a descendant of \((\phi, \psi)\), or more precisely, the descendant via \( \pi \); on the other hand \((\phi, \psi)\) is an ancestor of \((\tilde{\phi}, \tilde{\psi})\).

Theorem 1.15. [7] Let \( u \in \mathbb{P} \); if \((\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathbb{P}^2_u \) then for \( \phi = \gcd\{\phi_1, \phi_2\} \) there exists a \( \psi \in \mathbb{P} \) such that \((\phi, \psi) \in \mathbb{P}^2_u, \ord(\phi, \psi) \leq \min\{\ord(\phi_1, \psi_1), \ord(\phi_2, \psi_2)\}\).

Corollary 1.16. The lattice \((\mathbb{P}^2_u, \prec)\) has a minimum element – the minimum pair compatible with \( u \), \((\phi, \psi)\) – and \( \ord(\phi, \psi) < \ord(\phi, \psi) \) for all \((\phi, \psi) \in \mathbb{P}^2_u \), provided that \((\phi, \psi) \) is not equivalent to \((\tilde{\phi}, \tilde{\psi})\). All pairs belonging to \( \mathbb{P}^2_u \) are descendants of the minimum pair.

Corollary 1.17. Let \( u \in \mathbb{P}^* \) with \((\phi, \psi) \in \mathbb{P}^2_u, \ord(\phi, \psi) = s, i.e., u satisfies the order s equation \( D(\phi x) = \psi x \), or equivalently, \( \phi Dx = \xi x, \xi = \psi - D\phi \). Then \( u \) admits a lower order equation if and only if

(a) \( \phi \) and \( \xi \) have common zeros, \( \phi \not\parallel \xi \): \( \phi = \tau \cdot \phi_\tau, \xi = \tau \cdot \xi_\tau, \phi_\tau, \xi_\tau, \tau \in \mathbb{P}, \deg \tau > 0; \)

(b) \( u \) satisfies the lower order equation \( \tau \cdot \phi, Du = \tau \cdot \xi, u = \phi_\tauDu = \xi_\tau u \), or equivalently, \( D(\phi_\tau u) = (\xi_\tau + D\phi_\tau)u \). The order of the new equation is \( s - \deg \tau \).

Definition 1.18. Let \( u \in \mathbb{P}^* \) be a semi-classical functional; we define the class of \( u \), \( \text{cl } u \), as the order of the minimum pair of \( u \).

Remark 1.19. All semi-classical functionals have associated an unique non negative class number. A classical functional/OPS is a semi-classical functional/OPS of class 0.

The distributional equation of a functional leads us to the distributional equation of certain polynomial modifications of it, the standard modifications mentioned in the introduction.

Corollary 1.20. Let \( u \in \mathbb{P}^* \) and \((\phi, \psi) \in \mathbb{P}^2_u, \ord(\phi, \psi) = s \); then \( v := \phi u \) satisfies the order s equation of polynomials \( \phi \) and \( \psi + D\phi \), i.e., \((\phi, \psi + D\phi) \in \mathbb{P}^2_u \), \( \ord(\phi, \psi + D\phi) = s \). In general, for \( v^{(k)} = \phi^k u \) we have \((\phi, \psi + kD\phi) \in \mathbb{P}^2_{(k)} \), \( \ord(\phi, \psi + kD\phi) = s \).
Proof. From Proposition 1.13 with $\pi = \phi$. It is straightforward to see that $\text{ord}(\phi, \psi + D\phi) = \text{ord}(\phi, \psi)$. Analogously for $v$ we obtain the equation for $v^{(2)} = \phi v = \phi^2 u$ and so on. □

1.3. Distributional equation and difference equation

The distributional equation of order $s$ of a normalized functional can be interpreted as a $s+2$ order linear difference equation for the moments of the functional $u_n = \langle u, x^n \rangle$ depending from the initial moments $u_1, \ldots, u_s \mid u_{-1} = 0, u_0 = 1$.

Proposition 1.21. Let $\phi, \psi \in \mathbb{P}$, if $\text{ord}(\phi, \psi) = s$, i.e., $\phi = \sum_{i=0}^{s+2} a_it^i$ and $\psi = \sum_{i=0}^{s+1} b_ix^i$ with $a_{s+2} \neq 0$ or $b_{s+1} \neq 0$, then the distributional equation $D(\phi) = \psi x$ and the difference equation: $(na_{s+2} + b_{s+1})x_{n+s+1} + (nb_{s+1} + a_s)x_n + \ldots + (na_1 + b_0)x_1 + n\alpha_0 x_{n-1} = 0$, $n \geq 0$, $x_{-1} = 0$, are equivalent, where $x_n := \langle x, x^n \rangle$ denote the positions of the moments of a functional that satisfies the distributional equation $D(\phi) = \psi x$.

Proof. It follows from the fact that $\langle D(\phi), x^n \rangle = \langle \psi x, x^n \rangle \iff \langle x, nx^{n-1}\phi + x^n \psi \rangle = 0$. □

Definition 1.22. We say that a polynomial pair $(\phi, \psi)$, or the corresponding distributional equation $D(\phi) = \psi x$, of order $s$ with $\phi = \sum_{i=0}^{s+2} a_it^i$ and $\psi = \sum_{i=0}^{s+1} b_ix^i$, $a_{s+2} \neq 0$ or $b_{s+1} \neq 0$, has a $n_0$-singularity if there exists an $n_0 \in \mathbb{N}$ such that $n_0a_{s+2} + b_{s+1} = 0$. Otherwise the pair or the equation is said to be regular.

Remark 1.23. Increasing the order of the distributional equation multiplying their polynomials by $\pi \in \mathbb{P}, \deg \pi \leq n_0$, (Proposition 1.13) the new equation has a $(n_0 - \deg \pi)$-singularity, and for $\deg \pi > n_0$, the singularity disappears.

Definition 1.24. We say that a functional is singular, or more precisely, that it has a $n_0$-singularity, if and only if the minimum pair has a $n_0$-singularity. Otherwise we say that the functional is regular.

Proposition 1.25. If $\phi(x) := a_2x^2 + a_1x + a$ and $\psi(x) := b_1x + b_0$ are the polynomials of the equations of Theorem 1.9 of a classical functional/OPS, $(P_n) = \text{mops} u$, then $(Q_n) = \frac{1}{n+1}DP_{n+1}$ is the monic OPS of $v = \phi u$, i.e., $\langle v, Q_nQ_m \rangle = k_n\delta_{nm}$, $k_n \neq 0, n \geq 0$ and the regularity condition holds: $na_2 + b_1 \neq 0, \forall n \geq 0$.

Proof. See the proof of Proposition 1.34 with $s = 0$. □

Remark 1.26. In the classical case, the distributional/difference equation does not have any singularity: all the classical functionals are regular functionals. The dif-
ference equation corresponding to the classical distributional equation of a classical functional $u$. $D(\phi x) = \psi x$, with $\phi(x) = a_2 x^2 + a_1 x + a_0$, $\psi(x) = b_1 x + b_0, b_1 \neq 0$,

$$(10) \quad (na_2 + b_1)x_{n+2} + (na_1 + b_0)x_{n+1} + x_n = 0, \quad x_{-1} = 0,$$

depends from the moment in position $x_0$. If we normalize the functional, $u_0 = 1$, only the functionals $u$ and $0$ satisfy the equation $D(\phi x) = \psi x$ and we can associate an unique non-zero functional to the polynomial pair $(\phi, \psi)$ of class $0$. Proposition 1.20 says that $v$ satisfies also the distributional equation of order $0$ with $\phi$ and $\psi + D\phi$, $(\phi, \psi + D\phi) \in \mathbb{P}^2_u$; the derivatives of a classical OPS are classical, and so on.

**Corollary 1.27.** Let $u \in \mathbb{P}^*$ be a classical functional with $(\phi, \psi) \in \mathbb{P}^2_u$, ord$(\phi, \psi) = 0$, and let $(P_n) = \text{mops} u$ and $(Q^{(k)}_n)$ the monic sequence of $k$-order derivatives, $Q^{(k)}_n = \frac{1}{(n+k)!} D^k P_{n+k}$; then $(Q^{(k)}_n)$ constitutes a classical OPS: $(Q^{(k)}_n) = \text{mops} v^{(k)}$, $v^{(k)} = \phi^k u$ and $(\phi, \psi + kD\phi) \in \mathbb{P}^2_{v^{(k)}}$.

This result can be obtained by successive derivation of the Sturm-Liouville equation, $\phi^2 P_n + \psi^2 P_n = \lambda_n P_n$, but this procedure is not applicable to semi-classical OPS/functionals.

**Proposition 1.28.** Let $u \in \mathbb{P}^*$ be a classical functional with $(\phi, \psi) \in \mathbb{P}^2_u$, ord$(\phi, \psi) = 0$; then $\phi$ and $\psi + kD\phi$ are coprime for $k \geq 0$, coprimality condition.

**Proof.** For $k = 0$: if $\psi \not\equiv \phi^{\deg \psi = 1}$, $\psi | \phi \implies \phi = \psi \varphi$, $\varphi \in \mathbb{P}$, $\deg \varphi > 0 \implies S - L \text{eq.}$

$$\psi(\varphi^2 P_n + \psi^2 P_n) = \lambda_n P_n, \quad n \geq 1 \implies \psi | P_n, \quad n \geq 1 \implies \text{(P_n) is not a OPS.}$$

Similarly for $k > 0$. □

**Remark 1.29.** The regularity and the coprimality conditions are both sufficient conditions for the quasi-definite character of the non-zero functional that satisfies the distributional equation.

In this frame we need the following generalization of the orthogonality for the derivatives of a semi-classical OPS.

**Definition 1.30.** Let $(R_n)$ be a basis sequence of polynomials and $v$ a functional; we say that $(R_n)$ is strictly quasi-orthogonal of order $s \geq 0$ with respect to the functional $v$ if and only if

(a) $\langle v, R_n R_m \rangle = 0$ for all $m, n \geq 0$ such that $|n - m| > s$ and

(b) $\langle v, R_m R_n \rangle \neq 0$ for all $m, n \geq 0$ such that $|n - m| = s$.

**Remark 1.31.** The definition of quasi-orthogonality [6], QO in short, of order $s$ weakens the condition (b) of the strict QO. It requires only that there exists at least an index $n_0$ such that $\langle v, R_{n_0} R_{n_0+s} \rangle \neq 0$. We shall need in Section 2 a not so weak
We admit that the condition (b) of the strict QO does not hold for a finite number of indices.

**Definition 1.32.** Let \((R_n)\) be a basis sequence of polynomials and \(v\) a functional; we say that \((R_n)\) is strictly quasi-orthogonal of order \(s \geq 0\) with respect to the functional \(v\) with singularities at \(n_i, 1 \leq i \leq k\), if and only if condition (a) holds, and for

(b) \(\langle v, R_m R_n \rangle \neq 0\) for all \(m, n \geq 0\) such that \(|n - m| = s\), and \(m, n \neq n_i, 1 \leq i \leq k\), and \(\langle v, R_{n_i} R_{n_i+s} \rangle = 0, 1 \leq i \leq k\). This is said a \(n_i\)-singularity in the strict QO.

The following criterion is easy to check.

**Criterion 1.33.** Let \((R_n)\) be a basis sequence of polynomials and \(v\) a functional; the QO conditions of order \(s\) with respect to \(v\), Definition 1.30, are equivalent to these

(a) \(\leftrightarrow \langle v, x^n R_n \rangle = 0\) for all \(m, n \geq 0\) such that \(|n - m| > s\) and

(b) \(\leftrightarrow \langle u, x^n R_{n+s} \rangle \neq 0\) for all \(n \geq 0\) in the strict QO, and for \(n \neq n_i, 1 \leq i \leq k\), in the strict QO with singularities, where \(\langle v, x_n R_{n_i+s} \rangle = 0, 1 \leq i \leq k\).

**Proposition 1.34.** Let \(u \in \mathbb{P}_s^*\) be a semi-classical functional with a regular pair \((\phi, \psi) \in \mathbb{P}_t^2, \text{ord}(\phi, \psi) = s\) and let \((P_n) = \text{mops} u\); if \((Q_n)\) is the monic sequence \(Q_n = \frac{1}{n+1} D P_{n+1}\), then \((Q_n)\) is strictly QO of order \(s\) with respect to the functional \(\phi u\).

**Proof.** We use the criterion

\[
\langle \phi u, x^n Q_n \rangle = \frac{1}{n+1} \langle \phi u, x^n D P_{n+1} \rangle
= \frac{1}{n+1} \langle \phi u, D(x^n P_{n+1}) - P_{n+1} D x^n \rangle
= \frac{-1}{n+1} \langle (D(\phi u), x^n P_{n+1}) + \langle u, m x^{n-1} P_{n+1} \rangle \rangle
= \frac{-1}{n+1} \langle (u, x^n \psi \cdot P_{n+1}) + m \langle u, x^{n-1} \psi \phi \cdot P_{n+1} \rangle \rangle.
\]

If \(\text{ord}(\phi, \psi) = s\), then \(\text{deg} \phi \leq s + 2, \phi = \sum_{i=0}^{s+2} a_i x^i, \text{deg} \psi \leq s + 1, \psi = \sum_{i=0}^{s+1} b_i x^i, \) and \(\langle \phi u, x^n Q_n \rangle = 0\) for \(n - m > s\). Condition (a) holds. In the other hand, if \(m = n - s\)

\[
\langle \phi u, x^{n-s} Q_n \rangle = \frac{-1}{n+1} \langle (u, b_{s+1} x^{n+1} P_{n+1}) + \langle u, a_{s+2} x^{n+1} P_{n+1} \rangle \rangle
= \frac{-1}{n+1} \langle (n - s) a_{s+2} + b_{s+2} \rangle \langle u, x^{n+1} P_{n+1} \rangle,
\]

and condition (b) holds if the functional is regular, i.e., \((n - s) a_{s+2} + b_{s+2} \neq 0\).

**Remark 1.35.** It is easy to generalize this proposition for \((Q_n^{(k)})\) strictly QO of order \(ks\) with respect to \(v^{(k)} := \phi^k u\), that satisfies the order \(s\) equation of polynomials \((\phi, \psi + kD \phi)\). In Proposition 2.2 we study the case of singular pairs.
In [9] we have built a family of semi-classical functionals with singularities in 
\( n_0 = 1, 3, 5, \ldots \). So our assumption is that there exist semi-classical functionals 
with \( n_0 \)-singularities, \( n_0 \geq 0 \), for any class \( s > 0 \). The \( n_0 \)-singularity of the 
distributional equation yields a \( n_0 \)-singularity in the strictly QO of the derivatives, 
see Remark 1.31. The corresponding distributional equation of the derivative 
functional has a \((n_0 - 1)\)-singularity, if \( n_0 > 0 \). The second derivatives would 
have a \((n_0 - 1)\)-singularity in the strictly QO but also a failure in \( n_0 \) corre-
sponding to the \( n_0 \)-singularity in the strictly QO of the first derivatives.

**Lemma 2.1.** Let \( v \in \mathbb{P}^s \), not necessary quasi-definite, with \( (\phi, \psi) \in \mathbb{P}_v^2 \), 
ord\((\phi, \psi) = s \), and \( n_0 \)-singular, let \( R_n \in \mathbb{P} \) be a basis sequence, i.e., 
\( \deg R_n = n, n \geq 0 \), and \( (O_n) \) the sequence of their monic derivatives, 
\( O_n = \frac{1}{n+1} DR_{n+1} \).

If \( (R_n) \) is strictly QO of order \( r \) with respect to \( v \) with singularities in \( \{n_1, \ldots, n_k\} \), 
then \( (O_n) \) is strictly QO of order \( r + s \) with respect to \( \phi v \) with singularities in \( n_0 \),
and in \( n'_i = n_i - (s + 1) \) for all \( n'_i \geq 0, 0 \leq i \leq k \).

\( (O_n) \) is also strictly QO related to \( r \pi, \pi \in \mathbb{P}, \deg \pi > 0 \), of order \( r + s + \deg \pi \)
with singularities in \( n_0 - \deg \pi \) if \( \deg \pi \leq n_0 \) and in \( n'_i = n_i - (s + 1 + \deg \pi) \) for 
all \( n'_i \geq 0, 0 \leq i \leq k \).

**Proof.** Without loss of generality we suppose that \( (R_n) \) is a monic sequence:

\[
\langle \phi v, x^m O_n \rangle = \frac{1}{n+1} \langle \phi v, x^m, D R_{n+1} \rangle - \frac{1}{n+1} \langle \phi v, D(x^m R_{n+1}) - R_{n+1} D x^m \rangle
\]

\[
= \langle \phi v, D(x^m R_{n+1}) + m(\phi v, x^m - 1 R_{n+1}) \rangle
\]

\[
= \langle \phi v, x^m R_{n+1} + m(\phi v, x^m - 1 R_{n+1}) \rangle.
\]

Both terms vanish if \( n + 1 - (m + s + 1) > r \iff n - m > r + s \). If \( n - m = r + s \)
and \( \phi = \sum_{k=0}^{s+1} a_k x^k, \psi = \sum_{k=0}^{s+1} b_k x^k \),

\[
\langle \phi v, x^m O_{(r+s)+n} \rangle = \frac{-1}{(r+s)+n+1} \left( \langle v, (x^m \psi) R_{(r+s)+n+1} \rangle + n(\psi, (x^{n-1} \phi) R_{(r+s)+n+1} \rangle \right)
\]

\[
= \frac{-b_{s+1} a_{s+1}^2}{(r+s)+n+1} \langle v, x^{s+n+1} R_{(r+s)+n+1} \rangle.
\]

\( (O_n) \) has a singularity with respect to \( \phi v \) as a consequence of

(a) the \( n_0 \)-singularity of \( (\phi, \psi) \) \( n_0 a_{s+2} + b_{s+1} = 0 \implies \langle \phi v, x^{n_0} O_{(r+s)+n} \rangle = 0 \);

(b) the singularities of the strictly QO of \( (R_n) \) with respect to \( v \),
\( \langle v, x^{n_i} R_{n_i} \rangle = 0, 0 \leq i \leq k \), if \( n_i \geq (s + 1) \):

\[
\langle \phi v, x^{n_i-(s+1)} O_{n_i+r-1} \rangle = \frac{(n_i-(s+1)) a_{s+2} + b_{s+1}}{(n_i+r)} \langle v, x^{n_i} R_{n_i+r} \rangle = 0, 0 \leq i \leq k.
\]
The singularities appear for \( (\phi, \psi), \phi^r = \pi \phi \) and \( \psi^r = \pi \psi + \phi \psi \) is a descendant pair from \((\phi, \psi)\) of order \(s + \deg \pi\) which also belongs to \( \mathbb{P}^2_0 \), according to Proposition 1.13. If \( \deg \pi \leq n_0 \) the descendant pair is singular in \( n'_0 = n_0 - \deg \pi \). In this case
\[
(11) \quad \langle \phi^r, \psi^m O_n \rangle = \frac{-1}{n+1} \left( \langle \psi, \left( x^m \psi^r \right) R_{n+1} \rangle + m \langle \psi, \left( x^{m-1} \phi^r \right) R_{n+1} \rangle \right).
\]
Both terms vanish if \( n - m > r + s + \deg \pi \). If \( n - m = r + s + \deg \pi \), \( (12) \)
\[
\langle \phi^r, \psi^r O_{(r+s+\deg \pi)+n} \rangle = - \frac{b_{r+1} + n a_{r+2}}{(r+s+\deg \pi)+n+1} \langle \psi, x^{r+n+1} R_{r+s+n+1} \rangle.
\]
vanishes for \( n'_0 \) and for \( n''_0 = n_i - (s + 1) - \deg \pi \geq 0, \ 0 \leq i \leq k \).

We use the distributional equation, \( D(\psi) = \psi x \) satisfied by a quasi-definite functional, \( u \), to prove the QO of the derivatives of the OPS of \( u \) related with the functional \( \phi u \). With the polynomial \( \phi \), see Proposition 1.13, we build a distributional equation for this functional \( \phi u \). So we can prove the QO of the second-order derivatives with respect to \( \phi^2 u \) with the help of Lemma 2.1. We apply again the Proposition 1.13 to the equation of \( \phi v \) and we obtain an equation satisfied by \( \phi^2 v \) that proves the QO of the third-order derivatives related to \( \phi^3 u \), and so on.

**Theorem 2.2.** Let \( u \in \mathbb{P}^*, \) be a quasi-definite functional with \((\phi, \psi) \in \mathbb{P}^2_0, \) \( \text{ord}(\phi, \psi) = s, \) and let \((P_n) = \text{mops} u \) with the \( k \)-order monic derivatives \( Q^{(k)}_n = \frac{1}{(n+1)_k} D^k P_{n+k}, \ k \geq 0; \)

(a) If \((\phi, \psi)\) does not have singularity, then \( (Q^{(k)}_n) \) is strictly QO of order \( ks \) with respect to \( \phi^k u \).

(b) If \((\phi, \psi)\) has a singularity in \( n_0 \geq 0 \) we have following cases depending from the indices
\[
k_0 = 1 + \left\lfloor \frac{n_0}{s+2} \right\rfloor, \quad k_1 = 1 + \left\lfloor \frac{n_0}{s+1} \right\rfloor.
\]

(α) If \( k \leq k_0 \), then \( (Q^{(k)}_n) \) is strictly QO of order \( ks \) with respect to \( \phi^k u \) with \( k \) singularities in:
\[
m_{kj} = n_0 - (k - 1)(s + 2) + j, \ 0 \leq j \leq k - 1;
\]
(β) If \( k_0 \neq k_1 \) and \( k_0 < k \leq k_1 \), then \( (Q^{(k)}_n) \) is strictly QO of order \( ks \) with respect to \( \phi^k u \) with \( n_0 - (k - 1)(s + 1) \) singularities:
\[
m_{kj} = n_0 - (k - 1)(s + 2) + j, \quad (k - 1)(s + 2) - n_0 \leq j \leq k - 1;
\]
(γ) If \( k > k_1 \), then \( (Q^{(k)}_n) \) is strictly QO of order \( ks \) with respect to \( \phi^k u \) without singularities.
Proof. The distributional equation, \( D(\phi x) = \psi x \) satisfied by a quasi-definite functional, \( u \), prove the QO of the derivatives of the OPS of \( u \) related with the functional \( v^{(1)} := \phi u \). Proposition 1.31 allows to build an equation for \( v^{(1)} \):

\[
D(\phi u) = \psi u \iff D(\phi(\phi u)) = (\psi + D\phi)\phi u
\]

\[
\iff D(\phi v^{(1)}) = \psi^{(1)}v^{(1)}, \quad \psi^{(1)} = \psi + D\phi,
\]

and so we can prove the QO of order \( s + s \) of the second order derivatives with the help of Lemma 2.1. We repeat this procedure with (13) and we obtain an equation satisfied by \( v^{(2)} := \phi v^{(1)} \):

\[
D(\phi v^{(1)}) = \psi^{(1)}v^{(1)} \iff D(\phi(\phi v^{(1)})) = (\psi^{(1)} + D\phi)\phi v^{(1)}
\]

\[
\iff D(\phi v^{(2)}) = \psi^{(2)}v^{(2)}, \quad \psi^{(2)} = (\psi^{(1)} + D\phi) = (\psi + 2D\phi).
\]

In this way we have a chain of functionals \( v^{(k)} := \phi v^{(k-1)} = \phi^{2} v^{(k-2)} = \ldots = \phi^{k-1} v^{(1)} = \phi^{k} u \), so that \( (Q_{k}) \), \( k \geq 1 \), is QO of order \( ks \) related to \( v^{(k)} \). We use following notation, \( \phi^{(k)} := \phi, \psi^{(k)} := \psi + kD\phi \), each of these functionals satisfies the distributional equation with \( (\phi^{(k)}, \psi^{(k)}) \), i.e., \( (\phi^{(k)}, \psi^{(k)}) \in \mathbb{P}_{s}^{2} \). The generation of the singularities follows the schema

\[
(\phi, \psi) \text{ singular in } n_{0} \quad \text{produces these singularities in the strictly QO of the sequence of derivatives of order}
\]

\[
(\phi^{(1)}, \psi^{(1)}) \text{ in } n_{0} - (s + 2) = n_{1,0}, \quad m_{1,0} := n_{0}, \quad m_{1,1} := n_{0} - (s + 1)
\]

\[
(Q_{1}^{(s)})
\]

\[
(\phi^{(2)}, \psi^{(2)}) \text{ in } n_{0} - 2(s + 2) = n_{2,0}, \quad m_{2,0} := n_{0}, \quad m_{2,1} := n_{0} - (s + 1), \quad m_{2,2} := m_{2,1} - (s + 1)
\]

\[
(Q_{2}^{(s)})
\]

\[
(\phi^{(k-1)}, \psi^{(k-1)}) \text{ in } n_{0} - (k - 1)(s + 2), \quad m_{k,0} := n_{0} - (k - 1)(s + 2), \ldots, \quad m_{k,j} = m_{k, j-1} - (s + 1), \quad 0 \leq j \leq k - 1
\]

\[
(Q_{k}^{(s)})
\]

The possible singularities in the strictly QO of \( (Q_{n}^{(k)}) \) with respect to \( v^{(k)} \) have two different origins \( m_{k,0} = n_{0} - (k - 1)(s + 2) \) and \( m_{k,j} = m_{k-1,j-1} - (s + 1) \), \( 1 \leq j \leq k - 1 \), but

\[
m_{k,j} = m_{k-1,j-1} - (s + 1) = \ldots = m_{k-j,0} - j(s + 1)
\]

\[
= n_{0} - (k - j - 1)(s + 2) - j(s + 1).
\]

With this representation we can summarize both types of possible singularities:

\[
m_{k,j} = n_{0} - (k - j - 1)(s + 2) - j(s + 1), \quad 0 \leq j \leq k - 1.
\]

Moreover

\[
m_{k,j} = n_{0} - (k - j - 1)(s + 2) - j(s + 1)
\]

\[
= n_{0} - (k - j)(s + 2) - (j - 1)(s + 1) + 1 = m_{k,j-1} + 1
\]

shows us that they are consecutive \( m_{k,j} = m_{k,0} + j \), \( 0 \leq j \leq k - 1 \). Furthermore we have \( k \) singularities if and only if

\[
m_{k,j} \geq 0 \iff m_{k,k-1} \geq 0 \iff k \leq 1 + \left\lfloor \frac{n_{0}}{s + 2} \right\rfloor = k_{0} \iff (\alpha),
\]

and do not exist singularities if and only if

\[
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\]
If \( k_0 < k < k_1 \) then \( m_{kj} \geq 0 \iff m_{kj} + j \geq 0 \iff j \geq -m_{kj} = -n_0 + (k - 1)(s + 2) \) and they are singularities for \( m_{kj} \), \( j_0 \leq j \leq k - 1 \) (\( \beta \)). The number of singularities is \((k - 1) - j_0 = (k - 1) - (k - 1)(s + 2) + n_0 = n_0 - (k - 1)(s + 1)\). □

**Remark 2.3.** We have used always the same polynomial \( \phi \) to built the new equations. We can also use multiples of this polynomial \( \varphi = \pi \phi \), \( \pi \in \mathbb{P} \). In this case the resulting equation is an equation of higher order for \( \psi \)

\[
D(\phi u) = \psi u \implies D(\varphi u) = (\varphi \psi + \phi D\varphi)u
\]

(14)

\[
\iff D(\varphi(\phi u)) = (\pi \psi + D\pi)(\phi u)
\]

\[
\iff D(\varphi \psi) = \varsigma \psi, \quad \varsigma = \pi \psi + D\pi.
\]

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\]

\[
\iff D(\varphi \psi) = \varsigma \psi, \quad \varsigma = \pi \psi + D\pi.
\]

We can obtain the second pair \((\varphi, \varsigma)\) from the first, \((\phi, \psi + D\phi)\), multiplying the equation by \( \pi \):  

\[
\pi(\psi + D\phi) + \phi D\pi = \pi \psi + D(\phi \pi) \quad (14) \Rightarrow \varsigma.
\]

*Remark 2.3.* We have used always the same polynomial \( \phi \) to built the new equations. We can also use multiples of this polynomial \( \varphi = \pi \phi \), \( \pi \in \mathbb{P} \). In this case the resulting equation is an equation of higher order for \( \varphi \)

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(14)

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\]

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\]

\[
\iff D(\varphi \psi) = \varsigma \psi, \quad \varsigma = \pi \psi + D\pi.
\]

![Figure 1. The singularities of the k-order derivatives of a singular semi-classical functional of class 1 with a singularity in n_0 = 11 related to the standard modifications with the minimal polynomial.](image)
The second part of Lemma 2.1 shows that the singularities of the QO, if there exist, follows the same schema as for $\phi$ adding $-\deg \pi$ to $k_0$ and $k_1$, i.e., they are displaced $\deg \pi$ units to the left.

**Remark 2.4.** In order to find the lowest order of quasi-orthogonality of the $k$-order derivatives ($Q^{(k)}$) and the corresponding functional, we can take the minimum order equation of $u$ with the pair $(\phi, \psi)$ and build the new equations with the polynomial $\phi$. Proceeding in this way the minimum order of $QO$ of $(Q^{(k)})$ is $k$, related to the functional $\phi^k u$. Applying this procedure starting with an equation of $u$ of polynomials $(\phi, \psi)$, such that they are descendants of $(\phi, \psi)$, i.e., $\phi = \pi \phi$, $\psi = \pi \phi + \psi D\pi$, and modifying it with polynomials in the form $\pi \phi$, $\deg \pi \geq 0$, last we obtain an equation for a functional that is a polynomial modification of $\phi^k u$. All functionals obtained in this way are polynomial modifications of it. Proposition 2.2 says that the $QO$ of the $k$-order derivatives are strictly with respect to these functionals if $u$ is regular, and that can appear a finite number of consecutive singularities if the functional $u$ is singular, see Remark 2.3.

Figure 1 shows the behaviour of the $QO$ of the derivatives of the quasi-definite functional $w$, that satisfies the equation $D(x^3 x) = (-11x^2 + 4)x$ (see [9], Proposition 2.7). This equation does not admit a lower order equation, see Corollary 1.17. Thus, the pair $(x^3, -11x^2 + 4)$ is the minimum pair. The functional/OPS is of class 1, $s = 1$ with a singularity in $n_0 = 11$. With respect to the modified functionals $x^3 w$, the $k$-order derivative sequences are strictly $QO$ with singularity in 11 for $k = 1$, in 8 and 9 for $k = 2$, and so on.

3. QUASI-ORTHOGONALITY OF THE $k$-ORDER DERIVATIVES

The next question that arises is about the functionals with respect to which the $k$-order derivatives of an OPS are quasi-orthogonal, say $w$. We shall present the problem in three steps:

(i) $w$ is a polynomial modification of $u$: $w = \pi u$.

(ii) $\pi$ can be decomposed in a product of $k$ polynomials: $\pi = \phi_k \ldots \phi_1$.

(iii) These polynomials are multiples of the minimum polynomial $\phi$, i.e., $\phi_i = \rho_i \phi$, $\rho_i \in \mathbb{P}$, $i = 1, \ldots, k$.

Step (iii) is still open. Our assumption is that this is not necessarily so. A consequence of this assumption is that the coprime condition does not hold for semi-classical functional/OPS.

We answer question (i) for an OPS. First, we give the $QO$ criterion for an OPS.

**Criterion 3.1.** Let $(P_n)$ be an OPS and $w$ a functional. The $QO$ conditions of order $s$ for $(P_n)$ with respect to $w$, see Criterion 1.33, are equivalent to the following:

(a) $\langle w, x^m P_n \rangle = 0$ if $n-m > s$ $\iff$ $\langle w, P_n \rangle = 0$ if $n > s$,

(b) $\langle w, x^n P_{n+s} \rangle \neq 0$ if $n \geq 0$ $\iff$ $\langle w, P_s \rangle \neq 0$.
Proof. From the TTRR we get
\[ x^m P_n = x^{m-1}(xP_n) = x^{m-1}(P_{n+1} + b_n P_n + g_n P_{n-1}) \]
\[ = x^{m-2}(xP_{n+1} + d_n xP_n + g_n xP_{n-1}) = \ldots = \sum_{i=n-m}^{n+m} a_{ni} P_i, \quad a_{n,n-m} = g_n g_{n-1} \ldots g_{n-m+1} \neq 0, \quad n > m > 0. \]
But \( \langle w, x^m P_n \rangle = \sum_{i=n-m}^{n+m} a_{ni} \langle w, P_i \rangle, \quad a_{n,n-m} \neq 0 \), yields
\[ \langle w, P_n \rangle = 0, \quad n > s \implies \langle w, x^m P_n \rangle = 0, \quad n - m > s; \]
\[ \langle w, P_n \rangle = 0, \quad n > s \quad \text{and} \quad \langle w, P_s \rangle \neq 0 \implies \langle w, x^s P_{n+s} \rangle = a_{n+s,n} \langle w, P_s \rangle \neq 0, \quad n \geq 0. \]
The converse result is straightforward. \( \square \)

Remark 3.2. The QO of an OPS is always strict.

Proposition 3.3. [6] Let \( u \in \mathbb{P}^* \) be quasi-definite, \( (P_n) \) the corresponding monic
OPS, and \( w \) another functional; then \( \langle w, P_n \rangle = 0 \) for \( n > s \iff \exists \phi \in \mathbb{P}, \deg \phi \leq s \)
such that \( w = \phi u \). Moreover, \( \langle w, P_s \rangle \neq 0 \iff \deg \phi = s \).

Now we answer the questions (i) and (ii) for the derivatives of an OPS.

Lemma 3.4. Let \( u \in \mathbb{P}^* \) be a quasi-definite functional and \( (P_n) \) the corresponding
monic OPS.

If \( Q_n^{(j)} = \frac{1}{(n+1)_j} D^j P_{n+j}, \quad 0 \leq j \leq k \), then \( Q_n^{(j)} = \sum_{i=0}^{2(k-j)} \alpha_{n-i,j}^{(k)} Q_n^{(k)} \)
where \( Q_n^{(k)} = 0 \) if \( i > n \) and \( \alpha_{n-i,j}^{(k)} \) are polynomials of \( \deg \alpha_{n-i}^{(k)} \leq \min(i, 2(k-j) - i) \).

Proof. We apply the \( k \)-derivative to the TTRR:
\[ D^k(xP_m) = D^k P_{m+1} + d_m D^k P_m + g_m D^k P_{m-1}, \quad k \leq m + 1. \]
From the Leibniz rule \( D^k(xP_m) = x D^k P_m + k D^k P_{m+1} \), and thus we can re-
present \( D^k P_m \) as a polynomial combination of \( D^k P_{m+1}, D^k P_m \) and \( D^k P_{m-1} \):
\[ k D^k P_m = D^k P_{m+1} + (d_m - x) D^k P_m + g_m D^k P_{m-1}. \]
For the monic \( k \)-derivatives, if \( n = m - k + 1 \), we get
\[ Q_n^{(k-1)} = a_{n+k-1}^{k-1} \frac{n+k-1}{k} (d_{n+k-1} - x) Q_{n-1}^{(k)} + g_{n+k-1} \frac{n(n-1)}{k(n+k-1)} Q_{n-2}^{(k)}. \]
Concerning the degrees of the polynomial coefficients, with the notation of this
Lemma, we get \( \deg \alpha_{n,n-1}^{(k)} = 0 \), \( \deg \alpha_{n,n-1}^{(k-1)} = 1 \), \( \deg \alpha_{n,n-2}^{(k-1)} = 0 \). In the same
way we have
\[ Q_n^{(k-2)} = a_{n,n}^{k-2} \frac{k-2}{2} Q_n^{(k-1)} + \alpha_{n,n-1}^{k-2} \frac{k-2}{2} Q_{n-1}^{(k-1)} + \alpha_{n,n-2}^{k-2} Q_{n-2}^{(k-1)}. \]
Substituting the \( (k-1) \)-derivatives through the \( k \)-derivatives
\[ Q_n^{(k-2)} = \alpha_{n,n}^{k-2,k} Q_n^{(k)} + \alpha_{n,n-1}^{k-2,k} Q_{n-1}^{(k)} + \alpha_{n,n-2}^{k-2,k} Q_{n-2}^{(k)} + \alpha_{n,n-3}^{k-2,k} Q_{n-3}^{(k)} + \alpha_{n,n-4}^{k-2,k} Q_{n-4}^{(k)} \]

and \( \deg \alpha_{n,n}^{k-2,k} = 0 \leq \deg \alpha_{n,n-1}^{k-2,k} \leq 1 \leq \deg \alpha_{n,n-3}^{k-2,k} \leq 2. \) Continuing in this way we obtain the lemma. \( \square \)

**Proposition 3.5.** Let \( u \in \mathbb{P}^* \) be quasi-definite, \((P_n)\) the corresponding monic OPS, and \((Q_n^{(j)})\) the monic sequence of \( j \)-order derivatives of \((P_n)\), i.e.: \( Q_n^{(j)} = \frac{1}{(n+j)!} D^j P_{n+j} \); if \((Q_n^{(k)})\) is QO of order \( s \) with respect to \( w \in \mathbb{P}^* \), then \((Q_n^{(j)})\) is QO of order \( s^{(j)} \), \( 0 \leq s^{(j)} \leq s + 2(k-j) \) related to \( w \) for \( 0 \leq j \leq k \).

**Proof.** We study the QO of \((Q_n^{(j)})\) with the help of the QO of \((Q_n^{(k)}) \) with respect to \( w \):

\[ \langle w, x^m Q_n^{(j)} \rangle = \langle w, x^m \sum_{i=0}^{2(k-j)} \alpha_{n,i}^{j,k} Q_n^{(i)} \rangle = \sum_{i=0}^{2(k-j)} \langle w, x^m \alpha_{n,i}^{j,k} \cdot Q_n^{(i)} \rangle, \]

We have the following bounds for the degrees:

\[ \deg \alpha_{n,n-i}^{j,k} \leq i, \quad 0 \leq i \leq k-j, \]

\[ \deg \alpha_{n,n-i}^{j,k} \leq 2(k-j) - i, \quad k-j \leq i \leq 2(k-j), \]

The terms \( \langle w, x^m \alpha_{n,n-i}^{j,k} \cdot Q_n^{(i)} \rangle \) vanish

(a) \( 0 \leq i \leq k-j, \quad \text{if} \quad n-m > s + 2i \)

(b) \( k-j \leq i \leq 2(k-j), \quad \text{if} \quad n-m > s + 2(k-j) \)

On the other hand \((Q_n^{(j)}) \), \( 0 \leq j \leq k \), are basis sequences of \( \mathbb{P} \). Thus, \( \langle w, Q_n^{(j)} \rangle = 0, n \geq 0 \) yields \( w = 0 \). This is not true and there exists an index \( s_0 \geq 0 \) so that \( \langle w, Q_n^{(j)} \rangle \neq 0 \). Thus \( 0 \leq s_0 \leq s^{(j)} \). \( \square \)

**Remark 3.6.** Our purpose is to build a chain of polynomial modifications from \( u \) to \( w \) such that

\[
\begin{array}{cccccccc}
\text{u} & \varphi_1 & w_1 & \varphi_2 & w_2 & \ldots & \varphi_{k-1} & w_{k-1} & \varphi_k & w \\
(P_n) & \downarrow \text{OR} & \downarrow \text{QO} & \downarrow \text{QO} & \ldots & \downarrow \text{QO} & \downarrow \text{QO} & \downarrow \text{QO} & (P_n) \\
\end{array}
\]

We search for a polynomial \( \phi \) that transforms \( u \) in \( w \), \( w = \phi u \), and that can be factorized, \( \phi = \varphi_k \ldots \varphi_2 \varphi_1 \), in such a way that \((Q_n^{(j)})\) is QO related to \( w_j = \varphi_j \ldots \varphi_1 u \).

**Corollary 3.7.** With the conditions of the aforementioned proposition:
(a) \((P_n)\) is QO of order \(s_j\), \(0 \leq s_j \leq s + 2k - j\) related to \(D^j w\), \(0 \leq j \leq k\);
(b) there exist \(\phi_j \in \mathbb{P}\), \(0 < \deg \phi_j \leq s + 2k - j\) such that \(D^j w = \phi_j u\), \(0 \leq j \leq k - 1\);
(c) there exist \(k\) different distributional equations for \(u\) of pairs \((\phi_{j-1}, \phi_j)\), \(1 \leq j \leq k\) of order: \(\text{ord}(\phi_{j-1}, \phi_j) \leq s + 2k - j - 1\).

**Proof.** Transposing derivatives from \(\mathbb{P}\) to \(\mathbb{P}^*\):
\[
(w, Q_n^{(j)}) = 0, \; n > s + 2(k - j) \iff \frac{(-1)^j}{(n+j)} (D^j w, P_{n+j}) = 0, \; n > s + 2(k - j),
\]
and thus the order of strictly QO must be \(s_j \leq s + 2(k - j) + j = s + 2k - j\).
From Proposition 3.3
\[
\exists \phi_j \in \mathbb{P}, \; D^j w = \phi_j u, \; \deg \phi_j \leq s + 2k - j.
\]
On the other hand \(\deg \phi_j \geq 0, \; 0 \leq j \leq k: \deg \phi_j \leq 0 \implies w_j = 0\).
We can built \(k\) different distributional equations for \(u\):
\[
D^j w = \phi_j u \iff D(D^{j-1} w) = \phi_j u \iff D(\phi_{j-1} u) = \phi_j u, \; 1 \leq j \leq k,
\]
of order: \(\text{ord}(\phi_{j-1}, \phi_j) = \max\{\deg \phi_{j-1} - 2, \deg \phi_j - 1\} \leq s + 2k - j - 1\). □

<table>
<thead>
<tr>
<th>((Q_n^{(k)}))</th>
<th>(w)</th>
<th>(Dw)</th>
<th>(D^2 w)</th>
<th>(\ldots)</th>
<th>(D^{k-2} w)</th>
<th>(D^{k-1} w)</th>
<th>(D^k w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((Q_n^{(k-1)}))</td>
<td>(s)</td>
<td>(s)</td>
<td>(s + 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((Q_n^{(k-2)}))</td>
<td>(s + 2)</td>
<td>(s + 3)</td>
<td>(s + 2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((Q_n^{(k-3)}))</td>
<td>(s + 2k - 4)</td>
<td>(s + 2k - 5)</td>
<td>(s + 2k - 6)</td>
<td>(s + k - 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((Q_n))</td>
<td>(s + 2k - 2)</td>
<td>(s + 2k - 3)</td>
<td>(s + 2k - 4)</td>
<td>(s + k)</td>
<td>(s + k - 1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((P_n))</td>
<td>(s + 2k)</td>
<td>(s + 2k - 1)</td>
<td>(s + 2k - 2)</td>
<td>(s + k + 2)</td>
<td>(s + k + 1)</td>
<td>(s + k)</td>
<td></td>
</tr>
</tbody>
</table>

| Table 1. The \(w\)-column is the consequence of the Proposition 3.5 and the diagonals starting from this column and ending on the \(P_n\)-row are consequences of Corollary 3.7. The table shows the upper bounds of the order of QO. |

We need two different kind of arguments for the construction of the chain for \(k = 2\) and \(k = 3\). We shall apply Theorem 1.15. Notice that if \(\phi_{i+1} = \gcd(\phi_i, \phi_{i+1})\), then there exist polynomials \(\tau_i\) and \(\sigma_{i+1}\) such that \(\phi_i = \tau_i \phi_i \sigma_{i+1}\) and \(\phi_{i+1} = \sigma_{i+1} \phi_{i+1}\). The most important remark is that \(\tau_i\) and \(\sigma_{i+1}\) are coprime, \(\tau_i \sigma_{i+1}\), in this case \(\gcd(\tau_i, \sigma_{i+1}) = 1\).

- For \(k = 2\) we have two equations:
\[
\begin{align*}
w &= \phi_0 u, \quad \deg \phi_0 \leq s + 4 \quad \{ \text{w = } \psi_0 u \}_{s + 3 - 0} & D(\phi_0 u) = \psi_0 u, \; \psi_0 := \phi_1 & \text{ord}(\phi_0, \psi_0) \leq s + 3 - 0 \\
Dw &= \phi_1 u, \quad \deg \phi_1 \leq s + 3 \quad \{ \text{Dw = } \psi_1 u \}_{s + 3 - 1} & D(\phi_1 u) = \psi_1 u, \; \psi_1 := \phi_2 & \text{ord}(\phi_1, \psi_1) \leq s + 3 - 1 \\
D^2 w &= \phi_2 u, \quad \deg \phi_2 \leq s + 2 \quad \{ \text{D^2 w = } \psi_2 u \}_{s + 3 - 1} &
\end{align*}
\]
These equations prove the QO of \((Q_n)\) related to \(\phi_0 u (= w)\), and to
The first case is the trivial case, Remark 3.6, and in the second, it is not sure that there exists a polynomial modification from $Dw$ to $w$, for example, if $\deg \phi_0 > \deg \phi_1$—this is the case when the values of Table 1 are the order of $QO$. We must follow another way. For $\phi_{01} := \gcd(\phi_0, \phi_1)$ exists a polynomial $\psi_{01}$ such that $(\phi_{01}, \psi_{01}) \in \mathbb{P}^2_u$, according to Theorem 1.15. On the other hand $\phi_0 = \tau_0 \phi_{01}$, $\phi_1 = \sigma_1 \phi_{01}$ with $\tau_0 \uparrow \sigma_1$. We have a lattice of polynomial pairs and the corresponding lattice of polynomial modifications of $u$.

We define $v := \phi_{01} u$, $(\phi_{01}, \psi_{01}) \in \mathbb{P}^2_u$ yields that $(Q_n)$ are $QO$ with respect to $v$ of order $s$, $s = \text{ord}(\phi_{01}, \psi_{01})$, Proposition 1.34.

- For $k = 3$ suppose that $(Q_n^{(3)})$ is $QO$ with respect to the functional $w$:

$$
\begin{align*}
    w &= \phi_0 u, & \deg \phi_0 &\leq s + 6 \\
    Dw &= \phi_1 u, & \deg \phi_1 &\leq s + 5 \\
    D^2 w &= \phi_2 u, & \deg \phi_2 &\leq s + 4 \\
    D^3 w &= \phi_3 u, & \deg \phi_3 &\leq s + 3
\end{align*}
$$

Applying the Theorem 1.15

$$
\begin{align*}
    \phi_{01} &= \gcd(\phi_0, \phi_1) \implies \exists \psi_{01} \in \mathbb{P}, (\phi_{01}, \psi_{01}) \in \mathbb{P}^2_u, \\
    &\exists \tau_0 \uparrow \sigma_1 \in \mathbb{P}, \quad \tau_0 \phi_{01} = \phi_0 \quad \text{and} \quad \sigma_1 \phi_{01} = \phi_1; \\
    \phi_{12} &= \gcd(\phi_1, \phi_2) \implies \exists \psi_{12} \in \mathbb{P}, (\phi_{12}, \psi_{12}) \in \mathbb{P}^2_u, \\
    &\exists \tau_1 \uparrow \sigma_2 \in \mathbb{P}, \quad \tau_1 \phi_{12} = \phi_1 \quad \text{and} \quad \sigma_2 \phi_{12} = \phi_2.
\end{align*}
$$

Applying the same theorem to the new equations

$$
\begin{align*}
    \phi_{012} &= \gcd(\phi_{01}, \phi_{12}) \implies \exists \psi_{012} \in \mathbb{P}, (\phi_{012}, \psi_{012}) \in \mathbb{P}^2_u, \\
    &\exists \tau_0 \uparrow \sigma_{12} \in \mathbb{P}, \quad \tau_0 \phi_{012} = \phi_{01} \quad \text{and} \quad \sigma_{12} \phi_{012} = \phi_{12}.
\end{align*}
$$

We define $v := \phi_{012} u$, $w := \tau_0 v$, thus $w = \tau_0 w$. The $QO$ of $(Q_n)$ related to $v$ is obvious from the equation $D(\phi_{012} u) = \psi_{012} u$. To prove the $QO$ of $(Q_n^{(3)})$ related with $w$, we built a distributional equation for $v$ in following way:

$$
\begin{align*}
    w &= \tau_0 \tau_1 v, & Dw &= \sigma_1 \tau_0 v, & D^2 w &= \sigma_2 \sigma_{12} v,
\end{align*}
$$

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yields two distributional equations for $v$:

$$D(\tau_0 \tau_1 v) = \sigma_1 \tau_0 v, \quad D(\sigma_1 \tau_0 v) = \sigma_2 \tau_1 v.$$  

On the other hand

$$\tau_0 \parallel \sigma_1 \Rightarrow \tau_0 = \gcd(\tau_0, \sigma_1 \tau_0)$$

up to a non-zero constant factor, and there exists a polynomial $\omega_0$ so that $v$ satisfies the equation $D(\tau_0 v) = \omega_0 v$ (Theorem 1.15).

Now we give the generalization for arbitrary $k$.

**Theorem 3.8.** Let $u \in P^*$ be quasi-definite, $(P_n)$ the corresponding monic OPS, and $(Q_n^{(j)})$ the monic sequence of $j$-order derivatives of $(P_n)$, i.e.:

$$Q_n^{(j)} = \left(\frac{1}{n + 1}\right)^j P_{n + j}.$$  

If $(Q_n^{(k)})$ is $\mathcal{O}O$ of order $s \geq 0$ with respect to $w \in P^*$, then there exist $\varphi_j \in P$, $1 \leq j \leq k - 1$ such that $(Q_n^{(j)})$ is $\mathcal{O}O$ with respect to $w_j$, $0 \leq j \leq k$ where $w_j := \varphi_j w_{j-1}$, $w_0 = u$.

Moreover, for each $\varphi_j$ there exists a polynomial $c_j$ such that $(\varphi_j, c_j) \in P_w$.

**Proof.** We consider the $k$ equations of pairs $(\phi_0, \psi_0), (\phi_1, \psi_1), \ldots, (\phi_{k-1}, \psi_{k-1})$ with $\psi_i := \phi_{i+1}$ from Corollary 3.7. In this order we apply Theorem 1.15 to two consecutive pairs and we obtain a first level of new pairs of $P^2_u$

$$\phi_{i,i+1} := \gcd(\phi_i, \phi_{i+1}), \quad \exists \psi_{i,i+1} \in P : (\phi_{i,i+1}, \psi_{i,i+1}) \in P^2_u,$$

$0 \leq i \leq k - 1$.

There exist polynomials $\tau_i$ and $\sigma_{i+1}$, $\tau_i \parallel \sigma_{i+1}$, without common zeros that for $0 \leq i \leq k - 1$ satisfy

$$\tau_i \phi_{i,i+1} = \phi_i, \quad \sigma_{i+1} \phi_{i,i+1} = \phi_{i+1}.$$  

We apply the same theorem to the resulting $k - 1$ pairs $(\phi_{i,i+1}, \psi_{i,i+1})$, $0 \leq i \leq k - 2$ and we obtain a second level. For $0 \leq i \leq k - 2$
\[
gcd(\phi_{i,i+1}, \phi_{i+1,i+2}) = \phi_{i,i+1,i+2} = \gcd(\phi_i, \phi_{i+1}, \phi_{i+2}),
\]
\[
\exists \psi_{i,i+1,i+2} : (\phi_{i,i+1,i+2}, \psi_{i,i+1,i+2}) \in P^2_u,
\]
\[
\exists \tau_{i,i+1,i+2} : \tau_{i,i+1,i+2} \in P : \tau_{i,i+1,i+2} = \phi_{i,i+1,i+2}
\]

and so on. To obtain the \( k - l \) pairs of the \( l \) level, for \( 0 \leq i \leq k - l - 1 \)

\[
gcd(\phi_{i,i+1,...,i+l-1}, \phi_{i+1,i+2,...,i+l}) = \phi_{i,i+1,...,i+l} = \gcd(\phi_i,...,\phi_{i+l}),
\]
\[
\exists \psi_{i,i+1,...,i+l} : (\phi_{i,i+1,...,i+l}, \psi_{i,i+1,...,i+l}) \in P^2_u,
\]
\[
\exists \tau_{i,i+1,...,i+l} : \tau_{i,i+1,...,i+l} \in P : \tau_{i,i+1,...,i+l} = \phi_{i,i+1,...,i+l}
\]
\[
\text{Figure 2. The first polynomials of the pairs lattice of } P^2_u.
\]

This way ends with a unique pair in the \( k - 1 \) level. Notice that \( \phi_k \) is not involved in this procedure,

\[
gcd(\phi_{0,...,k-2}, \phi_{1,...,k-1}) = \phi_{0,...,k-1} = \gcd(\phi_0,...,\phi_{k-1}),
\]
\[
\exists \psi_{0,...,k-1} : (\phi_{0,...,k-2}, \psi_{0,...,k-1}) \in P^2_u,
\]
\[
\exists \tau_{0,...,k-1} : \tau_{0,...,k-1} \in P : \tau_{0,...,k-1} = \phi_{0,...,k-1}
\]

Figure 2 shows the result of this procedure. In the next, we simplify the nota-
tion of the indices and we conserve from the consecutive indices only the first and the last index: $\phi_{i,i+l} := \phi_{i,...,i+l}$, $\tau_{i,i+l} := \tau_{i,...,i+l}$, $\sigma_{i,i+l} := \sigma_{i,...,i+l}$.

Applying (15), we obtain

\[
\phi_i = \sigma_i \phi_{i-1,i} = \sigma_i \sigma_{i-1,i} \phi_{i-2,i-1,i} = \sigma_i \sigma_{i-1,i} \ldots \sigma_1 \phi_{0,i} = \sigma_i \sigma_{i-1,i} \ldots \sigma_1 \tau_{0,i} \phi_{0,i+1} = \sigma_i \sigma_{i-1,i} \ldots \sigma_1 \tau_{0,i} \tau_{0,i+1} \ldots \tau_{0,0} \phi_{0,i}, \quad 0 \leq i \leq l.
\]

We have climbed to the left from $\phi_i$ to $\phi_{0,i}$ and then to the right to $\phi_{0,l}$. We shall need in the next this factorization (16), $\phi_i = \varphi^{(l)}_i \phi_{0,l}$, $0 \leq i \leq l$, where $\phi_{0,l} = \gcd(\phi_0, \phi_1, \ldots, \phi_l)$ and $\varphi^{(l)}_i = \sigma_i \phi_{i-1,i} \ldots \sigma_1 \tau_{0,i} \tau_{0,i+1} \ldots \tau_{0,0}$.

We have build a lattice of polynomial pairs of $u$ (Figure 2). Now we consider the modifications of $u$ with the first polynomial of these pairs (Figure 3):

\[
\psi_{i}^{(l)} = \phi_{i,i} u, \quad 0 \leq l \leq k - 1, \ 1 \leq i \leq k - 1 - l.
\]

We select the chain of functionals between $u$ and $w$, i.e., $w_{0}^{(l)}$, $1 \leq l \leq k - 1$. The order of the levels is the inverse of the order of the functionals in Theorem 3.8, and also in Remark 3.6, $w_j = w_{0}^{(k-j)}$.

Now we present the core of the proof. We shall built a distributional.

\[
\begin{array}{c}
\phi_0 u = w \xrightarrow{\phi_1} w_{0}^{(1)} \xrightarrow{\phi_2} w_{0}^{(2)} \xrightarrow{\phi_3} w_{0}^{(3)} \xrightarrow{\phi_4} w_{0}^{(4)} \xrightarrow{\phi_5} \cdots \xrightarrow{\phi_{k-1}} w_{0}^{(k-1)} \xrightarrow{\phi_k} w_{0}^{(k)} \\
\end{array}
\]

Figure 3. The corresponding functional lattice of polynomial modifications of $u$.

We have build a lattice of polynomial pairs of $u$ (Figure 2). Now we consider the modifications of $u$ with the first polynomial of these pairs (Figure 3):

\[
\psi_{i}^{(l)} = \phi_{i,i} u, \quad 0 \leq l \leq k - 1, \ 1 \leq i \leq k - 1 - l.
\]

We select the chain of functionals between $u$ and $w$, i.e., $w_{0}^{(l)}$, $1 \leq l \leq k - 1$. The order of the levels is the inverse of the order of the functionals in Theorem 3.8, and also in Remark 3.6, $w_j = w_{0}^{(k-j)}$.

Now we present the core of the proof. We shall built a distributional.
equation for each functional $w^{(l)}_0 = w_j$, $1 \leq l \leq k - 1$, $j = k - l$, including $u (= w^{(k)}_0 = w_0)$ of pairs $(\varphi_{j+1}, \zeta_{j+1})$ with the first polynomial $\varphi_{j+1}$ such that $\varphi_{j+1} w_j = w_{j+1}$, $0 \leq j \leq k - 1$ and $w_k = w$. In this way we can apply the Lemma 2.1 and prove for $0 \leq j \leq k - 1$ the QO of the derivatives $(Q_{j+1})$ with respect to $w_{j+1}$. For the $k - 1$ level it is immediate. We have the pair $(\phi_{0,k-1}, \psi_{0,k-1}) \in P^2_\mathbf{v}$ and the $\phi_{0,k-1}$-modification of $u$ leads to $w_1$: $\phi_{0,k-1} u = w_{0(k-1)} = w_1$.

For all other levels $1 \leq l \leq k - 2$ we select the pairs $(\varphi_i, \varphi_{i+1})$, $0 \leq i \leq l - 1$ belonging to the base level of the lattice of $P^2_\mathbf{v}$. We transform it in pairs belonging to $P^2_\mathbf{w}$ in following way

$$
\varphi_i u^{(16)} \varphi_i^{(l)} \varphi_0 u^{(17)} \varphi_i^{(l)} w_0^{(l)}, \quad 0 \leq i \leq l.
$$

Thus, for $0 \leq i \leq l - 1$, corresponding to the thick lines of Figure 2,

$$
D(\varphi_i u) = \varphi_{i+1} u \iff D(\varphi_i^{(l)} w_0^{(l)}) = \varphi_i^{(l)} w_0^{(l)}.
$$

We have the pairs $(\varphi_i, \varphi_{i+1})$, $0 \leq i \leq l - 1$, of the lattice of $P^2_\mathbf{w}$. We apply Theorem 1.15 to these $l$ equations of $w^{(l)}_0 = w_j$, $j = k - l$. To determine the gcd$(\varphi_i^{(l)}, \ldots, \varphi_{i-1}^{(l)})$, we take into account that

$$
gcd(\varphi_0^{(l)}, \ldots, \varphi_{i-1}^{(l)}) = \tau_{0,i-1} = \gcd(\varphi_0^{(l)}, \ldots, \varphi_{i-1}^{(l)}) = \tau_{0,i-1}.
$$

There exists a polynomial $\zeta_0$ such that $(\tau_{0,i-1}, \zeta_{0,i-1}) \in P_\mathbf{w}$. On the other hand, the corresponding modification leads to the following functional in the chain

$$
\tau_{0,i-1} w_j = \tau_{0,i-1} w_0^{(l)} = \tau_{0,i-1} \phi_0 u = \phi_0 u = w^{(l-1)} = w_{j+1},
$$

thus $\varphi_{j+1} = \tau_{0,i-1}$ and $\zeta_{j+1} = \zeta_{0,i-1}$. \hfill \Box

This is our best approximation to question (iii). We have build a chain as in Remark 3.6 with the polynomials $\varphi_j$ and for each of these polynomials we have a distributional equation that satisfies the corresponding functional, $D(\varphi_j w_{j-1}) = \zeta_j w_{j-1}$. In (iii), the aim is to prove that the polynomials $\varphi_j$ are multiples of the minimum polynomial of the lattice of $u$. In this case every functional $w$, such that the $k$-order derivatives are quasi-orthogonal with respect to it, is a polynomial modification of $\phi^k u$: $w = \pi \phi^k u$. Thus, the schema of singularities is the described in Theorem 2.2 adding $-\deg \pi$ to $k_0$ and $k_1$.

4. CONCLUDING RESULTS

We must study the minimal pairs of the lattices of the standard modifications of $u$ to see the conditions for which (iii) holds. For the next discussion we need the following

**Proposition 4.1.** Let $v \in P^*$ be a functional not necessary quasi-definite satisfying
a distributional equation of minimal pair \((\psi, \psi)\), let \(w := \psi \nu\) so that \((\phi, \psi + D\phi) \in \mathbb{P}_w^2\), see Corollary 1.20. If this equation admits a lower order equation, i.e., \(\gcd(\phi, \psi) = \tau, \deg \tau > 0\), then the order of the minimal pair of the lattice \(\mathbb{P}^2_w\) is less than the order of the minimal pair of \(\mathbb{P}^2\).

**Proof.** Suppose that \(\phi = \tau \varphi, \psi = \tau \varsigma, \varphi, \varsigma \in \mathbb{P}\).

\[
D(\phi x) = (\psi + D\psi)x \quad (\text{1.10}) \quad \iff \quad \phi Dx = \psi x \iff \tau \varphi Dx = \tau \varsigma x \quad \iff \quad \varphi Dx = \varsigma x \quad (\text{1.10}) \quad \iff \quad D(\varphi x) = (\varsigma + D\varphi)x.
\]

To prove (*) we consider the descendant of \((\phi, \psi) \in \mathbb{P}_v^2\) via \(\varphi\)

\[
(\phi, \psi) \in \mathbb{P}_v^2 \xrightarrow{\varphi} (\varphi \phi \psi + \phi D \varphi) \in \mathbb{P}_v^2.
\]

We can obtain a new equation for \(w\) based on \(\varphi \psi = \varphi \tau \varsigma = \varphi \varsigma\)

\[
(\varphi \phi, \varphi \psi + \phi D \varphi) = (\varphi \phi, \varphi \varsigma + \phi D \varphi) \in \mathbb{P}_w^2 \xrightarrow{\psi} (\varphi, \varsigma + D \varphi) \in \mathbb{P}_w^2.
\]

Obviously, we can obtain \((\varphi, \varsigma + D \varphi)\) via \(\tau\), and \(\text{ord}(\varphi, \varsigma) = \text{ord}(\varphi, \varsigma + D \varphi)\). For the minimum pair of \(\mathbb{P}_w^2\) we get \(\text{ord}(\varphi, \varsigma) \leq \text{ord}(\varphi, \varsigma + D \varphi)\).

\[\square\]

**Remark 4.2.** In these cases, the compatibility of \(\phi u\) with the new equation of reduced order is automatically fulfilled, see condition (b) of Corollary 1.17.

**Remark 4.3.** The only that we can assure is that \((\phi, \psi + D\phi) \in \mathbb{P}_w^2\). Thus, the first polynomial which belongs to the minimal pair of the lattice of \(w\), \((\phi, \psi) \in \mathbb{P}_w^2\), divides \(\phi, \phi \psi\). Suppose a functional \(\nu\) that satisfies the equation of polynomials \(\phi = x(x^2 + 1), \psi = x(x - 2); \phi\) and \(\psi - D\phi\) are coprime, the pair is the minimal pair of the lattice of \(\nu\). On the other hand \(\phi\) and \(\psi\) have the common factor \(x\), thus \(x^2 - 1, x - 2) \in \mathbb{P}_w^2\), \(w = \varphi \nu\) and for these polynomials, \(\varphi = x^2 + 1\) and \(\nu = x - 2\), holds \(\phi \not\mid (\psi - D\phi)\) and the pair \((\phi, \psi)\) is the minimum pair of the lattice of \(w\). The question is if a quasi-definite functional can satisfy a distributional equation with this behaviour.

**Definition 4.4.** Let \(u \in \mathbb{P}^*\) be a functional, not necessarily quasi-definite, satisfying a distributional equation of polynomials \(\phi, \psi\), and let \((\phi, \psi)\) be the minimal pair of \(\mathbb{P}_w^2\); the functional is said to be coprime if and only if the coprimality condition holds, see Proposition 1.28 and Remark 1.29: \(\phi \not\mid (\psi + nD\phi), n \geq 0\).

For a semi-classical functional \(u\) we consider the first polynomial of the minimal pair of \(u\) to modify \(w_1 := \phi \nu\). Just now, we consider the first polynomial of the minimal pair of \(w_1\), \((\phi, \psi)\), to modify \(w_1 := \phi \nu\), and so on. We obtain the following chain of modified functionals
If the coprime condition holds for a quasi-definite functional, all first polynomials of the aforementioned minimal pairs are equal to \( o \), i.e.,
\[ o = o, \ldots = o. \]
In this case, we can assure that the decomposition presented in (iii) is always possible, and we can apply Theorem 2.2. We summarize the results in the next Corollary. Notice that the order of the minimal pair of \( u \) is the class of \( u \), \( \text{cl} u \), Definition 1.18.

**Corollary 4.5.** Let \( u \in \mathbb{P}^* \) be a coprime semi-classical functional with minimal pair \( (\phi, \psi) \), and let \( (P_n) = \text{mops} u \) and \( (Q_n^{(k)}) \) the sequence of \( k \)-order monic derivatives of \( (P_n) \). If \( (Q_n^{(k)}) \) is \( \text{QO} \) with respect to a functional \( w \in \mathbb{P}^* \), then there exists a polynomial \( \pi \) such that \( w = \pi \phi^k u \). Moreover, \( (Q_n^{(k)}) \) presents the minimum order of \( \text{QO} \) for the functional \( \tilde{w}_k := \phi^k u \) and the order is \( k \cdot \text{cl} u \). The \( \text{QO} \) with respect \( \tilde{w}_k \) is strict if \( u \) is regular; \( \text{cl} u \), \( n_0 \geq 0 \), then the \( \text{QO} \) is strict for \( k > 1 + \left[ \frac{n_0}{\text{cl} u + 1} \right] \), otherwise, is strict with at most \( k \) consecutive singularities (see Theorem 2.2).

If the coprimality condition does not hold, then we can factorize the polynomials \( \varphi_i \) of the Theorem 3.8 as follows

\[ \varphi_1 = \pi_1 \phi, \varphi_2 = \pi_2 \phi, \varphi_3 = \pi_3 \phi, \ldots, \varphi_k = \pi_k \phi, \]

where \( \phi, \phi, \ldots, \phi, \phi \) are the first polynomial of the minimal pairs of \( \tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{k-1} \) (18). We can say that \( \phi, \phi, \ldots, \phi, \phi, \phi \) and always is possible the factorization \( w = \pi \phi^k u \), \( \phi := \phi \). In both cases, within and without the coprime condition, \( \phi, \phi, \ldots, \phi, \phi \) and strictly divides \( \phi^k \).

The factorization (19) allows to determine the minimum order of \( \text{QO} \). We follow Theorems 2.2 and 3.8 in order to obtain the minimum order of \( \text{QO} \). The best choice for the polynomials \( \varphi_i \) of (19) is \( \varphi_i = \psi, 1 \geq i \geq n \). The order of \( \text{QO} \) of \( (Q_n^{(k)}) \) related to

\[ \tilde{w}_k = \phi \ldots \phi u \]
is \( \text{cl} \tilde{w}_0 + \text{cl} \tilde{w}_1 + \ldots + \text{cl} \tilde{w}_{n-1} \), where \( \tilde{w}_i \) are the polynomials in (18). For a \( n_0 \)-singular semi-classical functional, Theorem 2.2 says that increasing the derivative order \( k \), in each step, the singularity in the equation generates a new singularity in the \( \text{QO} \) of the corresponding derivatives, but also a displacement to the left of the set of singularities, see Figure 1. For the functional (20) the negative displacement in the first step is \( \text{cl} \tilde{w}_0 + 2 \), in the second \( \text{cl} \tilde{w}_1 + 2 \), and so on. Last, all singularities disappear, at least in \( 2 + \left[ \frac{n_0}{\text{cl} u + 1} \right] \) steps (coprime functional), and at most in \( 2 + n_0 \).

**Corollary 4.6.** Let \( u \in \mathbb{P}^* \) be a semi-classical functional with minimal pair \( (\phi, \psi) \), and let \( (P_n) = \text{mops} u \) and \( (Q_n^{(k)}) \) the sequence of \( k \)-order monic derivatives of \( (P_n) \). If \( (Q_n^{(k)}) \) is \( \text{QO} \) with respect to a functional \( w \in \mathbb{P}^* \), then there exist polynomials \( \pi \) and \( \phi \), \( \phi, \phi, \ldots, \phi \) such that \( w = \pi \phi^k u \) and \( (Q_n^{(k)}) \) presents the minimum order
of QO related to the functional \( \hat{w}_k := \phi_k^u; \ k \cdot cl \ u \) is an upper bound of the order of QO with respect \( \hat{w}_k \). This QO is strict if \( u \) is regular; if \( u \) is no-singular, then the QO is strict for all \( k \geq k' \), where \( 2 + \left\lfloor \frac{n_0}{cl u + 1} \right\rfloor \leq k' \leq 2 + n_0 \); otherwise, is strict with at most \( k \) consecutive singularities.

An open question is if the coprime condition holds for semi-classical functionals. In the literature we have not found a case, but our conjecture is that the coprime condition does not hold.

**Conjecture 4.7.** There exist semi-classical functionals \( u \) with minimal pair \( (\phi, \psi) \) such that the coprime condition does not hold, i.e., \( \exists n_0 \geq 0, \ \phi \nmid (\psi + n_0 D\phi) \).

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