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Kac–Moody groups and cluster algebras

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Abstract

Let Q be a finite quiver without oriented cycles, let Λ be the associated preprojective algebra, let $\mathfrak{g}$ be the associated Kac–Moody Lie algebra with Weyl group W, and let $\mathfrak{n}$ be the positive part of $\mathfrak{g}$. For each Weyl group element w, a subcategory $\mathcal{C}_w$ of $\mathrm{mod}(\Lambda)$ was introduced by Buan, Iyama, Reiten and Scott. It is known that $\mathcal{C}_w$ is a Frobenius category and that its stable category $\underline{\mathcal{C}}_w$ is a Calabi–Yau category of dimension two. We show that $\mathcal{C}_w$ yields a cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent group $N(w) := N \cap (w^{-1} N_- w)$. Here N is the pro-unipotent pro-group with Lie algebra the completion $\widehat{\mathfrak{n}}$ of $\mathfrak{n}$. One can identify $\mathbb{C}[N(w)]$ with a subalgebra of $U(\mathfrak{n})^*_{\mathrm{gr}}$, the graded dual of the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. Let $\mathcal{S}^*$ be the dual of Lusztig's semicanonical basis $\mathcal{S}$ of $U(\mathfrak{n})$. We show that all cluster monomials of $\mathbb{C}[N(w)]$ belong to $\mathcal{S}^*$, and that $\mathcal{S}^* \cap \mathbb{C}[N(w)]$ is a $\mathbb{C}$-basis of $\mathbb{C}[N(w)]$. Moreover, we show that the cluster algebra obtained from $\mathbb{C}[N(w)]$ by formally inverting the generators of the coefficient ring is isomorphic to the algebra $\mathbb{C}[N^w]$ of regular functions on the unipotent cell N^w of the Kac–Moody group with Lie algebra $\mathfrak{g}$. We obtain a corresponding dual semicanonical basis of $\mathbb{C}[N^w]$. As one application we obtain a basis for each acyclic cluster algebra, which contains all cluster monomials in a natural way.

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Keywords: Cluster algebra; Kac–Moody group; Quiver; Semicanonical basis; Frobenius category; Preprojective algebra

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Contents

1. Introduction

1.1. This is the continuation of an extensive project to obtain a better understanding of the relations between the following topics:

(i) Representation theory of quivers.
(ii) Representation theory of preprojective algebras.
(iii) Lusztig's (semi)canonical basis of universal enveloping algebras.
(iv) Fomin and Zelevinsky's theory of cluster algebras.
(v) Frobenius categories and 2-Calabi–Yau categories.
(vi) Cluster algebra structures on coordinate algebras of unipotent groups, Bruhat cells and flag varieties.

The topics (i) and (iii) are closely related. The numerous connections have been studied by many authors. Let us just mention Lusztig's work on canonical bases of quantum groups, and Ringel's Hall algebra approach to quantum groups. An important link between (ii) and (iii), due to Lusztig [50,51] and Kashiwara and Saito [45] is that the elements of the (semi)canonical basis are naturally parametrized by the irreducible components of the varieties of nilpotent representations of a preprojective algebra.

Cluster algebras were invented by Fomin and Zelevinsky [9,23,24], with the aim of providing a new algebraic and combinatorial setting for canonical bases and total positivity. One important breakthrough was the insight that the class of acyclic cluster algebras with a skew-symmetric exchange matrix can be categorified using the so-called cluster categories. Cluster categories were introduced by Buan, Marsh, Reineke, Reiten and Todorov [12], see also [46]. In a series of papers by some of these authors and also by Caldero and Keller [15,14], it was established that cluster categories have all necessary properties to provide the mentioned categorification. We refer to the nice overview article [11] for more details on the development of this beautiful theory which established a strong connection between the topics (i), (iv) and (v). More recently, a different and more general type of categorification using representations of quivers with potentials was developed by Derksen, Weyman and Zelevinsky [19,20]. This provides another strong link between topics (i) and (iv).

In [33] we showed that the representation theory of preprojective algebras Λ of Dynkin type (*i.e.* type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$) is also closely related to cluster algebras. We proved that $\mathrm{mod}(\Lambda)$ can be regarded as a categorification of a natural (upper) cluster structure on the polynomial algebra $\mathbb{C}[N]$. Here N is a maximal unipotent subgroup of a complex Lie group of the same type as Λ. Let $\mathfrak{n}$ be its Lie algebra, and let $U(\mathfrak{n})$ be the universal enveloping algebra of $\mathfrak{n}$. The graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$ can be identified with the coordinate algebra $\mathbb{C}[N]$. By means of our categorification, we were able to prove that all the cluster monomials of $\mathbb{C}[N]$ belong to the dual of Lusztig's semicanonical basis of $U(\mathfrak{n})$. Note that the cluster algebra $\mathbb{C}[N]$ is in general not acyclic.

The aim of this article is a vast generalization of these results to the more general setting of symmetric Kac–Moody groups and their unipotent cells. We also provide additional tools for studying the associated categories and cluster structures. For many cluster algebras we construct a basis (called *dual semicanonical basis*) which contains all cluster monomials in a natural way. In particular, we obtain such a basis for all acyclic cluster algebras. Also, we construct a dual PBW-basis of the cluster algebras involved. This provides another close link between Lie theory and the representation theory of preprojective algebras. We show that the coordinate rings $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$ are genuine cluster algebras in a natural way, and not just upper cluster algebras in the sense of [9].

Let us give some more details. We consider preprojective algebras $\Lambda = \Lambda_Q$ attached to quivers Q which are not necessarily of Dynkin type. These algebras are therefore infinite-dimensional in general. The category $\mathrm{nil}(\Lambda)$ of all finite-dimensional nilpotent representations of Λ is then too large to be related to a cluster algebra of finite rank. Moreover, it does not have projective or injective objects, and it lacks an Auslander–Reiten translation. However, Buan, Iyama, Reiten and Scott [13] have attached to each element w of the Weyl group $W = W_Q$ of Q a subcategory $\mathcal{C}_w$ of $\mathrm{nil}(\Lambda)$. They show that the categories $\mathcal{C}_w$ are Frobenius categories and the corresponding stable categories $\underline{\mathcal{C}}_w$ are Calabi–Yau categories of dimension two. (These results were also discovered and proved independently in [36] in the special case when w is an *adaptable* element of W.) Each subcategory $\mathcal{C}_w$ contains a distinguished maximal rigid Λ-module $V_{\mathbf{i}}$ associated to each reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of w. (A module X is called *rigid* if $\mathrm{Ext}^1_\Lambda(X, X) = 0$.)

Special attention is given to the algebra $B_{\mathbf{i}} := \mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$, which turns out to be quasi-hereditary. There is an equivalence between $\mathcal{C}_w$ and the category of Δ-filtered $B_{\mathbf{i}}$-modules. This allows us to describe mutations of maximal rigid Λ-modules in $\mathcal{C}_w$ in terms of the Δ-dimension vectors of the corresponding $B_{\mathbf{i}}$-modules.

To the subcategory $\mathcal{C}_w$ we associate a cluster algebra $\mathcal{A}(\mathcal{C}_w)$ which in general is not acyclic, and we show that $\mathcal{C}_w$ can be seen as a categorification of the cluster algebra $\mathcal{A}(\mathcal{C}_w)$. Each of the modules $V_{\mathbf{i}}$ provides an initial seed of this cluster algebra. (As a very special case, we also obtain in this way a new categorification of every acyclic cluster algebra with a skew-symmetric exchange matrix and a certain choice of coefficients.) The proof relies on the fact that the algebra $\mathcal{A}(\mathcal{C}_w)$ has a natural realization as a certain subalgebra of the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$, where $\mathfrak{n}$ is now the positive part of the symmetric Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ of the same type as Λ. We show that again all the cluster monomials belong to the dual of Lusztig's semicanonical basis of $U(\mathfrak{n})$.

Next, we prove that $\mathcal{A}(\mathcal{C}_w)$ has a simple monomial basis coming from the objects of the additive closure $\mathrm{add}(M_{\mathbf{i}})$, where $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$ is another Λ-module in $\mathcal{C}_w$ associated to a reduced expression $\mathbf{i}$ of w. The modules M_k are rigid, but $M_{\mathbf{i}}$ is not rigid, except in some trivial cases. We call it the *dual PBW-basis* of $\mathcal{A}(\mathcal{C}_w)$, and regard it as a generalization (in the dual setting) of the bases of $U(\mathfrak{n})$ constructed by Ringel in terms of quiver representations, when $\mathfrak{g}$ is finite-dimensional [58]. We use this to prove that $\mathcal{A}(\mathcal{C}_w)$ is spanned by a subset of the dual semicanonical basis of $U(\mathfrak{n})^*_{\mathrm{gr}}$. Thus, we obtain a natural basis of $\mathcal{A}(\mathcal{C}_w)$ containing all the cluster monomials. We call it the *dual semicanonical basis* of $\mathcal{A}(\mathcal{C}_w)$. We prove that $\mathcal{A}(\mathcal{C}_w)$ is isomorphic to the coordinate ring of the finite-dimensional unipotent subgroup $N(w)$ of the symmetric Kac–Moody group attached to $\mathfrak{g}$. Moreover, we show that the cluster algebra obtained from $\mathcal{A}(\mathcal{C}_w)$ by formally inverting the generators of the coefficient ring is isomorphic to the algebra of regular functions on the unipotent cell N^w of the Kac–Moody group. This solves Conjecture IV.3.1 of [13].

Note also that in the Dynkin case the unipotent cells N^w are closely related to the double Bruhat cells of type (e, w), whose coordinate ring is known to be an upper cluster algebra by a result of [9]. However, our proof is different and shows that $\mathbb{C}[N^w]$ is not only an upper cluster algebra but a genuine cluster algebra.

Finally, we explain how the results of this paper are related to those of [37], in which a cluster algebra structure on the coordinate ring of the unipotent radical N_K of a parabolic subgroup of a complex simple algebraic group of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$ was introduced. We give a proof of Conjecture 9.6 of [37].

1.2. Remark

Our preprint [36] contains special cases of the main results of this article: When w is an *adaptable* Weyl group element, we constructed and studied the subcategories $\mathcal{C}_w$ independently of [13], using different methods. For this case, [36] contains a proof of [13, Conjecture IV.3.1]. Since [36] is already cited in several published articles, we decided to keep it on the arXiv as a convenient reference, but it will not be published in a journal.

1.3. Notation

Throughout let K be an algebraically closed field. For a K-algebra A let $\mathrm{mod}(A)$ be the category of finite-dimensional left A-modules. By a *module* we always mean a finite-dimensional left module. Often we do not distinguish between a module and its isomorphism class. Let $D := \mathrm{Hom}_K(-, K) : \mathrm{mod}(A) \to \mathrm{mod}(A^{\mathrm{op}})$ be the usual duality.

For a quiver Q let $\mathrm{rep}(Q)$ be the category of finite-dimensional representations of Q over K. It is well known that we can identify $\mathrm{rep}(Q)$ and $\mathrm{mod}(KQ)$.

By a *subcategory* we always mean a full subcategory. For an A-module M let $\mathrm{add}(M)$ be the subcategory of all A-modules which are isomorphic to finite direct sums of direct summands of M. A subcategory $\mathcal{U}$ of $\mathrm{mod}(A)$ is an *additive subcategory* if any finite direct sum of modules in $\mathcal{U}$ is again in $\mathcal{U}$. By $\mathrm{Fac}(M)$ (resp. $\mathrm{Sub}(M)$) we denote the subcategory of all A-modules X such that there exists some $t \geqslant 1$ and some epimorphism $M^t \to X$ (resp. monomorphism $X \to M^t$).

For an A-module M let $\Sigma(M)$ be the number of isomorphism classes of indecomposable direct summands of M. An A-module is called *basic* if it can be written as a direct sum of pairwise non-isomorphic indecomposable modules.

For an A-module M and a simple A-module S let $[M : S]$ be the Jordan–Hölder multiplicity of S in a composition series of M. Let $\underline{\dim}(M) := \underline{\dim}_A(M) := ([M : S])_S$ be the *dimension vector* of M, where S runs through all isomorphism classes of simple A-modules.

For a set U we denote its cardinality by $|U|$. If $f : X \to Y$ and $g : Y \to Z$ are maps, then the composition is denoted by $gf = g \circ f : X \to Z$.

If U is a subset of a K-vector space V, then let $\mathrm{Span}_K\langle U\rangle$ be the subspace of V generated by U.

By $K(X_1, \dots, X_r)$ (resp. $K[X_1, \dots, X_r]$) we denote the field of rational functions (resp. the polynomial ring) in the variables $X_1, \dots, X_r$ with coefficients in K.

Let $\mathbb{C}$ be the field of complex numbers, and let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the natural numbers, including 0. Set $\mathbb{N}_1 := \mathbb{N} \setminus \{0\}$.

Recommended introductions to representation theory of finite-dimensional algebras and Auslander–Reiten theory are the books [7,2,29,55].

2. Definitions and known results

2.1. Preprojective algebras and nilpotent varieties

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver without oriented cycles. (As usual, Q_0 is the set of vertices, Q_1 is the set of arrows, an arrow $a \in Q_1$ starts in a vertex $s(a)$ and terminates in $t(a)$.) Let

$$\Lambda = \Lambda_Q = K\overline{Q}/(c)$$

be the associated *preprojective algebra*. We assume that Q is connected and has vertices $Q_0 = \{1, \dots, n\}$. Here K is an algebraically closed field, $K\overline{Q}$ is the path algebra of the *double quiver* $\overline{Q}$ of Q which is obtained from Q by adding to each arrow $a : i \to j$ in Q an arrow $a^* : j \to i$ pointing in the opposite direction, and (c) is the ideal generated by the element

$$c = \sum_{a \in Q_1} \left(a^* a - a a^*\right).$$

Clearly, the path algebra KQ is a subalgebra of Λ. Let $\pi_Q : \mathrm{mod}(\Lambda) \to \mathrm{mod}(KQ)$ be the corresponding restriction functor.

A Λ-module M is called *nilpotent* if a composition series of M contains only the simple modules $S_1, \dots, S_n$ associated to the vertices of Q. Let $\mathrm{nil}(\Lambda)$ be the abelian category of finite-dimensional nilpotent Λ-modules.

Let $d = (d_1, \dots, d_n) \in \mathbb{N}^n$. By

$$\mathrm{rep}(Q, d) = \prod_{a \in Q_1} \mathrm{Hom}_K\left(K^{d_{s(a)}}, K^{d_{t(a)}}\right)$$

we denote the affine space of representations of Q with dimension vector d. Furthermore, let $\operatorname{mod}(\Lambda, d)$ be the affine variety of elements

$$(f_a, f_{a^*})_{a\in Q_1} \in \prod_{a\in Q_1} \left(\operatorname{Hom}_K\left(K^{d_{s(a)}}, K^{d_{t(a)}}\right) \times \operatorname{Hom}_K\left(K^{d_{t(a)}}, K^{d_{s(a)}}\right)\right)$$

such that the following holds:

(i) For all $i \in Q_0$ we have

$$\sum_{a\in Q_1: s(a)=i} f_{a^*} f_a = \sum_{a\in Q_1: t(a)=i} f_a f_{a^*}.$$

By Λ_d we denote the variety of all $(f_a, f_{a^*})_{a\in Q_1} \in \operatorname{mod}(\Lambda, d)$ such that the following condition holds:

(ii) There exists some N such that for each path $a_1 a_2 \cdots a_N$ of length N in the double quiver $\overline{Q}$ of Q we have $f_{a_1} f_{a_2} \cdots f_{a_N} = 0$.

(It is not difficult to check that Λ_d is indeed an affine variety, namely for a fixed d we can choose $N = d_1 + \cdots + d_n$ in condition (ii) above.) If Q is a Dynkin quiver, then (ii) follows already from condition (i). One can regard (ii) as a nilpotency condition, which explains why the varieties Λ_d are often called *nilpotent varieties*. Note that $\operatorname{rep}(Q, d)$ can be considered as a subvariety of Λ_d. In fact $\operatorname{rep}(Q, d)$ forms an irreducible component of Λ_d. Lusztig [50, Section 12] proved that all irreducible components of Λ_d have the same dimension, namely

$$\dim \operatorname{rep}(Q, d) = \sum_{a\in Q_1} d_{s(a)} d_{t(a)}.$$

One can interpret Λ_d as the variety of nilpotent Λ-modules with dimension vector d. The group

$$\mathrm{GL}_d = \prod_{i=1}^{n} \mathrm{GL}_{d_i}(K)$$

acts on $\operatorname{mod}(\Lambda, d)$, Λ_d and $\operatorname{rep}(Q, d)$ by conjugation. Namely, for $g = (g_1, \ldots, g_n) \in \mathrm{GL}_d$ and $x = (f_a, f_{a^*})_{a\in Q_1} \in \operatorname{mod}(\Lambda, d)$ define

$$g.x := \left(g_{t(a)} f_a g_{s(a)}^{-1}, g_{s(a)} f_{a^*} g_{t(a)}^{-1}\right)_{a\in Q_1}.$$

The action on Λ_d and $\operatorname{rep}(Q, d)$ is obtained via restriction. The isomorphism classes of Λ-modules in $\operatorname{mod}(\Lambda, d)$ and Λ_d, and KQ-modules in $\operatorname{rep}(Q, d)$, respectively, correspond to the orbits of these actions. For a module M in $\operatorname{mod}(\Lambda, d)$, (resp. in Λ_d or in $\operatorname{rep}(Q, d)$), let $\mathrm{GL}_d.M$ denote its GL_d-orbit.

There is a bilinear form $\langle -, - \rangle = \langle -, - \rangle_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ associated to Q defined by

$$\langle d, e \rangle := \langle d, e \rangle_Q := \sum_{i\in Q_0} d_i e_i - \sum_{a\in Q_1} d_{s(a)} e_{t(a)}.$$

The dimension vector of a KQ-module M is denoted by $\underline{\dim}(M) = \underline{\dim}_Q(M)$. (Note that $\underline{\dim}_Q(M) = \underline{\dim}_\Lambda(M)$, since we can consider M also as a Λ-module.) For KQ-modules M and N set

$$\langle M, N\rangle := \langle M, N\rangle_Q := \dim \operatorname{Hom}_{KQ}(M, N) - \dim \operatorname{Ext}^1_{KQ}(M, N).$$

It is known that $\langle M, N\rangle = \langle \underline{\dim}(M), \underline{\dim}(N)\rangle$. Let $(-,-) = (-,-)_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the symmetrization of the bilinear form $\langle -,-\rangle$, *i.e.* $(d,e) := \langle d,e\rangle + \langle e,d\rangle$. For Λ-modules X and Y set

$$(X,Y)_Q := \big\langle \pi_Q(X), \pi_Q(Y)\big\rangle_Q + \big\langle \pi_Q(Y), \pi_Q(X)\big\rangle_Q.$$

Lemma 2.1. *(See [17, Lemma 1].) For any Λ-modules X and Y we have*

$$\dim \operatorname{Ext}^1_\Lambda(X,Y) = \dim \operatorname{Hom}_\Lambda(X,Y) + \dim \operatorname{Hom}_\Lambda(Y,X) - (X,Y)_Q.$$

In particular, $\dim \operatorname{Ext}^1_\Lambda(X,X)$ *is even, and* $\dim \operatorname{Ext}^1_\Lambda(X,Y) = \dim \operatorname{Ext}^1_\Lambda(Y,X)$.

Corollary 2.2. *For a nilpotent Λ-module X with dimension vector d the following are equivalent*:

- *The closure* $\overline{\mathrm{GL}_d.X}$ *of* $\mathrm{GL}_d.X$ *is an irreducible component of* Λ_d;
- *The orbit* $\mathrm{GL}_d.X$ *is open in* Λ_d;
- $\operatorname{Ext}^1_\Lambda(X,X) = 0$.

2.2. Semicanonical bases

We recall the definition of the dual semicanonical basis and its multiplicative properties, following [50,51,31,35]. From now on, assume that $K = \mathbb{C}$.

For each dimension vector $d = (d_1, \dots, d_n)$ we defined the affine variety Λ_d. A subset C of Λ_d is said to be constructible if it is a finite union of locally closed subsets. For a $\mathbb{C}$-vector space V, a function

$$f : \Lambda_d \to V$$

is *constructible* if the image $f(\Lambda_d)$ is finite and $f^{-1}(m)$ is a constructible subset of Λ_d for all $m \in V$.

The set of constructible functions $\Lambda_d \to \mathbb{C}$ is denoted by $M(\Lambda_d)$. This is a $\mathbb{C}$-vector space. Recall that the group GL_d acts on Λ_d by conjugation. By $M(\Lambda_d)^{\mathrm{GL}_d}$ we denote the subspace of $M(\Lambda_d)$ consisting of the constructible functions which are constant on the GL_d-orbits in Λ_d. Set

$$\widetilde{\mathcal{M}} := \bigoplus_{d \in \mathbb{N}^n} M(\Lambda_d)^{\mathrm{GL}_d}.$$

For $f' \in M(\Lambda_{d'})^{\mathrm{GL}_{d'}}$, $f'' \in M(\Lambda_{d''})^{\mathrm{GL}_{d''}}$ and $d = d' + d''$ we define a constructible function

$$f := f' \star f'' : \Lambda_d \to \mathbb{C}$$

in $M(\Lambda_d)^{\mathrm{GL}_d}$ by

$$f(X) := \sum_{m\in\mathbb{C}} m\chi_{\mathrm{c}}\bigl(\bigl\{U \subseteq X \bigm| f'(U)f''(X/U) = m\bigr\}\bigr)$$

for all $X \in \Lambda_d$, where U runs over the points of the Grassmannian of all submodules of X with $\underline{\dim}(U) = d'$. Here, for a constructible subset V of a complex variety we denote by $\chi_{\mathrm{c}}(V)$ its (topological) Euler characteristic with respect to cohomology with compact support. This turns $\widetilde{\mathcal{M}}$ into an associative $\mathbb{C}$-algebra.

Remark 2.3. Note that the product $\star$ defined here is opposite to the convolution product we have used in [31,32,35]. This new convention turns out to be better adapted to our choice of categorifying $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$ by categories closed under factor modules. It is also compatible with our choice in [37] of categorifying coordinate rings of partial flag varieties by categories closed under submodules.

For the canonical basis vector $e_i := \underline{\dim}(S_i)$ we know that Λ_{e_i} is just a point, which (as a Λ-module) is isomorphic to the simple module S_i. Define $\mathbf{1}_i : \Lambda_{e_i} \to \mathbb{C}$ by $\mathbf{1}_i(S_i) := 1$. By $\mathcal{M}$ we denote the subalgebra of $\widetilde{\mathcal{M}}$ generated by the functions $\mathbf{1}_i$ where $1 \leqslant i \leqslant n$. Set $\mathcal{M}_d := \mathcal{M} \cap M(\Lambda_d)^{\mathrm{GL}_d}$. It follows that

$$\mathcal{M} = \bigoplus_{d\in\mathbb{N}^n} \mathcal{M}_d$$

is an $\mathbb{N}^n$-graded $\mathbb{C}$-algebra. Let $U(\mathfrak{n})$ be the enveloping algebra of the positive part $\mathfrak{n}$ of the Kac–Moody Lie algebra $\mathfrak{g}$ associated to Q, see Sections 4.1 and 4.2.

Theorem 2.4. *(See Lusztig [51].) There is an isomorphism of $\mathbb{N}^n$-graded $\mathbb{C}$-algebras*

$$U(\mathfrak{n}) \to \mathcal{M}$$

defined by $E_i \mapsto \mathbf{1}_i$ for $1 \leqslant i \leqslant n$.

Let $\mathrm{Irr}(\Lambda_d)$ be the set of irreducible components of Λ_d.

Theorem 2.5. *(See Lusztig [51].) For each $Z \in \mathrm{Irr}(\Lambda_d)$ there is a unique $f_Z : \Lambda_d \to \mathbb{C}$ in $\mathcal{M}_d$ such that f_Z takes value 1 on some dense open subset of Z and value 0 on some dense open subset of any other irreducible component Z' of Λ_d. Furthermore, the set*

$$\mathcal{S} := \bigl\{f_Z \bigm| Z \in \mathrm{Irr}(\Lambda_d),\ d \in \mathbb{N}^n\bigr\}$$

is a $\mathbb{C}$-basis of $\mathcal{M}$.

The basis $\mathcal{S}$ is called the *semicanonical basis* of $\mathcal{M}$. By Theorem 2.4 we just identify $\mathcal{M}$ and $U(\mathfrak{n})$ and consider $\mathcal{S}$ also as a basis of $U(\mathfrak{n})$. Since $U(\mathfrak{n})$ is a cocommutative Hopf algebra, its graded dual

$$U(\mathfrak{n})^*_{\mathrm{gr}} = \bigoplus_{d\in\mathbb{N}^n} U_d^*$$

is a commutative $\mathbb{C}$-algebra. Let $\mathcal{M}_d^*$ be the dual space of $\mathcal{M}_d$, and set

$$\mathcal{M}^* := \bigoplus_{d \in \mathbb{N}^n} \mathcal{M}_d^*.$$

Again we identify $\mathcal{M}^*$ and $U(\mathfrak{n})_{\mathrm{gr}}^*$.

For $X \in \Lambda_d$ define an *evaluation function*

$$\delta_X : \mathcal{M}_d \to \mathbb{C}$$

by $\delta_X(f) := f(X)$.

It is not difficult to show that the map $X \mapsto \delta_X$ from Λ_d to $\mathcal{M}_d^*$ is constructible in the above sense. So on every irreducible component $Z \in \mathrm{Irr}(\Lambda_d)$ there is a Zariski open set on which this map is constant. Define $\rho_Z := \delta_X$ for any X in this open set. The $\mathbb{C}$-vector space $\mathcal{M}_d^*$ is spanned by the functions δ_X with $X \in \Lambda_d$. Then by construction

$$\mathcal{S}^* := \{\rho_Z \mid Z \in \mathrm{Irr}(\Lambda_d),\ d \in \mathbb{N}^n\}$$

is the basis of $\mathcal{M}^* \equiv U(\mathfrak{n})_{\mathrm{gr}}^*$ dual to Lusztig's semicanonical basis $\mathcal{S}$ of $U(\mathfrak{n})$.

In case X is a rigid Λ-module, the orbit of X in Λ_d is open, its closure is an irreducible component Z, and $\delta_X = \rho_Z$ belongs to $\mathcal{S}^*$.

For a module $X \in \Lambda_d$ and an m-tuple $\mathbf{i} = (i_1, \dots, i_m)$ with $1 \leqslant i_j \leqslant n$ for all j, let $\mathcal{F}_{\mathbf{i},X}$ denote the projective variety of composition series of type $\mathbf{i}$ of X. Thus an element in $\mathcal{F}_{\mathbf{i},X}$ is a flag

$$(0 = X_0 \subset X_1 \subset \cdots \subset X_m = X)$$

of submodules X_j of X such that for all $1 \leqslant j \leqslant m$ the subfactor X_j/X_{j-1} is isomorphic to the simple Λ-module S_{i_j} associated to the vertex i_j of Q. Let

$$d_{\mathbf{i}} : \Lambda_d \to \mathbb{C}$$

be the map which sends $X \in \Lambda_d$ to $\chi_c(\mathcal{F}_{\mathbf{i},X})$. It follows from the definition of $\star$ that $d_{\mathbf{i}} = \mathbf{1}_{i_1} \star \cdots \star \mathbf{1}_{i_m}$. The $\mathbb{C}$-vector space $\mathcal{M}_d$ is spanned by the maps $d_{\mathbf{i}}$. We have $\delta_X(d_{\mathbf{i}}) = \chi_c(\mathcal{F}_{\mathbf{i},X})$.

Theorem 2.6. *(See [31].) For $X, Y \in \mathrm{nil}(\Lambda)$ we have $\delta_X \delta_Y = \delta_{X \oplus Y}$.*

In [35] a more complicated formula than the one in Theorem 2.6 is given, expressing $\delta_X \delta_Y$ as a linear combination of δ_Z where Z runs over all possible middle terms of non-split short exact sequences with end terms X and Y. The formula is especially useful when $\dim \mathrm{Ext}_\Lambda^1(X, Y) = 1$. In this case, the following hold:

Theorem 2.7. *(See [35, Theorem 2].) Let $X, Y \in \mathrm{nil}(\Lambda)$. If $\dim \mathrm{Ext}_\Lambda^1(X, Y) = 1$ with*

$$0 \to X \to E' \to Y \to 0 \quad \text{and} \quad 0 \to Y \to E'' \to X \to 0$$

the corresponding non-split short exact sequences, then

$$\delta_X \delta_Y = \delta_{E'} + \delta_{E''}.$$

2.3. Frobenius categories

Let A be a K-algebra. Let $\mathcal{C}$ be a subcategory of a module category $\mathrm{mod}(A)$ which is closed under extensions. Clearly, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \mathrm{Ext}^1_A(X, Y)$$

for all modules X and Y in $\mathcal{C}$. An A-module C in $\mathcal{C}$ is called *$\mathcal{C}$-projective* (resp. *$\mathcal{C}$-injective*) if $\mathrm{Ext}^1_A(C, X) = 0$ (resp. $\mathrm{Ext}^1_A(X, C) = 0$) for all $X \in \mathcal{C}$. If C is $\mathcal{C}$-projective and $\mathcal{C}$-injective, then C is also called *$\mathcal{C}$-projective–injective*. We say that $\mathcal{C}$ has *enough projectives* (resp. *enough injectives*) if for each $X \in \mathcal{C}$ there exists a short exact sequence $0 \to Y \to C \to X \to 0$ (resp. $0 \to X \to C \to Y \to 0$) where C is $\mathcal{C}$-projective (resp. $\mathcal{C}$-injective) and $Y \in \mathcal{C}$. If $\mathcal{C}$ has enough projectives and enough injectives, and if these coincide (*i.e.* an object is $\mathcal{C}$-projective if and only if it is $\mathcal{C}$-injective), then $\mathcal{C}$ is called a *Frobenius subcategory* of $\mathrm{mod}(A)$. In particular, $\mathcal{C}$ is a Frobenius category in the sense of Happel [38]. Of course, for $A = \Lambda$, an A-module C in $\mathcal{C}$ is $\mathcal{C}$-projective if and only if it is $\mathcal{C}$-injective, see Lemma 2.1.

By definition the objects in the *stable category* $\underline{\mathcal{C}}$ are the same as the objects in $\mathcal{C}$, and the morphism spaces $\mathrm{Hom}_{\underline{\mathcal{C}}}(X, Y)$ are the morphism spaces in $\mathcal{C}$ modulo morphisms factoring through $\mathcal{C}$-projective–injective objects. The category $\underline{\mathcal{C}}$ is a triangulated category in a natural way [38], where the shift is given by the relative inverse syzygy functor

$$\Omega^{-1} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}.$$

For all X and Y in $\mathcal{C}$ there is a functorial isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{C}}}\bigl(X, \Omega^{-1}(Y)\bigr) \cong \mathrm{Ext}^1_{\mathcal{C}}(X, Y).$$

The category $\underline{\mathcal{C}}$ is a *2-Calabi–Yau category*, if for all $X, Y \in \underline{\mathcal{C}}$ there is a functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \cong D\,\mathrm{Ext}^1_{\mathcal{C}}(Y, X).$$

2.4. Frobenius categories associated to Weyl group elements

By $\hat{I}_1, \dots, \hat{I}_n$ we denote the indecomposable injective Λ-modules with socle $S_1, \dots, S_n$, respectively. Here $S_1, \dots, S_n$ are the 1-dimensional simple Λ-modules corresponding to the vertices of the quiver Q. (The modules $\hat{I}_i$ are infinite-dimensional if Q is not a Dynkin quiver.)

For a Λ-module X and a simple module S_j let $\mathrm{soc}_{(j)}(X) := \mathrm{soc}_{S_j}(X)$ be the sum of all submodules U of X with $U \cong S_j$. (In this definition, we do not assume that X is finite-dimensional.) For a sequence $(j_1, \dots, j_t)$ of indices with $1 \leqslant j_p \leqslant n$ for all p, there is a unique chain

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t \subseteq X$$

of submodules of X such that $X_p/X_{p-1} = \mathrm{soc}_{(j_p)}(X/X_{p-1})$. Define $\mathrm{soc}_{(j_1,\dots,j_t)}(X) := X_t$.

For the rest of this section, let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression of an element w of the Weyl group $W = W_Q$ of Q. (By definition, this is the Weyl group of the Kac–Moody Lie algebra $\mathfrak{g}$ associated to Q, see Section 4.1.) For $1 \leqslant k \leqslant r$ let

$$V_k := V_{\mathbf{i},k} := \mathrm{soc}_{(i_k,\dots,i_1)}(\hat{I}_{i_k}),$$

and set $V_{\mathbf{i}} := V_1 \oplus \cdots \oplus V_r$. (The module $V_{\mathbf{i}}$ is dual to the cluster-tilting object constructed in [13, Section III.2].) Define

$$\mathcal{C}_{\mathbf{i}} := \mathrm{Fac}(V_{\mathbf{i}}) \subseteq \mathrm{nil}(\Lambda).$$

For $1 \leqslant j \leqslant n$ let $k_j := \max\{1 \leqslant k \leqslant r \mid i_k = j\}$. Define $I_{\mathbf{i},j} := V_{\mathbf{i},k_j}$ and set

$$I_{\mathbf{i}} := I_{\mathbf{i},1} \oplus \cdots \oplus I_{\mathbf{i},n}.$$

The category $\mathcal{C}_{\mathbf{i}}$ and the module $I_{\mathbf{i}}$ depend only on w, and not on the chosen reduced expression $\mathbf{i}$ of w. Therefore, we define

$$\mathcal{C}_w := \mathcal{C}_{\mathbf{i}} \quad \text{and} \quad I_w := I_{\mathbf{i}}.$$

(If Q is a Dynkin quiver, and $w = w_0$ is the longest Weyl group element, then $\mathcal{C}_w = \mathrm{nil}(\Lambda) = \mathrm{mod}(\Lambda)$.) Without loss of generality, we assume that for each $1 \leqslant j \leqslant n$ there is some $1 \leqslant k \leqslant r$ with $i_k = j$. Otherwise, we could just replace Q by a quiver with fewer vertices. Note also that $\mathcal{C}_w = \mathrm{add}(I_w)$ if and only if $i_k \neq i_s$ for all $k \neq s$. In this case, most of our theory becomes trivial.

The following three theorems are proved in [13]. They were also obtained independently and by different methods in [36] in the case when w is adaptable.

Theorem 2.8. *For any Weyl group element w the following hold*:

(i) *$\mathcal{C}_w$ is a Frobenius category*;
(ii) *The stable category $\underline{\mathcal{C}}_w$ is a 2-Calabi–Yau category*;
(iii) *$\mathcal{C}_w$ has n indecomposable $\mathcal{C}_w$-projective–injective modules, namely the indecomposable direct summands of I_w*;
(iv) $\mathcal{C}_w = \mathrm{Fac}(I_w)$.

We denote the relative inverse syzygy functor of $\underline{\mathcal{C}}_w$ by Ω_w^{-1}.

Recall that a Λ-module T is *rigid* if $\mathrm{Ext}^1_\Lambda(T,T) = 0$. Let $\mathcal{C}$ be a subcategory of $\mathrm{mod}(\Lambda)$, and let $T \in \mathcal{C}$ be rigid. Recall that for all $X, Y \in \mathrm{mod}(\Lambda)$ we have $\dim \mathrm{Ext}^1_\Lambda(X,Y) = \dim \mathrm{Ext}^1_\Lambda(Y,X)$. We need the following definitions:

- T is *$\mathcal{C}$-maximal rigid* if $\mathrm{Ext}^1_\Lambda(T \oplus X, X) = 0$ with $X \in \mathcal{C}$ implies $X \in \mathrm{add}(T)$;
- T is a *$\mathcal{C}$-cluster-tilting module* if $\mathrm{Ext}^1_\Lambda(T, X) = 0$ with $X \in \mathcal{C}$ implies $X \in \mathrm{add}(T)$.

Theorem 2.9. *For a rigid Λ-module T in $\mathcal{C}_w$ the following are equivalent*:

(i) $\Sigma(T) = \mathrm{length}(w)$;
(ii) *T is $\mathcal{C}_w$-maximal rigid*;
(iii) *T is a $\mathcal{C}_w$-cluster-tilting module.*

For $1 \leqslant k \leqslant r$ let

$$k^- := \max\{0, 1 \leqslant s \leqslant k-1 \mid i_s = i_k\},$$
$$k^+ := \min\{k+1 \leqslant s \leqslant r, r+1 \mid i_s = i_k\}.$$

For $1 \leqslant i, j \leqslant n$ let q_{ij} be the number of edges between the vertices i and j of the underlying graph of our quiver Q.

Following Berenstein, Fomin and Zelevinsky we define a quiver $\Gamma_{\mathbf{i}}$ as follows: The vertices of $\Gamma_{\mathbf{i}}$ are just the numbers $1, \dots, r$. For $1 \leqslant s, t \leqslant r$ there are q_{i_s, i_t} arrows from s to t provided $t^+ \geqslant s^+ > t > s$. These are called the *ordinary arrows* of $\Gamma_{\mathbf{i}}$. Furthermore, for each $1 \leqslant s \leqslant r$ there is an arrow $s \to s^-$ provided $s^- > 0$. These are the *horizontal arrows* of $\Gamma_{\mathbf{i}}$.

On the other hand, let A be a K-algebra, and let $X = X_1^{n_1} \oplus \cdots \oplus X_t^{n_t}$ be a finite-dimensional A-module, where the X_i are pairwise non-isomorphic indecomposable modules and $n_i \geqslant 1$. Let $S_i = S_{X_i}$ be the simple $\mathrm{End}_A(X)^{\mathrm{op}}$-module corresponding to X_i. Then $\mathrm{Hom}_A(X, X_i)$ is the indecomposable projective $\mathrm{End}_A(X)^{\mathrm{op}}$-module with top S_i. The basic facts on the quiver Γ_X of the endomorphism algebra $\mathrm{End}_A(X)^{\mathrm{op}}$ are collected in [33, Section 3.2]. In particular, we have a 1-1 correspondence between the vertices of Γ_X and the modules $X_1, \dots, X_t$.

Theorem 2.10. *The module $V_{\mathbf{i}}$ is $\mathcal{C}_w$-maximal rigid, and we have $\Gamma_{V_{\mathbf{i}}} = \Gamma_{\mathbf{i}}$.*

For example, let Q be a quiver with underlying graph $1 \overline{\overline{}}\, 2 \,\frac{}{}\, 3$. Then $\mathbf{i} := (i_7, \dots, i_1) := (3, 1, 2, 3, 1, 2, 1)$ is a reduced expression of a Weyl group element $w \in W_Q$. The quiver $\Gamma_{\mathbf{i}}$ looks as follows:

$$\begin{tikzcd}
6 \arrow[rr] & & 3 \arrow[rr] \arrow[dl, shift left] \arrow[dl, shift right] & & 1 \arrow[dl, shift left] \arrow[dl, shift right] \\
& 5 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[rr] \arrow[dl] & & 2 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[dl] & \\
7 \arrow[rr] & & 4 \arrow[ul] & &
\end{tikzcd}$$

We often try to visualize Λ-modules. For example, let Q be the quiver

$$\begin{tikzcd}
1 \arrow[dr, "b"] \arrow[dd, "a"'] & \\
& 2 \arrow[dl, "c"] \\
3 &
\end{tikzcd}$$

and let $\mathbf{i} := (i_6, \dots, i_1) := (3, 2, 1, 3, 2, 1)$. Then the Λ-module $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_6$ looks as follows:

$$V_1 = 1 \qquad V_2 = \begin{smallmatrix} & 1 \\ 2 & \end{smallmatrix} \qquad V_3 = \begin{smallmatrix} &&&1\\ 1&&2&\\ &3&& \end{smallmatrix}$$

$$V_4 = \begin{smallmatrix} &&&&1\\ &1&&2&\\ 2&&3&&\\ &1&&& \end{smallmatrix} \qquad V_5 = \begin{smallmatrix} &&&&&&1\\ &&&1&&2&\\ 1&&2&&3&&\\ &3&&1&&&\\ &&2&&&& \end{smallmatrix} \qquad V_6 = \begin{smallmatrix} &&&&&&&1\\ &&&&1&&2&\\ &1&&2&&3&&\\ 2&&3&&1&&&\\ &1&&2&&&&\\ &&3&&&&& \end{smallmatrix}$$

The numbers can be interpreted as basis vectors or as composition factors. For example, the module V_5 is a 9-dimensional Λ-module with dimension vector $\underline{\dim}_\Lambda(V_5) = (d_1, d_2, d_3) = (4, 3, 2)$. More precisely, one could display V_5 as follows:

$$
\begin{tikzcd}
 & & & & & & 1 \arrow[dl, "b"] \\
 & & & 1 \arrow[dl, "b"'] \arrow[dr, "a"] & & 2 \arrow[dl, "c"] & \\
1 \arrow[dr, "a"'] & & 2 \arrow[dl, "c"'] \arrow[dr, "b^*"] & & 3 \arrow[dl, "a^*"] & & \\
 & 3 \arrow[dr, "c^*"'] & & 1 \arrow[dl, "b"] & & & \\
 & & 2 & & & &
\end{tikzcd}
$$

This picture shows how the different arrows of the quiver $\overline{Q}$ of Λ act on the 9 basis vectors of V_5. For example, one can see immediately that the socle of X is isomorphic to S_2, and the top is isomorphic to $S_1 \oplus S_1 \oplus S_1$.

2.5. *Relative homology for $\mathcal{C}_w$*

We recall some notions from relative homology theory which, for Artin algebras, was developed by Auslander and Solberg [4,5].

Let A be a K-algebra, and let $X, Y, Z, T \in \operatorname{mod}(A)$. Set

$$F_T := \operatorname{Hom}_A(T, -) : \operatorname{mod}(A) \to \operatorname{mod}\bigl(\operatorname{End}_A(T)^{\mathrm{op}}\bigr).$$

A short exact sequence

$$0 \to Z \to Y \to X \to 0$$

is *F_T-exact* if $0 \to F_T(Z) \to F_T(Y) \to F_T(X) \to 0$ is exact. By $F_T(X, Z)$ we denote the set of equivalence classes of F_T-exact sequences with end terms X and Z as above.

Let $\mathcal{Y}_T$ be the subcategory of all $X \in \operatorname{mod}(A)$ such that there exists an exact sequence

$$\cdots \xrightarrow{f_3} T_3 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} X \to 0 \tag{1}$$

where $T_i \in \operatorname{add}(T)$ for all i and the short exact sequences

$$0 \to \operatorname{Ker}(f_i) \to T_i \to \operatorname{Im}(f_i) \to 0$$

are F_T-exact for all $i \geqslant 0$. We call sequence (1) an $\operatorname{add}(T)$-*resolution* of X. We say that (1) has *length* at most d if $T_j = 0$ for all $j > d$. Note that

$$\operatorname{add}(T) \subseteq \mathcal{Y}_T.$$

Dually, one defines $\mathrm{add}(T)$-*coresolutions*

$$0 \to X \xrightarrow{g_0} T_0 \xrightarrow{g_1} T_1 \xrightarrow{g_2} T_2 \xrightarrow{g_3} \cdots$$

where we require now that the sequences

$$0 \to \mathrm{Im}(g_i) \to T_i \to \mathrm{Coker}(g_i) \to 0$$

are F^T-exact, where F^T is the contravariant functor $\mathrm{Hom}_A(-, T)$.

For $X \in \mathcal{Y}_T$ and $Z \in \mathrm{mod}(A)$ let $\mathrm{Ext}^i_{F_T}(X, Z)$, $i \geqslant 0$ be the cohomology groups of the cocomplex obtained by applying the functor $\mathrm{Hom}_A(-, Z)$ to the sequence

$$\cdots \xrightarrow{f_3} T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0.$$

Lemma 2.11. *(See [4].) For $X \in \mathcal{Y}_T$ and $Z \in \mathrm{mod}(A)$ there is a functorial isomorphism*

$$\mathrm{Ext}^1_{F_T}(X, Z) \cong F_T(X, Z).$$

Proposition 2.12. *(See [5, Proposition 3.7].) For $X \in \mathcal{Y}_T$ and $Z \in \mathrm{mod}(A)$ there is a functorial isomorphism*

$$\mathrm{Ext}^i_{F_T}(X, Z) \to \mathrm{Ext}^i_{\mathrm{End}_A(T)^{\mathrm{op}}}\big(\mathrm{Hom}_A(T, X), \mathrm{Hom}_A(T, Z)\big)$$

for all $i \geqslant 0$.

Corollary 2.13. *The functor*

$$\mathrm{Hom}_A(T, -) : \mathcal{Y}_T \to \mathrm{mod}\big(\mathrm{End}_A(T)^{\mathrm{op}}\big)$$

is fully faithful. In particular, $\mathrm{Hom}_A(T, -)$ *has the following properties*:

(i) *If $X \in \mathcal{Y}_T$ is indecomposable, then* $\mathrm{Hom}_A(T, X)$ *is indecomposable*;
(ii) *If* $\mathrm{Hom}_A(T, X) \cong \mathrm{Hom}_A(T, Y)$ *for some $X, Y \in \mathcal{Y}_T$, then $X \cong Y$.*

Note that Corollary 2.13 follows already from [3, Section 3], see also [6, Lemma 1.3 (b)].

Corollary 2.14. *Let $T \in \mathrm{mod}(A)$, and let $\mathcal{C}$ be an extension closed subcategory of $\mathcal{Y}_T$. If*

$$\psi : 0 \to \mathrm{Hom}_A(T, X) \xrightarrow{\mathrm{Hom}_A(T,f)} \mathrm{Hom}_A(T, Y) \xrightarrow{\mathrm{Hom}_A(T,g)} \mathrm{Hom}_A(T, Z) \to 0$$

is a short exact sequence of $\mathrm{End}_A(T)^{\mathrm{op}}$*-modules with $X, Y, Z \in \mathcal{C}$, then*

$$\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is a short exact sequence in $\mathrm{mod}(A)$.

Now we apply the above ideas to the category $\mathcal{C}_w$. The following proposition is proved in [36] for adaptable w and in [13] for arbitrary w. In a more general framework it is proved in [47].

Proposition 2.15. *Let T be a $\mathcal{C}_w$-maximal rigid module, and let $X \in \mathcal{C}_w$. Then there exists an* $\operatorname{add}(T)$*-resolution of the form*

$$0 \to T_1 \to T_0 \to X \to 0$$

and an $\operatorname{add}(T)$*-coresolution of the form*

$$0 \to X \to T_0' \to T_1' \to 0.$$

Corollary 2.16. *For each $\mathcal{C}_w$-maximal rigid module T we have $\mathcal{C}_w \subseteq \mathcal{Y}_T$.*

Corollary 2.17. *For each $X \in \mathcal{C}_w$ the projective dimension of the* $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$*-module* $\operatorname{Hom}_\Lambda(T, X)$ *is at most one.*

Corollary 2.18. *(See [40, Theorem 5.3.2].) If T and R are $\mathcal{C}_w$-maximal rigid Λ-modules, then the* $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$*-module* $\operatorname{Hom}_\Lambda(T, R)$ *is a classical tilting module, and*

$$\operatorname{End}_{\operatorname{End}_\Lambda(T)^{\mathrm{op}}}\big(\operatorname{Hom}_\Lambda(T, R)\big) \cong \operatorname{End}_\Lambda(R).$$

2.6. The cluster algebra $\mathcal{A}(\mathcal{C}_w, T)$

We refer to [25] for an excellent survey on cluster algebras. Here we only recall the main definitions and introduce a cluster algebra $\mathcal{A}(\mathcal{C}_w, T)$ associated to a Weyl group element w and a $\mathcal{C}_w$-maximal rigid Λ-module T.

If $\widetilde{B} = (b_{ij})$ is any $r \times (r-n)$-matrix with integer entries, then the *principal part* B of $\widetilde{B}$ is obtained from $\widetilde{B}$ by deleting the last n rows. Given some $1 \leqslant k \leqslant r-n$ define a new $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b_{ij}')$ by

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases}$$

where $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant r-n$. One calls $\mu_k(\widetilde{B})$ a *mutation* of $\widetilde{B}$. If $\widetilde{B}$ is an integer matrix whose principal part is skew-symmetric, then it is easy to check that $\mu_k(\widetilde{B})$ is also an integer matrix with skew-symmetric principal part. In this case, Fomin and Zelevinsky define a cluster algebra $\mathcal{A}(\widetilde{B})$ as follows. Let $\mathcal{F} = \mathbb{C}(y_1, \dots, y_r)$ be the field of rational functions in r commuting variables $y_1, \dots, y_r$. Define $\mathbf{y} := (y_1, \dots, y_r)$. One calls $(\mathbf{y}, \widetilde{B})$ the *initial seed* of $\mathcal{A}(\widetilde{B})$. For $1 \leqslant k \leqslant r-n$ define

$$y_k^* := \frac{\prod_{b_{ik}>0} y_i^{b_{ik}} + \prod_{b_{ik}<0} y_i^{-b_{ik}}}{y_k}.$$

The pair $(\mu_k(\mathbf{y}), \mu_k(\widetilde{B}))$, where $\mu_k(\mathbf{y})$ is obtained from $\mathbf{y}$ by replacing y_k by y_k^*, is the *mutation in direction k* of the seed $(\mathbf{y}, \widetilde{B})$.

Now one can iterate this process of mutation and obtain inductively a set of seeds. Thus each seed consists of an r-tuple of algebraically independent elements of $\mathcal{F}$ called a *cluster* and of a matrix called the *exchange matrix*. The elements of a cluster are its *cluster variables*. Given a cluster $(f_1, \dots, f_r)$, the monomials $f_1^{m_1} f_2^{m_2} \cdots f_r^{m_r}$ where $m_k \geqslant 0$ for all k are called *cluster monomials*. A seed has $r - n$ neighbours obtained by mutation in direction $1 \leqslant k \leqslant r - n$. One does not mutate the last n elements of a cluster, they serve as "coefficients" and belong to every cluster. The *cluster algebra* $\mathcal{A}(\widetilde{B})$ is by definition the subalgebra of $\mathcal{F}$ generated by the set of all cluster variables appearing in all seeds obtained by iterated mutation starting with the initial seed.

It is often convenient to define a cluster algebra using an oriented graph, as follows. Let Γ be a quiver without loops or 2-cycles with vertices $\{1, \dots, r\}$. We can define an $r \times r$-matrix $B(\Gamma) = (b_{ij})$ by setting

$$b_{ij} = (\text{number of arrows } j \to i \text{ in } \Gamma) - (\text{number of arrows } i \to j \text{ in } \Gamma).$$

Let $B(\Gamma)^\circ$ be the $r \times (r - n)$-matrix obtained by deleting the last n columns of $B(\Gamma)$. The principal part of $B(\Gamma)^\circ$ is skew-symmetric, hence this yields a cluster algebra $\mathcal{A}(B(\Gamma)^\circ)$.

We apply this procedure to our subcategory $\mathcal{C}_w$. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module with T_k indecomposable for all k. Without loss of generality assume that $T_{r-n+1}, \dots, T_r$ are $\mathcal{C}_w$-projective–injective. By Γ_T we denote the quiver of the endomorphism algebra $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$. We then define the cluster algebra

$$\mathcal{A}(\mathcal{C}_w, T) := \mathcal{A}\big(B(\Gamma_T)^\circ\big).$$

In particular, we denote by $\mathcal{A}(\mathcal{C}_w)$ the cluster algebra $\mathcal{A}(\mathcal{C}_w, V_{\mathbf{i}})$ attached to the $\mathcal{C}_w$-maximal rigid module $V_{\mathbf{i}}$ of Section 2.4. Thus $\mathcal{A}(\mathcal{C}_w) := \mathcal{A}(B(\Gamma_{\mathbf{i}})^\circ)$. (Up to isomorphism of cluster algebras, this definition does not depend on the choice of $\mathbf{i}$, see Section 3.1.)

2.7. Mutation of rigid modules

The results of this section are straightforward generalizations of results in [33], see [36, Sections 12, 13, 14] and [13].

Let A be a K-algebra, and M be an A-module. A homomorphism $f : X \to M'$ in $\operatorname{mod}(A)$ is a *left* $\operatorname{add}(M)$*-approximation* of X if $M' \in \operatorname{add}(M)$ and the induced map

$$\operatorname{Hom}_A(f, M) : \operatorname{Hom}_A\big(M', M\big) \to \operatorname{Hom}_A(X, M)$$

is surjective. A morphism $f : V \to W$ is called *left minimal* if every morphism $g : W \to W$ with $gf = f$ is an isomorphism. Dually, one defines right $\operatorname{add}(M)$-approximations and right minimal morphisms. Some well known basic properties of approximations can be found in [33, Section 3.1].

Proposition 2.19. *Let T be a basic $\mathcal{C}_w$-maximal rigid Λ-module, and let X be an indecomposable direct summand of T which is not $\mathcal{C}_w$-projective–injective. Then there are short exact sequences*

$$0 \to X \xrightarrow{f'} T' \xrightarrow{g'} Y \to 0$$

and

$$0 \to Y \xrightarrow{f''} T'' \xrightarrow{g''} X \to 0$$

such that the following hold:

(i) f' *and* f'' *are minimal left* $\operatorname{add}(T/X)$*-approximations, and* g' *and* g'' *are minimal right* $\operatorname{add}(T/X)$*-approximations*;
(ii) $Y \oplus T/X$ *is a basic* $\mathcal{C}_w$*-maximal rigid* Λ*-module* (*in particular* Y *is indecomposable*), *and* $X \ncong Y$;
(iii) $\dim \operatorname{Ext}^1_\Lambda(Y, X) = \dim \operatorname{Ext}^1_\Lambda(X, Y) = 1$;
(iv) *We have* $\operatorname{add}(T') \cap \operatorname{add}(T'') = 0$;
(v) *The quiver* Γ_T *of* $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ *has no loops and no* 2*-cycles*;
(vi) *We have*

$$\operatorname{gl.dim}\bigl(\operatorname{End}_\Lambda(T)^{\mathrm{op}}\bigr) = \begin{cases} 3 & \mathcal{C}_w \neq \operatorname{add}(I_w), \\ 1 & \mathcal{C}_w = \operatorname{add}(I_w) \text{ and } n > 1, \\ 0 & \mathcal{C}_w = \operatorname{add}(I_w) \text{ and } n = 1. \end{cases}$$

In the situation of the above proposition, we call $\{X, Y\}$ an *exchange pair associated to* T/X, and we write

$$\mu_X(T) = Y \oplus T/X.$$

We say that $Y \oplus T/X$ is the mutation of T in direction X. The short exact sequence

$$0 \to X \xrightarrow{f'} T' \xrightarrow{g'} Y \to 0$$

is the *exchange sequence* starting in X and ending in Y. Thus, we have

$$\mu_Y\bigl(\mu_X(T)\bigr) = T.$$

Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module with T_k indecomposable for all k. Without loss of generality we assume that $T_{r-n+1}, \dots, T_r$ are $\mathcal{C}_w$-projective–injective. As in Section 2.6 write $B(T) := B(\Gamma_T) = (t_{ij})_{1 \leqslant i,j \leqslant r}$, and let $B(T)^\circ = (t_{ij})$ be the $r \times (r-n)$-matrix obtained from $B(T)$ by deleting the last n columns.

For $1 \leqslant k \leqslant r-n$ let

$$0 \to T_k \to T' \to T_k^* \to 0$$

and

$$0 \to T_k^* \to T'' \to T_k \to 0$$

be exchange sequences associated to the direct summand T_k of T. It follows that

$$T' = \bigoplus_{t_{ik}<0} T_i^{-t_{ik}} \quad \text{and} \quad T'' = \bigoplus_{t_{ik}>0} T_i^{t_{ik}}.$$

Set

$$T^* = \mu_{T_k}(T) = T_k^* \oplus T/T_k.$$

The quivers of the endomorphism algebras $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$ and $\mathrm{End}_\Lambda(\mu_{T_k}(T))^{\mathrm{op}}$ are related via Fomin and Zelevinsky's mutation rule:

Theorem 2.20. *Let w be a Weyl group element. For a basic $\mathcal{C}_w$-maximal rigid Λ-module T as above and $1 \leqslant k \leqslant r - n$ we have*

$$B\big(\mu_{T_k}(T)\big)^\circ = \mu_k\big(B(T)^\circ\big).$$

2.8. Categorification

An (additive) *categorification* of a cluster algebra $\mathcal{A}(\widetilde{B})$ as in Section 2.6 is given by the following:

(A) A $\mathbb{C}$-linear, hom-finite Frobenius category $\mathcal{E}$ with a cluster structure in the sense of [13, II.1] on the basic $\mathcal{E}$-maximal rigid objects.
(B) A basic $\mathcal{E}$-maximal rigid objects T such that $B(\Gamma_T)^\circ = \widetilde{B}$.
(C) A cluster character $\chi_? : \mathrm{obj}(\mathcal{E}) \to \mathbb{C}(y_1, \dots, y_r)$ in the sense of Palu [52, Definition 1.2], with triangles replaced by short exact sequences.
(D) The cluster character $\chi_?$ induces a bijection between basic, T-reachable $\mathcal{E}$-maximal rigid objects and clusters in $\mathcal{A}(\widetilde{B})$.

Remark 2.21. (1) Conditions (A)–(C) imply obviously that each cluster monomial in $\mathcal{A}(\widetilde{B})$ is of the form χ_R for some $\mathcal{E}$-rigid object R. Thus condition (D) is a kind of injectivity requirement for $\chi_?$.

(2) By the results in Section 2.7 we have a cluster structure on $\mathcal{C}_w$. We can take $T = V_{\mathbf{i}}$, for which we know $\Gamma_{V_{\mathbf{i}}}$ by Theorem 2.10. By Theorems 2.6 and 2.7 our $\delta_?$ is a good candidate for a cluster character. In fact, by Theorems 3.1 and 3.2 below, we know that $\delta_X \in \mathcal{A}(B(\Gamma_{V_{\mathbf{i}}})^\circ)$ for *all* $X \in \mathcal{C}_w$. (By Theorem 3.1 the algebra $\mathcal{A}(B(\Gamma_{V_{\mathbf{i}}})^\circ)$ is (up to isomorphism) a subalgebra of $\mathcal{M}^*$.) Property (D) holds in our situation because of the construction of the dual semicanonical basis. For this reason we call $(\mathcal{C}_w, V_{\mathbf{i}})$ a categorification of $\mathcal{A}(B(\Gamma_{V_{\mathbf{i}}})^\circ)$.

3. Main results

In this section, let $K = \mathbb{C}$ be the field of complex numbers.

3.1. The cluster algebra $\mathcal{A}(\mathcal{C}_w)$ as a subalgebra of $\mathcal{M}^* \equiv U(\mathfrak{n})^*_{\mathrm{gr}}$

For a reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of a Weyl group element w let $\mathcal{T}(\mathcal{C}_w)$ be the graph with vertices the isomorphism classes of basic $\mathcal{C}_w$-maximal rigid Λ-modules and with edges given by mutations. Let $T = T_1 \oplus \cdots \oplus T_r$ be a vertex of $\mathcal{T}(\mathcal{C}_w)$, and let $\mathcal{T}(\mathcal{C}_w, T)$ denote the connected component of $\mathcal{T}(\mathcal{C}_w)$ containing T. Two modules in $\mathcal{T}(\mathcal{C}_w)$ are *mutation equivalent* if they belong to the same connected component. A Λ-module X is called T-*reachable* if $X \in \mathrm{add}(R)$ for some vertex R of $\mathcal{T}(\mathcal{C}_w, T)$. Denote by $\mathcal{R}(\mathcal{C}_w, T)$ the subalgebra of $\mathcal{M}^*$ generated by the δ_{R_i} $(1 \leqslant i \leqslant r)$ where $R = R_1 \oplus \cdots \oplus R_r$ runs over all vertices of $\mathcal{T}(\mathcal{C}_w, T)$. The following theorem is our first main result. The proof is given in Section 15.1.

Theorem 3.1. *Let w be a Weyl group element. Then the following hold*:

(i) *There is a unique isomorphism $\iota : \mathcal{A}(\mathcal{C}_w, T) \to \mathcal{R}(\mathcal{C}_w, T)$ such that*

$$\iota(y_i) = \delta_{T_i} \quad (1 \leqslant i \leqslant r).$$

(ii) *If we identify the two algebras $\mathcal{A}(\mathcal{C}_w, T)$ and $\mathcal{R}(\mathcal{C}_w, T)$ via ι, then the clusters of $\mathcal{A}(\mathcal{C}_w, T)$ are identified with the r-tuples $\delta(R) = (\delta_{R_1}, \ldots, \delta_{R_r})$, where R runs over the vertices of the graph $\mathcal{T}(\mathcal{C}_w, T)$. In particular, $\{\delta_X \mid X \text{ is } T\text{-reachable}\}$ is the set of cluster monomials in $\mathcal{R}(\mathcal{C}_w, T)$, and all cluster monomials belong to the dual semicanonical basis $\mathcal{S}^*$ of $\mathcal{M}^* \equiv U(\mathfrak{n})^*_{\mathrm{gr}}$.*

The proof of Theorem 3.1 relies on Theorem 2.20 and the multiplication formula in Theorem 2.7.

Write $\mathcal{R}(\mathcal{C}_w) := \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. (The algebra $\mathcal{R}(\mathcal{C}_w)$ and its cluster algebra structure do not depend on $\mathbf{i}$, since all $\mathcal{C}_w$-maximal rigid modules of the form $V_{\mathbf{i}}$ are mutation equivalent, see [13, Proposition III.4.3].) Theorem 3.1 shows that the cluster algebra $\mathcal{A}(\mathcal{C}_w)$ is canonically isomorphic to the subalgebra $\mathcal{R}(\mathcal{C}_w)$ of $U(\mathfrak{n})^*_{\mathrm{gr}}$.

As an application, our theory provides an algorithm which computes the Euler characteristics $\chi_c(\mathcal{F}_{\mathbf{k},R})$ for all cluster monomials δ_R in $\mathcal{R}(\mathcal{C}_w)$ and all composition series types $\mathbf{k} = (k_1, \ldots, k_s)$, see Section 18.2. This is quite remarkable, since starting from the definitions this seems to be an impossible task in almost all cases.

3.2. Dual PBW-bases and dual semicanonical bases

Let $\mathbf{i} = (i_r, \ldots, i_1)$ be a reduced expression of a Weyl group element w. Let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ be defined as before. For each $1 \leqslant k \leqslant r$ there is a canonical embedding

$$\iota_k : V_{k^-} \to V_k.$$

Here we set $V_0 := 0$. Let M_k be the cokernel of ι_k, and define

$$M_{\mathbf{i}} := M_1 \oplus \cdots \oplus M_r.$$

These modules play an important role in our theory. (In case w is adaptable and $\mathbf{i}$ is Q^{op}-adapted, the module $M_{\mathbf{i}}$ is a terminal KQ-module in the sense of [36].)

In the spirit of Ringel's construction of PBW-bases for quantum groups [58], we construct dual PBW-bases for our cluster algebras $\mathcal{A}(\mathcal{C}_w)$. The following theorem is our second main result. The proof will be given in Section 15.

Theorem 3.2. *Let $\mathbf{i} = (i_r, \ldots, i_1)$ be a reduced expression of a Weyl group element w, and let $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$ be defined as above.*

(i) *The cluster algebra $\mathcal{R}(\mathcal{C}_w)$ is a polynomial ring in r variables. More precisely, we have*

$$\mathcal{R}(\mathcal{C}_w) = \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}] = \operatorname{Span}_{\mathbb{C}}\langle \delta_X \mid X \in \mathcal{C}_w \rangle.$$

(ii) *The set $\{\delta_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$ is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w)$.*

(iii) *The subset $\mathcal{S}_w^* := \mathcal{S}^* \cap \mathcal{R}(\mathcal{C}_w)$ of the dual semicanonical basis is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w)$ containing all cluster monomials.*

Let $\widetilde{\mathcal{R}}(\mathcal{C}_w)$ be the algebra obtained from $\mathcal{R}(\mathcal{C}_w)$ by formally inverting the elements δ_P for all $\mathcal{C}_w$-projective–injectives P. In other words, $\widetilde{\mathcal{R}}(\mathcal{C}_w)$ is the cluster algebra obtained from $\mathcal{R}(\mathcal{C}_w)$ by inverting the generators of its coefficient ring. Similarly, let $\underline{\mathcal{R}}(\mathcal{C}_w)$ be the cluster algebra obtained from $\mathcal{R}(\mathcal{C}_w)$ by specializing the elements δ_P to 1. For both cluster algebras $\widetilde{\mathcal{R}}(\mathcal{C}_w)$ and $\underline{\mathcal{R}}(\mathcal{C}_w)$ we get a $\mathbb{C}$-basis which is easily obtained from the dual semicanonical basis $\mathcal{S}_w^*$ and again contains all cluster monomials, see Sections 15.5 and 15.6.

3.3. The shift functor in $\underline{\mathcal{C}}_w$

As mentioned before, the category $\underline{\mathcal{C}}_w$ is a triangulated category with shift functor Ω_w^{-1}. Recall that $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ is a basic $\mathcal{C}_w$-maximal rigid module. Set $T_{\mathbf{i}} := I_w \oplus \Omega_w^{-1}(V_{\mathbf{i}})$. In Section 13.1 we construct a sequence of mutations which starts in $V_{\mathbf{i}}$ and ends in $T_{\mathbf{i}}$. This mutation sequence is crucial for the proof of some of our results. (For example, it helps to show that the coordinate rings $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$ are generated by the set of cluster variables.)

Now let $R = R_1 \oplus \cdots \oplus R_r$ be any $\mathcal{C}_w$-maximal rigid Λ-module, which is mutation equivalent to $V_{\mathbf{i}}$. Suppose that we know a sequence of mutations starting in $V_{\mathbf{i}}$ and ending in R. Then we can use the mutation sequence from $V_{\mathbf{i}}$ to $T_{\mathbf{i}}$ to obtain a mutation sequence between R and $I_w \oplus \Omega_w^{-1}(R)$, and between R and $I_w \oplus \Omega_w(R)$, see Section 13.3.

3.4. Unipotent subgroups and cells

Let w be a Weyl group element and put $\Delta_w^+ := \{\alpha \in \Delta^+ \mid w(\alpha) < 0\}$. Let

$$\mathfrak{n}(w) = \bigoplus_{\alpha \in \Delta_w^+} \mathfrak{n}_\alpha$$

be the corresponding sum of root subspaces of $\mathfrak{n}$, see Section 4.3. This is a finite-dimensional nilpotent Lie algebra. Let $N(w)$ be the corresponding finite-dimensional unipotent group, see Section 5.2.

The maximal Kac–Moody group attached to $\mathfrak{g}$ as in [48, Chapter 6] comes with a pair of subgroups N and N_- (denoted by $\mathcal{U}$ and $\mathcal{U}_-$ in [48]). Note that later on for the definition of generalized minors in Section 7 we also have to work with the minimal Kac–Moody group (denoted by $\mathcal{G}^{\min}$ in [48, 7.4]). We have

$$N(w) = N \cap \left(w^{-1} N_- w\right).$$

We also define the *unipotent cell*

$$N^w := N \cap (B_- w B_-)$$

where B_- is the standard negative Borel subgroup of the Kac–Moody group.

Every Λ-module X in $\mathcal{C}_w$ gives rise to a linear form $\delta_X \in \mathcal{M}^* \equiv U(\mathfrak{n})^*_{\mathrm{gr}}$ and by means of the identification $U(\mathfrak{n})^*_{\mathrm{gr}} \equiv \mathbb{C}[N]$ to a regular function φ_X on N.

The following theorem, proved in Section 8, is our third main result.

Theorem 3.3. *The algebras $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$ of regular functions on $N(w)$ and N^w, respectively, have a cluster algebra structure. For each reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of w, the tuple $((\varphi_{V_{\mathbf{i},1}}, \dots, \varphi_{V_{\mathbf{i},r}}), B(\Gamma_{V_{\mathbf{i}}})^\circ)$ provides an initial seed of these cluster algebra structures. The functions $\varphi_{V_{\mathbf{i},k}} \in \mathbb{C}[N]$ can be interpreted as generalized minors. We obtain natural cluster algebra isomorphisms*

$$\mathbb{C}\bigl[N(w)\bigr] \cong \mathcal{R}(\mathcal{C}_w) \quad \textit{and} \quad \mathbb{C}\bigl[N^w\bigr] \cong \widetilde{\mathcal{R}}(\mathcal{C}_w).$$

As a result, we have obtained a categorification in the sense of 2.8 of the cluster algebra structure on $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$.

3.5. Example

We are going to illustrate some of the previous results on an example. Let Q be a quiver with underlying graph $1 \;\text{——}\; 2 \;\text{——}\; 3 \;\text{——}\; 4$ and let $\mathbf{i} := (3, 4, 2, 1, 3, 4, 2, 1)$. This is a reduced expression of the Weyl group element $w := s_3s_4s_2s_1s_3s_4s_2s_1$. The category $\mathcal{C}_w$ contains 18 indecomposable modules, and 4 of these are $\mathcal{C}_w$-projective–injective. The stable category $\underline{\mathcal{C}}_w$ is triangle equivalent to the cluster category $\mathcal{C}_Q$.

The maximal rigid module $V_{\mathbf{i}}$ has 8 indecomposable direct summands, namely

$$V_1 = 1 \qquad V_2 = \begin{smallmatrix} 1 & \\ & 2 \end{smallmatrix} \qquad V_3 = 4 \qquad V_4 = \begin{smallmatrix} 1 & & & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}$$

$$V_5 = I_{\mathbf{i},1} = \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} \qquad V_6 = I_{\mathbf{i},2} = \begin{smallmatrix} & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \qquad V_7 = I_{\mathbf{i},4} = \begin{smallmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{smallmatrix} \qquad V_8 = I_{\mathbf{i},3} = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}.$$

Similarly, $T_{\mathbf{i}}$ has 4 non-$\mathcal{C}_w$-projective–injective indecomposable direct summands, namely

$$T_1 = 2 \qquad T_2 = \begin{smallmatrix} 2 & & 4 \\ & 3 & \end{smallmatrix} \qquad T_3 = \begin{smallmatrix} 1 & & \\ & 2 & \\ & & 3 \end{smallmatrix} \qquad T_4 = \begin{smallmatrix} 2 & \\ & 3 \end{smallmatrix}.$$

Here we set $T_k := \Omega_w^{-1}(V_k)$ for $1 \leqslant k \leqslant 4$.

The group N can be taken to be the group of upper unitriangular 5×5 matrices with complex coefficients. Given two subsets I and J of $\{1, 2, \dots, 5\}$ with $|I| = |J|$, we denote by $D_{IJ} \in \mathbb{C}[N]$ the regular function mapping an element $x \in N$ to its minor $D_{IJ}(x)$ with row subset I and column subset J. We get

$$\begin{array}{llll} \varphi_{V_1} = D_{\{1\},\{2\}} & \varphi_{V_2} = D_{\{12\},\{23\}} & \varphi_{V_3} = D_{\{1234\},\{1235\}} & \varphi_{V_4} = D_{\{123\},\{235\}} \\ \varphi_{V_5} = D_{\{1\},\{3\}} & \varphi_{V_6} = D_{\{12\},\{35\}} & \varphi_{V_7} = D_{\{1234\},\{2345\}} & \varphi_{V_8} = D_{\{123\},\{345\}} \\ \varphi_{T_1} = D_{\{12\},\{13\}} & \varphi_{T_2} = D_{\{123\},\{135\}} & \varphi_{T_3} = D_{\{123\},\{234\}} & \varphi_{T_4} = D_{\{123\},\{134\}}. \end{array}$$

The unipotent subgroup $N(w)$ consists of all 5×5 matrices of the form

$$\begin{bmatrix} 1 & u_1 & u_2 & u_7 & u_4 \\ 0 & 1 & u_5 & u_8 & u_6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (u_1, \dots, u_8 \in \mathbb{C}).$$

The unipotent cell N^w is a locally closed subset of N defined by the following equations and inequalities:

$$N^w = \bigl\{x \in N \bigm| D_{\{1\},\{4\}}(x) = D_{\{1\},\{5\}}(x) = D_{\{12\},\{45\}}(x) = 0,\ D_{\{1\},\{3\}}(x) \neq 0,$$
$$D_{\{12\},\{35\}}(x) \neq 0,\ D_{\{123\},\{345\}}(x) \neq 0,\ D_{\{1234\},\{2345\}}(x) \neq 0\bigr\}.$$

Note that the 4 inequalities are given by the non-vanishing of the 4 regular functions $\varphi_{I_{\mathbf{i},j}}$, $1 \leqslant j \leqslant 4$ attached to the indecomposable $\mathcal{C}_w$-projective–injective modules. We have

$$D_{\{1\},\{4\}} = \varphi_{\begin{smallmatrix} & & 3 \\ & 2 & \\ 1 & & \end{smallmatrix}} \qquad D_{\{1\},\{5\}} = \varphi_{\begin{smallmatrix} & & & 4 \\ & & 3 & \\ & 2 & & \\ 1 & & & \end{smallmatrix}} \qquad D_{\{12\},\{45\}} = \varphi_{\begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}}.$$

Our results show that the polynomial algebra $\mathbb{C}[N(w)]$ has a cluster algebra structure, of which $(\varphi_{V_1}, \varphi_{V_2}, \varphi_{V_3}, \varphi_{V_4}, \varphi_{I_{\mathbf{i},1}}, \varphi_{I_{\mathbf{i},2}}, \varphi_{I_{\mathbf{i},3}}, \varphi_{I_{\mathbf{i},4}})$ is a distinguished cluster. Its coefficient ring is the polynomial ring in the four variables $(\varphi_{I_{\mathbf{i},1}}, \varphi_{I_{\mathbf{i},2}}, \varphi_{I_{\mathbf{i},3}}, \varphi_{I_{\mathbf{i},4}})$. The cluster mutations of this algebra come from mutations of the basic $\mathcal{C}_w$-maximal rigid Λ-modules. Moreover, if we replace the coefficient ring by the ring of Laurent polynomials in the four variables $(\varphi_{I_{\mathbf{i},1}}, \varphi_{I_{\mathbf{i},2}}, \varphi_{I_{\mathbf{i},3}}, \varphi_{I_{\mathbf{i},4}})$, we obtain the coordinate ring $\mathbb{C}[N^w]$.

4. Kac–Moody Lie algebras

From now on, let $K = \mathbb{C}$ be the field of complex numbers. In this section we recall known results on Kac–Moody Lie algebras.

4.1. Kac–Moody Lie algebras

Let $\Gamma = (\Gamma_0, \Gamma_1, \gamma)$ be a finite graph (without loops). It has as set of vertices Γ_0, edges Γ_1 and $\gamma : \Gamma_1 \to \mathcal{P}_2(\Gamma_0)$ determining the adjacency of the edges; here $\mathcal{P}_2(\Gamma_0)$ denotes the set of two-element subsets of Γ_0. If $\Gamma_0 = \{1, 2, \dots, n\}$ we can assign to Γ a *symmetric generalized Cartan matrix* $C_\Gamma = (c_{ij})_{1 \leqslant i,j \leqslant n}$, which is an $n \times n$-matrix with integer entries

$$c_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -|\gamma^{-1}(\{i, j\})| & \text{if } i \neq j. \end{cases}$$

Obviously, the assignment $\Gamma \mapsto C_\Gamma$ induces a bijection between isomorphism classes of graphs with vertex set $\{1, 2, \dots, n\}$ and symmetric generalized Cartan matrices in $\mathbb{Z}^{n \times n}$ up to simultaneous permutation of rows and columns.

For a quiver $Q = (Q_0, Q_1, s, t)$ as defined in Section 2.1, its underlying graph $|Q| := (Q_0, Q_1, q)$ is given by $q(a) = \{s(a), t(a)\}$ for all $a \in Q_1$ *i.e.* it is obtained by "forgetting" the orientation of the edges. We write $C_Q := C_{|Q|} := (c_{ij})_{i,j}$.

Let $\mathfrak{g} := \mathfrak{g}_Q := \mathfrak{g}(C_Q)$ be the (symmetric) *Kac–Moody Lie algebra* (see [43]) associated to Q, which is defined as follows: Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space of dimension $2n - \mathrm{rank}(C_Q)$, and let $\Pi := \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ be linearly independent subsets of the vector spaces $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, such that

$$\alpha_i\big(\alpha_j^\vee\big) = c_{ij}$$

for all i, j.

Let $\mathfrak{h}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*$ be a vector space decomposition, where $\mathfrak{h}_1^*$ is just the subspace with basis Π, and $\mathfrak{h}_2^*$ is any direct complement of $\mathfrak{h}_1^*$ in $\mathfrak{h}^*$. Let $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the standard bilinear form, defined by $(\alpha_j, \alpha_j) := \alpha_i(\alpha_j^\vee)$, $(\alpha_i, x) := (x, \alpha_i) := x(\alpha_i^\vee)$, and $(x, y) := 0$ for all $x, y \in \mathfrak{h}_2^*$ and $1 \leqslant i, j \leqslant n$. Note that $\alpha_i(\alpha_j^\vee) = (\underline{\dim}(S_i), \underline{\dim}(S_j))_Q$, where $(-,-)_Q$ is the bilinear form defined in Section 2.1.

Now $\mathfrak{g} = (\mathfrak{g}, [-,-])$ is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{h}$ and the symbols e_i and f_i $(1 \leqslant i \leqslant n)$ satisfying the following defining relations:

(L1) $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$,
(L2) $[h, e_i] = \alpha_i(h)e_i$, and $[h, f_i] = -\alpha_i(h) f_i$,
(L3) $[e_i, f_i] = \alpha_i^\vee$ and $[e_i, f_j] = 0$ for all $i \neq j$,
(L4) $(\mathrm{ad}(e_i)^{1-c_{ij}})(e_j) = 0$ for all $i \neq j$,
(L5) $(\mathrm{ad}(f_i)^{1-c_{ij}})(f_j) = 0$ for all $i \neq j$.

(For $x, y \in \mathfrak{g}$ and $m \geqslant 1$ we set $\mathrm{ad}(x)(y) := \mathrm{ad}(x)^1(y) := [x, y]$ and $\mathrm{ad}(x)^{m+1}(y) := \mathrm{ad}(x)^m([x, y])$.)

The Lie algebra $\mathfrak{g}$ is finite-dimensional if and only if Q is a Dynkin quiver. In this case, this is the usual Serre presentation of the simple Lie algebra associated to Q.

Conversely, if $\mathfrak{g} = \mathfrak{g}(C)$ is a Kac–Moody Lie algebra defined by a symmetric generalized Cartan matrix C, we say that $\mathfrak{g}$ is of type Γ if $C = C_\Gamma$. This is well defined for symmetric Kac–Moody Lie algebras. We call Γ the *Dynkin graph* of $\mathfrak{g}$.

For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha := \big\{x \in \mathfrak{g} \bigm| [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\big\}.$$

One can show that $\dim \mathfrak{g}_\alpha < \infty$ for all α. By

$$R := \sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

we denote the *root lattice* of $\mathfrak{g}$. Define $R^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$. The *roots* of $\mathfrak{g}$ are defined as the elements in

$$\Delta := \big\{\alpha \in R \setminus \{0\} \bigm| \mathfrak{g}_\alpha \neq 0\big\}.$$

Set $\Delta^+ := \Delta \cap R^+$ and $\Delta^- := \Delta \cap (-R^+)$. One can show that $\Delta = \Delta^+ \cup \Delta^-$. The elements in Δ^+ and Δ^- are the *positive roots* and the *negative roots*, respectively. The elements in $\{\alpha_1, \dots, \alpha_n\}$ are positive roots of $\mathfrak{g}$ and are called *simple roots*.

One has the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ with

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha} \quad \text{and} \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

The Lie algebra $\mathfrak{n}$ is generated by $e_1, \dots, e_n$ with defining relations (L4). Set $\mathfrak{n}_\alpha := \mathfrak{g}_\alpha$ if $\alpha \in R^+ \setminus \{0\}$.

For $1 \leqslant i \leqslant n$ define an element s_i in the automorphism group $\mathrm{Aut}(\mathfrak{h}^*)$ of $\mathfrak{h}^*$ by

$$s_i(\alpha) := \alpha - \alpha\big(\alpha_i^\vee\big)\alpha_i$$

for all $\alpha \in \mathfrak{h}^*$. The subgroup $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ generated by $s_1, \dots, s_n$ is the *Weyl group* of $\mathfrak{g}$. The elements s_i are called *Coxeter generators* of W. The identity element of W is denoted by 1. The *length* $l(w)$ of some $w \neq 1$ in W is the smallest number $t \geqslant 1$ such that $w = s_{i_t} \cdots s_{i_2} s_{i_1}$ for some $1 \leqslant i_j \leqslant n$. In this case $(i_t, \dots, i_2, i_1)$ is a *reduced expression* for w. Let $R(w)$ be the set of all reduced expressions for w. We set $l(1) = 0$.

A root $\alpha \in \Delta$ is a *real root* if $\alpha = w(\alpha_i)$ for some $w \in W$ and some i. It is well known that $\dim \mathfrak{g}_\alpha = 1$ if α is a real root. By Δ_{re} we denote the set of real roots of $\mathfrak{g}$. Define $\Delta_{\mathrm{re}}^+ := \Delta_{\mathrm{re}} \cap \Delta^+$.

Finally, let us fix a basis $\{\varpi_j \mid 1 \leqslant j \leqslant 2n - \mathrm{rank}(C_Q)\}$ of $\mathfrak{h}^*$ such that

$$\varpi_j\big(\alpha_i^\vee\big) = \delta_{ij} \quad \big(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant 2n - \mathrm{rank}(C_Q)\big).$$

The ϖ_j are the *fundamental weights*. We denote by

$$P := \big\{\nu \in \mathfrak{h}^* \bigm| \nu\big(\alpha_i^\vee\big) \in \mathbb{Z} \text{ for all } 1 \leqslant i \leqslant n\big\}$$

the *integral weight lattice*, and we set

$$P^+ := \big\{\nu \in P \bigm| \nu\big(\alpha_i^\vee\big) \geqslant 0 \text{ for all } 1 \leqslant i \leqslant n\big\}.$$

The elements in P^+ are called *integral dominant weights*. We have

$$P = \bigoplus_{j=1}^{n} \mathbb{Z}\varpi_j \oplus \bigoplus_{j=n+1}^{2n-\mathrm{rank}(C_Q)} \mathbb{C}\varpi_j \quad \text{and} \quad P^+ = \bigoplus_{j=1}^{n} \mathbb{N}\varpi_j \oplus \bigoplus_{j=n+1}^{2n-\mathrm{rank}(C_Q)} \mathbb{C}\varpi_j.$$

Define

$$\overline{P} := \bigoplus_{j=1}^{n} \mathbb{Z}\varpi_j \quad \text{and} \quad \overline{P}^+ := \bigoplus_{j=1}^{n} \mathbb{N}\varpi_j.$$

The lattice $\overline{P}$ can be naturally identified with the weight lattice of the derived subalgebra $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

4.2. *The universal enveloping algebra $U(\mathfrak{n})$*

The universal enveloping algebra $U(\mathfrak{n})$ of the Lie algebra $\mathfrak{n}$ is the associative $\mathbb{C}$-algebra defined by generators $E_1, \dots, E_n$ and relations

$$\sum_{k=0}^{1-c_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-c_{ij}-k)} = 0$$

for all $i \neq j$, where the c_{ij} are the entries of the generalized Cartan matrix C_Q, and let

$$E_i^{(k)} := E_i^k / k!.$$

We have a canonical embedding $\iota : \mathfrak{n} \to U(\mathfrak{n})$ which maps e_i to E_i for all $1 \leqslant i \leqslant n$. We consider $\mathfrak{n}$ as a subspace of $U(\mathfrak{n})$, and we also identify e_i and E_i.

Let

$$J = \begin{cases} \mathbb{N}_1 & \text{if } \dim(\mathfrak{n}) = \infty, \\ \{1, 2, \dots, d\} & \text{if } \dim(\mathfrak{n}) = d. \end{cases}$$

Let $\mathrm{P} := \{p_i \mid i \in J\}$ be a $\mathbb{C}$-basis of $\mathfrak{n}$ such that $\mathrm{P} \cap \mathfrak{n}_\alpha$ is a basis of $\mathfrak{n}_\alpha$ for all positive roots α. We assume that $\{e_1, \dots, e_n\} \subset \mathrm{P}$. Thus e_i is a basis vector of the (1-dimensional) space $\mathfrak{n}_{\alpha_i}$. For $k \geqslant 0$ define

$$p_i^{(k)} := p_i^k / k!.$$

Let $\mathbb{N}^{(J)}$ be the set of tuples $(m_i)_{i \in J}$ of natural numbers m_i such that $m_i = 0$ for all but finitely many m_i. For $\mathbf{m} = (m_i)_{i \geqslant 1} \in \mathbb{N}^{(J)}$ define

$$p_{\mathbf{m}} := p_1^{(m_1)} p_2^{(m_2)} \cdots p_s^{(m_s)}$$

where s is chosen such that $m_j = 0$ for all $j > s$.

Theorem 4.1 *(Poincaré–Birkhoff–Witt). The set*

$$\mathcal{P} := \{ p_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^{(J)} \}$$

is a $\mathbb{C}$-basis of $U(\mathfrak{n})$.

The basis $\mathcal{P}$ is called a *PBW-basis* of $U(\mathfrak{n})$. For $d = (d_1, \dots, d_n) \in \mathbb{N}^n$ let U_d be the subspace of $U(\mathfrak{n})$ spanned by the elements of the form $e_{i_1} e_{i_2} \cdots e_{i_m}$, where for each $1 \leqslant i \leqslant n$ the set $\{k \mid i_k = i,\ 1 \leqslant k \leqslant m\}$ contains exactly d_i elements. It follows that

$$U(\mathfrak{n}) = \bigoplus_{d \in \mathbb{N}^n} U_d.$$

This turns $U(\mathfrak{n})$ into an $\mathbb{N}^n$-graded algebra.

Furthermore, $U(\mathfrak{n})$ is a cocommutative Hopf algebra with comultiplication

$$\Delta : U(\mathfrak{n}) \to U(\mathfrak{n}) \otimes U(\mathfrak{n})$$

defined by $\Delta(x) := 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{n}$. It is easy to check that

$$\Delta(p_{\mathbf{m}}) = \sum_{\mathbf{k}} p_{\mathbf{k}} \otimes p_{\mathbf{m}-\mathbf{k}}, \tag{2}$$

where the sum is over all tuples $\mathbf{k} = (k_i)_{i \geqslant 1}$ with $0 \leqslant k_i \leqslant m_i$ for every i.

By U_d^* we denote the vector space dual of U_d. Define the *graded dual* of $U(\mathfrak{n})$ by

$$U(\mathfrak{n})^*_{\mathrm{gr}} := \bigoplus_{d \in \mathbb{N}^n} U_d^*.$$

It follows that $U(\mathfrak{n})^*_{\mathrm{gr}}$ is a commutative associative $\mathbb{C}$-algebra with multiplication defined via the comultiplication Δ of $U(\mathfrak{n})$: For $f', f'' \in U(\mathfrak{n})^*_{\mathrm{gr}}$ and $x \in U(\mathfrak{n})$, we have

$$\left(f' \cdot f''\right)(x) = \sum_{(x)} f'(x_{(1)}) f''(x_{(2)}),$$

where (using the Sweedler notation) we write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

Let $\mathcal{P}^* := \{p^*_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^{(J)}\}$ be the dual PBW-basis of $U(\mathfrak{n})^*_{\mathrm{gr}}$, where

$$p^*_{\mathbf{m}}(p_{\mathbf{n}}) := \begin{cases} 1 & \text{if } \mathbf{m} = \mathbf{n}, \\ 0 & \text{otherwise.} \end{cases}$$

The element in $\mathcal{P}^*$ corresponding to $p_i \in \mathrm{P}$ is denoted by p_i^*. It follows from (2) that

$$p^*_{\mathbf{m}} \cdot p^*_{\mathbf{n}} = p^*_{\mathbf{m}+\mathbf{n}},$$

that is, each element $p^*_{\mathbf{m}}$ in $\mathcal{P}^*$ is equal to a monomial in the p_i^*'s. Hence, the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$ can be identified with the polynomial algebra $\mathbb{C}[p_1^*, p_2^*, \ldots]$ (with countably many variables p_i^*).

4.3. *The Lie algebra* $\mathfrak{n}(w)$

Let

$$\widehat{\mathfrak{n}} := \prod_{\alpha \in \Delta^+} \mathfrak{n}_\alpha$$

be the completion of $\mathfrak{n}$. A subset $\Theta \subseteq \Delta^+$ is *bracket closed* if for all $\alpha, \beta \in \Theta$ with $\alpha + \beta \in \Delta^+$ we have $\alpha + \beta \in \Theta$. In this case, we define

$$\widehat{\mathfrak{n}}(\Theta) := \prod_{\alpha \in \Theta} \mathfrak{n}_\alpha.$$

Since Θ is bracket closed, $\widehat{\mathfrak{n}}(\Theta)$ is a Lie subalgebra of $\widehat{\mathfrak{n}}$. One calls Θ *bracket coclosed* if $\Delta^+ \setminus \Theta$ is bracket closed.

For $w \in W$ set $\Delta_w^+ := \{\alpha \in \Delta^+ \mid w(\alpha) < 0\}$. It is well known that for each reduced expression $(i_r, \ldots, i_2, i_1) \in R(w)$ we have

$$\Delta_w^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r})\}.$$

For $1 \leqslant k \leqslant r$ set

$$\beta_{\mathbf{i}}(k) := \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{otherwise.} \end{cases}$$

The set Δ_w^+ contains $l(w)$ positive roots, all of these are real roots, see for example [48, 1.3.14]. The next lemma is also well known.

Lemma 4.2. *For every $w \in W$, the set Δ_w^+ is bracket closed and bracket coclosed.*

Let $\mathfrak{n}(w) := \widehat{\mathfrak{n}}(\Delta_w^+)$ be the *nilpotent Lie algebra associated to w*. We have

$$\mathfrak{n}(w) = \bigoplus_{\alpha \in \Delta_w^+} \mathfrak{n}_\alpha$$

and $\dim \mathfrak{n}(w) = l(w)$.

Again, let $\mathbf{i} = (i_r, \ldots, i_1)$ be a reduced expression. As in Section 4.2 we choose a $\mathbb{C}$-basis $\mathrm{P} = \{p_j \mid j \in J\}$ such that $\mathrm{P} \cap \mathfrak{n}_\alpha$ is a basis of $\mathfrak{n}_\alpha$ for all positive roots α. The resulting PBW-basis $\mathcal{P} = \{p_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^{(J)}\}$ of $U(\mathfrak{n})$ is called **i**-*compatible* provided the vector p_k belongs to $\mathfrak{n}_{\beta_{\mathbf{i}}(k)}$ for all $1 \leqslant k \leqslant r$. In this case

$$\mathcal{P}_{\mathbf{i}} := \{p_1^{(m_1)} p_2^{(m_2)} \cdots p_r^{(m_r)} \mid m_k \geqslant 0 \text{ for all } 1 \leqslant k \leqslant r\}$$

is a PBW-basis of the universal enveloping algebra $U(\mathfrak{n}(w))$ of $\mathfrak{n}(w)$, and

$$\mathcal{P}_{\mathbf{i}}^* := \{(p_1^*)^{m_1} (p_2^*)^{m_2} \cdots (p_r^*)^{m_r} \mid m_k \geqslant 0 \text{ for all } 1 \leqslant k \leqslant r\}$$

is the corresponding dual PBW-basis of the graded dual $U(\mathfrak{n}(w))_{\mathrm{gr}}^*$.

4.4. Highest weight modules

A $U(\mathfrak{g})$-module M is a *weight module* or $\mathfrak{h}$-*diagonalizable* if

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

where

$$M_\mu := \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}.$$

For each vector $v \in M_\mu$ let $\mathrm{wt}(v) := \mu$ be its *weight*. Analogously, one defines when a right $U(\mathfrak{g})$-module is a *weight module*.

A $U(\mathfrak{g})$-module M is a *highest weight module* if the following hold:

- M is a weight module;
- There is a vector $v \in M$ with $U(\mathfrak{g}) \cdot v = M$;
- $e_i \cdot v = 0$ for all i.

A right $U(\mathfrak{g})$-module M is a *lowest weight module* if the following hold:

- M is a weight module;
- There is a vector $u \in M$ with $u \cdot U(\mathfrak{g}) = M$;
- $u \cdot f_i = 0$ for all i.

When we work with right $U(\mathfrak{g})$-modules, we invert the usual ordering on weights. So if M is a lowest weight right $U(\mathfrak{g})$-module, then the vector u (which is uniquely determined up to a non-zero scalar) has actually the lowest weight of M. Indeed, if $m \in M_\mu$ and $h \in \mathfrak{h}$, then we have

$$(m \cdot e_i) \cdot h = \big(\mu(h)m\big) \cdot e_i - \big(\alpha_i(h)m\big) \cdot e_i = (\mu - \alpha_i)(h)(m \cdot e_i).$$

Here we used that $[h, e_i] = he_i - e_ih = \alpha_i(h)e_i$. So $m \cdot e_i$ has weight $\mu - \alpha_i$.

4.5. *Construction of highest weight modules*

In this section we present some of our results from [32] in a form convenient for our present purpose. For $\nu \in P^+$ we write

$$\hat{I}_\nu := \bigoplus_{i=1}^{n} \hat{I}_i^{\nu(\alpha_i^\vee)}.$$

For $1 \leqslant i \leqslant n$ and a nilpotent Λ-module X we denote by $\mathcal{G}(i, X)$ the variety of submodules Y of X such that $X/Y \cong S_i$. Similarly, if

$$\mathrm{soc}(X) = \bigoplus_{i=1}^{n} S_i^{m_i}$$

and $\nu \in P^+$ is such that $\nu(\alpha_i^\vee) \geqslant m_i$ for $1 \leqslant i \leqslant n$, then we have an embedding $X \hookrightarrow \hat{I}_\nu$. In this case, we denote by $\mathcal{G}(i, \nu, X)$ the variety of submodules Y of $\hat{I}_\nu$ such that $X \subset Y$ and $Y/X \cong S_i$. Hence, if $\underline{\dim}(X) = \beta$ and $f \in \mathcal{M}_{\beta-\alpha_i}$, we can form the following sum

$$\Sigma := \sum_{m \in \mathbb{C}} m\chi_c\big(\{Y \in \mathcal{G}(i, X) \mid f(Y) = m\}\big).$$

For convenience we shall denote such an expression by an integral, for example,

$$\Sigma = \int_{Y \in \mathcal{G}(i,X)} f(Y).$$

Similarly, there exists a partition

$$\mathcal{G}(i,X) = \bigsqcup_{j=1}^{m} A_j$$

into constructible subsets such that $\delta_Y = \delta_{Y'}$ for all $Y, Y' \in A_j$. Then, choosing arbitrary $Y_j \in A_j$ for $j = 1, \dots, m$, we can also denote by an integral the following element of $\mathcal{M}^*_{\beta-\alpha_i}$,

$$\int_{Y \in \mathcal{G}(i,X)} \delta_Y = \sum_{j=1}^{m} \chi_{\mathrm{c}}(A_j)\delta_{Y_j}.$$

Theorem 4.3. *Let $\lambda \in P$ be an integral weight, and let $M_{\mathrm{low}}(\lambda)$ be the lowest weight Verma right $U(\mathfrak{g})$-module (with underlying vector space $U(\mathfrak{n})$) with lowest weight λ. Under the identifications*

$$M_{\mathrm{low}}(\lambda) \equiv U(\mathfrak{n}) \equiv \mathcal{M}$$

the corresponding right $U(\mathfrak{g})$-module structure on $\mathcal{M}$ is described as follows: The generators $e_i \in \mathfrak{n}$, $f_i \in \mathfrak{n}_-$, $h \in \mathfrak{h}$ act on $g \in \mathcal{M}_\beta$ by

$$(g \cdot e_i)\bigl(X'\bigr) = \int_{Y \in \mathcal{G}(i,X')} g(Y),$$

$$(g \cdot f_i)(X) = \int_{Y \in \mathcal{G}(i,\nu,X)} g(Y) - (\nu - \lambda)\bigl(\alpha_i^\vee\bigr) g(X \oplus S_i),$$

$$g \cdot h = (\lambda - \beta)(h) g,$$

where $X' \in \Lambda_{\beta+\alpha_i}$, $X \in \Lambda_{\beta-\alpha_i}$ and $\nu \in P^+$ are as above.

Note that $g \cdot e_i = g * \mathbf{1}_i$ by our convention for the multiplication in $\mathcal{M}$. Moreover, the formula for $g \cdot f_i \in \mathcal{M}_{\beta-\alpha_i}$ is in fact independent of the choice of ν.

For each $\mathfrak{h}$-diagonalizable right $U(\mathfrak{g})$-module

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

one can consider the *dual* representation

$$M^* = \bigoplus_{\mu \in \mathfrak{h}^*} M^*_\mu$$

defined by $M_\mu^* := \mathrm{Hom}_{\mathbb{C}}(M_\mu, \mathbb{C})$. It acquires the structure of a *left* $U(\mathfrak{g})$-module via

$$(x \cdot \phi)(m) := \phi(m \cdot x) \quad \bigl(x \in U(\mathfrak{g}),\ m \in M\bigr).$$

Consider the canonical epimorphism from the Verma module $M_{\mathrm{low}}(\lambda)$ to the irreducible lowest weight right $U(\mathfrak{g})$-module $L_{\mathrm{low}}(\lambda)$. For the corresponding dual representations we obtain an inclusion

$$L_{\mathrm{low}}^*(\lambda) \hookrightarrow M_{\mathrm{low}}^*(\lambda).$$

It is well known that $L_{\mathrm{low}}^*(\lambda)$ is isomorphic to the irreducible highest weight left $U(\mathfrak{g})$-module $L(\lambda)$ with highest weight λ. This yields the following realization of the integrable module $L(\lambda)$ in terms of δ-functions.

Theorem 4.4. *Let $\lambda \in P^+$ be an integral dominant weight. The subspace*

$$U(\lambda) := \mathrm{Span}_{\mathbb{C}}\langle \delta_X \mid X \text{ submodule of } \hat{I}_\lambda \rangle$$

of $U(\mathfrak{n})_{\mathrm{gr}}^*$ carries the above-mentioned structure of an irreducible highest weight left $U(\mathfrak{g})$-module $L(\lambda)$. For such X with $\underline{\dim}(X) = \beta$ the action of the Chevalley generators of $U(\mathfrak{g})$ is given by

$$e_i \cdot \delta_X = \int_{Y \in \mathcal{G}(i,X)} \delta_Y,$$

$$f_i \cdot \delta_X = \int_{Y' \in \mathcal{G}(i,\lambda,X)} \delta_{Y'},$$

$$h \cdot \delta_X = (\lambda - \beta)(h)\delta_X.$$

Note that $U(\mathfrak{n})_{\mathrm{gr}}^*$ carries also a *right* $U(\mathfrak{n})$-module structure coming from the left regular representation of $U(\mathfrak{n})$. In order to describe it, we introduce the following definition. For $X \in \Lambda_\beta$ we denote by $\mathcal{G}'(i, X)$ the variety of submodules Y of X such that $\underline{\dim}(Y) = \alpha_i$. Each element of this space is isomorphic to S_i and clearly $\mathcal{G}'(i, X)$ is a projective space. It is easy to see that

$$\delta_X \cdot e_i = \int_{S \in \mathcal{G}'(i,X)} \delta_{X/S}.$$

Under the above identification $M_{\mathrm{low}}^*(\lambda) \equiv U(\mathfrak{n})_{\mathrm{gr}}^*$, the subspace of $U(\mathfrak{n})_{\mathrm{gr}}^*$ carrying the $U(\mathfrak{g})$-module $L(\lambda)$ can be described as follows.

Corollary 4.5. *For $\lambda \in P^+$ we have*

$$U(\lambda) = \bigl\{\phi \in U(\mathfrak{n})_{\mathrm{gr}}^* \bigm| \phi \cdot e_i^{\lambda(\alpha_i^\vee)+1} = 0 \text{ for all } 1 \leqslant i \leqslant n\bigr\}.$$

Proof. The nilpotent Λ-module X is isomorphic to a submodule of $\hat{I}_\lambda$ if and only if

$$\delta_X \cdot e_i^{\lambda(\alpha_i^\vee)+1} = 0$$

for every i. The claim then follows from Theorem 4.4. $\square$

Note that for $\lambda, \mu \in P^+$ we have $U(\lambda) = U(\mu)$ if and only if

$$\lambda - \mu \in \bigoplus_{j=n+1}^{2n-\mathrm{rank}(C_Q)} \mathbb{C}\varpi_j.$$

5. Unipotent groups

5.1. The group N and its coordinate ring $\mathbb{C}[N]$

The completion $\widehat{\mathfrak{n}}$ of $\mathfrak{n}$ defined in 4.3, is a pro-nilpotent pro-Lie algebra, see [48, Section 6.1.1]. Let N be the pro-unipotent pro-group with Lie algebra $\widehat{\mathfrak{n}}$. We refer to Kumar's book [48, Section 4.4] for all missing definitions.

We can assume that $N = \widehat{\mathfrak{n}}$ as a set and that the multiplication of N is defined via the Baker–Campbell–Hausdorff formula. Hence the exponential map $\mathrm{Exp} : \widehat{\mathfrak{n}} \to N$ is just the identity map.

Put $\mathcal{H} := U(\mathfrak{n})^*_{\mathrm{gr}}$. This is a commutative Hopf algebra. We can regard $\mathcal{H}$ as the coordinate ring $\mathbb{C}[N]$ of N, that is, we can identify N with the set

$$\max \mathrm{Spec}(\mathcal{H}) \equiv \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathbb{C})$$

of $\mathbb{C}$-algebra homomorphisms $\mathcal{H} \to \mathbb{C}$. An element $f \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathbb{C})$ is determined by the images $c_i := f(p_i^*)$ for all $i \geqslant 1$.

It is well known (see *e.g.* [1, Section 3.4]) that $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathbb{C})$ can also be identified with the group $G(\mathcal{H}^\circ)$ of all group-like elements of the dual Hopf algebra $\mathcal{H}^\circ$ of $\mathcal{H}$, by mapping $f \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathbb{C})$ to

$$y_f = \sum_{\mathbf{m}} \left(\prod_i c_i^{m_i} \right) p_{\mathbf{m}} \in G\left(\mathcal{H}^\circ\right).$$

Note that the map $f \mapsto y_f$ does not depend on the choice of the PBW-basis $\mathcal{P} = \{p_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^{(J)}\}$. Note also that $G(\mathcal{H}^\circ)$ is contained in the vector space dual $\mathcal{H}^*$ of $\mathcal{H}$, which is the completion $\widehat{U(\mathfrak{n})}$ of $U(\mathfrak{n})$ with respect to its natural grading. When we use this second identification, an element $x \in N = \widehat{\mathfrak{n}}$ is simply represented by the group-like element

$$\exp(x) := \sum_{k \geqslant 0} x^k / k!$$

in $\widehat{U(\mathfrak{n})}$. To summarize, we have $\mathcal{H} = U(\mathfrak{n})^*_{\mathrm{gr}} \equiv \mathbb{C}[N]$ and

$$N \equiv \max \mathrm{Spec}(\mathcal{H}) \equiv \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}, \mathbb{C}) \equiv G\left(\mathcal{H}^\circ\right) \subset \mathcal{H}^\circ \subset \mathcal{H}^* \equiv \widehat{U(\mathfrak{n})}.$$

5.2. The unipotent groups $N(w)$ and $N'(w)$

Let Θ be a bracket closed subset of Δ^+, and let

$$N(\Theta) := \operatorname{Exp}\bigl(\widehat{\mathfrak{n}}(\Theta)\bigr)$$

be the corresponding pro-unipotent pro-group. For example, if $\alpha \in \Delta^+_{\mathrm{re}}$, then $\Theta_\alpha := \{\alpha\}$ is bracket closed. In this case, $N(\alpha)$ is called the *one-parameter subgroup* of N associated to α. We have an isomorphism of groups $N(\alpha) \cong (\mathbb{C}, +)$.

If Θ is bracket closed and bracket coclosed, then set $N'(\Theta) := N(\Delta^+ \setminus \Theta)$. In this case, the multiplication in N yields a bijection [48, Lemma 6.1.2]

$$m : N(\Theta) \times N'(\Theta) \to N.$$

For $w \in W$ let $N(w) := N(\Delta^+_w)$. This is a unipotent algebraic group of dimension $l(w)$, and its Lie algebra is $\mathfrak{n}(w)$. Again we can identify $U(\mathfrak{n}(w))^*_{\mathrm{gr}} \equiv \mathbb{C}[N(w)]$. Similarly, define $N'(w) := N'(\Delta^+_w)$.

6. Evaluation functions and generating functions of Euler characteristics

Recall the identifications $\mathcal{M}^* \equiv U(\mathfrak{n})^*_{\mathrm{gr}} = \mathbb{C}[N]$. To every $X \in \operatorname{nil}(\Lambda)$, we have associated a linear form $\delta_X \in U(\mathfrak{n})^*_{\mathrm{gr}}$. We shall also denote the evaluation function δ_X by φ_X when we regard it as a function on N. For $1 \leqslant i \leqslant n$ define $x_i : \mathbb{C} \to N$ by

$$x_i(t) := \exp(te_i) = \sum_{k \geqslant 0} \frac{(te_i)^k}{k!}$$

The following formula shows how to evaluate φ_X on a product of $x_i(t)$'s.

Proposition 6.1. *Let $X \in \operatorname{nil}(\Lambda)$, and let $\mathbf{i} = (i_1, \dots, i_k)$ be any sequence with $1 \leqslant i_j \leqslant n$ for all $1 \leqslant j \leqslant k$. We have*

$$\varphi_X\bigl(x_{i_1}(t_1) \cdots x_{i_k}(t_k)\bigr) = \sum_{\mathbf{a}=(a_1,\dots,a_k) \in \mathbb{N}^k} \chi_{\mathrm{c}}(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X}) \frac{t_1^{a_1} \cdots t_k^{a_k}}{a_1! \cdots a_k!}.$$

Here $\mathbf{i}^{\mathbf{a}}$ is short for the sequence $(i_1, \dots, i_1, \dots, i_k, \dots, i_k)$ consisting of a_1 letters i_1 followed by a_2 letters i_2, etc.

Proof. By Section 5.1 we can regard $x_{i_1}(t_1) \cdots x_{i_k}(t_k)$ as an element of $\widehat{U(\mathbf{n})}$, namely,

$$x_{i_1}(t_1) \cdots x_{i_k}(t_k) = \sum_{\mathbf{a}=(a_1,\dots,a_k) \in \mathbb{N}^k} \frac{t_1^{a_1} \cdots t_k^{a_k}}{a_1! \cdots a_k!} e_{i_1}^{a_1} \cdots e_{i_k}^{a_k}.$$

It follows from the identification of φ_X with δ_X that

$$\varphi_X\big(x_{i_1}(t_1)\cdots x_{i_k}(t_k)\big) = \sum_{\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k} \frac{t_1^{a_1}\cdots t_k^{a_k}}{a_1!\cdots a_k!}\delta_X\big(e_{i_1}^{a_1}\cdots e_{i_k}^{a_k}\big).$$

Now, in the geometric realization $\mathcal{M}$ of the enveloping algebra $U(\mathfrak{n})$ in terms of constructible functions, $e_{i_1}^{a_1}\cdots e_{i_k}^{a_k}$ becomes the convolution product $\mathbf{1}_{i_1}^{a_1}\star\cdots\star\mathbf{1}_{i_k}^{a_k}$ and it is easy to see that

$$\delta_X\big(\mathbf{1}_{i_1}^{a_1}\star\cdots\star\mathbf{1}_{i_k}^{a_k}\big) = \chi_{\mathrm{c}}(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X}).$$

This finishes the proof. □

Remark 6.2. The formula for φ_X given in [33, Section 9] involves descending flags instead of ascending flags of submodules of X. This is because in the present paper we have taken a convolution product $\star$ opposite to that of our previous papers, see Remark 2.3.

Proposition 6.1 says that we can think of the φ-functions φ_X as generating functions of Euler characteristics.

For $\mathbf{i} = (i_1,\dots,i_k)$ and $\mathbf{a} = (a_1,\dots,a_k)$ as above and $X \in \mathrm{nil}(\Lambda)$ let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of partial composition series of type $(\mathbf{i},\mathbf{a})$ of X. Thus an element of $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is a chain

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X$$

of submodules of X such that $X_j/X_{j-1} \cong S_{i_j}^{a_j}$ for all $1 \leqslant j \leqslant k$. There is an obvious surjective morphism $\pi_{\mathbf{i},\mathbf{a}} : \mathcal{F}_{\mathbf{i}^{\mathbf{a}},X} \to \mathcal{F}_{\mathbf{i},\mathbf{a},X}$ whose fibers are all isomorphic to

$$\mathcal{F}\big(\mathbb{C}^{a_1}\big)\times\cdots\times\mathcal{F}\big(\mathbb{C}^{a_k}\big),$$

where $\mathcal{F}(\mathbb{C}^m)$ is the variety of complete flags of subspaces in $\mathbb{C}^m$. In particular, we have

$$\chi_{\mathrm{c}}(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X}) = \chi_{\mathrm{c}}(\mathcal{F}_{\mathbf{i},\mathbf{a},X})a_1!\cdots a_k!.$$

Summarizing, we get

$$\varphi_X\big(x_{i_1}(t_1)\cdots x_{i_k}(t_k)\big) = \sum_{\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k} \chi_{\mathrm{c}}(\mathcal{F}_{\mathbf{i},\mathbf{a},X})t_1^{a_1}\cdots t_k^{a_k}.$$

7. Generalized minors

7.1. Generalized minors

We start with some generalities on Kac–Moody groups. Let $G^{\min}$ be the Kac–Moody group with $\mathrm{Lie}(G^{\min}) = \mathfrak{g}$ defined in [48, 7.4]. It has a refined Tits system

$$\big(G^{\min}, \mathrm{Norm}_{G^{\min}}(H), N\cap G^{\min}, N_-, H\big).$$

Write $N^{\min} := G^{\min}\cap N$. Moreover, $G^{\min}$ is an affine ind-variety in a unique way [48, 7.4.8].

For any real root α of $\mathfrak{g}$, the one-parameter subgroup $N(\alpha)$ is contained in $G^{\min}$, and the $N(\alpha)$ together with H generate $G^{\min}$ as a group. We have an anti-automorphism $g \mapsto g^T$ of $G^{\min}$ which maps $N(\alpha)$ to $N(-\alpha)$ for each real root α, and fixes H. We have another anti-automorphism $g \mapsto g^\iota$ which fixes $N(\alpha)$ for every real root α, and $h^\iota = h^{-1}$ for every $h \in H$.

For each $\gamma \in \mathfrak{h}^*$ there is a character $H \to \mathbb{C}^*$, $a \mapsto a^\gamma$ defined by $\exp(h)^\gamma := e^{\gamma(h)}$ for all $h \in \mathfrak{h}$.

For $1 \leqslant i \leqslant n$ we have a unique homomorphism $\varphi_i : SL_2(\mathbb{C}) \to G^{\min}$ satisfying

$$\varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(te_i), \qquad \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(tf_i) \quad (t \in \mathbb{C}).$$

We define

$$\bar{s}_i := \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $w \in W$, we define $\overline{w} := \bar{s}_{i_r} \cdots \bar{s}_{i_1}$, where $(i_r, \dots, i_1)$ is a reduced expression for w. Thus, we choose for every $w \in W$ a particular representative $\overline{w}$ of w in the normalizer $\mathrm{Norm}_{G^{\min}}(H)$.

Let $L(\lambda)$ denote the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. Let u_λ be a highest weight vector of $L(\lambda)$. This is an integrable module, so it is also a representation of $G^{\min}$. For a reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of a Weyl group element w, the vector

$$\bar{s}_{i_1} \cdots \bar{s}_{i_r}(u_\lambda) \in L(\lambda)$$

is an *extremal weight vector* of $L(\lambda)$, *i.e.* it belongs to the extremal weight space $L(\lambda)_{w(\lambda)}$. For a $U(\mathfrak{g})$-module V and a weight vector $v \in V_\mu$ define

$$f_i^{\max} v := f_i^{(m)} v$$

where $m \geqslant 0$ is maximal such that $f_i^{(m)} v \neq 0$. Similarly, define $e_i^{\max}$. The following results can be found in [42, Section 4.4.3]: We have

$$\bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_\lambda) = f_{i_1}^{\max} f_{i_2}^{\max} \cdots f_{i_r}^{\max}(u_\lambda)$$

and

$$e_{i_1} f_{i_2}^{\max} \cdots f_{i_r}^{\max}(u_\lambda) = 0.$$

Furthermore,

$$\mathrm{wt}\big(\bar{s}_{i_1} \cdots \bar{s}_{i_r}(u_\lambda)\big) = \mathrm{wt}\big(\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_\lambda)\big) - b_1 \alpha_{i_1}$$

where $b_1 := -(s_{i_1} \cdots s_{i_r}(\lambda), \alpha_{i_1}) = (s_{i_2} \cdots s_{i_r}(\lambda), \alpha_{i_1})$.

We have the following analogue of the Gaussian decomposition.

Proposition 7.1. *Let G_0 be the subset $N_- \cdot H \cdot N^{\min}$ of $G^{\min}$.*

(i) *The subset G_0 is dense open in $G^{\min}$ and each element $g \in G_0$ admits a unique factorization $g = [g]_-[g]_0[g]_+$ with $[g]_- \in N_-$, $[g]_0 \in H$ and $[g]_+ \in N^{\min}$.*
(ii) *The map $g \mapsto [g]_+$ (resp. $g \mapsto [g]_0$) is a morphism of ind-varieties from G_0 to $N^{\min}$ (resp. to H).*

Part (i) follows from the fundamental properties of a refined Tits system [48, Theorem 5.2.3]. For part (ii), see [48, Proposition 7.4.11].

Following Fomin and Zelevinsky [22] we can now define for each ϖ_j a generalized minor $\Delta_{\varpi_j,\varpi_j}$ as the regular function on $G^{\min}$ such that

$$\Delta_{\varpi_j,\varpi_j}(g) = [g]_0^{\varpi_j} \quad (g \in G_0).$$

For $w \in W$, we also define $\Delta_{\varpi_j,w(\varpi_j)}$ by

$$\Delta_{\varpi_j,w(\varpi_j)}(g) := \Delta_{\varpi_j,\varpi_j}(g\overline{w}).$$

The generalized minors $\Delta_{\varpi_j,\varpi_j}(g)$ have the following alternative description.

Proposition 7.2. *Let $g \in G^{\min}$. The coefficient of u_{ϖ_j} in the projection of gu_{ϖ_j} on the weight space $L(\varpi_j)_{\varpi_j}$ is equal to $\Delta_{\varpi_j,\varpi_j}(g)$.*

Proof. Set $u_j := u_{\varpi_j}$. Let $g = [g]_-[g]_0[g]_+ \in G_0$. We have $[g]_+u_j = u_j$, and $[g]_0u_j = [g]_0^{\varpi_j}u_j$. The result then follows from the fact that $[g]_-u_j$ is equal to u_j plus elements in lower weights. □

Proposition 7.3. *We have*

$$G_0 = \left\{g \in G^{\min} \mid \Delta_{\varpi_j,\varpi_j}(g) \neq 0 \text{ for all } 1 \leqslant j \leqslant n\right\}.$$

Proof. Set $u_j := u_{\varpi_j}$. We use the Birkhoff decomposition [48, Theorem 5.2.3]

$$G^{\min} = \bigsqcup_{w \in W} N_-\overline{w}HN^{\min},$$

where G_0 is the subset of the right-hand side corresponding to $w = e$. If $g = [g]_-[g]_0[g]_+ \in G_0$, then $\Delta_{\varpi_j,\varpi_j}(g) = [g]_0^{\varpi_j} \neq 0$. Conversely, if $g \notin G_0$ we have $g = n_-whn$ for some $n_- \in N_-$, $n \in N^{\min}$, $h \in H$ and $w \neq e$. Then for some j we have $w(\varpi_j) \neq \varpi_j$ and $\overline{w}hnu_j$ is a multiple of the extremal weight vector $\overline{w}u_j$. Since the projection of $n_-\overline{w}u_j$ on the highest weight space of $L(\varpi_j)$ is zero, it follows that $\Delta_{\varpi_j,\varpi_j}(g) = 0$. Finally, note that for any $j > n$ the minor $\Delta_{\varpi_j,\varpi_j}$ does not vanish on $G^{\min}$. Indeed, the corresponding highest weight irreducible module $L(\varpi_j)$ is one-dimensional since $\varpi_j(\alpha_i^\vee) = 0$ for any i. Hence in the above description of G_0, we may omit the minors $\Delta_{\varpi_j,\varpi_j}$ with $j > n$. □

7.2. The module $L(\lambda)$ as a subspace of $\mathbb{C}[N]$

For $w \in W$ and $1 \leqslant j \leqslant n$, we denote by

$$D_{\varpi_j, w(\varpi_j)}$$

the restriction of the generalized minor $\Delta_{\varpi_j, w(\varpi_j)}$ to $N^{\min}$. For example, D_{ϖ_j, ϖ_j} is equal to the constant function 1. In Section 9.1 we are going to show that each (restricted) generalized minor $D_{\varpi_j, w(\varpi_j)}$ can be identified with a generating function φ_X for a certain Λ-module X. In order to do this, we need to recall some results on Kac–Moody groups.

Let $G' := [G^{\min}, G^{\min}]$ be the group constructed by Kac and Peterson [44], see [48, Section 7.4.E (1)]. The associated Lie algebra is $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Let $\mathbb{C}[G']_{\text{s.r.}}$ denote the algebra of strongly regular functions on G' [44, Section 2C]. Define the invariant ring

$$\mathbb{C}\big[N_- \backslash G'\big]_{\text{s.r.}} := \big\{ f \in \mathbb{C}\big[G'\big]_{\text{s.r.}} \bigm| f(ng) = f(g) \text{ for all } n \in N_-,\ g \in G' \big\}.$$

This ring is endowed with the usual left action of G' given by

$$(g \cdot f)\big(g'\big) := f\big(g'g\big) \quad \big(f \in \mathbb{C}\big[N_- \backslash G'\big]_{\text{s.r.}},\ g, g' \in G'\big).$$

It was proved by Kac and Peterson [44, Corollary 2.2] that as a left G'-module, it decomposes as follows

$$\mathbb{C}\big[N_- \backslash G'\big]_{\text{s.r.}} = \bigoplus_{\lambda \in \overline{P}^+} L(\lambda).$$

This is a multiplicity-free decomposition, in which the irreducible highest weight module $L(\lambda)$ is carried by the subspace

$$S(\lambda) = \big\{ f \in \mathbb{C}\big[N_- \backslash G'\big]_{\text{s.r.}} \bigm| f(hg) = \Delta_\lambda(h) f(g) \text{ for all } h \in H,\ g \in G' \big\},$$

where we denote

$$\Delta_\lambda := \prod_{j=1}^{n} \Delta_{\varpi_j, \varpi_j}^{\lambda(\alpha_j^\vee)}.$$

Clearly, Δ_λ is contained in $S(\lambda)$, and it is a highest weight vector. Moreover, for any $w \in W$, the 1-dimensional extremal weight space of $S(\lambda)$ with weight $w(\lambda)$ is spanned by

$$\Delta_{w(\lambda)} := \prod_{j=1}^{n} \Delta_{\varpi_j, w(\varpi_j)}^{\lambda(\alpha_j^\vee)}.$$

Now consider the restriction map

$$\rho : \mathbb{C}\big[N_- \backslash G'\big]_{\text{s.r.}} \to \mathbb{C}\big[N^{\min}\big]_{\text{s.r.}}$$

given by restriction of functions from G' to $N^{\min}$.

Lemma 7.4. *For every $\lambda \in \overline{P}^+$, the restriction*

$$\rho_\lambda : S(\lambda) \to \mathbb{C}\big[N^{\min}\big]_{\text{s.r.}}$$

of ρ to $S(\lambda)$ is injective.

Proof. Let B'_- be the Borel subgroup of G' with unipotent radical N_-. We have

$$N^{\min} \subset G_0 \cap G' = B'_- N^{\min}.$$

It follows that the natural projection from G' onto $B'_-\backslash G'$ restricts to an embedding of $N^{\min}$, with image the open subset of the flag variety $\mathcal{X} = B'_-\backslash G'$ defined by the non-vanishing of the minors $\Delta_{\varpi_j,\varpi_j}$. Now $\mathbb{C}[N_-\backslash G']_{\text{s.r.}}$ can be regarded as the multi-homogeneous coordinate ring of $\mathcal{X}$ with homogeneous components $S(\lambda)$, where λ runs through $\overline{P}^+$. It follows that $\mathbb{C}[N^{\min}]$ can be identified with the subring of degree 0 homogeneous elements of the localized ring obtained from $\mathbb{C}[N_-\backslash G']_{\text{s.r.}}$ by formally inverting the element

$$\Delta := \prod_{j=1}^{n} \Delta_{\varpi_j,\varpi_j}.$$

Therefore, the restriction ρ_λ of ρ to every homogeneous piece $S(\lambda)$ is an embedding. □

It follows that we can transport the G'-module structure from $S(\lambda)$ to $\rho(S(\lambda))$ by setting

$$g \cdot \varphi := \rho\big(g \cdot \rho_\lambda^{-1}(\varphi)\big) \quad \big(g \in G',\ \varphi \in \rho\big(S(\lambda)\big)\big).$$

In this way, we can identify the highest weight module $L(\lambda)$ with the subspace $\rho(S(\lambda))$ of $\mathbb{C}[N^{\min}]_{\text{s.r.}}$. The highest weight vector is now $\rho(\Delta_\lambda) = 1$, and the extremal weight vectors are the (restricted) generalized minors

$$D_{w(\lambda)} := \prod_{j=1}^{n} D_{\varpi_j,w(\varpi_j)}^{\lambda(\alpha_j^\vee)},$$

for $w \in W$.

At this point, we note that a strongly regular function on $N^{\min}$ is just the same as an element of $U(\mathfrak{n})^*_{\text{gr}}$. Indeed, the elements of $\mathbb{C}[N^{\min}]_{\text{s.r.}}$ are the restrictions to $N^{\min}$ of the linear combinations of matrix coefficients of the irreducible integrable representations $L(\lambda)$ with $\lambda \in \overline{P}^+$ of G', see [44, Lemma 4.2]. Now, by Theorem 4.4, we can realize every $L(\lambda)$ as a subspace of $U(\mathfrak{n})^*_{\text{gr}}$, and every $f \in U(\mathfrak{n})^*_{\text{gr}}$ belongs to such a subspace for $\lambda = \sum_{i=1}^n l_i\varpi_i$ with the $l_i \gg 0$. It follows that each element of $U(\mathfrak{n})^*_{\text{gr}}$ can be seen as a matrix coefficient for some $L(\lambda)$, and vice versa. We can therefore identify

$$\mathbb{C}\big[N^{\min}\big]_{\text{s.r.}} \equiv U(\mathfrak{n})^*_{\text{gr}} \equiv \mathbb{C}[N].$$

Moreover, these two ways of embedding $L(\lambda)$ in $\mathbb{C}[N]$ coincide.

Lemma 7.5. *Let $\lambda \in \overline{P}^+$. Under the identification $U(\mathfrak{n})^*_{\mathrm{gr}} \equiv \mathbb{C}[N^{\min}]_{\mathrm{s.r.}}$, the subspace $U(\lambda)$ defined in Theorem* 4.4 *coincides with $\rho(S(\lambda))$.*

Proof. The natural right action of $U(\mathfrak{n})$ on $U(\mathfrak{n})^*_{\mathrm{gr}}$ defined before Corollary 4.5 coincides with the right action of $U(\mathfrak{n})$ on $\mathbb{C}[N^{\min}]_{\mathrm{s.r.}}$ obtained by differentiating the right regular representation of $N^{\min}$:

$$(f \cdot n)(x) = f(nx) \quad \bigl(x, n \in N^{\min},\ f \in \mathbb{C}\bigl[N^{\min}\bigr]_{\mathrm{s.r.}}\bigr).$$

Consider first the case of a fundamental weight $\lambda = \varpi_j$. It is easy to check that

$$\Delta_{\varpi_j,\varpi_j}\bigl(x_i(t)g\bigr) = \begin{cases} \Delta_{\varpi_j,\varpi_j}(g) & \text{if } i \neq j, \\ \Delta_{\varpi_j,\varpi_j}(g) + t\Delta_{\varpi_j,\varpi_j}(\bar{s}_j g) & \text{if } i = j. \end{cases}$$

Now, the subspace $\rho(S(\lambda))$ is spanned by the functions $n \mapsto \Delta_{\varpi_j,\varpi_j}(ng)$, $(n \in N_-,\ g \in G')$. By differentiating the previous equation with respect to t and setting $t = 0$, we obtain that

$$\rho\bigl(S(\lambda)\bigr) \subseteq \bigl\{f \in \mathbb{C}\bigl[N^{\min}\bigr]_{\mathrm{s.r.}} \bigm| f \cdot e_i = 0 \text{ for } i \neq j,\ f \cdot e_j^2 = 0\bigr\}.$$

Hence, using Corollary 4.5, we see that $\rho(S(\lambda))$ is contained in the embedding of $L(\varpi_j)$ into the dual Verma module $M^*_{\mathrm{low}}(\varpi_j)$. Since these spaces have the same graded dimensions, they must coincide. The case of a general $\lambda \in \overline{P}^+$ follows using the fact that

$$\Delta_\lambda = \prod_{j=1}^{n} \Delta_{\varpi_j,\varpi_j}^{\lambda(\alpha_j^\vee)}$$

and that the e_i's act as derivations on $\mathbb{C}[N^{\min}]_{\mathrm{s.r.}}$. □

8. The coordinate rings $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$

8.1. The coordinate ring $\mathbb{C}[N(w)]$ as a ring of invariants

Again, we fix a reduced expression $\mathbf{i} = (i_r, \ldots, i_1)$ of a Weyl group element w. Assume that

$$\mathcal{P} = \bigl\{p_{\mathbf{m}} \bigm| \mathbf{m} \in \mathbb{N}^{(J)}\bigr\}$$

is an $\mathbf{i}$-compatible PBW-basis of $U(\mathfrak{n})$. Note that this PBW-basis of $U(\mathfrak{n})$ and also the corresponding dual PBW-basis of $U(\mathfrak{n})^*_{\mathrm{gr}}$ are homogeneous with respect to the (root lattice) $\mathbb{N}^n$-grading of $U(\mathfrak{n})$. We write $|\mathbf{m}| = d \in \mathbb{N}^n$ in case $p_{\mathbf{m}}$ is a homogeneous element of degree $d \in \mathbb{N}^n$. Let us denote by $(\mathbf{e}_i)_{i \in J}$ the usual coordinate vectors of $\mathbb{Z}^{(J)}$. For example, $|\mathbf{e}_k| = \beta_{\mathbf{i}}(k)$ for $1 \leqslant k \leqslant r$.

The multiplication $\mu : U(\mathfrak{n}) \otimes U(\mathfrak{n}) \to U(\mathfrak{n})$ is given by its effect on the PBW-basis, say

$$p_{\mathbf{m}} \cdot p_{\mathbf{n}} = \sum_{|\mathbf{k}| = |\mathbf{m}+\mathbf{n}|} c^{\mathbf{k}}_{\mathbf{m},\mathbf{n}}\, p_{\mathbf{k}}.$$

Next, the comultiplication $\mu^* : \mathbb{C}[N] \to \mathbb{C}[N] \otimes \mathbb{C}[N]$ is a ring homomorphism, so it is determined by the value on the generators $p_i^* = p_{\mathbf{e}_i}^*$. By construction, we have

$$\mu^*(p_i^*) = \sum_{|\mathbf{m}+\mathbf{n}|=|\mathbf{e}_i|} c_{\mathbf{m},\mathbf{n}}^{\mathbf{e}_i} (p_{\mathbf{m}}^* \otimes p_{\mathbf{n}}^*).$$

Lemma 8.1. *Let $1 \leqslant i \leqslant r$ and $0 \neq \mathbf{n} \in \mathbb{N}^{(J)}$ such that $n_j = 0$ for $1 \leqslant j \leqslant r$. Then $c_{\mathbf{m},\mathbf{n}}^{\mathbf{e}_i} = 0$.*

Proof. Let $\mathbf{m} = \mathbf{m}^< + \mathbf{m}^>$ such that $m_j^< = 0$ for $j > r$ and $m_j^> = 0$ for $1 \leqslant j \leqslant r$, so $p_{\mathbf{m}} = p_{\mathbf{m}^<} \cdot p_{\mathbf{m}^>}$. Since Δ_w^+ is bracket closed and coclosed we have

$$p_{\mathbf{m}^>} \cdot p_{\mathbf{n}} = \sum_{|\mathbf{k}'|=|\mathbf{m}^>+\mathbf{n}|} c_{\mathbf{m}^>,\mathbf{n}}^{\mathbf{k}'} p_{\mathbf{k}'}$$

with $k_j' = 0$ for $1 \leqslant j \leqslant r$. Thus

$$p_{\mathbf{m}} \cdot p_{\mathbf{n}} = \sum_{|\mathbf{k}'|=|\mathbf{m}^>+\mathbf{n}|} c_{\mathbf{m}^>,\mathbf{n}}^{\mathbf{k}'} p_{\mathbf{k}'+\mathbf{m}^<}.$$

Putting $\mathbf{k} = \mathbf{k}' + \mathbf{m}^<$ we get $c_{\mathbf{m},\mathbf{n}}^{\mathbf{k}} = c_{\mathbf{m}^>,\mathbf{n}}^{\mathbf{k}'}$. Thus, if in our situation $c_{\mathbf{m},\mathbf{n}}^{\mathbf{k}} \neq 0$ then $k_j \neq 0$ for some $k > r$. □

Now, let us turn to the subgroups $N(w)$ and $N'(w)$. Consider the ideals

$$I(w) := (p_{r+1}^*, p_{r+2}^*, \ldots), \qquad I'(w) := (p_1^*, \ldots, p_r^*)$$

in $\mathbb{C}[N]$. Then we have

$$N(w) = \{\nu \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[N], \mathbb{C}) \mid \nu(I(w)) = 0\}, \quad \text{and}$$
$$N'(w) = \{\nu' \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[N], \mathbb{C}) \mid \nu'(I'(w)) = 0\}.$$

In other words we have canonically $\mathbb{C}[N(w)] = \mathbb{C}[N]/I(w)$ and $\mathbb{C}[N'(w)] = \mathbb{C}[N]/I'(w)$.

We consider the action of $N'(w)$ on N via *right* multiplication. By definition, this comes from the left action of $N'(w)$ on $\mathbb{C}[N]$ given by

$$\nu' \cdot f = (\mathrm{id} \otimes \nu')\mu^*(f)$$

for $f \in \mathbb{C}[N]$ and $\nu' \in N'(w)$. (Here we identify $\mathbb{C}[N] \otimes \mathbb{C} \equiv \mathbb{C}[N]$ in the canonical way.)

We denote by $\mathbb{C}[N]^{N'(w)}$ the invariant subring for this group action.

Proposition 8.2. *Consider the injective ring homomorphism*

$$\tilde{\pi}_w^* : \mathbb{C}[N(w)] \to \mathbb{C}[N]$$

defined by $p_i^* + I(w) \mapsto p_i^*$ for $1 \leqslant i \leqslant r$. The corresponding morphism (of schemes) $\tilde{\pi}_w : N \to N(w)$ is $N'(w)$-invariant and is a retraction for the inclusion of $N(w)$ into N. As a consequence, $\tilde{\pi}_w^*$ identifies $\mathbb{C}[N(w)]$ with $\mathbb{C}[N]^{N'(w)} = \mathbb{C}[p_1^*, \dots, p_r^*]$.

Proof. We have

$$\mu^*(p_i^*) = 1 \otimes p_i^* + p_i^* \otimes 1 + \sum_{|\mathbf{m}+\mathbf{n}|=|\mathbf{e}_i|} c_{\mathbf{m},\mathbf{n}}^{\mathbf{e}_i}(p_{\mathbf{m}}^* \otimes p_{\mathbf{n}}^*)$$

where in the last sum $|\mathbf{m}| \neq 0 \neq |\mathbf{n}|$. Thus for $1 \leqslant i \leqslant r$ and $\nu' \in N'(w)$ we get

$$\nu' \cdot p_i^* = 1 \cdot 0 + p_i^* \cdot 1 + \sum_{|\mathbf{m}+\mathbf{n}|=|\mathbf{e}_i|} c_{\mathbf{m},\mathbf{n}}^{\mathbf{e}_i} p_{\mathbf{m}}^* \cdot \nu'(p_{\mathbf{n}}^*)$$

with the last sum vanishing by Lemma 8.1 and the definition of $N'(w)$. In other words, $p_i^* \in \mathbb{C}[N]^{N'(w)}$ for $1 \leqslant i \leqslant r$. Thus, $\tilde{\pi}_w : N \to N(w)$ is $N'(w)$-invariant, that is, $\tilde{\pi}_w(nn') = \tilde{\pi}_w(n)$ for any $n' \in N'(w)$.

Now, since the multiplication map $N(w) \times N'(w) \to N$ is bijective, each $N'(w)$-orbit on N is of the form $n \cdot N'(w)$ for a unique $n \in N(w)$. We conclude that the inclusion $N(w) \hookrightarrow N$ is a section for $\tilde{\pi}_w$. Our claim follows. $\square$

8.2. The coordinate ring $\mathbb{C}[N^w]$ as a localization of $\mathbb{C}[N]^{N'(w)}$

Let us now consider the groups $N(w)$ and $N'(w)$ introduced in Section 5.2.

Lemma 8.3. *We have*

$$N(w) = N \cap (w^{-1} N_- w),$$
$$N'(w) = N \cap (w^{-1} N w),$$
$$N'(w) \cap N^{\min} = N^{\min} \cap (w^{-1} N^{\min} w).$$

Proof. This follows from [48, 5.2.3] and [48, 6.2.8]. $\square$

It follows that $\Delta_{\varpi_j, w^{-1}(\varpi_j)}$ is invariant under the action of $N'(w) \cap N^{\min}$ on $G^{\min}$ via right multiplication. Indeed, for $g \in G^{\min}$ and $n' \in N'(w) \cap N^{\min}$, we have $n'\overline{w}^{-1} = \overline{w}^{-1} n''$ for some $n'' \in N'(w) \cap N^{\min}$, hence

$$\begin{aligned}\Delta_{\varpi_j, w^{-1}(\varpi_j)}(gn') &= \Delta_{\varpi_j, \varpi_j}(gn'\overline{w}^{-1}) = \Delta_{\varpi_j, \varpi_j}(g\overline{w}^{-1}n'') \\ &= \Delta_{\varpi_j, \varpi_j}(g\overline{w}^{-1}) = \Delta_{\varpi_j, w^{-1}(\varpi_j)}(g).\end{aligned}$$

Define

$$O_w := \{n \in N^{\min} \mid \Delta_{\varpi_j, w^{-1}(\varpi_j)}(n) \neq 0 \text{ for all } 1 \leqslant j \leqslant n\}.$$

This is the open subset of $N^{\min}$ consisting of elements n such that $\overline{w}n^T \in G_0$. Indeed,

$$\Delta_{\varpi_j, w^{-1}(\varpi_j)}(n) = \Delta_{\varpi_j,\varpi_j}\big(n\overline{w}^{-1}\big) = \Delta_{\varpi_j,\varpi_j}\big(\big(n\overline{w}^{-1}\big)^T\big) = \Delta_{\varpi_j,\varpi_j}\big(\overline{w}n^T\big),$$

since $\overline{w}^{-1} = \overline{w}^T$. Following [8, Section 5], we can now define the map $\tilde{\eta}_w : O_w \to N^{\min}$ given by

$$\tilde{\eta}_w(z) := \big[\overline{w}z^T\big]_+.$$

Recall that $N^w = N \cap (B_- w B_-)$, see Section 3.4.

Proposition 8.4. *The following properties hold*:

(i) *The map $\tilde{\eta}_w$ is a morphism of ind-varieties.*
(ii) *The image of $\tilde{\eta}_w$ is N^w.*
(iii) *$\tilde{\eta}_w(x) = \tilde{\eta}_w(y)$ if and only if $x = yn'$ for some $n' \in N'(w) \cap N^{\min}$.*
(iv) *$\tilde{\eta}_w$ restricts to a bijective morphism $N(w) \cap O_w \to N^w$.*
(v) *We have $N^w \subset O_w$, and $\tilde{\eta}_w$ restricts to a bijection $\eta_w : N^w \to N^w$.*
(vi) *The inverse of η_w is given by $\eta_w^{-1}(x) = \eta_{w^{-1}}(x^\iota)^\iota$ for $x \in N^w$. It follows that η_w is an automorphism of N^w.*

Proof. Property (i) follows from Proposition 7.1(ii). Next, we have

$$\big[\overline{w}z^T\big]_+ = \big(\big[\overline{w}z^T\big]_0^{-1}\big[\overline{w}z^T\big]_-^{-1}\big)\overline{w}z^T \in B_-\overline{w}B_-.$$

This shows that the image of $\tilde{\eta}_w$ is contained in N^w. The rest of (ii) and (iii) is proved as in [8, Proposition 5.1]. Property (iv) follows from (ii), (iii), and the decomposition $N^{\min} = N(w) \times (N'(w) \cap N^{\min})$. Finally, (v) and (vi) are proved exactly in the same way as in [8, Propositions 5.1, 5.2]. $\square$

Proposition 8.5. *The map $\tilde{\pi}_w$ restricts to a morphism $\pi_w : N^w \to O_w \cap N(w)$. This is an isomorphism with inverse*

$$\eta_w^{-1}\tilde{\eta}_w : O_w \cap N(w) \to N^w.$$

In particular, N^w is an affine variety with coordinate ring identified to the localized ring $\mathbb{C}[N]_{\Delta_w}^{N'(w)}$, where

$$\Delta_w := \prod_{j=1}^{n} \Delta_{\varpi_j, w^{-1}(\varpi_j)}.$$

Proof. By Proposition 8.4 (iv) and (v), we know that $\eta_w^{-1}\tilde{\eta}_w$ is a bijection. On the other hand $\tilde{\pi}_w(N^w) \subseteq O_w \cap N(w)$ because $N^w \subset O_w$. Now, by Proposition 8.4(iii), we have

$$\tilde{\eta}_w\big(\pi_w(x)\big) = \tilde{\eta}_w(x) = \eta_w(x)$$

for every $x \in N^w$. Hence $\eta_w^{-1}\tilde{\eta}_w\pi_w(x) = x$ for every x in N^w. So we have $\eta_w^{-1}\tilde{\eta}_w\pi_w = \mathrm{id}_{N^w}$, and this proves that π_w is the inverse of $\eta_w^{-1}\tilde{\eta}_w$.

These maps are morphisms of varieties so they induce isomorphisms

$$\mathbb{C}\big[N^w\big] \xrightarrow{\sim} \mathbb{C}\big[N(w) \cap O_w\big] = \mathbb{C}\big[N(w)\big]_{\Delta_w} \xrightarrow{\sim} \mathbb{C}[N]_{\Delta_w}^{N'(w)}. \qquad \square$$

The following commutative diagram displays the different morphisms appearing in Propositions 8.4 and 8.5:

$$\begin{tikzcd}
N \arrow[rr, "\tilde{\pi}_w"] & & N(w) & & O_w \arrow[rr, "\tilde{\eta}_w"] & & N^{\min} \\
& N^w \arrow[ul, "\iota"] \arrow[rr, "\sim", "\pi_w"'] \arrow[rrrr, bend right, "\eta_w"'] & & N(w) \cap O_w \arrow[ul, "\iota"'] \arrow[ur, "\iota"] \arrow[rr, "\sim"] & & N^w \arrow[ur, "\iota"'] &
\end{tikzcd}$$

(The arrows labeled with ι are inclusion maps.)

9. The modules V_k and M_k

For the entire section, we fix a reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of a Weyl group element w, and as before let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ and $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$. Recall that for each $1 \leqslant k \leqslant r$ there is a short exact sequence

$$0 \to V_{k^-} \to V_k \to M_k \to 0$$

of Λ-modules.

9.1. Generalized minors as φ-functions

For $1 \leqslant k \leqslant r$ set

$$w_{\leqslant k}^{-1} := s_{i_1} \cdots s_{i_k}.$$

Proposition 9.1. *For $1 \leqslant k \leqslant r$ we have*

$$\varphi_{V_k} = D_{\varpi_{i_k}, w_{\leqslant k}^{-1}(\varpi_{i_k})}.$$

In particular, we have $\varphi_{I_{\mathbf{i},j}} = D_{\varpi_j, w^{-1}(\varpi_j)}$ for every $1 \leqslant j \leqslant n$.

Proof. Using Lemma 7.5, we can realize the fundamental module $L(\varpi_{i_k})$ as the subspace $\rho(S(\varpi_{i_k}))$ of $\mathbb{C}[N]$. Then using Theorem 4.4, the definition of V_k (see Section 2.4) and the discussion in Section 7.1, we can check that the function φ_{V_k} is an extremal weight vector of weight $w_{\leqslant k}^{-1}(\varpi_{i_k})$ in $L(\varpi_{i_k})$, hence it coincides with $D_{\varpi_{i_k}, w_{\leqslant k}^{-1}(\varpi_{i_k})}$ up to a scalar. Moreover, its image under $e_{i_k}^{\max} \cdots e_{i_1}^{\max}$ is equal to 1, so the normalizations agree and we have $\varphi_{V_k} = D_{\varpi_{i_k}, w_{\leqslant k}^{-1}(\varpi_{i_k})}$. □

Corollary 9.2. *For $1 \leqslant k \leqslant r$ we have $\underline{\dim}(V_k) = \varpi_{i_k} - s_{i_1} s_{i_2} \cdots s_{i_k}(\varpi_{i_k})$.*

Proof. The statement follows from the following general fact: Assume that $\delta_X \in U(\lambda)$ for some weight $\lambda \in P^+$ and some Λ-module X. When we consider δ_X as an element of $L(\lambda) \equiv U(\lambda)$, Theorem 4.4 implies that $\mathrm{wt}(\delta_X) = \lambda - \underline{\dim}(X)$. □

Recall that for $1 \leqslant k \leqslant r$ we defined

$$\beta_{\mathbf{i}}(k) = \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{otherwise.} \end{cases}$$

Corollary 9.3. *For $1 \leqslant k \leqslant r$ we have $\underline{\dim}(M_k) = \beta_{\mathbf{i}}(k)$.*

Proof. By Corollary 9.2 we know that $\underline{\dim}(V_k) = \varpi_{i_k} - s_{i_1} s_{i_2} \cdots s_{i_k}(\varpi_{i_k})$ for each $1 \leqslant k \leqslant r$. By the definition of M_k we have

$$\begin{aligned} \underline{\dim}(M_k) &= \underline{\dim}(V_k) - \underline{\dim}(V_{k^-}) \\ &= s_{i_1} s_{i_2} \cdots s_{i_{k^-}}(\varpi_{i_k}) - s_{i_1} s_{i_2} \cdots s_{i_k}(\varpi_{i_k}) \\ &= s_{i_1} s_{i_2} \cdots s_{i_{k^-}}\big(\varpi_{i_k} - s_{i_{k^-+1}} \cdots s_{i_k}(\varpi_{i_k})\big). \end{aligned}$$

But

$$s_j(\varpi_{i_k}) = \begin{cases} \varpi_{i_k} & \text{if } j \neq i_k, \\ \varpi_{i_k} - \alpha_{i_k} & \text{if } j = i_k. \end{cases}$$

It follows that

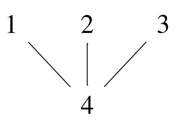
$$\begin{aligned} \underline{\dim}(M_k) &= s_{i_1} s_{i_2} \cdots s_{i_{k^-}}\big(\varpi_{i_k} - \varpi_{i_k} + s_{i_{k^-+1}} \cdots s_{i_{k-1}}(\alpha_{i_k})\big) \\ &= s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}). \end{aligned}$$

This finishes the proof. □

Corollary 9.4. *We have $\Delta_w^+ = \{\underline{\dim}(M_1), \ldots, \underline{\dim}(M_r)\}$.*

9.2. Example

Let Q be a quiver with underlying graph



Let w be the Weyl group element $s_3s_4s_2s_1s_4$. The set of reduced expressions for w is $R(w) = \{(3,4,2,1,4), (3,4,1,2,4)\}$. We have

$$\Delta_w^+ = \left\{ \begin{smallmatrix}0&0&0\\&1&\end{smallmatrix}, \begin{smallmatrix}1&0&0\\&1&\end{smallmatrix}, \begin{smallmatrix}0&1&0\\&1&\end{smallmatrix}, \begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}, \begin{smallmatrix}1&1&1\\&2&\end{smallmatrix} \right\}.$$

Let $\mathbf{i} = (3,4,2,1,4)$. We get

$$V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_5 = 4 \oplus \begin{smallmatrix}&4\\1&\end{smallmatrix} \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}&4\\1&2\\&4\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\1&2&\\&4&\\&&3\end{smallmatrix}$$

and

$$M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_5 = 4 \oplus \begin{smallmatrix}&4\\1&\end{smallmatrix} \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}&4\\1&2\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\1&2&\\&4&\\&&3\end{smallmatrix}.$$

Note that $\operatorname{add}(M_{\mathbf{i}})$ is neither closed under factor modules nor under submodules. We have

$$\mathcal{C}_w = \operatorname{add}\left(V_{\mathbf{i}} \oplus \begin{smallmatrix}&4\\1&2\end{smallmatrix}\right).$$

We can think of $\mathcal{C}_w$ as a categorification of a cluster algebra of type $\mathbb{A}_1$ with four coefficients.

9.3. Example

Let Q be a quiver with underlying graph $1 \,=\!=\, 2 \,—\, 3$. Then $\mathbf{i} := (i_7, \dots, i_1) := (3,1,2,3,1,2,1)$ is a reduced expression of a Weyl group element $w \in W_Q$. The indecomposable direct summands of $V_{\mathbf{i}}$ are

$$V_1 = 1 \qquad V_2 = \begin{smallmatrix}1&&1\\&2&\end{smallmatrix} \qquad V_3 = \begin{smallmatrix}1&&1&&1\\&2&&2&\\&&1&&\end{smallmatrix}$$

$$V_4 = \begin{smallmatrix}1&&1\\&2&\\&3&\end{smallmatrix} \qquad V_5 = \begin{smallmatrix}1&&1&&1&&1&&1&&1\\&2&&2&&&&2&&2&\\&&1&&&3&&&1&&\\&&&&&2&&&&&\end{smallmatrix}$$

$$V_6 = \begin{smallmatrix}1&&1&&1&&1&&1&&1&&1&&1&&1\\&2&&2&&&&2&&2&&&&2&&2&\\&&1&&&3&&&1&&&3&&&1&&\\&&&&&2&&&&&&2&&&&&\\&&&&&&&&1&&&&&&&&\end{smallmatrix} \qquad V_7 = \begin{smallmatrix}1&&1&&1&&1\\&2&&2&&2&\\&&1&&1&&\\&&&2&&&\\&&&3&&&\end{smallmatrix}.$$

Here, the Λ-modules are represented by their socle filtration. The indecomposable $\mathcal{C}_w$-projective–injective modules are V_5, V_6 and V_7. The corresponding functions φ_{V_k} are given by

$$\varphi_{V_1} = D_{\varpi_1, s_1(\varpi_1)} \qquad \varphi_{V_2} = D_{\varpi_2, s_1s_2(\varpi_2)} \qquad \varphi_{V_3} = D_{\varpi_1, s_1s_2s_1(\varpi_1)}$$

$$\varphi_{V_4} = D_{\varpi_3, s_1s_2s_1s_3(\varpi_3)} \qquad \varphi_{V_5} = D_{\varpi_2, s_1s_2s_1s_3s_2(\varpi_2)}$$

$$\varphi_{V_6} = D_{\varpi_1, s_1s_2s_1s_3s_2s_1(\varpi_1)} \qquad \varphi_{V_7} = D_{\varpi_3, s_1s_2s_1s_3s_2s_1s_3(\varpi_3)}.$$

9.4. Example

We continue to discuss the example from Section 3.5. Thus Q is a quiver with underlying graph $1 \text{ —— } 2 \text{ —— } 3 \text{ —— } 4$. Note that the Weyl group W_Q is the symmetric group S_5, and the generators s_i are the transpositions $(i, i+1)$. The generalized minors become ordinary minors. More precisely, for $w \in S_5$ and $i \in \{1,2,3,4,5\}$ we have

$$\Delta_{\varpi_i, w(\varpi_i)} = \Delta_{\{1,2,\dots,i\}, w(\{1,2,\dots,i\})},$$

since we may identify S_5 with the group of permutation matrices in GL_5. Here $\Delta_{I,J}$ denotes the minor in $\mathbb{C}[\mathrm{SL}_5]$ with row set I and column set J. As in Section 3.5 let $w := s_3 s_4 s_2 s_1 s_3 s_4 s_2 s_1$ and $\mathbf{i} := (i_8, \dots, i_1) := (3,4,2,1,3,4,2,1)$. We get

$$\begin{aligned} x_{\mathbf{i}}(t) &:= x_3(t_8)x_4(t_7)x_2(t_6)x_1(t_5)x_3(t_4)x_4(t_3)x_2(t_2)x_1(t_1) \\ &= \begin{pmatrix} 1 & t_5+t_1 & t_5t_2 & 0 & 0 \\ 0 & 1 & t_6+t_2 & t_6t_4 & t_6t_4t_3 \\ 0 & 0 & 1 & t_8+t_4 & t_8(t_7+t_3)+t_4t_3 \\ 0 & 0 & 0 & 1 & t_7+t_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

A straightforward calculation shows:

$$\begin{aligned} D_{\varpi_1, w_{\leqslant 1}^{-1}(\varpi_1)} &= D_{\{1\},\{2\}} = t_5 + t_1, \\ D_{\varpi_2, w_{\leqslant 2}^{-1}(\varpi_2)} &= D_{\{1,2\},\{2,3\}} = t_6(t_5+t_1) + t_2t_1, \\ D_{\varpi_4, w_{\leqslant 3}^{-1}(\varpi_4)} &= D_{\{1,2,3,4\},\{1,2,3,5\}} = t_7 + t_3, \\ D_{\varpi_3, w_{\leqslant 4}^{-1}(\varpi_3)} &= D_{\{1,2,3\},\{2,3,5\}} = t_8\big(t_7\big(t_6(t_5+t_1)+t_2t_1\big) + t_6t_3(t_5+t_1) + t_3t_2t_1\big) + t_4t_3t_2t_1, \\ D_{\varpi_1, w_{\leqslant 5}^{-1}(\varpi_1)} &= D_{\{1\},\{3\}} = t_5t_2, \\ D_{\varpi_2, w_{\leqslant 6}^{-1}(\varpi_2)} &= D_{\{1,2\},\{3,5\}} = t_6t_5t_4t_3t_2, \\ D_{\varpi_4, w_{\leqslant 7}^{-1}(\varpi_4)} &= D_{\{1,2,3,4\},\{2,3,4,5\}} = t_7t_4t_2t_1, \\ D_{\varpi_3, w_{\leqslant 8}^{-1}(\varpi_3)} &= D_{\{1,2,3\},\{3,4,5\}} = t_8t_7t_6t_5t_4t_2. \end{aligned}$$

Here the evaluation of the minors is always on $x_{\mathbf{i}}(t)$. Due to the structure of the modules V_k described in Section 3.5, we could also use Proposition 6.1 and calculate directly that

$$\varphi_{V_k}\big(x_{\mathbf{i}}(t)\big) = D_{\varpi_{i_k}, w_{\leqslant k}^{-1}(\varpi_{i_k})}\big(x_{\mathbf{i}}(t)\big)$$

for all $1 \leqslant k \leqslant 8$.

9.5. Refined socle and top series

For any Λ-module $X \in \mathcal{C}_w$ there exists a unique chain

$$0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X$$

of submodules of X such that $X_{k-1}/X_k = \mathrm{soc}_{S_{i_k}}(X/X_k)$. This is called the *refined socle series* of type $\mathbf{i}$ of X. Define

$$\mathbf{s_i}(X) := (p_r, \ldots, p_1)$$

where $p_k := \dim(X_{k-1}/X_k)$ for $1 \leqslant k \leqslant r$. Similarly, there exists a unique chain

$$0 = Y_r \subseteq \cdots \subseteq Y_1 \subseteq Y_0 = X$$

of submodules of X such that $Y_{k-1}/Y_k = \mathrm{top}_{S_{i_k}}(Y_{k-1})$ for all $1 \leqslant k \leqslant r$. This is called the *refined top series* of type $\mathbf{i}$ of X. Define

$$\mathbf{t_i}(X) := (q_r, \ldots, q_1)$$

where $q_k := \dim(Y_{k-1}/Y_k)$ for $1 \leqslant k \leqslant r$. (For a simple module S and a module M let $\mathrm{top}_S(M)$ be the intersection of all submodules U of M with $M/U \cong S$.)

The existence of refined socle and top series of type $\mathbf{i}$ of $X \in \mathcal{C}_w$ comes from the fact that $V_\mathbf{i}$ generates the category $\mathcal{C}_w$. It follows directly from the definitions that each module V_k has a refined socle and top series of type $\mathbf{i}$. Now one easily checks that this property also holds for factor modules of modules in $\mathrm{add}(V_\mathbf{i})$.

The uniqueness of refined socle and top series of type $\mathbf{i}$ implies the following result:

Lemma 9.5. *Let* $\mathbf{i} = (i_r, \ldots, i_1)$ *be a reduced expression of* w, *and let* $X \in \mathcal{C}_w$. *Set* $\mathbf{s} := \mathbf{s_i}(X) = (p_r, \ldots, p_1)$ *and* $\mathbf{t} := \mathbf{t_i}(X) = (q_r, \ldots, q_1)$. *Then the following hold*:

(i) *We have*

$$\mathcal{F}_{\mathbf{i}^\mathbf{s}, X} \cong \prod_{k=1}^{r} \mathcal{F}\big(\mathbb{C}^{p_k}\big) \quad \textit{and} \quad \mathcal{F}_{\mathbf{i}^\mathbf{t}, X} \cong \prod_{k=1}^{r} \mathcal{F}\big(\mathbb{C}^{q_k}\big).$$

In particular,

$$\chi_\mathrm{c}(\mathcal{F}_{\mathbf{i}^\mathbf{s}, X}) = \prod_{k=1}^{r} p_k! \quad \textit{and} \quad \chi_\mathrm{c}(\mathcal{F}_{\mathbf{i}^\mathbf{t}, X}) = \prod_{k=1}^{r} q_k!.$$

(ii) $\mathcal{F}_{\mathbf{i},\mathbf{s},X}$ *and* $\mathcal{F}_{\mathbf{i},\mathbf{t},X}$ *both consist of a single point. In particular,* $\chi_\mathrm{c}(\mathcal{F}_{\mathbf{i},\mathbf{s},X}) = 1$ *and* $\chi_\mathrm{c}(\mathcal{F}_{\mathbf{i},\mathbf{t},X}) = 1$.

Observe that $(i_k, \dots, i_t)$ is a reduced expression for the Weyl group element $w_{k,t} := s_{i_k} s_{i_{k-1}} \cdots s_{i_t}$ for all $1 \leqslant t \leqslant k \leqslant r$. Set $\mathbf{j} := (i_r, \dots, i_2)$. For $1 \leqslant k \leqslant r$ define

$$b_k := b_{\mathbf{i},k} := -\big(s_{i_k} \cdots s_{i_r}(\varpi_{i_r}), \alpha_{i_k}\big) = \big(s_{i_{k+1}} \cdots s_{i_r}(\varpi_{i_r}), \alpha_{i_k}\big),$$

and set $\mathbf{b_i} := (b_r, \dots, b_1)$.

Proposition 9.6. *For* $\mathbf{i}$ *and* $\mathbf{j}$ *as above, the following hold*:

(i) $\mathrm{top}_{S_{i_1}}(V_{\mathbf{j},r-1}) = 0$;
(ii) $\mathrm{top}_{S_{i_1}}(V_{\mathbf{i},r}) = S_{i_1}^{b_1}$;
(iii) $\mathbf{s_i}(V_{\mathbf{i},r}) = \mathbf{t_i}(V_{\mathbf{i},r}) = b_{\mathbf{i}}$.

Proof. For $r = 1$ the statements are obvious. Thus assume $r \geqslant 2$. Let $u_{\varpi_{i_r}}$ be a highest weight vector in $L(\varpi_{i_r})$. Since $\mathbf{i} = (\mathbf{j}, i_1)$ is a reduced expression, we know from Section 7.1 that

$$e_{i_1}\big(\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}})\big) = 0. \tag{3}$$

By Proposition 9.1 we can identify $\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}})$ with $\varphi_{V_{\mathbf{j},r-1}}$. We have $\mathrm{top}_{S_{i_1}}(V_{\mathbf{j},r-1}) \cong S_{i_1}^c$ for some $c \geqslant 0$. Let U be the unique submodule such that $V_{\mathbf{j},r-1}/U = \mathrm{top}_{S_{i_1}}(V_{\mathbf{j},r-1})$. We get

$$e_{i_1}^{(c)} \varphi_{V_{\mathbf{j},r-1}} = \varphi_U \neq 0.$$

But if $c \geqslant 1$, then Eq. (3) yields $e_{i_1}^{(c)} \varphi_{V_{\mathbf{j},r-1}} = 0$, a contradiction. This implies $c = 0$. So we proved (i). To show (ii) we use that $\varphi_{V_{\mathbf{i},r}}$ can be identified with

$$\bar{s}_{i_1}\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}}) = f_{i_1}^{\max}\big(\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}})\big) = f_{i_1}^{\max}(\varphi_{V_{\mathbf{j},r-1}}).$$

We have $\mathrm{wt}(\bar{s}_{i_1} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}})) = \mathrm{wt}(\bar{s}_{i_2} \cdots \bar{s}_{i_r}(u_{\varpi_{i_r}})) - b_1\alpha_{i_1}$, see Section 7.1. This implies (ii). Finally, it follows by induction on r that $\mathbf{s_i}(V_{\mathbf{i},r}) = \mathbf{t_i}(V_{\mathbf{i},r}) = b_{\mathbf{i}}$. This finishes the proof. □

9.6. *Computation of the Euler characteristics* $\chi_c(\mathcal{F}_{\mathbf{k},V_k})$

By Proposition 6.1, to evaluate φ_{V_k} on $x_{j_1}(t_1) \cdots x_{j_p}(t_p)$, we need to know the Euler characteristic $\chi_c(\mathcal{F}_{\mathbf{k},V_k})$ for arbitrary types $\mathbf{k}$ of composition series. These Euler characteristics can in turn be calculated via a simple algorithm that we shall now describe.

To this end, it will be convenient to embed $U(\mathfrak{n})^*_{\mathrm{gr}} \equiv \mathbb{C}[N]$ in the shuffle algebra F^*, as explained in [49, Section 2.8]. As a $\mathbb{C}$-vector space, F^* has a basis consisting of all words

$$w[\mathbf{k}] := w[k_1, k_2, \dots, k_s] := w_{k_1} w_{k_2} \cdots w_{k_s} \quad (1 \leqslant k_1, \dots, k_s \leqslant n,\ s \geqslant 0),$$

in the letters $w_1, \dots, w_n$. The multiplication in F^* is the classical commutative shuffle product $\sqcup\!\sqcup$ of words with unit the empty word $w[]$, see *e.g.* [53] and [49, Section 2.5]. By [49, Propositions 9 and 10], for any $X \in \mathrm{nil}(\Lambda)$ the image of φ_X in this embedding is just the *generating function*

$$g_X := \sum_{\mathbf{k}} \chi_c(\mathcal{F}_{\mathbf{k},X}) w[\mathbf{k}]$$

of the Euler characteristics $\chi_c(\mathcal{F}_{\mathbf{k},X})$ for all types $\mathbf{k}$ of composition series. (The Euler characteristic $\chi_c(\mathcal{F}_{\mathbf{k},X})$ is equal to the coefficient of $t_1 \cdots t_s$ in $\varphi_X(x_{k_1}(t_1) \cdots x_{k_s}(t_s))$.)

Let $\lambda \in P^+$ and $1 \leqslant i \leqslant n$. Define endomorphisms $\rho_\lambda(e_i)$, $\rho_\lambda(f_i)$ of the vector space F^* by

$$\rho_\lambda(e_i)\big(w[j_1, \dots, j_k]\big) := \delta_{i,j_k} w[j_1, \dots, j_{k-1}],$$

$$\rho_\lambda(f_i)\big(w[j_1, \dots, j_k]\big) := \sum_{l=0}^{k} (\lambda - \alpha_{j_1} - \cdots - \alpha_{j_l})\big(\alpha_i^\vee\big) w[j_1, \dots, j_l, i, j_{l+1}, \dots, j_k].$$

Proposition 9.7. *The formulas above extend to a representation $\rho_\lambda : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(F^*)$ of $U(\mathfrak{g})$. This turns F^* into a $U(\mathfrak{g})$-module. The image of $\mathbb{C}[N]$ in its embedding in F^* is a $U(\mathfrak{g})$-submodule isomorphic to the dual Verma module $M^*_{\mathrm{low}}(\lambda)$, see Section* 4.5. *In particular the set*

$$\big\{\rho_\lambda(f_{i_1} \cdots f_{i_s})\big(w[]\big) \bigm| 1 \leqslant i_1, \dots, i_s \leqslant n,\ s \geqslant 0\big\}$$

spans the irreducible module $L(\lambda)$, considered as a submodule of $M^*_{\mathrm{low}}(\lambda)$.

The above formulas for $\rho_\lambda(e_i)$ and $\rho_\lambda(f_i)$ can be obtained by specializing q to 1 in the formulas of the proof of [49, Proposition 50]. We omit the details.

By Proposition 9.1, for $1 \leqslant k \leqslant r$ we have

$$\varphi_{V_k} = D_{\varpi_{i_k}, w_{\leqslant k}^{-1}(\varpi_{i_k})}.$$

By Section 7.1 we know that φ_{V_k} is obtained by acting on the highest weight vector $u_{\varpi_{i_k}}$ of $L(\varpi_{i_k})$ with the product $f_{i_1}^{(b_1)} \cdots f_{i_k}^{(b_k)}$ of divided powers of the Chevalley generators, where $b_k = b_{\mathbf{i},k}$ is defined as in Section 9.5. Therefore we have

$$g_{V_k} = \rho_{\varpi_{i_k}}\big(f_{i_1}^{(b_1)} \cdots f_{i_k}^{(b_k)}\big)\big(w[]\big). \tag{4}$$

Hence to calculate the generating function g_{V_k} one only needs to apply $b_1 + \cdots + b_k = \dim(V_k)$ times the above combinatorial formula for $\rho_{\varpi_{i_k}}(f_i)$. Thus we have obtained an algorithm for calculating all Euler characteristics $\chi_c(\mathcal{F}_{\mathbf{k},V_k})$.

9.7. Example

We continue the example of Section 9.3. Clearly, we have

$$g_{V_1} = \rho_{\varpi_1}(f_1)\big(w[]\big) = \varpi_1\big(\alpha_1^\vee\big) w[1] = w[1].$$

Similarly

$$g_{V_2} = \rho_{\varpi_2}\big(f_1^{(2)} f_2\big)\big(w[]\big).$$

Now we calculate successively

$$\begin{aligned}
\rho_{\varpi_2}(f_2)\big(w[]\big) &= \varpi_2\big(\alpha_2^\vee\big)w[2] = w[2],\\
\rho_{\varpi_2}(f_1)\big(w[2]\big) &= \varpi_2\big(\alpha_1^\vee\big)w[1,2] + (\varpi_2-\alpha_2)\big(\alpha_1^\vee\big)w[2,1] = 2w[2,1],\\
\rho_{\varpi_2}(f_1)\big(2w[2,1]\big) &= 2\big(\varpi_2\big(\alpha_1^\vee\big)w[1,2,1] + (\varpi_2-\alpha_2)\big(\alpha_1^\vee\big)w[2,1,1]\\
&\quad + (\varpi_2-\alpha_2-\alpha_1)\big(\alpha_1^\vee\big)w[2,1,1]\big)\\
&= 4w[2,1,1].
\end{aligned}$$

Hence, taking into account that $f_1^{(2)} = f_1^2/2$, we get

$$g_{V_2} = 2w[2,1,1].$$

Similar applications of formula (4) yield the following results

$$\begin{aligned}
g_{V_3} &= \rho_{\varpi_1}\big(f_1^{(3)} f_2^{(2)} f_1\big)\big(w[]\big) = 4w[1,2,1,2,1,1] + 12w[1,2,2,1,1,1],\\
g_{V_4} &= \rho_{\varpi_3}\big(f_1^{(2)} f_2 f_3\big)\big(w[]\big) = 2w[3,2,1,1],\\
g_{V_7} &= \rho_{\varpi_3}\big(f_1^{(4)} f_2^{(3)} f_1^{(2)} f_2 f_3\big)\big(w[]\big)\\
&= 288w[3,2,1,1,2,2,2,1,1,1,1] + 144w[3,2,1,1,2,2,1,2,1,1,1]\\
&\quad + 96w[3,2,1,2,1,2,2,1,1,1,1] + 48w[3,2,1,1,2,2,1,1,2,1,1]\\
&\quad + 48w[3,2,1,2,1,1,2,2,1,1,1] + 48w[3,2,1,2,1,2,1,2,1,1,1]\\
&\quad + 48w[3,2,1,1,2,1,2,2,1,1,1] + 16w[3,2,1,2,1,2,1,1,2,1,1]\\
&\quad + 16w[3,2,1,2,1,1,2,1,2,1,1] + 16w[3,2,1,1,2,1,2,1,2,1,1].
\end{aligned}$$

The generating functions g_{V_5} and g_{V_6} are too large to be included here. For example g_{V_5} is a linear combination of 402 words.

9.8. The modules $M[b,a]$

For $1 \leqslant k \leqslant r$ let

$$\begin{aligned}
k^- &:= \max\{0, 1 \leqslant s \leqslant k-1 \mid i_s = i_k\},\\
k^+ &:= \min\{k+1 \leqslant s \leqslant r, r+1 \mid i_s = i_k\},\\
k_{\min} &:= \min\{1 \leqslant s \leqslant r \mid i_s = i_k\},\\
k_{\max} &:= \max\{1 \leqslant s \leqslant r \mid i_s = i_k\}.
\end{aligned}$$

Set $k^{(0)} := k$, and for an integer m define $k^{(m-1)} := (k^{(m)})^-$ and $k^{(m+1)} := (k^{(m)})^+$. For $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant r+1$ let

$$k^-(j) := \max\{0, 1 \leqslant s \leqslant k-1 \mid i_s = j\},$$

and

$$k[j] := \left|\{1 \leqslant s \leqslant k-1 \mid i_s = j\}\right|,$$

and set $t_j := (r+1)[j]$.

For $1 \leqslant a \leqslant b \leqslant r$ with $i_a = i_b$ define $M[b,a] := V_b / V_{a^-}$. (For convenience, we define $V_0 = V_{r+1} = 0$.) We have a short exact sequence

$$0 \to M\left[a^-, b_{\min}\right] \to M[b, b_{\min}] \to M[b,a] \to 0.$$

Note that $a_{\min} = b_{\min}$, since we assume $i_a = i_b$. For $1 \leqslant k \leqslant r$ we have $M[k, k_{\min}] = V_k$ and $M[k,k] = M_k$. One can visualize a module $M[b,a]$ by

$$\begin{array}{c} M_b \\ \hline M_{b^-} \\ \hline \cdots \\ \hline M_a \end{array}$$

We have

$$V_{\mathbf{i}} = \bigoplus_{k=1}^{r} M[k, k_{\min}].$$

For each k we have a short exact sequence

$$0 \to M[k, k_{\min}] \to M[k_{\max}, k_{\min}] \to M\left[k_{\max}, k^+\right] \to 0.$$

Note that $M[k_{\max}, k_{\min}] = I_{\mathbf{i}, i_k}$ is $\mathcal{C}_w$-projective–injective. Define

$$T_k := T_{\mathbf{i},k} := \begin{cases} V_k & \text{if } k^+ = r+1, \\ M[k_{\max}, k^+] & \text{otherwise.} \end{cases}$$

Thus if $k^+ \neq r+1$, then $\Omega_w^{-1}(V_k) = T_k$. Define $T_{\mathbf{i}} := T_1 \oplus \cdots \oplus T_r$. In other words, we have

$$T_{\mathbf{i}} = \bigoplus_{k=1}^{r} M[k_{\max}, k] = I_w \oplus \Omega_w^{-1}(V_{\mathbf{i}}).$$

9.9. Computation of $\dim \operatorname{Hom}_\Lambda(V_k, M_s)$

Lemma 9.8. *Let* $1 \leqslant k, s \leqslant r$.

(i) *If* $k \leqslant s$, *then we have*

$$\dim \operatorname{Hom}_\Lambda(V_k, M_s) = \dim \operatorname{Hom}_\Lambda(M_k, M_s) \cong \begin{cases} 0 & \text{if } k < s, \\ 1 & \text{if } k = s. \end{cases}$$

(ii) *If $k > s$, then*

$$\dim \operatorname{Hom}_\Lambda(V_k, M_s) = \begin{cases} \sum_{m \geqslant 0, k^{(-m)} > s} (M_{k^{(-m)}}, M_s)_Q & \text{if } i_k \neq i_s, \\ 1 + \sum_{m \geqslant 0, k^{(-m)} > s} (M_{k^{(-m)}}, M_s)_Q & \text{if } i_k = i_s. \end{cases}$$

(iii) *We have*

$$\dim \operatorname{Hom}_\Lambda(V_k, V_s) = \dim \operatorname{Hom}_\Lambda(V_k, M_s \oplus M_{s^-} \oplus \cdots \oplus M_{s_{\min}}).$$

Proof. We have short exact sequences

$$\eta : 0 \to V_{k^-} \xrightarrow{\iota_k} V_k \xrightarrow{\pi_k} M_k \to 0 \quad \text{and} \quad \psi : 0 \to V_{s^-} \xrightarrow{\iota_s} V_s \xrightarrow{\pi_s} M_s \to 0.$$

First, assume that $k < s$. Then the module M_k is contained in $\mathcal{C}_{(i_s,\dots,i_1)}$ and also in $\mathcal{C}_{(i_{s-1},\dots,i_1)}$ Now V_s is $\mathcal{C}_{(i_s,\dots,i_1)}$-projective–injective and V_{s^-} is $\mathcal{C}_{(i_{s-1},\dots,i_1)}$-projective–injective. This implies

$$\dim \operatorname{Hom}_\Lambda(M_k, V_{s^-}) = \dim \operatorname{Hom}_\Lambda(M_k, V_s) \quad \text{and} \quad \dim \operatorname{Ext}^1_\Lambda(M_k, V_{s^-}) = 0.$$

Now apply $\operatorname{Hom}_\Lambda(M_k, -)$ to the sequence ψ and get $\operatorname{Hom}_\Lambda(M_k, M_s) = 0$. Next, apply $\operatorname{Hom}_\Lambda(-, M_s)$ to η. We have $\operatorname{Hom}_\Lambda(M_k, M_s) = 0$ and by induction we also get $\operatorname{Hom}_\Lambda(V_{k^-}, M_s) = 0$. This implies $\operatorname{Hom}_\Lambda(V_k, M_s) = 0$.

Next, let $k = s$. We apply $\operatorname{Hom}_\Lambda(-, M_k)$ to η. Since $\operatorname{Hom}_\Lambda(V_{k^-}, M_k) = 0$, we get $\dim \operatorname{Hom}_\Lambda(V_k, M_k) = \dim \operatorname{Hom}_\Lambda(M_k, M_k)$.

Applying $\operatorname{Hom}_\Lambda(V_k, -)$ to η gives an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(V_k, V_{k^-}) \xrightarrow{\operatorname{Hom}_\Lambda(V_k, \iota_k)} \operatorname{Hom}_\Lambda(V_k, V_k) \xrightarrow{\operatorname{Hom}_\Lambda(V_k, \pi_k)} \operatorname{Hom}_\Lambda(V_k, M_k) \to 0.$$

Here we use that V_{k^-} is contained in $\mathcal{C}_{(i_k,\dots,i_1)}$ and V_k is $\mathcal{C}_{(i_k,\dots,i_1)}$-projective–injective. Thus every homomorphism $h : V_k \to M_k$ factors through π_k. In other words, there exists some $g : V_k \to V_k$ such that $\pi_k \circ g = h$. Now V_k is indecomposable, so the endomorphism ring $\operatorname{End}_\Lambda(V_k)$ is local. Therefore $g = \lambda \mathrm{id}_{V_k} + g'$ for some nilpotent endomorphism g' and some $\lambda \in K$. Now we easily see that the image of g' is contained in $\iota_k(V_{k^-})$. Thus $h = \lambda \pi_k$. This implies $\dim \operatorname{Hom}_\Lambda(V_k, M_k) = 1$.

Finally, assume that $k > s$. Then Lemma 2.1 yields

$$\begin{aligned} \dim \operatorname{Ext}^1_\Lambda(V_k, M_s) &= \dim \operatorname{Hom}_\Lambda(V_k, M_s) + \dim \operatorname{Hom}_\Lambda(M_s, V_k) - (V_k, M_s)_Q \\ &= \dim \operatorname{Hom}_\Lambda(V_k, M_s) + \dim \operatorname{Hom}_\Lambda(M_s, V_k) - (V_{k^-}, M_s)_Q - (M_k, M_s)_Q \\ &= \dim \operatorname{Hom}_\Lambda(V_k, M_s) + \dim \operatorname{Hom}_\Lambda(M_s, V_k) + \dim \operatorname{Ext}^1_\Lambda(V_{k^-}, M_s) \\ &\quad - \dim \operatorname{Hom}_\Lambda(V_{k^-}, M_s) - \dim \operatorname{Hom}_\Lambda(M_s, V_{k^-}) - (M_k, M_s)_Q. \end{aligned}$$

Since $s < k$, we have $\dim \operatorname{Hom}_\Lambda(M_s, V_{k^-}) = \dim \operatorname{Hom}_\Lambda(M_s, V_k)$ and $\operatorname{Ext}^1_\Lambda(V_k, M_s) = \operatorname{Ext}^1_\Lambda(V_{k^-}, M_s) = 0$. Thus we get

$$\dim \operatorname{Hom}_\Lambda(V_k, M_s) = (M_k, M_s)_Q + \dim \operatorname{Hom}_\Lambda(V_{k^-}, M_s).$$

The result follows by induction.

To prove (iii) we just apply $\mathrm{Hom}_\Lambda(V_k, -)$ to the short exact sequence $0 \to V_{s^-} \to V_s \to M_s \to 0$, and then use induction. $\square$

Note that in general we have $\dim \mathrm{Hom}_\Lambda(V_k, M_s) \neq \dim \mathrm{Hom}_\Lambda(M_k, M_s)$.

Corollary 9.9. *For $1 \leqslant k \leqslant r$ we have* $\mathrm{Ext}^1_\Lambda(M_k, M_k) = 0$.

Proof. Again we use the short exact sequence

$$\eta : 0 \to V_{k^-} \to V_k \to M_k \to 0.$$

The three modules in this sequence are contained in $\mathcal{C}_{(i_k,\dots,i_1)}$. In particular, V_k is $\mathcal{C}_{(i_k,\dots,i_1)}$-projective–injective. This implies $\mathrm{Ext}^1_\Lambda(V_k, M_k) = 0$. We have $\mathrm{Hom}_\Lambda(V_{k^-}, M_k) = 0$ by Lemma 9.8. Thus, applying the functor $\mathrm{Hom}_\Lambda(-, M_k)$ to η we get $\mathrm{Ext}^1_\Lambda(M_k, M_k) = 0$. $\square$

Corollary 9.10. *For $1 \leqslant k \leqslant r$ with $k^- \neq 0$ we have* $\dim \mathrm{Ext}^1_\Lambda(M_k, V_{k^-}) = 1$.

Proof. Apply $\mathrm{Hom}_\Lambda(M_k, -)$ to the sequence η appearing in the proof of Corollary 9.9. $\square$

10. The $\mathrm{add}(M_{\mathbf{i}})$-stratification of $\mathcal{C}_w$

10.1. The stratification

Let $\mathbf{a} = (a_1, \dots, a_r)$ be a tuple of nonnegative integers, and let $\mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ be the category of all Λ-modules X such that there exists a chain

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

of submodules of X with $X_k/X_{k-1} \cong M_k^{a_k}$ for all $1 \leqslant k \leqslant r$.

Lemma 10.1. *If X is a module in $\mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ and $\mathcal{C}_{M_{\mathbf{i}},\mathbf{b}}$, then $\mathbf{a} = \mathbf{b}$.*

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r)$. There is a short exact sequence

$$0 \to X_{r-1} \to X \to M_r^{a_r} \to 0.$$

Lemma 9.8 and induction show that $\mathrm{Hom}_\Lambda(X_{r-1}, M_r) = 0$. Thus $\dim \mathrm{Hom}_\Lambda(X, M_r) = a_r$. Similarly, we get $\dim \mathrm{Hom}_\Lambda(X, M_r) = b_r$. Thus $a_r = b_r$, and by induction we get $a_k = b_k$ for all $1 \leqslant k \leqslant r$. $\square$

Define

$$\mathcal{C}_{M_{\mathbf{i}}} := \bigcup_{\mathbf{a} \in \mathbb{N}^r} \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}.$$

Lemma 10.2. *We have $\mathcal{C}_w = \mathcal{C}_{M_{\mathbf{i}}}$.*

Proof. The category $\mathcal{C}_w$ contains all M_k, and $\mathcal{C}_w$ is closed under extensions. This implies $\mathcal{C}_{M_{\mathbf{i}}} \subseteq \mathcal{C}_w$.

Vice versa, assume $X \in \mathcal{C}_w$. By Proposition 2.15 there exists a short exact sequence

$$\varepsilon : 0 \to V'' \xrightarrow{f} V' \xrightarrow{g} X \to 0$$

with $V', V'' \in \operatorname{add}(V_{\mathbf{i}})$ and g is a minimal right $\operatorname{add}(V_{\mathbf{i}})$-approximation. We call ε a *minimal* $\operatorname{add}(V_{\mathbf{i}})$*-resolution* of length at most one. Since V_r is $\mathcal{C}_w$-projective–injective, by the minimality of g we know that V'' does not contain a direct summand isomorphic to V_r. Let U be the unique submodule of V' such that $V'/U \cong M_r^{a_r}$ with a_r maximal. Clearly, we have

$$U \cong V_{r^-}^{a_r} \oplus V'/V_r^{a_r}.$$

By Lemma 9.8 and induction, the image of f is contained in U. We have $V'/\operatorname{Im}(f) \cong X$. Let $X_{r-1} := g(U)$. We get $X/X_{r-1} \cong M_r^{a_r}$, and by passing to the restriction maps, we obtain a short exact sequence

$$0 \to V'' \xrightarrow{f'} V_{r^-}^{a_r} \oplus V'/V_r^{a_r} \to X_{r-1} \to 0.$$

This is an $\operatorname{add}(V_{\mathbf{i}})$-resolution of X_{r-1}. By possibly deleting a direct summand of f' of the form $\mathrm{id} : V_{r^-}^a \to V_{r^-}^a$, this yields again a minimal $\operatorname{add}(V_{\mathbf{i}})$-resolution of length at most one of X_{r-1}. The result follows by induction. □

For $X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ set

$$M_{\mathbf{i}}(X) := M_1^{a_1} \oplus \cdots \oplus M_r^{a_r}.$$

Recall that $B_{\mathbf{i}} := \operatorname{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$.

For a Λ-module $X \in \mathcal{C}_w$ we want to compute the dimension vector of the $B_{\mathbf{i}}$-module $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)$. The indecomposable projective $B_{\mathbf{i}}$-modules are the modules $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$, $1 \leqslant k \leqslant r$. Thus the entries of the dimension vector $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X))$ are

$$\dim \operatorname{Hom}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k), \operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)\big)$$

where $1 \leqslant k \leqslant r$. By Corollaries 2.13 and 2.16 we have

$$\operatorname{Hom}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k), \operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)\big) \cong \operatorname{Hom}_\Lambda(V_k, X).$$

For $1 \leqslant k \leqslant r$ define

$$\Delta_k := \operatorname{Hom}_\Lambda(V_{\mathbf{i}}, M_k).$$

(In Section 11 we prove that $B_{\mathbf{i}}$ is a quasi-hereditary algebra and that the Δ_k are the corresponding standard modules.) The following result follows directly from Lemma 9.8.

Lemma 10.3. *The dimension vectors $\underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$, $1 \leqslant k \leqslant r$ are linearly independent.*

Lemma 10.4. *For all* $1 \leqslant k \leqslant r$ *we have*

$$\underline{\dim}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_{\Lambda}(V_{\mathbf{i}}, V_k)\big) = \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k) + \underline{\dim}_{B_{\mathbf{i}}}(\Delta_{k^-}) + \cdots + \underline{\dim}_{B_{\mathbf{i}}}(\Delta_{k_{\min}}).$$

Proof. Use the short exact sequence

$$0 \to V_{k^-} \to V_k \to M_k \to 0$$

and an induction on k. $\square$

The next result shows that Lemma 10.4 is just a special case of a general fact.

Proposition 10.5. *For a* Λ*-module* $X \in \mathcal{C}_w$ *and* $\mathbf{a} = (a_1, \dots, a_r)$ *the following are equivalent*:

(i) $X \in \mathcal{C}_{M_{\mathbf{i}}, \mathbf{a}}$;
(ii) *There exists a short exact sequence*

$$0 \to \bigoplus_{k=1}^{r} V_{k^-}^{a_k} \to \bigoplus_{k=1}^{r} V_k^{a_k} \to X \to 0;$$

(iii) $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_{\Lambda}(V_{\mathbf{i}}, X)) = \underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_{\Lambda}(V_{\mathbf{i}}, M_{\mathbf{i}}(X))) = \sum_{k=1}^{r} a_k \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$.

Proof. (i) $\Rightarrow$ (ii): Assume $X \in \mathcal{C}_{M_{\mathbf{i}}, \mathbf{a}}$ with $\mathbf{a} = (a_1, \dots, a_r)$. By induction we get the following diagram of morphisms with exact row and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \bigoplus_{k=1}^{r-1} V_{k^-}^{a_k} \arrow[d] & & V_{r^-}^{a_r} \arrow[d] & \\
 & \bigoplus_{k=1}^{r-1} V_k^{a_k} \arrow[d, "f"] & & V_r^{a_r} \arrow[d, "g"] & \\
0 \arrow[r] & X_{r-1} \arrow[r, "\iota"] \arrow[d] & X \arrow[r, "\pi"] & M_r^{a_r} \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

Since V_r is $\mathcal{C}_w$-projective–injective, there exists a homomorphism g' such that $\pi \circ g' = g$. Then $[f, g'] : \bigoplus_{k=1}^{r} V_k^{a_r} \to X$ is an epimorphism. Let $Z := \operatorname{Ker}([f, g'])$. The Snake Lemma yields an exact sequence

$$\bigoplus_{k=1}^{r-1} V_{k^-}^{a_k} \xrightarrow{h'} Z \xrightarrow{h''} V_{r^-}^{a_r}.$$

Clearly, h'' is an epimorphism, since f is an epimorphism. For dimension reasons h' is a monomorphism. Thus we get a short exact sequence

$$0 \to \bigoplus_{k=1}^{r-1} V_{k^-}^{a_k} \xrightarrow{h'} Z \xrightarrow{h''} V_{r^-}^{a_r} \to 0.$$

Applying $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, -)$ to this sequence yields an exact sequence of $B_{\mathbf{i}}$-modules with a projective end term. Thus this sequence splits, and we get $Z = \bigoplus_{k=1}^{r} V_{k^-}^{a_k}$. So we constructed a short exact sequence

$$\eta_X : 0 \to \bigoplus_{k=1}^{r} V_{k^-}^{a_k} \to \bigoplus_{k=1}^{r} V_{k}^{a_k} \to X \to 0.$$

(ii) $\Rightarrow$ (iii): Apply $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, -)$ to the short exact sequence η_X. Since $V_{\mathbf{i}}$ is rigid, this yields a short exact sequence of $B_{\mathbf{i}}$-modules, and we get

$$\begin{aligned}
\underline{\dim}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)\big) &= \underline{\dim}_{B_{\mathbf{i}}}\left(\operatorname{Hom}_\Lambda\left(V_{\mathbf{i}}, \bigoplus_{k=1}^{r} V_k^{a_k}\right)\right) - \underline{\dim}_{B_{\mathbf{i}}}\left(\operatorname{Hom}_\Lambda\left(V_{\mathbf{i}}, \bigoplus_{k=1}^{r} V_{k^-}^{a_k}\right)\right) \\
&= \sum_{k=1}^{r} a_k \left(\underline{\dim}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k)\big) - \underline{\dim}_{B_{\mathbf{i}}}\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_{k^-})\big)\right) \\
&= \sum_{k=1}^{r} a_k \,\underline{\dim}_{B_{\mathbf{i}}}(\Delta_k).
\end{aligned}$$

This implies (iii).

(iii) $\Rightarrow$ (i): Let $X \in \mathcal{C}_w$, and assume $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)) = \sum_{k=1}^{r} a_k \,\underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$. Set $\mathbf{a} = (a_1, \dots, a_r)$. We know that $X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{b}}$ for some $\mathbf{b} = (b_1, \dots, b_r)$. By the implication (i) $\Rightarrow$ (iii) we get $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)) = \sum_{k=1}^{r} b_k \,\underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$. Since the vectors $\underline{\dim}_{B_{\mathbf{i}}}(\Delta_1), \dots, \underline{\dim}_{B_{\mathbf{i}}}(\Delta_r)$ are linearly independent, we get $a_k = b_k$ for all k. $\square$

Corollary 10.6. *For $X, Y \in \mathcal{C}_w$ we have $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, X)) = \underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, Y))$ if and only if $X, Y \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ for some $\mathbf{a}$.*

Proof. By Lemma 10.3 the dimension vectors $\underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$ are linearly independent. Now use Proposition 10.5. $\square$

A short exact sequence $\eta : 0 \to X \to Y \to Z \to 0$ of Λ-modules is called $M_{\mathbf{i}}$-*split* if $M_{\mathbf{i}}(X) \oplus M_{\mathbf{i}}(Z) \cong M_{\mathbf{i}}(Y)$. Recall that $F_{V_{\mathbf{i}}} := \operatorname{Hom}_\Lambda(V_{\mathbf{i}}, -)$.

Corollary 10.7. *For a short exact sequence $\eta : 0 \to X \to Y \to Z \to 0$ of Λ-modules in $\mathcal{C}_w$ the following are equivalent*:

(i) *η is $F_{V_{\mathbf{i}}}$-exact*;
(ii) *η is $M_{\mathbf{i}}$-split.*

Proof. Clearly, η is $F_{V_{\mathbf{i}}}$-exact if and only if

$$\underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)\big) + \underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, Z)\big) = \underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, Y)\big).$$

By Proposition 10.5 this happens if and only if $M_{\mathbf{i}}(X) \oplus M_{\mathbf{i}}(Z) \cong M_{\mathbf{i}}(Y)$. $\square$

10.2. Example

Let Q be a quiver with underlying graph

$$1 \;\text{—}\; 2 \;\text{—}\; 3$$

and let w_0 be the longest Weyl group element in W_Q. Thus we have $\mathcal{C}_{w_0} = \mathrm{mod}(\Lambda)$. The short exact sequences

$$\eta' : 0 \to {\scriptstyle 2} \to \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}&3\\2&\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 0 \quad \text{and} \quad \eta'' : 0 \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \to {\scriptstyle 2} \to 0$$

are exchange sequences in $\mathrm{mod}(\Lambda)$. Let $\mathbf{i} = (1, 2, 1, 3, 2, 1)$ and $\mathbf{j} = (2, 1, 2, 3, 2, 1)$ be reduced expressions of w_0. We get

$$M_{\mathbf{i}} = {\scriptstyle 1} \oplus \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \oplus {\scriptstyle 2} \oplus \begin{smallmatrix}2&\\&3\end{smallmatrix} \oplus {\scriptstyle 3} \quad \text{and} \quad M_{\mathbf{j}} = {\scriptstyle 1} \oplus \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \oplus {\scriptstyle 3} \oplus \begin{smallmatrix}&3\\2&\end{smallmatrix} \oplus {\scriptstyle 2}.$$

Now one easily observes that η' is $M_{\mathbf{i}}$-split and not $M_{\mathbf{j}}$-split, and η'' is $M_{\mathbf{j}}$-split but not $M_{\mathbf{i}}$-split.

11. Quasi-hereditary algebras associated to reduced expressions

11.1. Quasi-hereditary algebras

Let A be a finite-dimensional algebra. By $P_1, \dots, P_r$ and $Q_1, \dots, Q_r$ and $S_1, \dots, S_r$ we denote the indecomposable projective, indecomposable injective and simple A-modules, respectively, where $S_i = \mathrm{top}(P_i) = \mathrm{soc}(Q_i)$.

For a class $\mathcal{U}$ of A-modules let $\mathcal{F}(\mathcal{U})$ be the class of all A-modules X which have a filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t = X$$

of submodules such that all factors X_j/X_{j-1} belong to $\mathcal{U}$ for all $1 \leqslant j \leqslant t$. Such a filtration is called a *$\mathcal{U}$-filtration* of X. We call these modules the *$\mathcal{U}$-filtered modules*.

Fix a bijective map $\omega : \{S_1, \dots, S_r\} \to \{1, \dots, r\}$. Let Δ_i be the largest factor module of P_i such that $[\Delta_i : S_j] = 0$ for all j with $\omega(S_j) > \omega(S_i)$, and set

$$\Delta = \{\Delta_1, \dots, \Delta_r\}.$$

The modules Δ_i are called *standard modules*. The algebra A is called *quasi-hereditary* if $\mathrm{End}_A(\Delta_i) \cong K$ for all i, and if ${}_AA$ belongs to $\mathcal{F}(\Delta)$. Quasi-hereditary algebras first occured in Cline, Parshall and Scott's [16] study of highest weight categories.

Note that the definition of a quasi-hereditary algebra depends on the chosen ordering of the simple modules. If we reorder them, it could happen that our algebra is no longer quasi-hereditary.

Now assume A is a quasi-hereditary algebra, and let $\mathcal{F}(\Delta)$ be the subcategory of Δ-filtered A-modules. For $X \in \mathcal{F}(\Delta)$ let $[X : \Delta_i]$ be the number of times that Δ_i occurs as a factor in a Δ-filtration of X. Then

$$\underline{\dim}_\Delta(X) = \big([X : \Delta_1], \dots, [X : \Delta_r]\big) \in \mathbb{N}^r$$

is the Δ-*dimension vector* of X. Let ∇_i be the largest submodule of Q_i such that $[\nabla_i : S_j] = 0$ for all j with $\omega(S_j) > \omega(S_i)$, and let

$$\nabla = \{\nabla_1, \dots, \nabla_r\}.$$

The modules ∇_i are called *costandard modules*. The following results (and the missing definitions) can be found in [56,57]:

(i) There is a unique (up to isomorphism) basic tilting module $T(\Delta \cap \nabla)$ over A such that

$$\operatorname{add}\big(T(\Delta \cap \nabla)\big) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

(ii) $\mathcal{F}(\Delta)$ is closed under extensions and under direct summands.
(iii) $[P_i : \Delta_j] = [\nabla_j : S_i]$ for all $1 \leqslant i, j \leqslant r$.
(iv) If $X \in \mathcal{F}(\Delta)$, then $[X : \Delta_i] = \dim \operatorname{Hom}_A(X, \nabla_i)$ for all i.
(v) $\operatorname{Hom}_A(\Delta_i, \Delta_j) = 0$ for all i, j with $\omega(S_i) > \omega(S_j)$.
(vi) $\operatorname{Ext}^1_A(\Delta_i, \Delta_j) = 0$ for all i, j with $\omega(S_i) \geqslant \omega(S_j)$.
(vii) The $\mathcal{F}(\Delta)$-projective modules are the projective A-modules. The $\mathcal{F}(\nabla)$-injective modules are the injective A-modules.
(viii) The $\mathcal{F}(\Delta)$-injective modules are the modules in $\operatorname{add}(T(\Delta \cap \nabla))$. The $\mathcal{F}(\nabla)$-projective modules are the modules in $\operatorname{add}(T(\Delta \cap \nabla))$.
(ix) If $\operatorname{Ext}^1_A(X, \nabla_i) = 0$ for all i, then $X \in \mathcal{F}(\Delta)$. Similarly, if $\operatorname{Ext}^1_A(\Delta_i, Y) = 0$ for all i, then $Y \in \mathcal{F}(\nabla)$.

The module $T(\Delta \cap \nabla)$ is called the *characteristic tilting module* of A. In general, $T(\Delta \cap \nabla)$ is not a classical tilting module. (Here a tilting module is called *classical* provided its projective dimension is at most one.) The endomorphism algebra $\operatorname{End}_A(T(\Delta \cap \nabla))$ is called the *Ringel dual* of A. It is again a quasi-hereditary algebra in a natural way, see [56].

Following Ringel [59], the finite-dimensional algebra A is *strongly quasi-hereditary* if there is a bijective map $\omega : \{S_1, \dots, S_r\} \to \{1, \dots, r\}$ such that for each $1 \leqslant k \leqslant r$ there is a short exact sequence

$$0 \to R_k \to P_k \to D_k \to 0$$

satisfying the following two properties:

(1) R_k is a direct sum of indecomposable projective A-modules P_j with $\omega(j) > \omega(k)$;
(2) $[D_k : S_j] = \begin{cases} 0 & \text{if } \omega(j) > \omega(k), \\ 1 & \text{if } j = k. \end{cases}$

Each strongly quasi-hereditary algebra is quasi-hereditary with $\Delta_k = D_k$ for all k. Furthermore, we have $\mathrm{proj.dim}(\Delta_k) \leqslant 1$ for all k. If each of the modules R_k is indecomposable, then one easily checks that A is Δ-serial, *i.e.* each P_k has a unique Δ-filtration.

11.2. The algebra $B_{\mathbf{i}}$ is quasi-hereditary

As before, let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ and $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$. Set $B_{\mathbf{i}} := \mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$. For $1 \leqslant k \leqslant r$ let $S(k)$ be the (simple) top of the indecomposable $B_{\mathbf{i}}$-modules $P_k := \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$. As before, define $\Delta_k := \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_k)$, and set

$$\Delta := \{\Delta_1, \ldots, \Delta_r\}.$$

Define $\omega : \{S(1), \ldots, S(r)\} \to \{1, \ldots, n\}$ by $\omega(S(k)) := r - k + 1$.

The following theorem was first proved in [36, Section 16] for adaptable Weyl group elements. Later the statement was generalized to arbitrary Weyl group elements by Iyama and Reiten [41]. Here we present a proof for the general case, which is very similar to our original proof of the adaptable case.

Theorem 11.1. *Let $\mathbf{i}$ be a reduced expression of a Weyl group element w. The following hold*:

(i) *The algebra $B_{\mathbf{i}} = \mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$ is strongly quasi-hereditary and Δ-serial with standard modules $\Delta = \{\Delta_1, \ldots, \Delta_r\}$*;
(ii) *The functor $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, -)$ yields an equivalence of categories $F_{\mathbf{i}} : \mathcal{C}_w \to \mathcal{F}(\Delta)$*;
(iii) $T(\Delta \cap \nabla) = \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, T_{\mathbf{i}})$.

Proof. (i): We know that for each $1 \leqslant k \leqslant r$ there is a short exact sequence

$$\eta : 0 \to V_{k^-} \xrightarrow{\iota_k} V_k \to M_k \to 0.$$

We apply the functor $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, -)$ to this sequence and obtain a short exact sequence

$$0 \to \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_{k^-}) \to P_k \xrightarrow{H} \Delta_k \to 0$$

of $B_{\mathbf{i}}$-modules. Let $\omega(S(j)) \geqslant \omega(S(k))$, and let $F : \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_j) \to \Delta_k$ be a homomorphism of $B_{\mathbf{i}}$-modules. Since $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_j)$ is a projective $B_{\mathbf{i}}$-module, there is a homomorphism $G : \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_j) \to P_k$ such that $H \circ G = F$. There exists a Λ-module homomorphism $g : V_j \to V_k$ such that $G = \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, g)$. Assume $\omega(S(j)) > \omega(S(k))$. Since $j < k$, we know that $\mathrm{Im}(g) \subseteq \iota_k(V_{k^-})$. Thus $\mathrm{Im}(G) \subseteq \mathrm{Im}(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, \iota_k)) = \mathrm{Ker}(H)$. But this implies $F = 0$. Therefore we have $[\Delta_k : S(j)] = 0$. Next, we consider the case $\omega(S(j)) = \omega(S(k))$. The endomorphism ring $\mathrm{End}_\Lambda(V_k)$ is local, and we work over an algebraically closed field. Thus $g = \lambda \mathrm{id}_{V_k} + g'$ with g' nilpotent and $\lambda \in K$. We have $\mathrm{soc}(V_k) \subseteq \mathrm{Ker}(g')$. This implies $\mathrm{Im}(g') \subseteq \iota_k(V_{k^-})$. Thus $F = H \circ G = H \circ \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, \lambda \mathrm{id}_{V_k})$. In other words, $\mathrm{Hom}_{B_{\mathbf{i}}}(P_k, \Delta_k)$ is 1-dimensional. This finishes the proof of (i).

(ii): For $X, Z \in \mathcal{C}_w$ we have a functorial isomorphism

$$\mathrm{Ext}^1_{F_{V_{\mathbf{i}}}}(X, Z) \to \mathrm{Ext}^1_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X), \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, Z)\big).$$

Thus the image of the functor

$$\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, -) : \mathcal{C}_w \to \mathrm{mod}(B_{\mathbf{i}})$$

is extension closed. Clearly, for all $1 \leqslant k \leqslant r$ the standard module Δ_k is in $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, \mathcal{C}_w)$. It follows that $\mathcal{F}(\Delta) \subseteq \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, \mathcal{C}_w)$.

Now let $X \in \mathcal{C}_w$. By Lemma 10.2 we know that $X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ for some $\mathbf{a} = (a_1, \dots, a_r)$. Thus there is a short exact sequence

$$\eta : 0 \to X_{r-1} \to X \to M_r^{a_r} \to 0.$$

We claim that η is $F_{V_{\mathbf{i}}}$-exact: Clearly, η is F_{V_r}-exact, since V_r is $\mathcal{C}_w$-projective–injective and $X_{r-1} \in \mathcal{C}_w$. Since $\mathrm{Hom}_\Lambda(V_k, M_r) = 0$ for all $k < r$, it follows that η is also F_{V_k}-exact for all such k. Clearly, $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_r^{a_r})$ is contained in $\mathcal{F}(\Delta)$. By induction also $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X_{r-1})$ is in $\mathcal{F}(\Delta)$. Since $\mathcal{F}(\Delta)$ is closed under extensions, and since η is $F_{V_{\mathbf{i}}}$-exact, we get that $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)$ is in $\mathcal{F}(\Delta)$. So we proved that $\mathcal{F}(\Delta) = \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, \mathcal{C}_w)$. Now Corollary 2.13 and Lemma 2.16 show that the restriction functor $F_{\mathbf{i}} : \mathcal{C}_w \to \mathcal{F}(\Delta)$ is an equivalence of categories.

(iii): It is enough to show that $\mathrm{Ext}^1_\Lambda(\Delta_k, T_{\mathbf{i}}) = 0$ for all $1 \leqslant k \leqslant r$, see Section 11.1. Recall that all indecomposable direct summands of $T_{\mathbf{i}}$ are of the form $M[s_{\max}, s]$ where $1 \leqslant s \leqslant r$. We fix such an s.

For each $1 \leqslant k \leqslant r$ there is a short exact sequence

$$\eta : 0 \to M\big[k^-, k_{\min}\big] \to M[k, k_{\min}] \to M_k \to 0.$$

Applying $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, -)$ yields a projective resolution

$$0 \to \mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, M\big[k^-, k_{\min}\big]\big) \to \mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, M[k, k_{\min}]\big) \to \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_k) \to 0$$

of $B_{\mathbf{i}}$-modules.

If $k \leqslant s$, then $\mathrm{Hom}_\Lambda(M[k^-, k_{\min}], M[s_{\max}, s]) = 0$. Since $F_{\mathbf{i}}$ is an equivalence, we get

$$\mathrm{Hom}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, M\big[k^-, k_{\min}\big]\big), \mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, M[s_{\max}, s]\big)\big) = 0.$$

This implies $\mathrm{Ext}^1_{B_{\mathbf{i}}}(\Delta_k, \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M[s_{\max}, s])) = 0$.

Next, assume that $k > s$. We have a short exact sequence

$$\psi : 0 \to M\big[s^-, s_{\min}\big] \to M[s_{\max}, s_{\min}] \to M[s_{\max}, s] \to 0.$$

Applying $\mathrm{Hom}_\Lambda(-, M_k)$ yields $\mathrm{Ext}^1_\Lambda(M[s_{\max}, s], M_k) = 0$. Thus $\mathrm{Ext}^1_\Lambda(M_k, M[s_{\max}, s]) = 0$. This implies

$$\mathrm{Ext}^1_{F_{V_{\mathbf{i}}}}\big(M_k, M[s_{\max}, s]\big) = \mathrm{Ext}^1_{B_{\mathbf{i}}}\big(\Delta_k, \mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, M[s_{\max}, s]\big)\big) = 0.$$

Here we used that $\mathrm{Ext}^1_\Lambda(M[s_{\max}, s_{\min}], M_k) = 0$ (since $M[s_{\max}, s_{\min}]$ is $\mathcal{C}_w$-projective–injective), and $\mathrm{Hom}_\Lambda(M[s^-, s_{\min}], M_k) = 0$ by Lemma 9.8. This finishes the proof of (iii). □

Corollary 11.2. *The modules* $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, I_{\mathbf{i},j})$, $1 \leqslant j \leqslant n$ *are the indecomposable* $\mathcal{F}(\Delta)$-*projective–injectives modules.*

Proof. This follows from Theorem 11.1(iii) and Section 11.1. $\square$

Each Δ-filtration of the indecomposable projective $B_{\mathbf{i}}$-module $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$ looks as follows:

$$\begin{array}{c} \Delta_k \\ \hline \Delta_{k^-} \\ \hline \cdots \\ \hline \Delta_{k^{\min}} \end{array}$$

(We just displayed the factors of the (unique) Δ-filtration of $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$.)

We can now reformulate parts of Proposition 10.5 as follows:

Proposition 11.3. *For a Λ-module $X \in \mathcal{C}_w$ and $\mathbf{a} = (a_1, \dots, a_r)$ the following are equivalent*:

(1) $X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$;
(2) $\underline{\dim}_\Delta(F_{\mathbf{i}}(X)) = (a_1, \dots, a_r)$.

Proof. Since $\Delta_k = \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_k)$, it is clear that (iii) in Proposition 10.5 and (2) are equivalent. $\square$

We know that $B_{\mathbf{i}}$ is an algebra of finite global dimension. Thus one can define the Ringel form

$$\langle X, Y\rangle_{B_{\mathbf{i}}} := \big\langle \underline{\dim}(X), \underline{\dim}(Y)\big\rangle_{B_{\mathbf{i}}} := \sum_{j \geqslant 0} (-1)^j \dim \mathrm{Ext}^j_{B_{\mathbf{i}}}(X, Y).$$

The next lemma gives the values of $\langle -, -\rangle_{B_{\mathbf{i}}}$ applied to standard modules.

Lemma 11.4. *For $1 \leqslant k, s \leqslant r$ we have*

$$\langle \Delta_k, \Delta_s\rangle_{B_{\mathbf{i}}} = \dim \mathrm{Hom}_{B_{\mathbf{i}}}(\Delta_k, \Delta_s) - \dim \mathrm{Ext}^1_{B_{\mathbf{i}}}(\Delta_k, \Delta_s) = \begin{cases} 0 & \textit{if } k < s, \\ 1 & \textit{if } k = s, \\ (M_k, M_s)_Q & \textit{if } k > s. \end{cases}$$

Proof. As before, for $1 \leqslant t \leqslant r$ we set $P_t := \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_t)$ and $\Delta_t := \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_t)$. We know that $\mathrm{proj.dim}(\Delta_t) \leqslant 1$ for all t. Thus

$$\langle \Delta_k, \Delta_s\rangle_{B_{\mathbf{i}}} = \dim \mathrm{Hom}_{B_{\mathbf{i}}}(\Delta_k, \Delta_s) - \dim \mathrm{Ext}^1_{B_{\mathbf{i}}}(\Delta_k, \Delta_s).$$

The cases $k < s$ and $k = s$ are clear, see Section 11.1. Thus, assume $k > s$. The short exact sequence

$$0 \to V_{k^-} \to V_k \to M_k \to 0$$

yields a projective resolution

$$0 \to P_{k^-} \to P_k \to \Delta_k \to 0$$

of Δ_k. We apply $\operatorname{Hom}_\Lambda(-, M_s)$ and obtain an exact sequence

$$0 \to \operatorname{Hom}_{B_{\mathbf{i}}}(\Delta_k, \Delta_s) \to \operatorname{Hom}_{B_{\mathbf{i}}}(P_k, \Delta_s) \to \operatorname{Hom}_{B_{\mathbf{i}}}(P_{k^-}, \Delta_s) \to \operatorname{Ext}^1_{B_{\mathbf{i}}}(\Delta_k, \Delta_s) \to 0.$$

This implies

$$\begin{aligned}
\langle \Delta_k, \Delta_s \rangle_{B_{\mathbf{i}}} &= \dim \operatorname{Hom}_{B_{\mathbf{i}}}(P_k, \Delta_s) - \dim \operatorname{Hom}_{B_{\mathbf{i}}}(P_{k^-}, \Delta_s) \\
&= \dim \operatorname{Hom}_\Lambda(V_k, M_s) - \dim \operatorname{Hom}_\Lambda(V_{k^-}, M_s) \\
&= (M_k, M_s)_Q.
\end{aligned}$$

For the third equality we use Lemma 9.8. $\square$

11.3. Example

For an arbitrary $\mathcal{C}_w$-maximal rigid Λ-module T, it seems to be difficult to determine when $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is quasi-hereditary and when not.

Even if Q is a quiver with underlying graph

$$1 \text{ --- } 2 \text{ --- } 3$$

there are maximal rigid modules whose endomorphism algebra is not quasi-hereditary: Let $w = w_0$ be the longest Weyl group element in W_Q. Let T be the $\mathcal{C}_w$-maximal rigid Λ-module

$$\begin{smallmatrix}2&\\&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix}.$$

The quiver of $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ looks as follows:

$$\begin{tikzcd}
\begin{smallmatrix}2&\\&3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \arrow[d] & \begin{smallmatrix}&2\\1&\end{smallmatrix} \arrow[l] \\
\begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \arrow[u] & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \arrow[l] \arrow[r] & \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix} \arrow[u]
\end{tikzcd}$$

It is not difficult to show that $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is not a quasi-hereditary algebra.

12. Mutations of clusters via dimension vectors

12.1. Dimension vectors of rigid modules

Let A be a finite-dimensional K-algebra. For $m \geqslant 0$ let A^m be the free A-module of rank m. By $\operatorname{mod}(A, m)$ we denote the affine variety of m-dimensional A-modules. (One can define $\operatorname{mod}(A, m)$ as the variety of K-algebra homomorphisms $A \to M_m(K)$.) If U is a submodule of A^m such that A^m/U is m-dimensional, then the *Richmond stratum* $\mathcal{S}(U, A^m)$ is the subset of $\operatorname{mod}(A, m)$ consisting of the modules X such that there exists a short exact sequence

$$0 \to U \to A^m \to X \to 0,$$

see [54]. A more general situation was studied by Bongartz [10].

Theorem 12.1. *(See [54, Theorem 1].) The Richmond stratum $\mathcal{S}(U, A^m)$ is a smooth, irreducible, locally closed subset of* $\mathrm{mod}(A, m)$*, and*

$$\dim \mathcal{S}\big(U, A^m\big) = \dim \mathrm{Hom}_A\big(U, A^m\big) - \dim \mathrm{End}_A(U).$$

Proposition 12.2. *Assume that* $\mathrm{gl.dim}(A) < \infty$. *Let M and N be rigid A-modules of projective dimension at most one. If $\underline{\dim}(M) = \underline{\dim}(N)$, then $M \cong N$.*

Proof. Let m be the K-dimension of M and N. Thus, there are projective resolutions

$$0 \to P \to A^m \to M \to 0 \quad \text{and} \quad 0 \to P' \to A^m \to N \to 0$$

of M and N, respectively. Here we used that the projective dimensions of M and N are at most one. Since $\underline{\dim}(M) = \underline{\dim}(N)$, we get $\underline{\dim}(P) = \underline{\dim}(P')$. Since A is a finite-dimensional algebra of finite global dimension, its Cartan matrix is invertible. In other words, the dimension vectors of the indecomposable projective A-modules are linearly independent. Thus we get $P \cong P'$.

Since M and N are rigid, their $\mathrm{GL}_m(K)$-orbits are open in $\mathrm{mod}(A, m)$. In particular, these orbits are open in the Richmond stratum $\mathcal{S}(P, A^m)$. But $\mathcal{S}(P, A^m)$ is irreducible, and therefore it can contain at most one open orbit. It follows that $M \cong N$. $\square$

Now, let $\mathcal{C}_w = \mathrm{Fac}(V_{\mathbf{i}})$ be defined as before, and let $T = T_1 \oplus \cdots \oplus T_r$ be a fixed basic $\mathcal{C}_w$-maximal rigid module and set $B := \mathrm{End}_A(T)^{\mathrm{op}}$.

Corollary 12.3. *Let X and Y be indecomposable rigid modules in $\mathcal{C}_w$. If*

$$\underline{\dim}_B\big(\mathrm{Hom}_A(T, X)\big) = \underline{\dim}_B\big(\mathrm{Hom}_A(T, Y)\big),$$

then $X \cong Y$.

Proof. Use Corollary 2.17 and Proposition 2.19(vi), and then apply Proposition 12.2. $\square$

12.2. Mutations via dimension vectors

We now explain how to calculate mutations of clusters via dimension vectors. We start with some notation: For $\mathbf{d} = (d_1, \ldots, d_r)$ and $\mathbf{f} = (f_1, \ldots, f_r)$ in $\mathbb{Z}^r$ define

$$\max\{\mathbf{d}, \mathbf{f}\} := (h_1, \ldots, h_r)$$

where $h_s = \max\{d_s, f_s\}$ for $1 \leqslant s \leqslant r$. Set $\mathrm{Max}\{\mathbf{d}, \mathbf{f}\} := \mathbf{d}$ if $d_s \geqslant f_s$ for all s. In this case, we write $\mathbf{d} \geqslant \mathbf{f}$. Of course, $\mathrm{Max}\{\mathbf{d}, \mathbf{f}\} = \mathbf{d}$ implies $\max\{\mathbf{d}, \mathbf{f}\} = \mathbf{d}$. By $|\mathbf{d}|$ we denote the sum of the entries of $\mathbf{d}$.

Let Γ be a quiver without loops and without 2-cycles and with vertices $1, \dots, r$. Some of these vertices can be considered as *frozen vertices*, *i.e.* one cannot perform a mutation at these vertices.

Now replace each vertex s of Γ by some $\mathbf{d}_s \in \mathbb{Z}^r$. Thus we obtain a new quiver Γ' whose vertices are elements in $\mathbb{Z}^r$.

For k not a frozen vertex, define the mutation $\mu_{\mathbf{d}_k}(\Gamma')$ of Γ' at the vertex $\mathbf{d}_k$ in two steps:

(1) Replace the vertex $\mathbf{d}_k$ of Γ' by

$$\mathbf{d}_k^* := -\mathbf{d}_k + \max\left\{ \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i, \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j \right\}$$

where the sums are taken over all arrows in Γ' which start, respectively end in the vertex $\mathbf{d}_k$.

(2) Change the arrows of Γ' following Fomin and Zelevinsky's quiver mutation rule for the vertex $\mathbf{d}_k$.

Thus starting with Γ' we can use iterated mutation and obtain quivers whose vertices are elements in $\mathbb{Z}^r$.

For example, if for each s we choose $\mathbf{d}_s = -\mathbf{e}_s$, where $\mathbf{e}_s$ is the sth canonical basis vector of $\mathbb{Z}^r$, then the resulting vertices (*i.e.* elements in $\mathbb{Z}^r$) are the denominator vectors of the cluster variables of the cluster algebra $\mathcal{A}(B(\Gamma)^\circ)$ associated to Γ, compare with [26, Section 7, Eq. (7.7)]. (The variables attached to the frozen vertices serve as (non-invertible) coefficients. To obtain the denominator vectors as defined in [26] one has to ignore the entries corresponding to these n coefficients.) It is an open problem, if these denominator vectors actually parametrize the cluster variables of $\mathcal{A}(B(\Gamma)^\circ)$.

We will show that for an appropriate choice of Γ and of the initial vectors $\mathbf{d}_s$, the quivers obtained by iterated mutation of Γ' are in bijection with the seeds and clusters of $\mathcal{A}(B(\Gamma)^\circ)$. All resulting vertices (including the $\mathbf{d}_s$) will be elements in $\mathbb{N}^r$, and we will show that for our particular choice of initial vectors, we can use "Max" instead of "max" in the formula above. (This holds for all iterated mutations.)

For the rest of this section let $T = T_1 \oplus \cdots \oplus T_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module, and set $B := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$.

Proposition 12.4. *Let $R = R_1 \oplus \cdots \oplus T_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module. Let*

$$\eta' : 0 \to R_k \xrightarrow{f'} R' \xrightarrow{g'} R_k^* \to 0 \quad \textit{and} \quad \eta'' : 0 \to R_k^* \xrightarrow{f''} R'' \xrightarrow{g''} R_k \to 0$$

be the two exchange sequences associated to an indecomposable direct summand R_k of R which is not $\mathcal{C}_w$-projective–injective. Then $\dim \mathrm{Hom}_\Lambda(T, R') \neq \dim \mathrm{Hom}_\Lambda(T, R'')$, *and we have*

$$\begin{aligned}&\underline{\dim}_B\big(\mathrm{Hom}_\Lambda(T, R_k)\big) + \underline{\dim}_B\big(\mathrm{Hom}_\Lambda\big(T, R_k^*\big)\big)\\ &\quad = \max\big\{\underline{\dim}_B\big(\mathrm{Hom}_\Lambda\big(T, R'\big)\big), \underline{\dim}_B\big(\mathrm{Hom}_\Lambda\big(T, R''\big)\big)\big\}.\end{aligned}$$

Furthermore, the following are equivalent:

(i) η' *is* F_T*-exact*;
(ii) $\dim \operatorname{Hom}_\Lambda(T, R') > \dim \operatorname{Hom}_\Lambda(T, R'')$;
(iii) $\underline{\dim}_B(\operatorname{Hom}_\Lambda(T, R')) \geqslant \underline{\dim}_B(\operatorname{Hom}_\Lambda(T, R''))$.

Proof. By Corollary 2.18 we know that $\operatorname{Hom}_\Lambda(T, R)$ is a classical tilting module over B. Thus we can apply [39, Lemma 2.2] and assume without loss of generality that

$$\operatorname{Ext}^1_B\big(\operatorname{Hom}_\Lambda(T, R_k), \operatorname{Hom}_\Lambda\big(T, R_k^*\big)\big) = 0.$$

By Proposition 2.12,

$$\begin{aligned} 1 = \dim \operatorname{Ext}^1_\Lambda\big(R_k^*, R_k\big) &\geqslant \dim \operatorname{Ext}^1_{F_T}\big(R_k^*, R_k\big) \\ &= \dim \operatorname{Ext}^1_B\big(\operatorname{Hom}_\Lambda\big(T, R_k^*\big), \operatorname{Hom}_\Lambda(T, R_k)\big) > 0. \end{aligned}$$

This implies $\operatorname{Ext}^1_\Lambda(R_k^*, R_k) = \operatorname{Ext}^1_{F_T}(R_k^*, R_k)$. Thus η' is F_T-exact, and

$$\eta : 0 \to \operatorname{Hom}_\Lambda(T, R_k) \xrightarrow{\operatorname{Hom}_\Lambda(T, f')} \operatorname{Hom}_\Lambda\big(T, R'\big) \xrightarrow{\operatorname{Hom}_\Lambda(T, g')} \operatorname{Hom}_\Lambda\big(T, R_k^*\big) \to 0$$

is a (non-split) short exact sequence. If we apply $\operatorname{Hom}_\Lambda(T, -)$ to η'', we obtain an exact sequence

$$0 \to \operatorname{Hom}_\Lambda\big(T, R_k^*\big) \xrightarrow{\operatorname{Hom}_\Lambda(T, f'')} \operatorname{Hom}_\Lambda\big(T, R''\big) \xrightarrow{\operatorname{Hom}_\Lambda(T, g'')} \operatorname{Hom}_\Lambda(T, R_k).$$

Now $\operatorname{Hom}_\Lambda(T, g'')$ cannot be an epimorphism, since that would yield a non-split extension and we know that $\operatorname{Ext}^1_B(\operatorname{Hom}_\Lambda(T, R_k), \operatorname{Hom}_\Lambda(T, R_k^*)) = 0$. Thus for dimension reasons we get $\dim \operatorname{Hom}_\Lambda(T, R') > \dim \operatorname{Hom}_\Lambda(T, R'')$. Using the functors $\operatorname{Hom}_B(P, -)$ where P runs through the indecomposable projective B-modules, it also follows that $\underline{\dim}_B(\operatorname{Hom}_\Lambda(T, R')) > \underline{\dim}_B(\operatorname{Hom}_\Lambda(T, R''))$. Finally, the formula for dimension vectors follows from the exactness of η. $\square$

Proposition 12.4 yields an easy combinatorial rule for the mutation of $\mathcal{C}_w$-maximal rigid modules. Let $R = R_1 \oplus \cdots \oplus R_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module. Without loss of generality we assume that $R_{r-n+1}, \ldots, R_r$ are $\mathcal{C}_w$-projective–injective. For $1 \leqslant s \leqslant r$ let $\mathbf{d}_s := \underline{\dim}_B(\operatorname{Hom}_\Lambda(T, R_s))$.

As before, let Γ_R be the quiver of $\operatorname{End}_\Lambda(R)^{\mathrm{op}}$. The vertices of Γ_R are labeled by the modules R_s. For each s we replace the vertex labeled by R_s by the dimension vector $\mathbf{d}_s$. The resulting quiver is denoted by Γ'_R.

For $1 \leqslant k \leqslant r - n$ let

$$0 \to R_k \to R' \to R_k^* \to 0 \quad \text{and} \quad 0 \to R_k^* \to R'' \to R_k \to 0$$

be the two resulting exchange sequences. We can now easily compute the dimension vector of the $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$-module $\operatorname{Hom}_\Lambda(T, R_k^*)$, namely Proposition 12.4 yields that

$$\mathbf{d}_k^* := \underline{\dim}_B\big(\operatorname{Hom}_\Lambda\big(T, R_k^*\big)\big) = \begin{cases} -\mathbf{d}_k + \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i & \text{if } \sum_{\mathbf{d}_i \to \mathbf{d}_k} |\mathbf{d}_i| > \sum_{\mathbf{d}_k \to \mathbf{d}_j} |\mathbf{d}_j|, \\ -\mathbf{d}_k + \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j & \text{otherwise}, \end{cases}$$

where the sums are taken over all arrows in Γ_R' which start, respectively end in the vertex $\mathbf{d}_k$. More precisely, we have

$$\mathbf{d}_k^* = -\mathbf{d}_k + \max\left\{ \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i, \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j \right\} \tag{5}$$

and we know that

$$\max\left\{ \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i, \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j \right\} = \mathrm{Max}\left\{ \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i, \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j \right\}. \tag{6}$$

Remark 12.5. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic $\mathcal{C}_w$-maximal rigid module, and let $B^{(T)} := (\langle S_i, S_j \rangle)_{1 \leqslant i,j \leqslant r}$ be the matrix of the Ringel form of the algebra $B := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Let X be a T-reachable Λ-module, see Section 3.1. Set $\mathbf{d} := \underline{\dim}_B(\mathrm{Hom}_\Lambda(T, X)) \in \mathbb{N}^r$. Define

$$\tilde{g}_T(X) := \mathbf{d} \cdot B^{(T)},$$

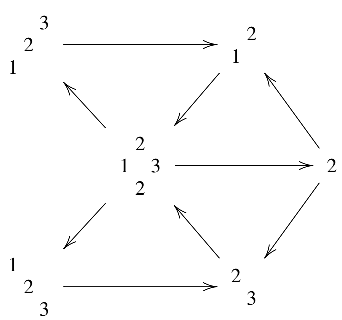
where $\mathbf{d}$ is considered as a row vector. As explained in [27, Section 4] the entries of $\tilde{g}_T(X)$, which correspond to the non-$\mathcal{C}_w$-projective–injective direct summands T_k of T form precisely the g-vector of φ_X with respect to the initial cluster $(\delta_{T_1}, \ldots, \delta_{T_r})$.

12.3. Examples (Dimension vectors of $B_{\mathbf{i}}$-modules)

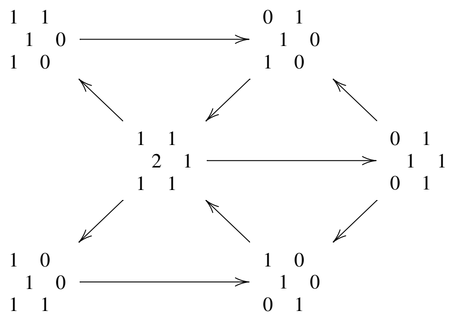
Let Q be a quiver with underlying graph $1 \,\text{—}\, 2 \,\text{—}\, 3$ and let $\mathbf{i} := (3, 1, 2, 3, 1, 2)$. Thus $\Gamma_{\mathbf{i}}$ looks as follows:

$$\begin{tikzcd}
5 \arrow[rr] & & 2 \arrow[dl] & \\
& 4 \arrow[ul] \arrow[rr] \arrow[dl] & & 1 \arrow[ul] \arrow[dl] \\
6 \arrow[rr] & & 3 \arrow[ul] &
\end{tikzcd}$$

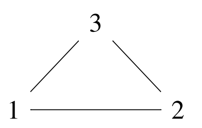
The following picture shows the quiver $\Gamma_{V_{\mathbf{i}}}$ of $\mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$ where the vertices corresponding to the modules V_k.



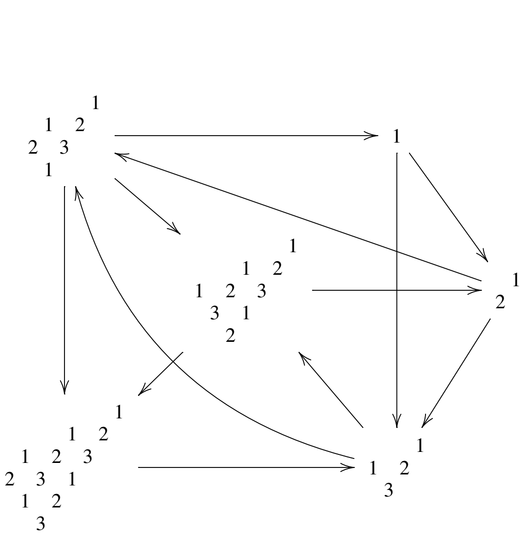
Here is the quiver $\Gamma'_{V_{\mathbf{i}}}$ whose vertices are the dimension vectors $\underline{\dim}_{B_{\mathbf{i}}}(\mathrm{Hom}_{\Lambda}(V_{\mathbf{i}}, V_k))$:



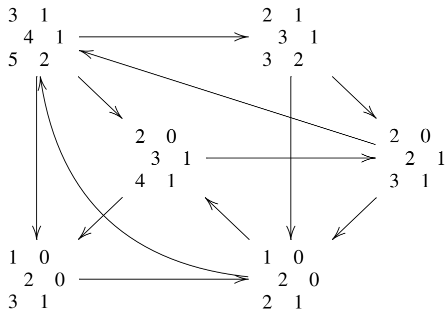
Next, let us look at an example of type $\widetilde{\mathbb{A}}_2$. Thus, let Q be a quiver with underlying graph



and let $\mathbf{i} := (3, 2, 1, 3, 2, 1)$. The quiver $\Gamma_{V_{\mathbf{i}}}$ of $\mathrm{End}_{\Lambda}(V_{\mathbf{i}})^{\mathrm{op}}$ looks as follows:



Here is the quiver $\Gamma'_{V_{\mathbf{i}}}$:



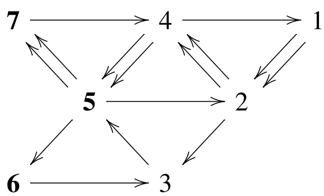
12.4. Example (Mutations via dimension vectors)

Let Q be a quiver with underlying graph $1 = 2 - 3$ and let $\mathbf{i} := (i_7, \dots, i_1) := (1,3,2,1,3,2,1)$ be a reduced expression. As before, let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_7$. The indecomposable $\mathcal{C}_w$-projective–injectives are V_5, V_6 and V_7. Let us compute the dimension vectors $\underline{\dim}_{B_{\mathbf{i}}}(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, M_k))$.

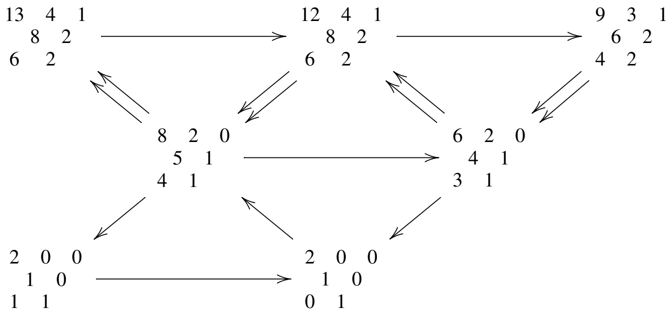
$$\underline{\dim}(\Delta_1) = \begin{smallmatrix} 9 && 3 && 1 \\ & 6 && 2 & \\ 4 && 2 && \end{smallmatrix} \qquad \underline{\dim}(\Delta_2) = \begin{smallmatrix} 6 && 2 && 0 \\ & 4 && 1 & \\ 3 && 1 && \end{smallmatrix} \qquad \underline{\dim}(\Delta_3) = \begin{smallmatrix} 2 && 0 && 0 \\ & 1 && 0 & \\ 0 && 1 && \end{smallmatrix} \qquad \underline{\dim}(\Delta_4) = \begin{smallmatrix} 3 && 1 && 0 \\ & 2 && 0 & \\ 2 && 0 && \end{smallmatrix}$$

$$\underline{\dim}(\Delta_5) = \begin{smallmatrix} 2 && 0 && 0 \\ & 1 && 0 & \\ 1 && 0 && \end{smallmatrix} \qquad \underline{\dim}(\Delta_6) = \begin{smallmatrix} 0 && 0 && 0 \\ & 0 && 0 & \\ 1 && 0 && \end{smallmatrix} \qquad \underline{\dim}(\Delta_7) = \begin{smallmatrix} 1 && 0 && 0 \\ & 0 && 0 & \\ 0 && 0 && \end{smallmatrix}.$$

Here is the quiver $\Gamma_{\mathbf{i}}$:



The following picture shows the quiver $\Gamma'_{V_{\mathbf{i}}}$. Its vertices are the dimension vectors of the $\operatorname{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$-modules $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$. These dimension vectors can be constructed easily using Lemma 9.8.



Now let us mutate the Λ-module V_4. We have

$$\underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, V_4)\big) = \begin{smallmatrix} 12 & & 4 & & 1 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix}.$$

We have to look at all arrows starting and ending in the corresponding vertex of $\Gamma'_{V_{\mathbf{i}}}$, and add up the entries of the attached dimension vectors, as explained in Section 12.2. Since

$$\left|\begin{smallmatrix} 13 & & 4 & & 1 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix}\right| + 2\cdot\left|\begin{smallmatrix} 6 & & 2 & & 0 \\ & 4 & & 1 & \\ 3 & & 1 & & \end{smallmatrix}\right| = 70 > 69 = \left|\begin{smallmatrix} 9 & & 3 & & 1 \\ & 6 & & 2 & \\ 4 & & 2 & & \end{smallmatrix}\right| + 2\cdot\left|\begin{smallmatrix} 8 & & 2 & & 0 \\ & 5 & & 1 & \\ 4 & & 1 & & \end{smallmatrix}\right|,$$

we get

$$\underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda\big(V_{\mathbf{i}}, V_4^*\big)\big) = \begin{smallmatrix} 13 & & 4 & & 1 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix} + 2\cdot \begin{smallmatrix} 6 & & 2 & & 0 \\ & 4 & & 1 & \\ 3 & & 1 & & \end{smallmatrix} - \begin{smallmatrix} 12 & & 4 & & 1 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix} = \begin{smallmatrix} 13 & & 4 & & 0 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix}$$

and the quiver $\Gamma'_{\mu_{V_4}(V_{\mathbf{i}})}$ looks as follows:

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{smallmatrix} 13 & & 4 & & 1 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix} \arrow[rrrr, bend left] \arrow[dddd, dash, dashed] \arrow[ddr, dash, dashed] \& \& \begin{smallmatrix} 13 & & 4 & & 0 \\ & 8 & & 2 & \\ 6 & & 2 & & \end{smallmatrix} \arrow[ll] \arrow[ddr, shift left] \arrow[ddr, shift right] \& \& \begin{smallmatrix} 9 & & 3 & & 1 \\ & 6 & & 2 & \\ 4 & & 2 & & \end{smallmatrix} \arrow[ll] \arrow[ddl, shift left] \arrow[ddl, shift right] \\
\& \& \& \& \\
\& \begin{smallmatrix} 8 & & 2 & & 0 \\ & 5 & & 1 & \\ 4 & & 1 & & \end{smallmatrix} \arrow[uur, shift left] \arrow[uur, shift right] \& \& \begin{smallmatrix} 6 & & 2 & & 0 \\ & 4 & & 1 & \\ 3 & & 1 & & \end{smallmatrix} \arrow[ll, shift left=2] \arrow[ll] \arrow[ll, shift right=2] \arrow[ddl] \& \\
\& \& \& \& \\
\begin{smallmatrix} 2 & & 0 & & 0 \\ & 1 & & 0 & \\ 1 & & 1 & & \end{smallmatrix} \arrow[rr] \arrow[uur, dash, dashed] \& \& \begin{smallmatrix} 2 & & 0 & & 0 \\ & 1 & & 0 & \\ 0 & & 1 & & \end{smallmatrix} \arrow[uul] \& \&
\end{tikzcd}$$

Note that we cannot control how the arrows between vertices corresponding to the three indecomposable $\mathcal{C}_w$-projective–injectives behave under mutation. But this does not matter, because these arrows are not needed for the mutation of seeds and clusters. In the picture, we indicate the missing information by lines of the form $- - -$. This process can be iterated, and our theory says that each of the resulting dimension vectors determines uniquely a cluster variable.

12.5. Mutations via Δ-dimension vectors

Using Lemma 9.8 we can explicitly compute the dimension vector of the $B_{\mathbf{i}}$-module $\Delta_s = \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_s)$ for all $1 \leqslant s \leqslant r$. Recall that the kth entry of this dimension vector is just $\dim \mathrm{Hom}_\Lambda(V_k, M_s)$. Thus, the K-dimension of Δ_s is

$$\dim(\Delta_s) = \dim \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, M_s) = \sum_{k=1}^{r} \dim \mathrm{Hom}_\Lambda(V_k, M_s).$$

Define

$$d_\Delta := \big(\dim(\Delta_1), \dots, \dim(\Delta_r)\big).$$

Now let $R = R_1 \oplus \cdots \oplus R_r$ be a basic $\mathcal{C}_w$-maximal rigid Λ-module, and suppose that R_k is not $\mathcal{C}_w$-projective–injective. Then we can mutate R in direction R_k. We obtain two exchange sequences

$$0 \to R_k \to R' \to R_k^* \to 0 \quad \text{and} \quad 0 \to R_k^* \to R'' \to R_k \to 0$$

with $R', R'' \in \operatorname{add}(R/R_k)$.

For brevity, set

$$\mathbf{d}_s := \underline{\dim}_\Delta\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, R_s)\big)$$

for all $1 \leqslant s \leqslant r$. Similarly to the definition of Γ_R' in Section 12.2 let Γ_R'' be the quiver which is obtained from the quiver of $\operatorname{End}_\Lambda(R)^{\mathrm{op}}$ by replacing the vertex corresponding to R_s by the Δ-dimension vector $\mathbf{d}_s$.

For $\mathbf{d} = (d_1, \dots, d_r)$ and $\mathbf{f} = (f_1, \dots, f_r)$ in $\mathbb{Z}^r$ define

$$\mathbf{d} \cdot \mathbf{f} := \sum_{i=1}^r d_i f_i.$$

Proposition 12.6. *The Δ-dimension vector of the $B_{\mathbf{i}}$-module* $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, R_k^*)$ *is*

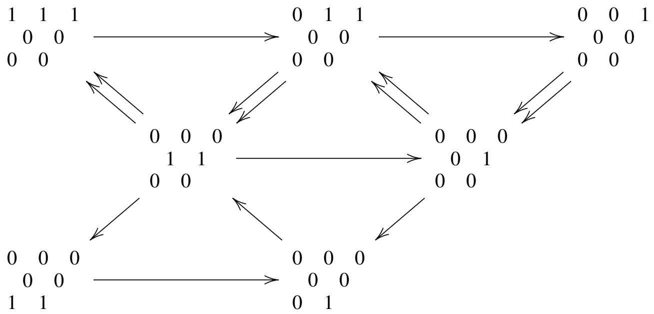
$$\mathbf{d}_k^* := \begin{cases} -\mathbf{d}_k + \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i & \text{if } \sum_{\mathbf{d}_i \to \mathbf{d}_k} \mathbf{d}_i \cdot d_\Delta > \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j \cdot d_\Delta, \\ -\mathbf{d}_k + \sum_{\mathbf{d}_k \to \mathbf{d}_j} \mathbf{d}_j & \text{otherwise.} \end{cases}$$

Here the sums are taken over all arrows of the quiver of Γ_R'' which start, respectively end in the vertex $\mathbf{d}_k$.

Proof. This follows immediately from our results in Section 12.2 □

12.6. Example (Mutations via Δ-dimension vectors)

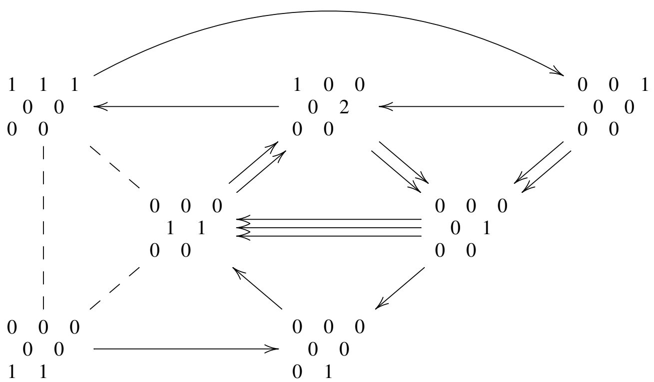
We repeat Example 12.4, but this time we work with Δ-dimension vectors. Let Q and $\mathbf{i}$ be as before. The following picture shows the quiver $\Gamma_{V_{\mathbf{i}}}''$. Its vertices are the Δ-dimension vectors of the $\operatorname{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$-modules $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_k)$.



Again, let us mutate the Λ-module V_4. We have

$$\underline{\dim}_\Delta\big(\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, V_4)\big) = \begin{smallmatrix} 0 & & 1 & & 1 \\ & 0 & & 0 & \\ 0 & & 0 & & \end{smallmatrix}.$$

We have to look at all arrows starting and ending in the corresponding vertex of $\Gamma''_{V_{\mathbf{i}}}$, and to add up the entries of the attached Δ-dimension vectors, as explained in the previous section. In this example it is clear that the ingoing arrows yield the required larger dimension, since the calculation with outgoing arrows would produce a Δ-dimension vector with negative entries, which is not possible. Thus the quiver $\Gamma''_{\mu_{V_4}(V_{\mathbf{i}})}$ looks as follows:



13. A sequence of mutations from $V_{\mathbf{i}}$ to $T_{\mathbf{i}}$

13.1. The algorithm

Let $\mathbf{i} := (i_r, \dots, i_1)$ be a reduced expression of a Weyl group element. For $1 \leqslant i, j \leqslant n$ set

$$q_{ij} := \begin{cases} -c_{ij} & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

(The c_{ij} are the entries of the Cartan matrix C of our Kac–Moody Lie algebra $\mathfrak{g}$, see Section 4.1. Note that this definition of q_{ij} is equivalent to the one in Section 2.4.) As before, we define a quiver $\Gamma_{\mathbf{i}}$ as follows: The vertices of $\Gamma_{\mathbf{i}}$ are $1, 2, \dots, r$. For $1 \leqslant s, t \leqslant r$ there are q_{i_s, i_t} arrows from s to t provided $t^+ \geqslant s^+ > t > s$. These are called the *ordinary arrows* of $\Gamma_{\mathbf{i}}$. Furthermore, for each $1 \leqslant s \leqslant r$ there is an arrow $s \to s^-$ provided $s^- > 0$. These are the *horizontal arrows* of $\Gamma_{\mathbf{i}}$.

As before, let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ and $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$. We know that the quiver $\Gamma_{\mathbf{i}}$ can be identified with the quiver $\Gamma_{V_{\mathbf{i}}}$ of the endomorphism algebra $B_{\mathbf{i}} = \mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$. The vertices of $\Gamma_{V_{\mathbf{i}}}$ are labeled by $V_1, \dots, V_r$. More precisely, the vertex s of $\Gamma_{\mathbf{i}}$ corresponds to the vertex $V_s = M[s, s_{\min}]$ of $\Gamma_{V_{\mathbf{i}}}$, where $1 \leqslant s \leqslant r$.

Recall that for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant r+1$, we defined

$$\begin{aligned} k[j] &:= \bigl|\{1 \leqslant s \leqslant k-1 \mid i_s = j\}\bigr|, \\ t_j &:= (r+1)[j], \\ k_{\min} &:= \min\{1 \leqslant s \leqslant r \mid i_s = i_k\}. \end{aligned}$$

Now we describe an algorithm which yields a sequence of mutations starting with $\Gamma_{V_{\mathbf{i}}}$ and ending with $\Gamma_{T_{\mathbf{i}}}$ (see Section 9.8 for the definition of $T_{\mathbf{i}}$). The proof is done by induction on $r - n$.

Before going into details let us describe the general idea of this algorithm. Assume that Q is the linearly oriented quiver

$$m \longrightarrow m-1 \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

of type $\mathbb{A}_m$. We would like to find a sequence of mutations which transforms Q into the quiver Q^{op}

$$m \longleftarrow m-1 \longleftarrow \cdots \longleftarrow 2 \longleftarrow 1$$

with opposite linear orientation. This can be done by applying the following $m-1$ sequences of mutations:

$$Q^1 := \mu_{m-1} \cdots \mu_2 \mu_1(Q), \qquad Q^2 := \mu_{m-2} \cdots \mu_2 \mu_1\bigl(Q^1\bigr), \qquad \dots, \qquad Q^{m-1} := \mu_1\bigl(Q^{m-2}\bigr).$$

Now one easily checks that $Q^{m-1} = Q^{\mathrm{op}}$. If we delete all ordinary arrows of $\Gamma_{\mathbf{i}}$ we obtain a disjoint union of linearly oriented quivers of type $\mathbb{A}_{m_i}$ for various $m_i \geqslant 1$. The main idea of the following algorithm is to apply a sequence of mutations to $\Gamma_{\mathbf{i}}$ which (in the same way as explained above) reverses the orientation of these subquivers of type $\mathbb{A}_{m_i}$ without causing too many changes for the remaining ordinary arrows.

In the following, we just ignore the symbols of the form $M[a, b]$ in case $a < b$.

Step 1. We mutate the following

$$r_1 := t_{i_1} - 1 - 1[i_1]$$

vertices of $\Gamma^0_{V_{\mathbf{i}}} := \Gamma_{V_{\mathbf{i}}}$ in the given order:

$$M\bigl[1_{\min}^{(1[i_1])}, 1_{\min}^{(1[i_1])}\bigr], M\bigl[1_{\min}^{(1[i_1]+1)}, 1_{\min}^{(1[i_1])}\bigr], M\bigl[1_{\min}^{(1[i_1]+2)}, 1_{\min}^{(1[i_1])}\bigr], \dots, M\bigl[1_{\min}^{(t_{i_1}-2)}, 1_{\min}^{(1[i_1])}\bigr].$$

Under the identification $\Gamma_{V_{\mathbf{i}}} \equiv \Gamma_{\mathbf{i}}$, this sequence of mutations corresponds to the sequence of mutations

$$\overrightarrow{\mu_1} := \mu_{1_{\min}^{(r_1-1)}} \circ \cdots \circ \mu_{1_{\min}^{(1)}} \circ \mu_{1_{\min}}.$$

We obtain a new quiver $\Gamma_{V_{\mathbf{i}}}^1$ with r_1 new vertices

$$M\big[1_{\min}^{(1[i_1]+1)}, 1_{\min}^{(1[i_1]+1)}\big], M\big[1_{\min}^{(1[i_1]+2)}, 1_{\min}^{(1[i_1]+1)}\big], M\big[1_{\min}^{(1[i_1]+3)}, 1_{\min}^{(1[i_1]+1)}\big], \dots, \\ M\big[1_{\min}^{(t_{i_1}-1)}, 1_{\min}^{(1[i_1]+1)}\big].$$

Step 2. We mutate the following

$$r_2 := t_{i_2} - 1 - 2[i_2]$$

vertices of $\Gamma_{V_{\mathbf{i}}}^1$ in the following order:

$$M\big[2_{\min}^{(2[i_2])}, 2_{\min}^{(2[i_2])}\big], M\big[2_{\min}^{(2[i_2]+1)}, 2_{\min}^{(2[i_2])}\big], M\big[2_{\min}^{(2[i_2]+2)}, 2_{\min}^{(2[i_2])}\big], \dots, M\big[2_{\min}^{(t_{i_2}-2)}, 2_{\min}^{(2[i_2])}\big].$$

This mutation sequence corresponds to

$$\overrightarrow{\mu_2} := \mu_{2_{\min}^{(r_2-1)}} \circ \cdots \circ \mu_{2_{\min}^{(1)}} \circ \mu_{2_{\min}}.$$

We obtain a new quiver $\Gamma_{V_{\mathbf{i}}}^2$ with r_2 new vertices

$$M\big[2_{\min}^{(2[i_2]+1)}, 2_{\min}^{(2[i_2]+1)}\big], M\big[2_{\min}^{(2[i_2]+2)}, 2_{\min}^{(2[i_2]+1)}\big], M\big[2_{\min}^{(2[i_2]+3)}, 2_{\min}^{(2[i_2]+1)}\big], \dots, \\ M\big[2_{\min}^{(t_{i_2}-1)}, 2_{\min}^{(2[i_2]+1)}\big].$$

Step k. We mutate the following

$$r_k := t_{i_k} - 1 - k[i_k]$$

vertices of $\Gamma_{V_{\mathbf{i}}}^{k-1}$ in the following order:

$$M\big[k_{\min}^{(k[i_k])}, k_{\min}^{(k[i_k])}\big], M\big[k_{\min}^{(k[i_k]+1)}, k_{\min}^{(k[i_k])}\big], M\big[k_{\min}^{(k[i_k]+2)}, k_{\min}^{(k[i_k])}\big], \dots, M\big[k_{\min}^{(t_{i_k}-2)}, k_{\min}^{(k[i_k])}\big].$$

This mutation sequence corresponds to

$$\overrightarrow{\mu_k} := \mu_{k_{\min}^{(r_k-1)}} \circ \cdots \circ \mu_{k_{\min}^{(1)}} \circ \mu_{k_{\min}}.$$

We obtain a new quiver $\Gamma_{V_{\mathbf{i}}}^k$ with r_k new vertices

$$M\big[k_{\min}^{(k[i_k]+1)}, k_{\min}^{(k[i_k]+1)}\big], M\big[k_{\min}^{(k[i_k]+2)}, k_{\min}^{(k[i_k]+1)}\big], M\big[k_{\min}^{(k[i_k]+3)}, k_{\min}^{(k[i_k]+1)}\big], \dots, \\ M\big[k_{\min}^{(t_{i_k}-1)}, k_{\min}^{(k[i_k]+1)}\big].$$

The algorithm stops when all vertices are of the form $M[k_{\max}, k]$. This will happen after

$$r(\mathbf{i}) := \sum_{j=1}^{n} \frac{t_j(t_j - 1)}{2}$$

mutations. Define

$$\mu_{\mathbf{i}} := \overrightarrow{\mu_r} \circ \cdots \circ \overrightarrow{\mu_2} \circ \overrightarrow{\mu_1}.$$

Thus we have

$$\mu_{\mathbf{i}}(V_{\mathbf{i}}) = T_{\mathbf{i}}.$$

As an example, assume Q is a Dynkin quiver of type $\mathbb{E}_8$. Thus the underlying graph of Q looks as follows:

$$\begin{array}{ccccccccccccc} & & & & 7 & & & & & & & & \\ & & & & | & & & & & & & & \\ 5 & — & 6 & — & 8 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

Let $c := s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1$. Then $w := c^{15}$ is the longest element in the Weyl group W of Q, and $\mathbf{i} := (8, \ldots, 2, 1, \ldots, 8, \ldots, 2, 1)$ is a reduced expression (with 120 entries) of w. We get $t_j = 15$ for all 8 vertices j of Q. Then our algorithm says that starting with $V_{\mathbf{i}}$ we reach $T_{\mathbf{i}}$ after $r(\mathbf{i}) = 8 \cdot 105 = 840$ mutations.

We now want to describe what happens to the quiver $\Gamma_{V_{\mathbf{i}}}^{k-1}$ when we apply the mutation sequence $\overrightarrow{\mu_k}$. First, we need some notation:

For each $1 \leqslant j \leqslant n$ let

$$p_j := \min\{1 \leqslant s \leqslant r \mid i_s = j\},$$
$$u_j := \min\{0, k \leqslant s \leqslant r \mid i_s = j\}.$$

Note that $p_j^{(0)} = p_j$. The sequence

$$\left(p_j^{(0)}, p_j^{(1)}, \ldots, p_j^{(r_{u_j}-1)}\right)$$

of vertices of $\Gamma_{V_{\mathbf{i}}}^{k-1}$ is called the *j-chain* of $\Gamma_{V_{\mathbf{i}}}^{k-1}$, provided $u_j \neq 0$. If $u_j = 0$, then we have an empty j-chain. The sequence

$$\left(p_j^{(0)}, p_j^{(1)}, \ldots, p_j^{(t_j-1)}\right)$$

is the *extended* j-chain.

Each full subgraph of $\Gamma_{V_{\mathbf{i}}}^{k-1}$ given by the vertices of a single extended j-chain looks as follows:

$$p_j^{(t_j-1)} \longleftarrow \cdots \longleftarrow p_j^{(r_{u_j}+1)} \longleftarrow p_j^{(r_{u_j})} \longrightarrow p_j^{(r_{u_j}-1)} \longrightarrow \cdots \longrightarrow p_j^{(2)} \longrightarrow p_j^{(1)} \longrightarrow p_j^{(0)}$$

The arrows of the extended j-chains ($1 \leqslant j \leqslant n$), are the *horizontal arrows* of $\Gamma_{V_{\mathbf{i}}}^{k-1}$. In the mutation sequence

$$\overrightarrow{\mu_r} \circ \cdots \circ \overrightarrow{\mu_{k+1}} \circ \overrightarrow{\mu_k}$$

there are no mutations at the vertices $p_j^{(r_{u_j})}, p_j^{(r_{u_j}+1)}, \dots, p_j^{(t_j-1)}$. These are called the *frozen vertices* of $\Gamma_{V_{\mathbf{i}}}^{k-1}$.

To describe the quiver $\Gamma_{V_{\mathbf{i}}}^{k}$, it is enough to study the effect of $\overrightarrow{\mu_k}$ on the $n-1$ full subgraphs of $\Gamma_{V_{\mathbf{i}}}^{k}$ which consist of the i_k-chain together with one extended j-chain, where $1 \leqslant j \leqslant n$ and $j \neq i_k$.

For brevity, set $s = s^{(0)} = k_{\min}$, $t = t^{(0)} = p_j$. Let $q = q_{i_k,j}$ be the number of edges between i_k and j in the underlying graph of Q. The following picture shows how the arrows between the i_k-chain and an extended j-chain in $\Gamma_{V_{\mathbf{i}}}^{k-1}$ look like (we have $1 \leqslant j \leqslant n$ with $i_k \neq j$, and we use the notation $u \xrightarrow{q} v$ if there are q arrows from u to v):

$$\begin{tikzcd}[column sep=small]
 & s^{(a_z)} \arrow[dl, "q"'] & & s^{(a_{z-1})} \arrow[dl, "q"'] & & \cdots & & s^{(a_2)} \arrow[dl, "q"'] & & s^{(a_1)} \arrow[dl, "q"'] \\
t^{(b_z)} & & t^{(b_{z-1})} \arrow[ul, "q"'] & & \cdots \arrow[ul, "q"'] & & t^{(b_2)} \arrow[ul, "q"'] & & t^{(b_1)} \arrow[ul, "q"'] &
\end{tikzcd}$$

Here $s^{(a_i)}$ belongs to the i_k-chain, and $t^{(b_i)}$ belongs to the extended j-chain for all $1 \leqslant i \leqslant z$. (The q arrows from $s^{(a_z)}$ to $t^{(b_z)}$ do not exist necessarily. But the first q arrows between the i_k-chain and the j-chain (counted from the right) always start at the i_k-chain. We do not display any arrows between frozen vertices, they don't play any role.)

The mutation sequence $\overrightarrow{\mu_k}$ consists of mutations at the vertices $s^{(0)}, s^{(1)}, \dots, s^{(r_k-1)}$. By definition,

$$\Gamma_{V_{\mathbf{i}}}^{k} := \overrightarrow{\mu_k}\big(\Gamma_{V_{\mathbf{i}}}^{k-1}\big).$$

After applying $\overrightarrow{\mu_k}$, the horizontal arrows of the i_k-chain stay the same, except the arrow $s^{(r_k)} \to s^{(r_k-1)}$ changes its orientation and becomes $s^{(r_k)} \leftarrow s^{(r_k-1)}$. The vertex $s^{(r_k-1)}$ becomes an additional frozen vertex of $\Gamma_{V_{\mathbf{i}}}^{k}$.

The arrows between the i_k-chain and the j-chain change as follows:

$$\begin{tikzcd}[column sep=small]
 & s^{(a_z-1)} \arrow[dl, "q"'] & & s^{(a_{z-1}-1)} \arrow[dl, "q"'] & & \cdots & & s^{(a_2-1)} \arrow[dl, "q"'] & & s^{(a_1-1)} \arrow[dl, "q"'] \\
t^{(b_z)} & & t^{(b_{z-1})} \arrow[ul, "q"'] & & \cdots \arrow[ul, "q"'] & & t^{(b_2)} \arrow[ul, "q"'] & & t^{(b_1)} \arrow[ul, "q"'] &
\end{tikzcd}$$

(In case $s^{(a_1)} = s$, the q arrows from $s^{(a_1-1)}$ to $t^{(b_1)}$ do not exist.)

We illustrate this again in a more explicit example: Here is a possible subgraph before we apply $\overrightarrow{\mu_k}$, where $r_k = 8$ and $r_{u_j} = 6$:

$$\begin{tikzcd}
s^{(t_{i_k}-1)} & \cdots \arrow[l] & s^{(8)} \arrow[l] \arrow[r] & s^{(7)} \arrow[r] \arrow[dl, "q"'] & s^{(6)} \arrow[r] & s^{(5)} \arrow[r] \arrow[dl, "q"'] & s^{(4)} \arrow[r] & s^{(3)} \arrow[r] & s^{(2)} \arrow[r] \arrow[dl, "q"'] & s^{(1)} \arrow[r] \arrow[dl, "q"'] & s^{(0)} \\
t^{(t_j-1)} & \cdots \arrow[l] & t^{(8)} \arrow[l] & t^{(7)} \arrow[l] & t^{(6)} \arrow[l] \arrow[r] \arrow[ul, "q"'] & t^{(5)} \arrow[r] & t^{(4)} \arrow[r] & t^{(3)} \arrow[r] \arrow[ull, "q"'] & t^{(2)} \arrow[r] \arrow[u, "q"] & t^{(1)} \arrow[r] & t^{(0)}
\end{tikzcd}$$

(The numbers r_k and r_{u_j} are determined by the orientation of the horizontal arrows in the above picture.)

This is how it looks like after we applied $\overrightarrow{\mu_k}$ to the r_k vertices of the i_k-chain:

$$\begin{tikzcd}
s^{(t_{i_k}-1)} & \cdots \arrow[l] & s^{(8)} \arrow[l] & s^{(7)} \arrow[l] \arrow[r] & s^{(6)} \arrow[r] \arrow[dll, "q"'] & s^{(5)} \arrow[r] & s^{(4)} \arrow[r] \arrow[dll, "q"'] & s^{(3)} \arrow[r] & s^{(2)} \arrow[r] & s^{(1)} \arrow[r] \arrow[dll, "q"'] & s^{(0)} \arrow[dll, "q"'] \\
t^{(t_j-1)} & \cdots \arrow[l] & t^{(8)} \arrow[l] & t^{(7)} \arrow[l] & t^{(6)} \arrow[l] \arrow[r] \arrow[u, "q"] & t^{(5)} \arrow[r] & t^{(4)} \arrow[r] & t^{(3)} \arrow[r] \arrow[ul, "q"'] & t^{(2)} \arrow[r] \arrow[ur, "q"] & t^{(1)} \arrow[r] & t^{(0)}
\end{tikzcd}$$

Again, possible arrows between frozen vertices are not shown.

Note that if we start with our initial $\mathcal{C}_w$-maximal rigid module $V_{\mathbf{i}}$, and if we only perform the $r(\mathbf{i})$ mutations described in the algorithm, then we obtain the subset

$$\bigl\{M[b,a] \bigm| 1 \leqslant a \leqslant b \leqslant r,\ i_a = i_b\bigr\}$$

of the set of indecomposable rigid modules of $\mathcal{C}_w$. In particular, this subset contains all modules $M_k = M[k,k]$ where $1 \leqslant k \leqslant r$. The next theorem describes the precise exchange relation obtained in each of the $r(\mathbf{i})$ steps of the algorithm above.

We use our description of mutations via Δ-dimension vectors from Section 12.5 in order to show that the mutation $M[s, s_{\min}^{(k[i_s])}]^*$ of $M[s, s_{\min}^{(k[i_s])}]$ is indeed $M[s^+, s_{\min}^{(k[i_s]+1)}]$.

In formula (7) below we just write $M[b,a]$ instead of $\delta_{M[b,a]}$. (Recall that for any Λ-module X and any constructible function $f \in \mathcal{M}$ we have $\delta_X(f) := f(X)$. This defines an element δ_X in $\mathcal{M}^*$.)

Theorem 13.1 *(Generalized determinantal identities). Let $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$. Then for $1 \leqslant k, s \leqslant r$ with $i_s = i_k$ we have*

$$\begin{aligned}
&M\bigl[s, s_{\min}^{(k[i_s])}\bigr] \cdot M\bigl[s^+, s_{\min}^{(k[i_s]+1)}\bigr] \\
&\quad = M\bigl[s^+, s_{\min}^{(k[i_s])}\bigr] \cdot M\bigl[s, s_{\min}^{(k[i_s]+1)}\bigr] \\
&\qquad + \prod_{t^+ \geqslant s^+ > t > s} M\bigl[t, t_{\min}^{(k[i_t])}\bigr]^{q_{i_s i_t}} \cdot \prod_{l^+ \geqslant s^+ > s > l > s_{\min}} M\bigl[l, l_{\min}^{(k[i_l])}\bigr]^{q_{i_s i_l}}.
\end{aligned} \tag{7}$$

Proof. Formula (7) is just an exchange relation corresponding to the mutation of the module $M[s, s_{\min}^{(k[i_s])}]$ with $M[s, s_{\min}^{(k[i_s])}]^* = M[s^+, s_{\min}^{(k[i_s]+1)}]$. More precisely, the mutation of $M[s, s_{\min}^{(k[i_s])}]$ happens during the mutation sequence $\overrightarrow{\mu_k}$, which is part of the mutation sequence $\mu_{\mathbf{i}}$. $\square$

Remark 13.2. It is not hard to see that the above theorem can be also stated as follows: For $1 \leqslant t < s \leqslant r$ with $i_s = i_j = i$ we have

$$M[s, t^+]\, M[s^-, t] = M[s, t]\, M[s^-, t^+] + \prod_{j \in I \setminus \{i\}} M[s^-(j), t^+(j)]^{q_{ij}},$$

where in addition to the notation in 9.8 we set $t^+(j) := \min\{r+1, t+1 \leqslant k \leqslant r \mid i_k = j\}$. Fomin and Zelevinsky [22, Theorem 1.17] prove generalized determinantal identities associated to pairs of Weyl group elements for all Dynkin cases (including the non-simply laced cases). Using the material of Section 7, formula (7) can be seen as a generalization of some of their identities to the symmetric Kac–Moody case.

Corollary 13.3. *The functions $\delta_{M_1}, \dots, \delta_{M_r}$ are algebraically independent. In particular, $\mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}]$, the subalgebra of $\mathcal{M}^*$ generated by the δ_{M_k}'s is just a polynomial ring in r variables.*

Proof. Clearly, the functions $\delta_{M[1,1_{\min}]}, \dots, \delta_{M[r,r_{\min}]}$ are algebraically independent, since $V_k = M[k, k_{\min}]$ and any product of the functions $\delta_{V_1}, \dots, \delta_{V_r}$ lies in the dual semicanonical basis. Here we use that $V_{\mathbf{i}}$ is rigid and then we apply [31, Theorem 1.1]. We claim that each function $\delta_{M[b,a]}$ with $1 \leqslant a \leqslant b \leqslant r$ and $i_a = i_b$ is a rational function in $\delta_{M_1}, \dots, \delta_{M_b}$. In particular, each δ_{V_k} is a rational function in $\delta_{M_1}, \dots, \delta_{M_r}$. This implies that $\delta_{M_1}, \dots, \delta_{M_r}$ are algebraically independent.

We prove our claim by induction on r and on the *length* $l([b, a]) := |\{a \leqslant k \leqslant b \mid i_k = i_b\}|$ of the *interval* $[b, a]$. For $r = 1$ the statement is clear. Also, if $l([b, a]) = 1$, then $M[b, a] = M_b$ and we are done as well. Thus assume by induction that our claim is true for all intervals $[d, c]$ of length at most m for some $m \geqslant 1$. All intervals of length $m + 1$ are of the form $[b^+, a]$ for some $1 \leqslant a \leqslant b \leqslant r$. We have $a = b_{\min}^{(k[i_b])}$ for some $1 \leqslant k \leqslant r$. We also assume by induction that our claim holds for all intervals $[d, c]$ with $b^+ > d$. Our formula (7) yields

$$\begin{aligned} M[b^+, a] = {} & \frac{1}{M[b, a^+]} \cdot \bigl(M[b, a] \cdot M[b^+, a^+]\bigr) \\ & - \frac{1}{M[b, a^+]} \cdot \Biggl(\prod_{t^+ \geqslant b^+ > t > b} M[t, t_{\min}^{(k[i_t])}]^{q_{i_b i_t}} \cdot \prod_{l^+ \geqslant b^+ > b > l > b_{\min}} M[l, l_{\min}^{(k[i_l])}]^{q_{i_b i_l}} \Biggr). \end{aligned} \tag{8}$$

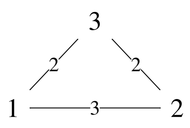
The intervals on the right hand side of this equation all have either length at most m, or they are of the form $[d, c]$ with $b^+ > d$. This finishes the proof. □

In fact, we will show that for any Λ-module $X \in \mathcal{C}_w$ we have $\delta_X \in \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}]$, see Theorem 15.1. In particular, for all $1 \leqslant k \leqslant r$ the rational function δ_{V_k} is a polynomial in $\delta_{M_1}, \dots, \delta_{M_r}$.

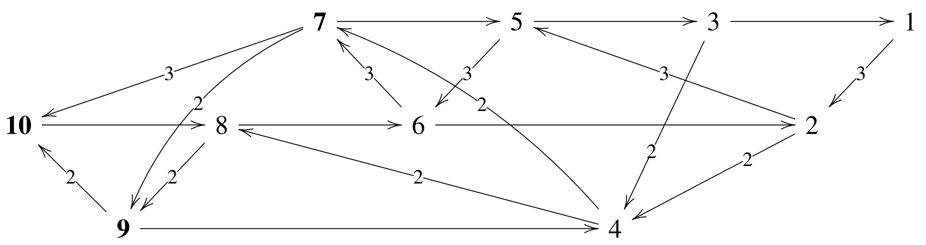
Another proof of the polynomiality of the functions $\delta_{M[b,a]}$ was found by Kedem and Di Francesco [21, Lemma B.7], using ideas of Fomin and Zelevinsky (in particular [9, Lemma 4.2]). We thank these four mathematicians for communicating their insights to us at MSRI in March 2008.

13.2. Example

Let Q be a quiver with underlying graph



Here we use the notation $i \overset{a}{\text{——}} j$ if there are a edges between i and j. Let $\mathbf{i} := (i_{10}, \dots, i_1) := (2, 3, 2, 1, 2, 1, 3, 1, 2, 1)$. This is a reduced expression for a Weyl group element in W_Q. The quiver $\Gamma_{\mathbf{i}}$ looks as follows:

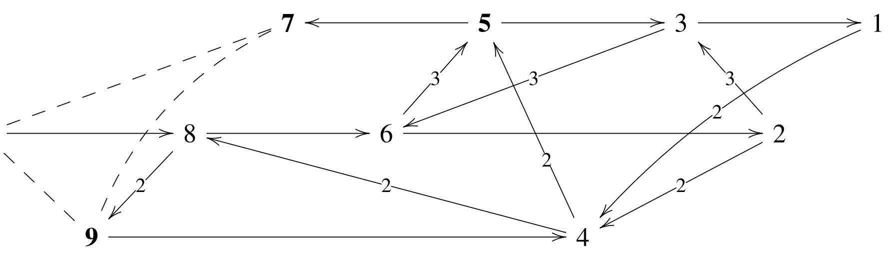


For the mutation sequence $\mu_{\mathbf{i}}$ we get

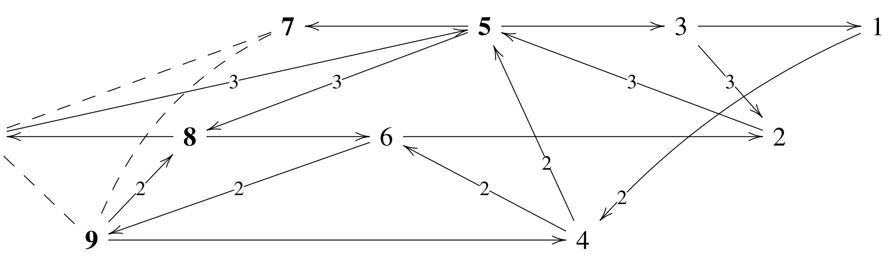
$$\begin{aligned} \mu_{\mathbf{i}} &= \overrightarrow{\mu_{10}} \circ \cdots \circ \overrightarrow{\mu_2} \circ \overrightarrow{\mu_1} \\ &= (\mathrm{id}) \circ (\mathrm{id}) \circ (\mu_2) \circ (\mathrm{id}) \circ (\mu_6\mu_2) \circ (\mu_1) \circ (\mu_4) \circ (\mu_3\mu_1) \circ (\mu_8\mu_6\mu_2) \circ (\mu_5\mu_3\mu_1). \end{aligned}$$

Here are the quivers $\Gamma_{\mathbf{i}}^k$:

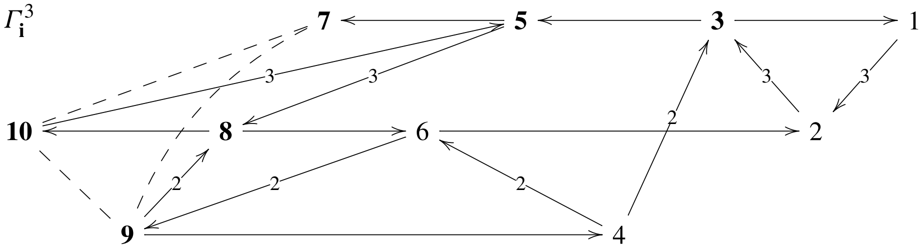
$\Gamma_{\mathbf{i}}^1$



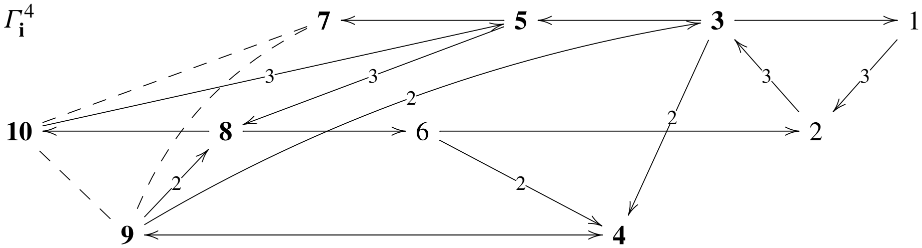
$\Gamma_{\mathbf{i}}^2$



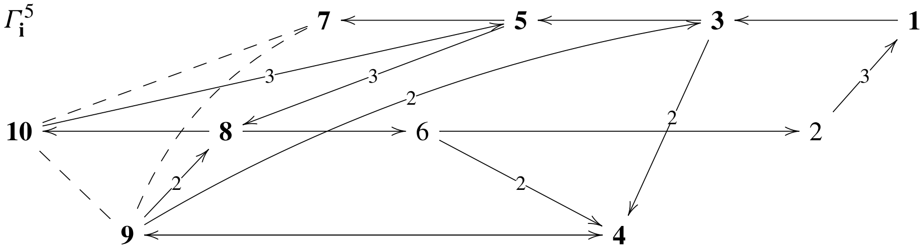
$\Gamma_{\mathbf{i}}^3$



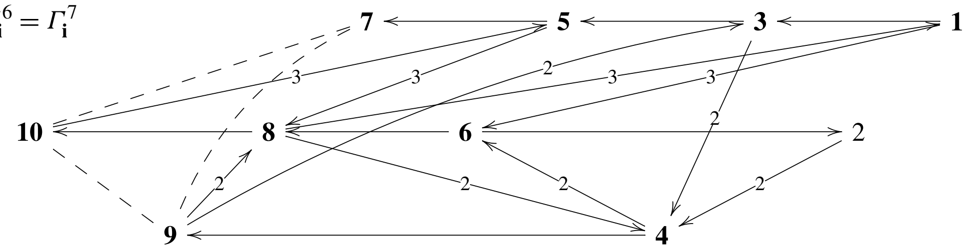
$\Gamma_{\mathbf{i}}^4$



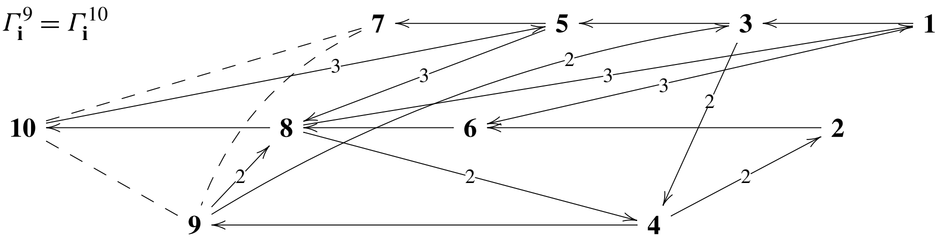
$\Gamma_{\mathbf{i}}^5$



$\Gamma_{\mathbf{i}}^6 = \Gamma_{\mathbf{i}}^7$



$\Gamma_{\mathbf{i}}^8 = \Gamma_{\mathbf{i}}^9 = \Gamma_{\mathbf{i}}^{10}$



Applying formula (7) to $M[s, s_{\min}^{(k[i_s])}] := M[6, 6_{\min}^{(2[i_6])}] = M[6,2]$ we get the following:

$$M[6,2] \cdot M[8,6] = M[8,2] \cdot M[6,6] + M[7,3]^3 \cdot M[4,4]^2 \quad (s = 6,\ k = 2).$$

Thus, we have

$$M[8,2] = \frac{1}{M[6,6]}\bigl(M[6,2] \cdot M[8,6] - M[7,3]^3 \cdot M[4,4]^2\bigr).$$

Similarly, we obtain

$$\begin{aligned}
&M[2,2] \cdot M[6,6] = M[6,2] + M[5,3]^3 \cdot M[4,4]^2 && (s = 2, k = 2),\\
&M[6,6] \cdot M[8,8] = M[8,6] + M[7,7]^3 && (s = 6, k = 6),\\
&M[5,3] \cdot M[7,5] = M[7,3] \cdot M[5,5] + M[6,6]^3 \cdot M[4,4]^2 && (s = 5, k = 3),\\
&M[3,3] \cdot M[5,5] = M[5,3] + M[4,4]^2 && (s = 3, k = 3),\\
&M[5,5] \cdot M[7,7] = M[7,5] + M[6,6]^3 && (s = 5, k = 5).
\end{aligned}$$

By our double induction (on r and on the length of the intervals $[b,a]$), in each of the above equations, we can write the functions $M[6,2]$, $M[8,6]$, $M[7,3]$, $M[5,3]$ and $M[7,5]$, respectively, as a rational function of the functions appearing in the same equation. Now one can use these equations to express $\delta_{M[8,2]}$ as a rational function in $\delta_{M_1}, \dots, \delta_{M_8}$. Remarkably, this rational function is a polynomial.

Finally, we display the dimension vectors of the modules $M_1, \dots, M_8$:

$$\begin{aligned}
&\beta_{\mathbf{i}}(1) = \begin{smallmatrix} & 0 & \\ 1 & & 0 \end{smallmatrix} && \beta_{\mathbf{i}}(2) = \begin{smallmatrix} & 1 & \\ 3 & & 0 \end{smallmatrix} && \beta_{\mathbf{i}}(3) = \begin{smallmatrix} & 3 & \\ 8 & & 0 \end{smallmatrix} && \beta_{\mathbf{i}}(4) = \begin{smallmatrix} & 8 & \\ 24 & & 1 \end{smallmatrix}\\
&\beta_{\mathbf{i}}(5) = \begin{smallmatrix} & 13 & \\ 40 & & 2 \end{smallmatrix} && \beta_{\mathbf{i}}(6) = \begin{smallmatrix} & 63 & \\ 189 & & 8 \end{smallmatrix} && \beta_{\mathbf{i}}(7) = \begin{smallmatrix} & 176 & \\ 527 & & 22 \end{smallmatrix} && \beta_{\mathbf{i}}(8) = \begin{smallmatrix} & 465 & \\ 1392 & & 58 \end{smallmatrix}.
\end{aligned}$$

As an exercise, the reader can compute $\beta_{\mathbf{i}}(9)$ and $\beta_{\mathbf{i}}(10)$.

Exchange equations are always homogeneous. For example,

$$M[5,3] \cdot M[7,5] = M[7,3] \cdot M[5,5] + M[6,6]^3 \cdot M[4,4]^2$$

is an equation of degree $\begin{smallmatrix} & 205 & \\ 615 & & 26 \end{smallmatrix}$.

13.3. The shift functor in $\underline{\mathcal{C}}_w$ via mutations

Fix a reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of some Weyl group element w. As before, let $T_{\mathbf{i}} := I_w \oplus \Omega_w^{-1}(V_{\mathbf{i}})$. Define

$$W_{\mathbf{i}} := I_w \oplus \Omega_w(V_{\mathbf{i}}).$$

In Section 13.1 we defined a sequence of mutations

$$\mu_{\mathbf{i}} = \overrightarrow{\mu_r} \circ \cdots \circ \overrightarrow{\mu_1} = \mu_{s_{r(\mathbf{i})}} \circ \cdots \circ \mu_{s_2} \circ \mu_{s_1}$$

where $1 \leqslant s_p \leqslant r$ for all p, such that

$$\mu_{\mathbf{i}}(V_{\mathbf{i}}) = \mu_{s_{r(\mathbf{i})}} \circ \cdots \circ \mu_{s_2} \circ \mu_{s_1}(V_{\mathbf{i}}) = T_{\mathbf{i}} \quad \text{and} \quad \mu_{\mathbf{i}}^{-1}(T_{\mathbf{i}}) = \mu_{s_1} \circ \mu_{s_2} \circ \cdots \circ \mu_{s_{r(\mathbf{i})}}(T_{\mathbf{i}}) = V_{\mathbf{i}}.$$

Clearly, if R is a basic $\mathcal{C}_w$-maximal rigid module such that $R = \mu_{p_t} \circ \cdots \circ \mu_{p_1}(V_{\mathbf{i}})$, then we have $R = \mu_{p_t} \circ \cdots \circ \mu_{p_1} \circ \mu_{\mathbf{i}}^{-1}(T_{\mathbf{i}})$.

Now define an involution

$$(-)^* : \{1, \dots, r\} \setminus \{1 \leqslant k \leqslant r \mid k^+ = r+1\} \to \{1, \dots, r\} \setminus \{1 \leqslant k \leqslant r \mid k^+ = r+1\}$$

by

$$\left(k_{\min}^{(m)}\right)^* := k_{\min}^{(t_j-2-m)},$$

where $j := i_{k_{\min}}$. Observe that every $1 \leqslant s \leqslant r$ can be written as $k_{\min}^{(m)}$ for some unique k (namely $k = s$) and some unique $0 \leqslant m \leqslant t_j - 1$. The following picture illustrates how $(-)^*$ permutes the vertices of $\Gamma_{\mathbf{i}}$:

$$\begin{tikzcd}
k_{\min}^{(t_j-1)} \arrow[r] & k_{\min}^{(t_j-2)} \arrow[r] \arrow[rrrrr, <->, bend left] & k_{\min}^{(t_j-3)} \arrow[r] \arrow[rrr, <->, bend left] & k_{\min}^{(t_j-4)} \arrow[r] \arrow[r, <->, bend left] & \cdots \arrow[r] & k_{\min}^{(2)} \arrow[r] & k_{\min}^{(1)} \arrow[r] & k_{\min}
\end{tikzcd}$$

Set

$$\left(\mu_{\mathbf{i}}^{-1}\right)^* := \mu_{s_1^*} \circ \mu_{s_2^*} \circ \cdots \circ \mu_{s_{r(\mathbf{i})}^*}.$$

Proposition 13.4. *Let R be a basic $\mathcal{C}_w$-maximal rigid module which is mutation equivalent to $V_{\mathbf{i}}$. Then $I_w \oplus \Omega_w^{-1}(R)$ and $I_w \oplus \Omega_w(R)$ are mutation equivalent to $V_{\mathbf{i}}$. More precisely, let*

$$R = \mu_{z_t} \circ \cdots \circ \mu_{z_1}(V_{\mathbf{i}}) \quad \textit{and} \quad R = \mu_{q_u} \circ \cdots \circ \mu_{q_1}(T_{\mathbf{i}}).$$

Then we have

$$I_w \oplus \Omega_w^{-1}(R) = \mu_{z_t^*} \circ \cdots \circ \mu_{z_1^*}(T_{\mathbf{i}}) \quad \textit{and} \quad I_w \oplus \Omega_w(R) = \mu_{q_u^*} \circ \cdots \circ \mu_{q_1^*}(V_{\mathbf{i}}).$$

Besides $\Omega_w^{-1}(R)$, we can also compute $\Omega_w(R)$ by just knowing a sequence of mutations from $V_{\mathbf{i}}$ to R. This works because $V_{\mathbf{i}}$ and $T_{\mathbf{i}}$ are connected via a known sequence of mutations, namely $\mu_{\mathbf{i}}$ if we start at $V_{\mathbf{i}}$, and $\mu_{\mathbf{i}}^{-1}$ if we start at $T_{\mathbf{i}}$. The following picture illustrates the situation:

$$\begin{tikzcd}
W_{\mathbf{i}} \arrow[d, rightsquigarrow, "\mu_{z_t^*} \circ \cdots \circ \mu_{z_1^*}"] & V_{\mathbf{i}} \arrow[l, rightsquigarrow, "(\mu_{\mathbf{i}}^{-1})^*"'] \arrow[r, rightsquigarrow, "\mu_{\mathbf{i}}"] \arrow[d, rightsquigarrow, "\mu_{z_t} \circ \cdots \circ \mu_{z_1}"] & T_{\mathbf{i}} \arrow[d, rightsquigarrow, "\mu_{z_t^*} \circ \cdots \circ \mu_{z_1^*}"] \\
I_w \oplus \Omega_w(R) & R & I_w \oplus \Omega_w^{-1}(R)
\end{tikzcd}$$

Proof. As before, for $1 \leqslant j \leqslant n$ set $p_j := \min\{1 \leqslant s \leqslant r \mid i_s = j\}$. Note that $p_j^{(t_j-1)} = p_{j_{\max}}$ and $p_j = p_{j_{\min}}$. In the following pictures we display only the relevant horizontal arrows. The quiver $\Gamma_{V_{\mathbf{i}}}$ looks as follows:

$$\begin{tikzcd}
M[p_1^{(t_1-1)}, p_1] \arrow[r] \arrow[d, dotted, no head] & M[p_1^{(t_1-2)}, p_1] \arrow[r] \arrow[d, dotted, no head] & \cdots \arrow[r] & M[p_1^{(1)}, p_1] \arrow[r] \arrow[d, dotted, no head] & M[p_1, p_1] \arrow[d, dotted, no head] \\
M[p_n^{(t_n-1)}, p_n] \arrow[r] & M[p_n^{(t_n-2)}, p_n] \arrow[r] & \cdots \arrow[r] & M[p_n^{(1)}, p_n] \arrow[r] & M[p_n, p_n]
\end{tikzcd}$$

Next, we display the quiver $\Gamma_{T_{\mathbf{i}}}$:

$$\begin{tikzcd}
M[p_{1_{\max}}, p_1] \arrow[d, dotted, no head] & M[p_{1_{\max}}, p_1^{(1)}] \arrow[l] \arrow[d, dotted, no head] & M[p_{1_{\max}}, p_1^{(2)}] \arrow[l] \arrow[d, dotted, no head] & \cdots \arrow[l] & M[p_{1_{\max}}, p_1^{(t_1-1)}] \arrow[l] \arrow[d, dotted, no head] \\
M[p_{n_{\max}}, p_n] & M[p_{n_{\max}}, p_n^{(1)}] \arrow[l] & M[p_{n_{\max}}, p_n^{(2)}] \arrow[l] & \cdots \arrow[l] & M[p_{n_{\max}}, p_n^{(t_n-1)}] \arrow[l]
\end{tikzcd}$$

We know that $I_w \oplus \Omega_w^{-1}(V_{\mathbf{i}}) = T_{\mathbf{i}}$. In particular, we have

$$\Omega_w^{-1}\big(M\big[p_j^{(s-1)}, p_j\big]\big) = M\big[p_{j_{\max}}, p_j^{(s)}\big]$$

for all $1 \leqslant j \leqslant n$ and $1 \leqslant s \leqslant t_j - 1$. Thus $\Gamma_{T_{\mathbf{i}}}$ looks like this:

$$\begin{tikzcd}
M[p_1^{(t_1-1)}, p_1] \arrow[d, dotted, no head] & \Omega_w^{-1}(M[p_1, p_1]) \arrow[l] \arrow[d, dotted, no head] & \Omega_w^{-1}(M[p_1^{(1)}, p_1]) \arrow[l] \arrow[d, dotted, no head] & \cdots \arrow[l] & \Omega_w^{-1}(M[p_1^{(t_1-2)}, p_1]) \arrow[l] \arrow[d, dotted, no head] \\
M[p_n^{(t_n-1)}, p_n] & \Omega_w^{-1}(M[p_n, p_n]) \arrow[l] & \Omega_w^{-1}(M[p_n^{(1)}, p_n]) \arrow[l] & \cdots \arrow[l] & \Omega_w^{-1}(M[p_n^{(t_n-2)}, p_n]) \arrow[l]
\end{tikzcd}$$

The n vertices of the form $M[p_j^{(t_j-1)}, p_j]$ at the "left" of both quivers $\Gamma_{V_{\mathbf{i}}}$ and $\Gamma_{T_{\mathbf{i}}}$ are frozen vertices, to all other vertices we can apply the mutation operation.

Now let

$$0 \to T_k \to T' \to T_k^* \to 0$$

be an exchange sequence associated to the cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. This yields an *exchange triangle* $T_k \to T' \to T_k^* \to T_k[1]$ in the stable category $\underline{\mathcal{C}}_w$. Note that $T_k[1] = \Omega_w^{-1}(T_k)$. It follows that $T_k[1] \to T'[1] \to T_k^*[1] \to T_k[2]$ is an exchange triangle as well. There is an associated exchange sequence

$$0 \to \Omega_w^{-1}(T_k) \to I \oplus \Omega_w^{-1}\big(T'\big) \to \Omega_w^{-1}\big(T_k^*\big) \to 0$$

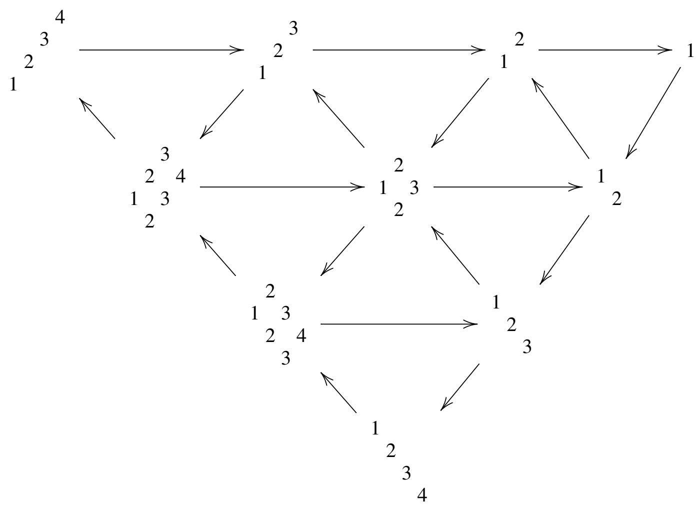
where I is some module in $\operatorname{add}(I_w)$. Thus, if we mutate the basic $\mathcal{C}_w$-maximal rigid module $I_w \oplus \Omega_w^{-1}(T)$ in direction $\Omega_w^{-1}(T_k)$, we obtain $(\Omega_w^{-1}(T_k))^* = \Omega_w^{-1}(T_k^*)$. We argue similarly to show that the mutation of $I_w \oplus \Omega_w(T)$ in direction $\Omega_w(T_k)$ gives $(\Omega_w(T_k))^* = \Omega_w(T_k^*)$. This finishes the proof. $\square$

Corollary 13.5. *If a Λ-module R is $V_{\mathbf{i}}$-reachable, then $\Omega_w^z(R)$ is $V_{\mathbf{i}}$-reachable for all $z \in \mathbb{Z}$.*

13.4. Example

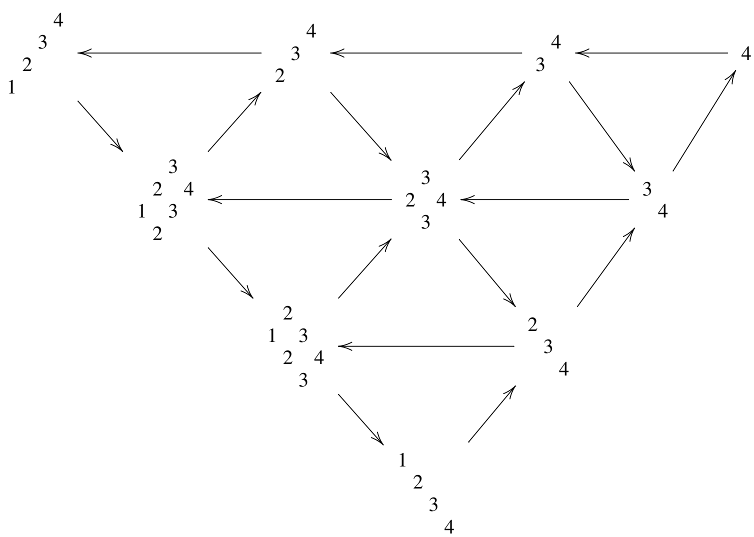
Let Q be a quiver with underlying graph $1 \;\text{——}\; 2 \;\text{——}\; 3 \;\text{——}\; 4$ and let $\mathbf{i} := (i_{10}, \dots, i_1) := (1, 2, 1, 3, 2, 1, 4, 3, 2, 1)$. Then we get

$\Gamma_{V_{\mathbf{i}}}$



and

$\Gamma_{T_{\mathbf{i}}}$



Again, we identify the vertices of $\Gamma_{V_\mathbf{i}}$ and $\Gamma_{T_\mathbf{i}}$ with the indecomposable direct summands of $V_\mathbf{i}$ and $T_\mathbf{i}$, respectively. As before, we identify $\Gamma_{V_\mathbf{i}}$ and the quiver $\Gamma_\mathbf{i}$, which looks as follows:

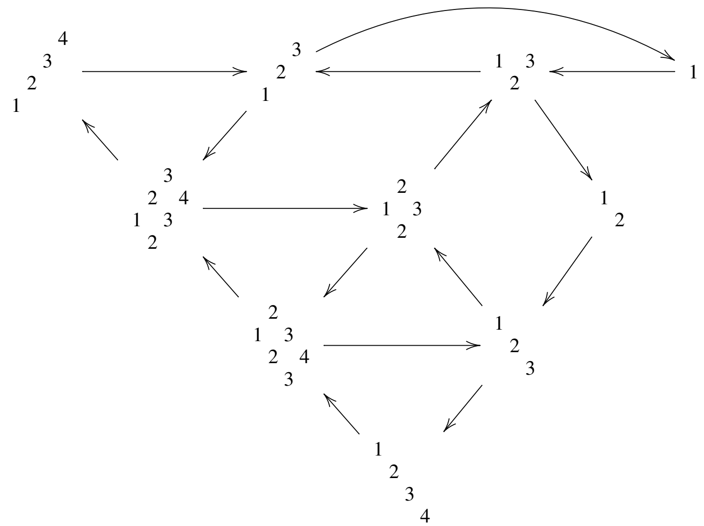
$$\Gamma_\mathbf{i}(\equiv \Gamma_{V_\mathbf{i}}) \qquad \begin{tikzcd}
10 \arrow[rr] & & 8 \arrow[rr] \arrow[dl] & & 5 \arrow[rr] \arrow[dl] & & 1 \arrow[dl] \\
& 9 \arrow[ul] \arrow[rr] & & 6 \arrow[ul] \arrow[rr] \arrow[dl] & & 2 \arrow[ul] \arrow[dl] & \\
& & 7 \arrow[ul] \arrow[rr] & & 3 \arrow[ul] \arrow[dl] & & \\
& & & 4 \arrow[ul] & & &
\end{tikzcd}$$

We have

$$\begin{aligned}
\mu_\mathbf{i} &= \overrightarrow{\mu_{10}} \circ \cdots \circ \overrightarrow{\mu_2} \circ \overrightarrow{\mu_1} \\
&= (\mathrm{id}) \circ (\mathrm{id}) \circ (\mu_1) \circ (\mathrm{id}) \circ (\mu_2) \circ (\mu_5 \circ \mu_1) \circ (\mathrm{id}) \circ (\mu_3) \circ (\mu_6 \circ \mu_2) \circ (\mu_8 \circ \mu_5 \circ \mu_1).
\end{aligned}$$

Mutation of $V_\mathbf{i}$ at V_5 yields the following quiver:

$\Gamma_{\mu_5(V_\mathbf{i})}$



The associated exchange sequences are

$$0 \to \left(\begin{smallmatrix}1&&3\\&2&\end{smallmatrix}\right) \to 1 \oplus \left(\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}\right) \to \left(\begin{smallmatrix}&2\\1&\end{smallmatrix}\right) \to 0 \quad \text{and} \quad 0 \to \left(\begin{smallmatrix}&2\\1&\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&\\&2\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&&3\\&2&\end{smallmatrix}\right) \to 0$$

Next, we mutate at V_6. The exchange sequences look as follows:

$$0 \to \left(\begin{smallmatrix}1&&3&\\&2&&4\\&&3&\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&&3\\&2&\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}&2&&\\1&&3&\\&2&&4\\&&3&\end{smallmatrix}\right) \to \left(\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}\right) \to 0 \quad \text{and}$$

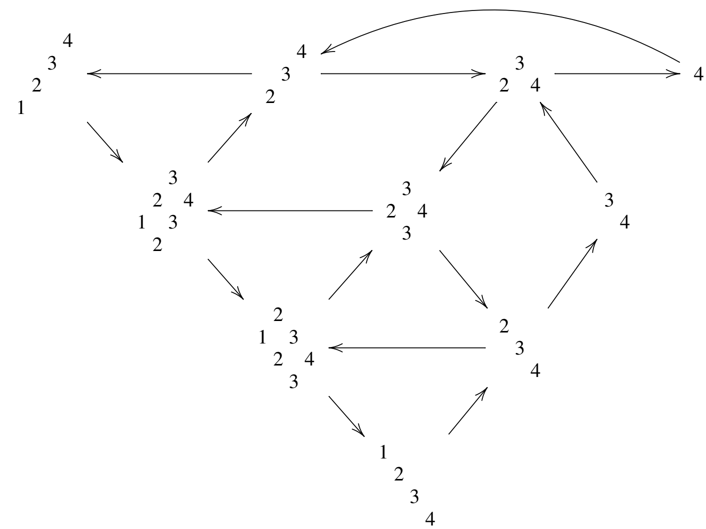
$$0 \to \left(\begin{smallmatrix} &2& \\ 1&&3 \\ &2& \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1&& \\ &2& \\ &&3 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} &&3& \\ &2&&4 \\ 1&&3& \\ &2&& \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1&&3& \\ &2&&4 \\ &&3& \end{smallmatrix}\right) \to 0$$

Set $R := (\mu_6 \circ \mu_5)(V_{\mathbf{i}})$. Thus we have $R = R_5 \oplus R_6 \oplus V_{\mathbf{i}}/(V_5 \oplus V_6)$ with

$$R_5 = \begin{smallmatrix} 1&&3 \\ &2& \end{smallmatrix} \quad \text{and} \quad R_6 = \begin{smallmatrix} 1&&3& \\ &2&&4 \\ &&3& \end{smallmatrix}.$$

To calculate $\Omega_w^{-1}(R)$, we have to compute $(\mu_{6^*} \circ \mu_{5^*})(T_{\mathbf{i}})$. Mutation of $T_{\mathbf{i}}$ at $5^* = 5$ yields the following quiver:

$\Gamma_{\mu_{5^*}(T_{\mathbf{i}})}$



The associated exchange sequences are

$$0 \to \begin{smallmatrix} &3& \\ 2&&4 \end{smallmatrix} \to \begin{smallmatrix} 3& \\ &4 \end{smallmatrix} \oplus \begin{smallmatrix} &&4 \\ &3& \\ 2&& \end{smallmatrix} \to \begin{smallmatrix} &4 \\ 3& \end{smallmatrix} \to 0 \quad \text{and} \quad 0 \to \begin{smallmatrix} &4 \\ 3& \end{smallmatrix} \to 4 \oplus \begin{smallmatrix} &3& \\ 2&&4 \\ &3& \end{smallmatrix} \to \begin{smallmatrix} &3& \\ 2&&4 \end{smallmatrix} \to 0.$$

Next, we mutate at $6^* = 2$. The exchange sequences look as follows:

$$0 \to 2 \to \begin{smallmatrix} &3& \\ 2&&4 \end{smallmatrix} \to \begin{smallmatrix} 3& \\ &4 \end{smallmatrix} \to 0 \quad \text{and} \quad 0 \to \begin{smallmatrix} 3& \\ &4 \end{smallmatrix} \to \begin{smallmatrix} 2&& \\ &3& \\ &&4 \end{smallmatrix} \to 2 \to 0.$$

We get $\Omega_w^{-1}(R) := \Omega_w^{-1}(R_5) \oplus \Omega_w^{-1}(R_6) \oplus T_{\mathbf{i}}/(T_5 \oplus T_6)$ with

$$\Omega_w^{-1}(R_5) = \begin{smallmatrix} &3& \\ 2&&4 \end{smallmatrix} \quad \text{and} \quad \Omega_w^{-1}(R_6) = 2.$$

14. Irreducible components associated to $\mathcal{C}_w$

14.1. Module varieties

Let $\Gamma := (\Gamma_0, \Gamma_1, s, t)$ be a finite quiver with vertex set $\Gamma_0 = \{1, \dots, r\}$, arrow set Γ_1 and maps $s, t : \Gamma_1 \to \Gamma_0$ which map an arrow a to its start vertex $s(a)$ and its terminal vertex $t(a)$, respectively. In this section, we interpret dimension vectors $\mathbf{f} = (f_1, \dots, f_r)$ for Γ as maps $\mathbf{f}\colon \Gamma_0 \to \mathbb{N}$. We consider the affine space

$$\operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f}) = \operatorname{rep}(\Gamma, \mathbf{f}) = \prod_{a \in \Gamma_1} \mathbb{C}^{\mathbf{f}(t(a)) \times \mathbf{f}(s(a))}$$

of representations of Γ with dimension vector $\mathbf{f}$. Here $\mathbb{C}^{p \times q}$ denotes the vector space of $(p \times q)$-matrices with entries in $\mathbb{C}$. This coincides with our definitions in Section 2.1, except that we now work with spaces $\mathbb{C}^{p \times q}$ of matrices rather than spaces $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^q, \mathbb{C}^p)$ of linear maps. So each element in $\operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f})$ is of the form $M = (M(a))_{a \in \Gamma_1}$ where $M(a)$ is a matrix of size $\mathbf{f}(t(a)) \times \mathbf{f}(s(a))$.

The group

$$\mathrm{GL}_{\mathbf{f}} := \prod_{i \in \Gamma_0} \mathrm{GL}_{\mathbf{f}(i)}(\mathbb{C})$$

acts from the left by conjugation on $\operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f})$, *i.e.* for $M = (M(a))_{a \in \Gamma_1} \in \operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f})$ and $g = (g(i))_{i \in \Gamma_0} \in \mathrm{GL}_{\mathbf{f}}$ we have

$$(g.M)(a) = g\big(t(a)\big)M(a)g\big(s(a)\big)^{-1}$$

for all $a \in \Gamma_1$. The orbits of $\mathrm{GL}_{\mathbf{f}}$ on $\operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f})$ correspond to the set of isomorphism classes of $\mathbb{C}\Gamma$-modules with dimension vector $\mathbf{f}$. Given a path $p = a_l \cdots a_2 a_1$ in Γ (*i.e.* $a_1, \dots, a_l$ are arrows with $s(a_{i+1}) = t(a_i)$ for $1 \leqslant i \leqslant l-1$) we define

$$M(p) := M(a_l) \cdots M(a_2)M(a_1)$$

for any $M \in \operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f})$. More generally, for any element $\rho \in e_i \mathbb{C}\Gamma e_j$ we have $M(\rho) \in \mathbb{C}^{\mathbf{f}(i) \times \mathbf{f}(j)}$, since ρ is a linear combination of paths from j to i. (For $k \in \Gamma_0$ we denote the associated path of length 0 by e_k.) Set $s(\rho) := j$ and $t(\rho) := i$. If $I \subset \mathbb{C}\Gamma$ is a finitely generated ideal contained in the ideal generated by all paths of length 2, we may assume that it is generated by elements $\rho_1, \dots, \rho_q$ with $\rho_k \in e_{t_k} \mathbb{C}\Gamma e_{s_k}$ for certain $s_k, t_k \in \Gamma_0$ where $1 \leqslant k \leqslant q$. Let $A := \mathbb{C}\Gamma/I$. We consider the affine $\mathrm{GL}_{\mathbf{f}}$-variety

$$\operatorname{mod}(A, \mathbf{f}) := \big\{M \in \operatorname{mod}(\mathbb{C}\Gamma, \mathbf{f}) \bigm| M(\rho_k) = 0 \text{ for } 1 \leqslant k \leqslant q\big\}.$$

Again, the $\mathrm{GL}_{\mathbf{f}}$-orbits correspond to the isomorphism classes of A-modules with dimension vector $\mathbf{f}$.

Given $M \in \operatorname{mod}(A, \mathbf{f})$ and $M' \in \operatorname{mod}(A, \mathbf{f}')$ we identify any homomorphism $\varphi \in \operatorname{Hom}_A(M, M')$ with a family of matrices

$$\big(\varphi(k)\big)_{k \in \Gamma_0} \in \prod_{k \in \Gamma_0} \mathbb{C}^{\mathbf{f}'(k) \times \mathbf{f}(k)}$$

such that

$$\varphi\big(t(b)\big)M(b) = M'(b)\varphi\big(s(b)\big)$$

for all $b \in \Gamma_1$. In other words, the diagram

$$\begin{array}{ccc} \mathbb{C}^{\mathbf{f}(s(b))} & \xrightarrow{\varphi(s(b))} & \mathbb{C}^{\mathbf{f}'(s(b))} \\ {\scriptstyle M(b)}\downarrow & & \downarrow{\scriptstyle M'(b)} \\ \mathbb{C}^{\mathbf{f}(t(b))} & \xrightarrow{\varphi(t(b))} & \mathbb{C}^{\mathbf{f}'(t(b))} \end{array}$$

commutes for all $b \in \Gamma_1$.

14.2. A stratification of Λ_d^w

Recall that for $X \in \mathrm{nil}(\Lambda)$ we have $X \in \mathcal{C}_w$ if and only if there is a (unique) filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

by submodules such that $X_k/X_{k-1} \cong M_k^{a_k}$ for some $a_k \geqslant 0$ for all $1 \leqslant k \leqslant r$, see Proposition 10.2. In this case, we have

$$\underline{\dim}_{B_{\mathbf{i}}}\big(\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)\big) = \sum_{k=1}^{r} a_k\, \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k),$$

i.e. $\mathbf{a} := (a_1, \dots, a_r)$ is the Δ-dimension vector of $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)$. Thus, with

$$\mu(\mathbf{a}) := \sum_{k=1}^{r} a_k\, \underline{\dim}_\Lambda(M_k)$$

we may consider

$$\Lambda^{\mathbf{a}} := \big\{X \in \Lambda_{\mu(\mathbf{a})} \bigm| X \text{ has a filtration } 0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X \\ \text{with } X_k/X_{k-1} \cong M_k^{a_k},\ 1 \leqslant k \leqslant r\big\}.$$

In other words, $\Lambda^{\mathbf{a}} = \{X \in \Lambda_{\mu(\mathbf{a})} \mid X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}\}$. Define

$$\Lambda_{\mathbf{d}}^w := \{X \in \Lambda_{\mathbf{d}} \mid X \in \mathcal{C}_w\}.$$

We get a finite decomposition

$$\Lambda_{\mathbf{d}}^w = \bigcup_{\mathbf{a} \in \mathbb{N}^r,\, \mu(\mathbf{a}) = \mathbf{d}} \Lambda^{\mathbf{a}}$$

into disjoint subsets.

Lemma 14.1. *$\Lambda^{\mathbf{a}}$ is an irreducible constructible subset of $\Lambda_{\mu(\mathbf{a})}$.*

Proof. We know from Proposition 10.5 that $X \in \Lambda^{\mathbf{a}}$ if and only if there exists a short exact sequence

$$0 \to \bigoplus_{k=1}^{r} V_{k^-}^{a_k} \to \bigoplus_{k=1}^{r} V_k^{a_k} \to X \to 0$$

with $V_{k^-} = 0$ if $k^- = 0$. Now the result follows from [10, Section 2.1]. $\square$

Remark 14.2. It is not hard to see that for $X \in \operatorname{nil}(\Lambda)$ the following are equivalent

(i) $X \in \mathcal{C}_w$;
(ii) $\operatorname{Hom}_\Lambda(D(J_w), X) = 0 = \operatorname{Ext}^1_\Lambda(D(\Lambda/J_w), X)$.

Here J_i is by definition the ideal of Λ which is as a $\mathbb{C}$-vector space generated by all paths p in $\overline{Q}$ with $p \neq e_i$, and we set $J_w := J_{i_r} \cdots J_{i_1}$. It follows that $\Lambda_{\mathbf{d}}^w$ is an open subset in $\Lambda_{\mathbf{d}}$ and it follows that $\Lambda^{\mathbf{a}}$ is a locally closed subset of $\Lambda_{\mu(\mathbf{a})}$. However, we will not need this fact.

14.3. Review of Bongartz's bundle construction

Following Bongartz [10, Section 4], we apply the above definitions and conventions in order to relate the varieties $\Lambda^{\mathbf{a}}$ and $\operatorname{mod}(B_{\mathbf{i}}, \mathbf{f})$. Assume that

$$\mathbf{f} = \sum_{k=1}^{r} a_k \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k).$$

Recall that this implies $\dim \operatorname{Hom}_\Lambda(V_k, X) = \mathbf{f}(k)$ for all $X \in \Lambda^{\mathbf{a}}$. It follows, that

$$\bigl\{(X, \varphi) \bigm| X \in \Lambda^{\mathbf{a}} \text{ and } \varphi \in \operatorname{Hom}_\Lambda(V_k, X)\bigr\}$$

is a (usually non-trivial) algebraic vector bundle of rank $\mathbf{f}(k)$ over $\Lambda^{\mathbf{a}}$. Thus, setting

$$I(\mathbf{f}) := \bigl\{(i, j) \in \mathbb{N}_1^2 \bigm| 1 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant \mathbf{f}(i)\bigr\}$$

we consider

$$\begin{aligned} H^{\mathbf{a}} := \bigl\{\bigl(X, \bigl(\varphi_j^{(i)}\bigr)_{(i,j)\in I(\mathbf{f})}\bigr) \bigm| X \in \Lambda^{\mathbf{a}} \text{ and } \bigl(\varphi_1^{(k)}, \dots, \varphi_{\mathbf{f}(k)}^{(k)}\bigr) \\ \text{is a basis of } \operatorname{Hom}_\Lambda(V_k, X),\ 1 \leqslant k \leqslant r\bigr\} \end{aligned}$$

equipped with a left $\mathrm{GL}_{\mathbf{d}}$-action given by

$$g.\bigl(X, \bigl(\varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr) = \bigl(g.X, \bigl(g \circ \varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr),$$

and with a right $\mathrm{GL}_{\mathbf{f}}$-action given by

$$\bigl(X, \bigl(\varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr).h = \left(X, \left(\sum_{t=1}^{\mathbf{f}(k)} \varphi_t^{(k)} h_{t,j}(k)\right)_{(k,j)\in I(\mathbf{f})}\right).$$

Here $h_{t,j}(k)$ denotes the entry in row t and column j of the matrix h_k. Clearly, the map

$$\pi_1 : H^{\mathbf{a}} \to \Lambda^{\mathbf{a}}$$

defined by

$$\bigl(X, \bigl(\varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr) \mapsto X$$

is a $\mathrm{GL}_{\mathbf{d}}$-equivariant $\mathrm{GL}_{\mathbf{f}}$-principal bundle.

In order to define a map $\pi_2 : H^{\mathbf{a}} \to \mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$ we write $B_{\mathbf{i}} = \mathbb{C}\Gamma/I$ for an admissible ideal I, and we identify the vertices $\Gamma_0 = \{1, 2, \dots, r\}$ with the summands $V_1, \dots, V_r$ of $V_{\mathbf{i}}$. Recall that $\Gamma \equiv \Gamma_{\mathbf{i}}$. Thus we may think of each arrow $b : i \to j$ in Γ_1 as a certain element $b \in \mathrm{Hom}_\Lambda(V_j, V_i)$. With these identifications

$$\pi_2\bigl(X, \bigl(\varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr) = \bigl(M(b)\bigr)_{b\in\Gamma_1}$$

is determined by

$$\varphi_j^{(s(b))} \circ b = \sum_{u=1}^{\mathbf{f}(t(b))} \varphi_u^{(t(b))} M_{u,j}(b).$$

Here $M_{u,j}(b)$ denotes the entry in row u and column j of the matrix $M(b)$. It is easy to verify that π_2 is a $\mathrm{GL}_{\mathbf{d}}$-invariant $\mathrm{GL}_{\mathbf{f}}$-equivariant morphism, if we view $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$ with the *right* $\mathrm{GL}_{\mathbf{f}}$-action induced from the usual left action via the anti-automorphism $h \mapsto h^{-1}$ of $\mathrm{GL}_{\mathbf{f}}$. Moreover, by construction

$$\pi_2\bigl(X, \bigl(\varphi_j^{(k)}\bigr)_{(k,j)\in I(\mathbf{f})}\bigr) \cong \mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)$$

as a $B_{\mathbf{i}}$-module. Thus, in fact $\mathrm{Im}(\pi_2) = \mathcal{F}(\Delta, \mathbf{f})$, where $\mathcal{F}(\Delta, \mathbf{f})$ is the subset of $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$ consisting of the Δ-filtered $B_{\mathbf{i}}$-modules with dimension vector $\mathbf{f}$. It is shown in [18, Corollary 1.5] that $\mathcal{F}(\Delta, \mathbf{f})$ is open in $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$. Since π_1 is a $\mathrm{GL}_{\mathbf{f}}$-principal bundle it follows from Lemma 14.1 that $\mathcal{F}(\Delta, \mathbf{f}) = \pi_2(\pi_1^{-1}(\Lambda^{\mathbf{a}}))$ is also irreducible. In particular, $\overline{\mathcal{F}(\Delta, \mathbf{f})}$ is an irreducible component of $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$.

Finally, for $\pi_2(X, (\varphi_j^{(k)})_{(k,j)\in I(\mathbf{f})}) = M$ we have

$$\begin{aligned}
\dim \mathrm{GL}_{\mathbf{d}}.X &= \dim \mathrm{GL}_{\mathbf{d}} - \dim \mathrm{End}_\Lambda(X),\\
\dim \pi_1^{-1}(\mathrm{GL}_{\mathbf{d}}.X) &= \dim \mathrm{GL}_{\mathbf{d}} - \dim \mathrm{End}_\Lambda(X) + \dim \mathrm{GL}_{\mathbf{f}},\\
\dim(\mathrm{GL}_{\mathbf{f}}.M) &= \dim \mathrm{GL}_{\mathbf{f}} - \dim \mathrm{End}_\Lambda(X).
\end{aligned}$$

The last equation holds, since the functor $F_{\mathbf{i}} : \mathcal{C}_w \to \mathcal{F}(\Delta)$ which maps X to $\mathrm{Hom}_\Lambda(V_{\mathbf{i}}, X)$ is an equivalence of additive categories. By the same token $\pi_2^{-1}(M.\mathrm{GL}_{\mathbf{f}}) = \pi_1^{-1}(\mathrm{GL}_{\mathbf{d}}.X)$. We conclude $\dim \pi_2^{-1}(M) = \dim \mathrm{GL}_{\mathbf{d}}$. Thus we proved the following:

Lemma 14.3. *For* $\mathbf{a} \in \mathbb{N}^r$ *and*

$$\mathbf{f} = \sum_{k=1}^{r} a_k \, \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$$

there exists a variety $H^{\mathbf{a}}$ *with a* $\mathrm{GL}_{\mathbf{d}}$–$\mathrm{GL}_{\mathbf{f}}$*-action together with two surjective morphisms*

$$\begin{tikzcd} & H^{\mathbf{a}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Lambda^{\mathbf{a}} & & \mathcal{F}(\Delta, \mathbf{f}) \end{tikzcd}$$

such that π_1 *is a* $\mathrm{GL}_{\mathbf{d}}$*-equivariant* $\mathrm{GL}_{\mathbf{f}}$*-principal bundle, and* π_2 *is a* $\mathrm{GL}_{\mathbf{f}}$*-equivariant and* $\mathrm{GL}_{\mathbf{d}}$*-invariant morphism. Moreover,* $\dim \pi_2^{-1}(M) = \dim \mathrm{GL}_{\mathbf{d}}$ *for all* $M \in \mathcal{F}(\Delta, \mathbf{f})$.

Since $\mathcal{C}_w = \mathrm{Fac}(V_{\mathbf{i}})$, it is easy to see that for $g \in \mathrm{GL}_{\mathbf{d}}$ and $h \in H^{\mathbf{a}}$ with $g.h = h$ we have $g = 1_{\mathrm{GL}_{\mathbf{d}}}$.

Remark 14.4. It seems plausible that with a dual bundle construction, as in [10, 4.3], one can show that π_2 is a $\mathrm{GL}_{\mathbf{d}}$-principal bundle.

14.4. Parametrization of components

Lemma 14.5. *For* $\mathbf{a} = (a_1, \dots, a_r)$*,* $\mathbf{d} = \mu(\mathbf{a})$ *and* $\mathbf{f} = \sum_{k=1}^{r} a_k \, \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k)$ *we have*

$$\dim \mathcal{F}(\Delta, \mathbf{f}) = \dim \mathrm{GL}_{\mathbf{f}} - \langle \mathbf{d}, \mathbf{d} \rangle_Q.$$

Proof. For any $N \in \mathcal{F}(\Delta, \mathbf{f})$ we have $\mathrm{proj.dim}_{B_{\mathbf{i}}}(N) \leqslant 1$, thus $\mathrm{Ext}^2_{B_{\mathbf{i}}}(N, N) = 0$, which implies that N is a smooth point [30, 3.7] of the *scheme* $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$. Recall that $\mu(\mathbf{a}) = \sum_{k=1}^{r} a_k \, \underline{\dim}(M_k)$. Now Voigt's Lemma [28, 1.3] and our Lemma 11.4 allow the calculation

$$\begin{aligned} \dim \mathcal{F}(\Delta, \mathbf{f}) &= \dim \mathrm{GL}_{\mathbf{f}} .N + \dim \mathrm{Ext}^1_{B_{\mathbf{i}}}(N, N) \\ &= \dim \mathrm{GL}_{\mathbf{f}} - \langle \mathbf{f}, \mathbf{f} \rangle_{B_{\mathbf{i}}} \\ &= \dim \mathrm{GL}_{\mathbf{f}} - \left(\sum_{k=1}^{r} a_k^2 \langle \Delta_k, \Delta_k \rangle_{B_{\mathbf{i}}} + \sum_{1 \leqslant s < k \leqslant r} a_k a_s \langle \Delta_k, \Delta_s \rangle_{B_{\mathbf{i}}} \right) \\ &= \dim \mathrm{GL}_{\mathbf{f}} - \left(\sum_{k=1}^{r} a_k^2 \langle M_k, M_k \rangle_Q + \sum_{1 \leqslant s < k \leqslant r} a_k a_s (M_k, M_s)_Q \right) \\ &= \dim \mathrm{GL}_{\mathbf{f}} - \langle \mathbf{d}, \mathbf{d} \rangle_Q. \end{aligned}$$

For the fourth equality we used Lemma 11.4 and the fact that

$$\langle \Delta_k, \Delta_k \rangle_{B_{\mathbf{i}}} = 1 = \langle M_k, M_k \rangle_Q.$$

This finishes the proof. $\square$

Proposition 14.6. *The (Zariski-) closure $Z^{\mathbf{a}}$ of $\Lambda^{\mathbf{a}}$ is an irreducible component of $\Lambda_{\mu(\mathbf{a})}$. In particular, $Z^{\mathbf{a}}$ is the unique irreducible component of $\Lambda_{\mu(\mathbf{a})}$ which contains a dense open subset which belongs to $\Lambda^{\mathbf{a}}$.*

Proof. We know from Lemma 14.1 that $\Lambda^{\mathbf{a}}$ is an irreducible constructible subset of $\Lambda_{\mu(\mathbf{a})}$. Thus, $Z^{\mathbf{a}}$ is an irreducible subvariety of $\Lambda_{\mu(\mathbf{a})}$. Since $\Lambda_{\mu(\mathbf{a})}$ is equi-dimensional, it remains to show that

$$\dim \Lambda^{\mathbf{a}} = \dim \mathrm{GL}_{\mathbf{d}} - \langle \mathbf{d}, \mathbf{d} \rangle_Q = \dim \Lambda_{\mathbf{d}}.$$

Recall that $\mathbf{d} = \mu(\mathbf{a})$ and

$$\mathbf{f} = \sum_{k=1}^{r} a_k \underline{\dim}_{B_{\mathbf{i}}}(\Delta_k).$$

As before, $\mathcal{F}(\Delta, \mathbf{f})$ denotes the irreducible open subset of Δ-good modules in the affine $\mathrm{GL}_{\mathbf{f}}$-variety $\mathrm{mod}(B_{\mathbf{i}}, \mathbf{f})$ of $B_{\mathbf{i}}$-modules with dimension vector $\mathbf{f}$. By Lemma 14.5 we know that

$$\dim \mathcal{F}(\Delta, \mathbf{f}) = \dim \mathrm{GL}_{\mathbf{f}} - \langle \mathbf{d}, \mathbf{d} \rangle_Q.$$

In Section 14.3 we constructed a $\mathrm{GL}_{\mathbf{d}}$–$\mathrm{GL}_{\mathbf{f}}$-variety $H^{\mathbf{a}}$ together with surjective morphisms

$$\begin{tikzcd} & H^{\mathbf{a}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Lambda^{\mathbf{a}} & & \mathcal{F}(\Delta, \mathbf{f}) \end{tikzcd}$$

with π_1 a $\mathrm{GL}_{\mathbf{d}}$-equivariant $\mathrm{GL}_{\mathbf{f}}$-principal bundle, and π_2 a $\mathrm{GL}_{\mathbf{f}}$-equivariant morphism with all fibers having the same dimension as $\mathrm{GL}_{\mathbf{d}}$. Our claim about the dimension of $\Lambda^{\mathbf{a}}$ follows. □

Let $M = M_1^{a_1} \oplus \cdots \oplus M_r^{a_r}$ for some $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$. We just proved that $Z^{\mathbf{a}}$ is an irreducible component of $\Lambda_{\mu(\mathbf{a})}$. Let us denote the corresponding dual semicanonical basis vector $\rho_{Z^{\mathbf{a}}}$ by s_M. Thus there is a dense open subset $U^{\mathbf{a}} \subseteq Z^{\mathbf{a}}$ such that $s_M = \delta_X$ for all $X \in U^{\mathbf{a}}$.

15. A dual PBW-basis and a dual semicanonical basis for $\mathcal{A}(\mathcal{C}_w)$

In this section we prove Theorem 3.1 and Theorem 3.2. We also deduce from these results the existence of semicanonical bases for the cluster algebras $\widetilde{\mathcal{R}}(\mathcal{C}_w, T)$ and $\underline{\mathcal{R}}(\mathcal{C}_w, T)$ obtained by inverting and specializing coefficients, respectively.

15.1. Proof of Theorem 3.1

By the definition of the cluster algebra $\mathcal{A}(\mathcal{C}_w, T)$, its initial seed is $(\mathbf{y}, B(T)^\circ)$ where $\mathbf{y} = (y_1, \ldots, y_r)$. In particular, $\mathcal{A}(\mathcal{C}_w, T)$ is a subalgebra of $\mathcal{F} := \mathbb{C}(y_1, \ldots, y_r)$. Since T is rigid, by Theorem 2.6 and [31, Theorem 1.1] every monomial in the δ_{T_k} belongs to the dual semicanonical basis $\mathcal{S}^*$, hence the δ_{T_k} are algebraically independent, and $(\delta_{T_1}, \ldots, \delta_{T_r})$ is a transcendence basis of the subfield $\mathcal{G}$ it generates inside the fraction field of $U(\mathfrak{n})^*_{\mathrm{gr}}$. Let $\iota : \mathcal{F} \to \mathcal{G}$ be the field isomorphism defined by $\iota(y_k) = \delta_{T_k}$ where $1 \leqslant k \leqslant r$. Combining Theorems 2.7 and 2.20 we see that the cluster variable z of $\mathcal{A}(\mathcal{C}_w, T)$ obtained from the initial seed $(\mathbf{y}, B(T)^\circ)$ through a sequence of seed mutations in successive directions $k_1, \dots, k_s$ will be mapped by ι to δ_X, where $X \in \mathcal{C}_w$ is the indecomposable rigid module obtained by the same sequence of mutations of rigid modules. It follows that ι restricts to an isomorphism from $\mathcal{A}(\mathcal{C}_w, T)$ to $\mathcal{R}(\mathcal{C}_w, T)$. This isomorphism is completely determined by the images of the elements y_k, hence the unicity. The cluster monomials are mapped to elements δ_R where R is a (not necessarily $\mathcal{C}_w$-maximal or basic) rigid module in $\mathcal{C}_w$, hence an element of $\mathcal{S}^*$. More precisely, the cluster monomials in $\mathcal{R}(\mathcal{C}_w, T)$ are the elements δ_R, where R runs through the set of all T-reachable modules (see Section 3.1 for the definition of T-reachable). This finishes the proof of Theorem 3.1.

15.2. Proof of Theorem 3.2

Let $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$ be as before. For $1 \leqslant k \leqslant r$ we proved that $\underline{\dim}(M_k) = \beta_{\mathbf{i}}(k)$. Set $\beta(k) := \beta_{\mathbf{i}}(k)$.

We have

$$\mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}] \subseteq \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}) \subseteq \operatorname{Span}_{\mathbb{C}} \langle \delta_X \mid X \in \mathcal{C}_w \rangle,$$

where the first inclusion follows from the observation that each of the Λ-modules M_k for $1 \leqslant k \leqslant r$ is the direct summand of a $\mathcal{C}_w$-maximal rigid module on the mutation path from $V_{\mathbf{i}}$ to $T_{\mathbf{i}}$, see Section 13. The second inclusion follows from the observation that $\operatorname{Span}_{\mathbb{C}} \langle \delta_X \mid X \in \mathcal{C}_w \rangle$ is an algebra. This follows from the fact that $\mathcal{C}_w$ is an additive category together with Theorem 2.6.

For each $M \in \operatorname{add}(M_{\mathbf{i}})$ we constructed a dual semicanonical basis vector s_M, see the explanation at the end of Section 14.4. If $M = M_k$ is an indecomposable direct summand of $M_{\mathbf{i}}$, then $s_M = \delta_{M_k}$. (For every rigid Λ-module $R \in \operatorname{nil}(\Lambda)$, the function δ_R belongs to the dual semicanonical basis. The modules M_k are rigid by Corollary 9.9.)

The following theorem is a slightly more explicit statement of Theorem 3.2:

Theorem 15.1. *Let w be a Weyl group element, and let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression of w. Then the following hold*:

(i) *We have*

$$\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}) = \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}] = \operatorname{Span}_{\mathbb{C}} \langle \delta_X \mid X \in \mathcal{C}_w \rangle.$$

(ii) *The set*

$$\{\delta_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$$

is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$.

(iii) *The subset*

$$\mathcal{S}_w^* := \{s_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$$

of the dual semicanonical basis is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$, and all cluster monomials of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ belong to $\mathcal{S}_w^*$.

The basis $\{\delta_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$ will be called *dual PBW-basis* of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$, and $\mathcal{S}_w^*$ the *dual semicanonical basis* of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. The proof of this theorem will be given after a series of lemmas.

Let

$$\mathfrak{n} = \bigoplus_{d \in \Delta^+} \mathfrak{n}_d$$

be the root space decomposition of $\mathfrak{n}$. We consider $\mathfrak{n}$ as a subspace of the universal enveloping algebra $U(\mathfrak{n})$. Since we identify $U(\mathfrak{n})$ and $\mathcal{M}$, we can think of an element f in $\mathfrak{n}_d$ as a constructible function $f : \Lambda_d \to \mathbb{C}$ in $\mathcal{M}_d$.

Lemma 15.2. *Let $f \in \mathfrak{n}_d$. If $d \notin \{\beta(k) \mid 1 \leqslant k \leqslant r\}$, then*

$$f(X) = 0 \textit{ for all } X \in \mathcal{C}_w.$$

Proof. Let $X \in \mathcal{C}_w$, and let $f \in \mathfrak{n}_d$ with $f(X) \neq 0$. In particular, $f \in \mathcal{M}_d$, and we have $d = \underline{\dim}(X) \in \Delta^+$. We know that $X \in \mathcal{C}_{M_{\mathbf{i}},\mathbf{a}}$ for some $\mathbf{a} = (a_1, \dots, a_r)$. Thus

$$\underline{\dim}(X) = \sum_{k=1}^{r} a_k \, \underline{\dim}(M_k).$$

By Lemma 4.2, $\Delta_w^+ = \{\beta(k) \mid 1 \leqslant k \leqslant r\}$ is a bracket closed subset of Δ^+. Thus $d = \beta(s)$ for some $1 \leqslant s \leqslant r$. This finishes the proof. $\square$

As before, let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression of a Weyl group element w, and let

$$\mathcal{P} = \{ p_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^{(J)} \}$$

be an $\mathbf{i}$-compatible PBW-basis of $U(\mathfrak{n})$, see Sections 4.2 and 4.3.

Lemma 15.3. *Let $p_{\mathbf{m}} \in \mathcal{P}$ where $\mathbf{m} = (m_j)_{j \in J}$. If $m_j > 0$ for some $j > r$, then*

$$p_{\mathbf{m}}(X) = 0 \quad \textit{for all } X \in \mathcal{C}_w.$$

Equivalently, $\delta_X(p_{\mathbf{m}}) = 0$ for all $X \in \mathcal{C}_w$.

Proof. We regard $p_{\mathbf{m}}$ as an element of $\mathcal{M}$, hence as a convolution product

$$p_{\mathbf{m}} = p_1^{(m_1)} \star p_2^{(m_2)} \star \cdots \star p_s^{(m_s)}.$$

Let us assume that $s > r$ and $m_s > 0$. It follows that $p_{\mathbf{m}} = p \star p_s$ where

$$p := \frac{1}{m_s} \big(p_1^{(m_1)} \star p_2^{(m_2)} \star \cdots \star p_{s-1}^{(m_{s-1})} \star p_s^{(m_s - 1)} \big).$$

Now let $X \in \mathcal{C}_w$. Then

$$p_{\mathbf{m}}(X) = (p \star p_s)(X) = \sum_{m \in \mathbb{C}} m \chi_{\mathrm{c}}\bigl(\bigl\{U \subseteq X \bigm| p(U) p_s(X/U) = m\bigr\}\bigr).$$

Since $\mathcal{C}_w$ is closed under factor modules, we get $X/U \in \mathcal{C}_w$ for all submodules U of X. Now Lemma 15.2 yields $p_s(X/U) = 0$ for all such U. Thus we proved that $p_{\mathbf{m}}(X) = 0$ for all $X \in \mathcal{C}_w$. $\square$

Recall from Section 4.3 that

$$\mathcal{P}_{\mathbf{i}}^* = \bigl\{\bigl(p_1^*\bigr)^{m_1} \cdots \bigl(p_r^*\bigr)^{m_r} \bigm| m_k \geqslant 0 \text{ for all } 1 \leqslant k \leqslant r\bigr\}$$

is a subset of the dual PBW-basis $\mathcal{P}^*$ of $U(\mathfrak{n})_{\mathrm{gr}}^*$. The following lemma is of central importance:

Lemma 15.4. *For $1 \leqslant k \leqslant r$ we have $p_k^* = \delta_{M_k}$ (up to rescaling of p_k).*

Proof. For each $1 \leqslant k \leqslant r$ there exists some $\mathbf{m} = (m_i)_{i \geqslant 1}$ such that $p_{\mathbf{m}}(M_k) \neq 0$, since $\delta_{M_k} \in \mathcal{M}^* \equiv U(\mathfrak{n})_{\mathrm{gr}}^*$ is a linear combination of elements in $\mathcal{P}^*$. Let s be the natural number with $m_s \geqslant 1$, but $m_j = 0$ for all $j > s$.

By Lemma 15.3, if $s > r$, then $p_{\mathbf{m}}(X) = 0$ for all modules $X \in \mathcal{C}_w$, a contradiction. Thus, we know that $s \leqslant r$. We even know that $s \leqslant k$, since M_k is an object of the subcategory $\mathcal{C}_u$ of $\mathcal{C}_w$, where $u = s_{i_k} \cdots s_{i_2} s_{i_1}$.

If $s = k$, then for dimension reasons $m_1 = \cdots = m_{k-1} = 0$ and $m_k = 1$. So we get $p_{\mathbf{m}} = p_k$.

Finally, assume $s < k$. Since $p_{\mathbf{m}}(M_k) \neq 0$, we know that M_k has a filtration

$$0 = U_{1,0} \subseteq U_{1,1} \subseteq \cdots \subseteq U_{1,m_1} = U_{2,0} \subseteq U_{2,1} \subseteq \cdots \subseteq U_{2,m_2} = U_{3,0} \subseteq \cdots \subseteq U_{s,m_s} = M_k$$

such that $p_i(U_{i,j}/U_{i,j-1}) \neq 0$ for all $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant m_i$. But we know that p_i lies in $\mathcal{M}_{\beta(i)}$. In other words, we have $\underline{\dim}(U_{i,j}/U_{i,j-1}) = \beta(i)$. This implies that $\beta(k)$, the dimension vector of M_k, is a positive integer linear combination of $\beta(i)$'s with $i < k$. More precisely,

$$\beta(k) = m_1 \beta(1) + \cdots + m_s \beta(s).$$

But $\beta(1), \dots, \beta(s)$ belong to the bracket closed set Δ_v^+ where $v := s_{i_s} \cdots s_{i_2} s_{i_1}$. Thus $\beta(k)$ is also in Δ_v^+, which is a contradiction, since $s < k$.

Summarizing, we proved that $p_{\mathbf{m}}(M_k) \neq 0$ if and only if $p_{\mathbf{m}} = p_k$. Now we can rescale our PBW-basis elements p_k, and we obtain without loss of generality that $p_k(M_k) = 1$. Thus we proved that

$$\delta_{M_k}(p_{\mathbf{m}}) := p_{\mathbf{m}}(M_k) = \begin{cases} 1 & \text{if } p_{\mathbf{m}} = p_k, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\delta_{M_k} = p_k^*$. $\square$

Corollary 15.5. *Under the identification $U(\mathfrak{n})_{\mathrm{gr}}^* \equiv \mathcal{M}^*$ we have*

$$\mathcal{P}_{\mathbf{i}}^* = \bigl\{\delta_M \bigm| M \in \mathrm{add}(M_{\mathbf{i}})\bigr\}.$$

Proof. By definition

$$\mathcal{P}_{\mathbf{i}}^* = \left\{ \left(p_1^*\right)^{m_1} \cdots \left(p_r^*\right)^{m_r} \;\middle|\; m_k \geqslant 0 \text{ for all } 1 \leqslant k \leqslant r \right\} \subseteq \mathcal{P}^*.$$

This implies the result, since $p_k^* = \delta_{M_k}$ and $\delta_X \cdot \delta_Y = \delta_{X \oplus Y}$ for all nilpotent Λ-modules X and Y. $\square$

Proof of Theorem 15.1. Let $X \in \mathcal{C}_w$. By Lemma 15.3 and Corollary 15.5, δ_X is a linear combination of dual PBW-basis vectors of the form δ_M with $M \in \operatorname{add}(M_{\mathbf{i}})$. Hence $\delta_X \in \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}]$, and

$$\operatorname{Span}_{\mathbb{C}}\langle \delta_X \mid X \in \mathcal{C}_w \rangle \subseteq \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_r}] \subseteq \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}).$$

Using the known reverse inclusions we get (i) and (ii) of Theorem 15.1.

Next, let $M = M_1^{a_1} \oplus \cdots \oplus M_r^{a_r}$ be a module in $\operatorname{add}(M_{\mathbf{i}})$. Set $\mathbf{a} := (a_1, \dots, a_r)$. Then $s_M = \delta_X$ for some module X in $Z^{\mathbf{a}}$. In particular, X is contained in $\mathcal{C}_w$. Thus, by what we proved up to now we get

$$s_M = \delta_X \in \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}).$$

For dimension reasons this implies that

$$\mathcal{S}_w^* := \left\{ s_M \;\middle|\; M \in \operatorname{add}(M_{\mathbf{i}}) \right\} = \mathcal{S}^* \cap \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$$

is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. By what we proved before, the set of cluster monomials of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ are a subset of $\mathcal{S}_w^*$. This finishes the proof of Theorem 15.1. $\square$

By Theorem 15.1, we know that

$$\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}) = \mathbb{C}\left[p_1^*, \dots, p_r^*\right].$$

Thus Proposition 8.2 yields the following result:

Proposition 15.6. *Under the identification $U(\mathfrak{n})_{\mathrm{gr}}^* \equiv \mathbb{C}[N]$ the cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ gets identified to the ring of invariants $\mathbb{C}[N]^{N'(w)}$, which is isomorphic to $\mathbb{C}[N(w)]$.*

Corollary 15.7. *Let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression of w. For $X \in \mathcal{C}_w$, the function $\varphi_X \in \mathbb{C}[N]$ is determined by its values on $\{x_{\mathbf{i}}(t) \mid t = (t_r, \dots, t_1) \in (\mathbb{C}^*)^r\}$ where $x_{\mathbf{i}}(t) := x_{i_r}(t_r) \cdots x_{i_2}(t_2) x_{i_1}(t_1)$.*

Proof. Let $\varphi, \psi \in \mathbb{C}[N]^{N'(w)}$. Then $\varphi = \psi$ if and only if $\varphi(x_{\mathbf{i}}(t)) = \psi(x_{\mathbf{i}}(t))$ for all $t \in (\mathbb{C}^*)^r$. Recall that each $x \in N$ can be written as $x = yy'$ for a unique $(y, y') \in N(w) \times N'(w)$. For $x \in N^w$ we have $\pi_w(x) = y$. Furthermore, the image of π_w is dense in $N(w)$, see Proposition 8.5. It is well known that the set $\{x_{\mathbf{i}}(t) \mid t \in (\mathbb{C}^*)^r\}$ contains a dense open subset of N^w. For $x = x_{\mathbf{i}}(t)$ we get

$$\varphi\big(\pi_w(x)\big) = \varphi(y) = \varphi\big(yy'\big) = \varphi(x).$$

For the second equality we used that φ is $N'(w)$-invariant. Since φ is a regular map, its values on the whole of $N(w)$ are already determined by its values on $\pi_w(x_{\mathbf{i}}(t))$, $t \in (\mathbb{C}^*)^r$. $\square$

15.3. Proof of Theorem 3.3

By Proposition 8.5, we know that $\mathbb{C}[N^w]$ is the localization of the ring $\mathbb{C}[N(w)]$ with respect to the minors $D_{\varpi_i, w^{-1}(\varpi_i)}$. By Proposition 15.6, $\mathbb{C}[N(w)]$ is equal to the cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. By Proposition 9.1, the minors $D_{\varpi_i, w^{-1}(\varpi_i)}$ coincide with the functions φ_X where X runs through the set of indecomposable $\mathcal{C}_w$-projective–injectives. In other words, the $D_{\varpi_i, w^{-1}(\varpi_i)}$ coincide with the generators of the coefficient ring of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. Hence $\mathbb{C}[N^w]$ is equal to the cluster algebra $\widetilde{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$.

15.4. Example

Let us discuss an example of base change between $\mathcal{P}_{\mathbf{i}}^*$ and $\mathcal{S}_w^*$. Let Q be a quiver with underlying graph $1 \text{ --- } 2 \text{ --- } 3$ and let $\mathbf{i} := (i_6, \dots, i_1) := (2, 3, 1, 2, 3, 1)$, which is a reduced expression of the Weyl group element $w := s_2 s_3 s_1 s_2 s_3 s_1$. As before, let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_6$ and $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_6$, where as always $M_k = M[k,k]$. The Λ-modules V_k are the following:

$$V_1 = M_1 = 1 \qquad V_2 = M_2 = 3 \qquad V_3 = M_3 = \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$$

$$V_4 = M[4,1] = \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix} \qquad V_5 = M[5,2] = \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \qquad V_6 = M[6,3] = \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}.$$

The initial cluster of our cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ looks as follows:

$$\begin{tikzcd}
& V_4 \arrow[rr] \arrow[dl] & & V_1 \arrow[dl] \\
V_6 \arrow[rr] & & V_3 \arrow[ul] \arrow[dl] & \\
& V_5 \arrow[ul] \arrow[rr] & & V_2 \arrow[ul]
\end{tikzcd}
=
\begin{tikzcd}
& \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix} \arrow[rr] \arrow[dl] & & 1 \arrow[dl] \\
\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \arrow[rr] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ul] \arrow[dl] & \\
& \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \arrow[ul] \arrow[rr] & & 3 \arrow[ul]
\end{tikzcd}$$

We have

$$M_4 = \begin{smallmatrix}&3\\2&\end{smallmatrix} \qquad M_5 = \begin{smallmatrix}1&\\&2\end{smallmatrix} \qquad M_6 = 2.$$

There are only three more indecomposable Λ-modules, namely

$$W_1 = \begin{smallmatrix}2&\\&3\end{smallmatrix} \qquad W_2 = \begin{smallmatrix}&2\\1&\end{smallmatrix} \qquad W_3 = \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}.$$

Observe that $\Omega(V_k) = W_k$ for $1 \leqslant k \leqslant 3$.

The functions δ_{M_k} can be computed easily. Indeed, for all $\mathbf{j}$ and k, the variety $\mathcal{F}_{\mathbf{j},M_k}$ is either empty or a single point, so $\chi_c(\mathcal{F}_{\mathbf{j},M_k})$ is either 0 or 1. Using Theorem 13.1 we get

$$\begin{aligned}\delta_{V_4} &= \delta_{M_1} \cdot \delta_{M_4} - \delta_{M_3},\\ \delta_{V_5} &= \delta_{M_2} \cdot \delta_{M_5} - \delta_{M_3},\\ \delta_{V_6} &= \delta_{M_3} \cdot \delta_{M_6} - \delta_{M_4} \cdot \delta_{M_5}.\end{aligned}$$

Some further exchange relations are

$$\begin{aligned}\delta_{V_3}\delta_{W_3} &= \delta_{V_4} \cdot \delta_{V_5} + \delta_{V_1}\delta_{V_2}\delta_{V_6},\\ \delta_{V_2}\delta_{W_2} &= \delta_{W_3} + \delta_{V_4},\\ \delta_{V_1}\delta_{W_1} &= \delta_{W_3} + \delta_{V_5}.\end{aligned}$$

The cluster variables in $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ are

$$\{\delta_{M_k},\ \delta_{V_s},\ \delta_{W_t} \mid 1 \leqslant k \leqslant 6,\ 4 \leqslant s \leqslant 6 \text{ and } 1 \leqslant t \leqslant 3\}.$$

(Here we consider the three coefficients δ_{V_k} with $4 \leqslant k \leqslant 6$ also as cluster variables.) Using the above formulas we get

$$\begin{aligned}\delta_{W_3} &= \delta_{M_1}\delta_{M_2}\delta_{M_6} - \delta_{M_1}\delta_{M_4} - \delta_{M_2}\delta_{M_5} + \delta_{M_3},\\ \delta_{W_2} &= \delta_{M_1}\delta_{M_6} - \delta_{M_5},\\ \delta_{W_1} &= \delta_{M_2}\delta_{M_6} - \delta_{M_4}.\end{aligned}$$

So we wrote all cluster variables as linear combinations of dual PBW-basis vectors.

15.5. Generalities on bases of algebras

A Λ-module $M = \bigoplus_{k=1}^r M_k^{a_k}$ in $\operatorname{add}(M_{\mathbf{i}})$ *has gaps* if for each $1 \leqslant j \leqslant n$ there is some $1 \leqslant k \leqslant r$ with $i_k = j$ and $a_k = 0$. In other words, M has gaps if and only if M has no direct summand of the form

$$M_{\mathbf{i}}(I_{\mathbf{i},j}) := M_{k_{\max}} \oplus \cdots \oplus M_{k_{\min}^+} \oplus M_{k_{\min}}$$

where $i_k = j$.

Lemma 15.8. *Let $M = M' \oplus M''$ be in* $\operatorname{add}(M_{\mathbf{i}})$ *such that*

$$M'' \cong M_{\mathbf{i}}(I_{\mathbf{i},j})$$

for some $1 \leqslant j \leqslant n$. Then we have $s_M = s_{M'} \cdot s_{M''}$.

Proof. We have $s_{M''} = \delta_{I_{\mathbf{i},j}}$, and $I_{\mathbf{i},j}$ is $\mathcal{C}_w$-projective–injective. The claim follows now easily from [31, Theorem 1.1] in combination with the explanations in [31, Section 2.6]. $\square$

Let $\mathrm{B} := \{b_i \mid i \geqslant 1\}$ be a K-basis of a commutative K-algebra A. For some fixed $n \geqslant 1$ let $\mathrm{C} := \{b_1, \dots, b_n\}$. A basis vector $b \in \mathrm{B}$ is called C-*free* if $b \notin b_i\mathrm{B}$ for some $b_i \in \mathrm{C}$. Assume that the following hold:

(i) For all $b_i \in \mathrm{C}$ we have $b_i\mathrm{B} \subseteq \mathrm{B}$.
(ii) If $b_1^{z_1} \cdots b_n^{z_n} b = b_1^{z_1'} \cdots b_n^{z_n'} b'$ for some $z_i, z_i' \geqslant 0$ and some C-free elements $b, b' \in \mathrm{B}$, then $b = b'$ and $z_i = z_i'$ for all i.

It follows that $\mathrm{B} = \{b_1^{z_1} \cdots b_n^{z_n} b \mid b \in \mathrm{B}$ is C-free, $z_i \geqslant 0\}$. Define

$$\underline{A} := A/(b_1 - 1, \dots, b_n - 1).$$

For $a \in A$, let $\underline{a}$ be the residue class of a in $\underline{A}$. Furthermore, let $A_{b_1,\dots,b_n}$ be the localization of A at $b_1, \dots, b_n$. The following lemma is easy to show.

Lemma 15.9. *With the notation above, the following hold*:

(1) *The set* $\underline{\mathrm{B}} := \{\underline{b} \mid b$ *is* C-*free*$\}$ *is a* K-*basis of* $\underline{A}$.
(2) *The set* $\mathrm{B}_{b_1,\dots,b_n} := \{b_1^{z_1} \cdots b_n^{z_n} b \mid b \in \mathrm{B}$ *is* C-*free*, $z_i \in \mathbb{Z}\}$ *is a* K-*basis of* $A_{b_1,\dots,b_n}$.

15.6. Inverting and specializing coefficients

One can rewrite the basis $\mathcal{S}_w^*$ appearing in Theorem 3.2 as

$$\mathcal{S}_w^* = \bigl\{(\delta_{I_{\mathbf{i},1}})^{z_1} \cdots (\delta_{I_{\mathbf{i},n}})^{z_n} s_M \bigm| M \in \operatorname{add}(M_{\mathbf{i}}),\ M \text{ has gaps, } z_i \geqslant 0\bigr\}.$$

The next two theorems deal with the situation of invertible coefficients and specialized coefficients.

Theorem 15.10 *(Invertible coefficients). The set*

$$\widetilde{\mathcal{S}}_w^* := \bigl\{(\delta_{I_{\mathbf{i},1}})^{z_1} \cdots (\delta_{I_{\mathbf{i},n}})^{z_n} s_M \bigm| M \in \operatorname{add}(M_{\mathbf{i}}),\ M \textit{ has gaps, } z_i \in \mathbb{Z}\bigr\}$$

is a $\mathbb{C}$-*basis of* $\widetilde{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$*, and* $\widetilde{\mathcal{S}}_w^*$ *contains all cluster monomials of the cluster algebra* $\widetilde{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$.

Next, we specialize all n coefficients $\delta_{I_{\mathbf{i},j}}$ of the cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ to 1. We obtain a new cluster algebra $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$ which does not have any coefficients. The residue class of $\delta_X \in \mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ is denoted by $\underline{\delta}_X$. The residue class of a dual semicanonical basis vector s_M is denoted by $\underline{s}_M$.

Theorem 15.11 *(No coefficients). The set*

$$\underline{\mathcal{S}}_w^* := \{\underline{s}_M \mid M \in \operatorname{add}(M_{\mathbf{i}}),\ M \text{ has gaps}\}$$

is a $\mathbb{C}$-basis of $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$, and $\underline{\mathcal{S}}_w^*$ contains all cluster monomials of the cluster algebra $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$.

Proof of Theorem 15.10 and Theorem 15.11. Let $\mathrm{B} := \{b_i \mid i \geqslant 1\} := \mathcal{S}_w^*$ be the dual semicanonical basis of $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. We can label the b_i such that

$$\{b_1, \dots, b_n\} = \{\delta_{I_{\mathbf{i},1}}, \dots, \delta_{I_{\mathbf{i},n}}\}.$$

Using Lemma 15.8 it is easy to check that the elements b_i satisfy the properties (i) and (ii) mentioned in Section 15.5. Then apply Lemma 15.9. □

16. Acyclic cluster algebras

In this section we will study the case of acyclic cluster algebras, which is of special interest. As before, let Q be an acyclic quiver with vertices $1, \dots, n$. Without loss of generality we assume that $i < j$ whenever there is an arrow $a : i \to j$ in Q. We define two Weyl group elements $c := s_n \cdots s_2 s_1$ and $w := c^2$. For simplicity we assume that Q is not a linearly oriented quiver of type $\mathbb{A}_n$. This implies that $\mathbf{i} := (n, \dots, 2, 1, n, \dots, 2, 1)$ is a reduced expression of w. Define $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_{2n}$ and $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_{2n}$ as before.

It follows that for $1 \leqslant j \leqslant n$ we have $M_j = I_j$ and $M_{n+j} = \tau_Q(I_j)$. Here I_j denotes the indecomposable injective KQ-module with socle S_j, and τ_Q is the Auslander–Reiten translation in $\operatorname{mod}(KQ)$.

Observe that $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$ is an acyclic cluster algebra associated to Q having n non-invertible coefficients, whereas $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$ is the acyclic cluster algebra associated to Q having no coefficients.

Theorem 16.1. *With w and $\mathbf{i}$ as above, the following hold*:

(i) $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}) = \mathbb{C}[\delta_{M_1}, \dots, \delta_{M_{2n}}] = \operatorname{Span}_{\mathbb{C}}\langle \delta_X \mid X \in \mathcal{C}_w\rangle$;
(ii) $\{\delta_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$ *and* $\{s_M \mid M \in \operatorname{add}(M_{\mathbf{i}})\}$ *are both a $\mathbb{C}$-basis of* $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$;
(iii) $\{\underline{s}_M \mid M \in \operatorname{add}(M_{\mathbf{i}}),\ M$ *has gaps*$\}$ *is a $\mathbb{C}$-basis of* $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$;
(iv) $\{\underline{\delta}_M \mid M \in \operatorname{add}(M_{\mathbf{i}}),\ M$ *has gaps*$\}$ *is a $\mathbb{C}$-basis of* $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}})$;
(v) *There is an isomorphism of cluster algebras $\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}}) \cong \mathcal{A}_Q$, where $\mathcal{A}_Q$ is the coefficient-free acyclic cluster algebra associated to Q.*

Proof. Parts (i), (ii) and (iii) were already proved before for arbitrary reduced expressions of arbitrary Weyl group elements. Part (v) is clear from our description of the initial seed (labeled by $V_{\mathbf{i}}$) for the cluster algebra $\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}})$. It remains to prove (iv): We have

$$\mathcal{R}(\mathcal{C}_w, V_{\mathbf{i}}) = \bigoplus_{d \in \mathbb{N}^n} \mathcal{R}_d$$

where $\mathcal{R}_d$ is the $\mathbb{C}$-vector space with basis $\{s_M \mid M \in \operatorname{add}(M_{\mathbf{i}}) \cap \operatorname{rep}(Q,d)\}$. We know that $\{\delta_M \mid M \in \operatorname{add}(M_{\mathbf{i}}) \cap \operatorname{rep}(Q,d)\}$ is a basis of $\mathcal{R}_d$ as well. After specializing the coefficients $\delta_{I_{\mathbf{i},j}}$, $1 \leqslant j \leqslant n$ to 1, we get

$$\underline{\mathcal{R}}(\mathcal{C}_w, V_{\mathbf{i}}) = \bigoplus_{d \in \mathbb{N}^n} \underline{\mathcal{R}}_d$$

where $\underline{\mathcal{R}}_d$ is the $\mathbb{C}$-vector space with basis

$$\{\underline{s}_M \mid M \in \operatorname{add}(M_{\mathbf{i}}) \cap \operatorname{rep}(Q,d),\ M \text{ has gaps}\}.$$

Now one can use the formula

$$\delta_{I_{\mathbf{i},i}} = \delta_{M_{n+i}} \cdot \delta_{M_i} - \prod_{j \to i} \delta_{M_{n+j}} \cdot \prod_{i \to k} \delta_{M_k}$$

(where the products are taken over all arrows of Q which start and end in i, respectively) and an induction on the number of vertices of Q to show that for every $M \in \operatorname{add}(M_{\mathbf{i}})$ which has gaps, the vector $\underline{s}_M$ is a linear combination of elements of the form $\underline{\delta}_{M'}$ where M' in $\operatorname{add}(M_{\mathbf{i}})$ has gaps and $|\underline{\dim}(M')| \leqslant |\underline{\dim}(M)|$. For dimension reasons we get that the vectors $\underline{\delta}_{M'}$ with M' having gaps form a linearly independent set. This implies (iv). $\square$

It is interesting to compare Theorem 16.1(iv) to Berenstein, Fomin and Zelevinsky's construction of a basis for the acyclic cluster algebra $\mathcal{A}_Q$. Let $\mathbf{y} := (y_1, \dots, y_n)$ be the initial cluster whose exchange matrix B_Q is encoded by Q, as in Section 2.6. Let $y_1^*, \dots, y_n^*$ be the n cluster variables obtained from $\mathbf{y}$ by one mutation in each of the n possible directions. Thus the n sets $\{y_1, \dots, y_n\} \setminus \{y_k\} \cup \{y_k^*\}$ form the neighboring clusters of our initial cluster $\mathbf{y}$. Using a simple Gröbner basis argument, the following is shown in [9]:

Theorem 16.2 *(Berenstein, Fomin, Zelevinsky). The monomials*

$$\{y_1^{p_1}(y_1^*)^{q_1} \cdots y_n^{p_n}(y_n^*)^{q_n} \mid p_i, q_i \geqslant 0,\ p_i q_i = 0\}$$

form a $\mathbb{C}$-basis of the acyclic cluster algebra $\mathcal{A}_Q$.

Starting with the initial seed $(\mathbf{y}, B_Q)$, which corresponds to $\Gamma_{\mathbf{i}} \equiv \Gamma_{V_{\mathbf{i}}}$, we perform the sequence of mutations $\mu_n \cdots \mu_2\mu_1$. In each step we obtain a new cluster variable which we denote by $y_k^\dagger$. Note that $y_1^\dagger = y_1^*$, but already $y_2^\dagger$ and y_2^* may be different. Observe that $\mu_n \cdots \mu_2\mu_1(B_Q) = B_Q$. We get that

$$((y_1^\dagger, \dots, y_n^\dagger), B_Q)$$

is a seed of the cluster algebra $\mathcal{A}_Q$ where

$$\{y_1, \dots, y_n\} \cap \{y_1^\dagger, \dots, y_n^\dagger\} = \varnothing.$$

Our version of Theorem 16.2 looks then as follows:

Theorem 16.3. *The monomials*

$$\left\{y_1^{p_1}\left(y_1^\dagger\right)^{q_1}\cdots y_n^{p_n}\left(y_n^\dagger\right)^{q_n}\mid p_i,q_i\geqslant 0,\ p_iq_i=0\right\}$$

form a $\mathbb{C}$-basis of the acyclic cluster algebra $\mathcal{A}_Q$.

Note that the initial cluster $(y_1,\dots,y_n)$ comes from $V_{\mathbf{i}}$ and the cluster $(y_1^\dagger,\dots,y_n^\dagger)$ comes from $T_{\mathbf{i}}$.

17. Coordinate rings of unipotent radicals

In this section, we assume that Q is of finite Dynkin type $\mathbb{A},\mathbb{D},\mathbb{E}$. We first recall some standard notation (we refer the reader to [37] for more details). The group G is now a simple complex algebraic group of the same type as Q. Let J be a subset of the set I of vertices of Q, and let K be the complementary subset. To K one can attach a standard parabolic subgroup B_K containing the Borel subgroup $B=HN$. We denote by N_K the unipotent radical of B_K. This is a subgroup of N. Let W_K be the subgroup of the Weyl group W generated by the reflections s_k with $k\in K$. This is a finite Coxeter group and we denote its longest element by w_0^K. The longest element of W is denoted by w_0.

In finite type, the preprojective algebra Λ is finite-dimensional and selfinjective. In agreement with [37], we shall denote by P_i the indecomposable projective Λ-module with top S_i and by Q_i the indecomposable injective module with socle S_i. We write

$$Q_J=\bigoplus_{j\in J}Q_j\quad\text{and}\quad P_J=\bigoplus_{j\in J}P_j.$$

In [37] we have shown that $\mathbb{C}[N_K]$ is naturally isomorphic to the subalgebra

$$R_K:=\operatorname{Span}_{\mathbb{C}}\left\langle\varphi_X\mid X\in\operatorname{Sub}(Q_J)\right\rangle$$

of $\mathbb{C}[N]$. As before, $\operatorname{Sub}(Q_J)$ is the full subcategory of $\operatorname{mod}(\Lambda)$ whose objects are submodules of direct sums of finitely many copies of Q_J. This allowed us to introduce a cluster algebra $\mathcal{A}_J\subseteq R_K$, whose cluster monomials are of the form φ_X with X a rigid module in $\operatorname{Sub}(Q_J)$. We conjectured that in fact $\mathcal{A}_J=R_K$, see [37, Conjecture 9.6].

We are going to prove that this conjecture follows from the results of this paper. Let $w:=w_0w_o^K$, and let $\mathbf{i}$ be a reduced expression for w.

Lemma 17.1. *We have $N_K=N'(w_0^K)=N(w_0w_0^K)$.*

Proof. We know that $N'(w_0^K)$ is the subgroup of N generated by the one-parameter subgroups $N(\alpha)$ with $\alpha>0$ and $w_0^K(\alpha)>0$. These are exactly the one-parameter subgroups of N which do not belong to the Levi subgroup of B_K, hence the first equality follows. Now, since $N=w_0N_-w_0$, we have

$$N'\left(w_0^K\right)=N\cap\left(w_0^KNw_0^K\right)=N\cap\left(w_0^Kw_0N_-w_0w_0^K\right)=N\left(w_0w_0^K\right).\qquad\square$$

As before, let $\operatorname{Fac}(P_J)$ be the subcategory of $\operatorname{mod}(\Lambda)$ whose objects are factor modules of direct sums of finitely many copies of P_J.

Lemma 17.2. *We have* $\mathcal{C}_{w_0 w_0^K} = \mathrm{Fac}(P_J)$.

Proof. By Proposition 9.1, we know that the indecomposable $\mathcal{C}_w$-projective–injective object $I_{\mathbf{i},i}$ with socle S_i satisfies

$$\varphi_{I_{\mathbf{i},i}} = D_{\varpi_i, w_0^K w_0(\varpi_i)} \quad (i \in I).$$

By [37, Section 6.2], it follows that $I_{\mathbf{i},i} = \mathcal{E}_{w_0^K} Q_i$, where $\mathcal{E}_{w_0^K}$ is the functor defined in [37, Section 5.2]. It readily follows that $I_{\mathbf{i},i}$ is the indecomposable projective–injective object of $\mathrm{Fac}(P_J)$ with simple socle S_i. Hence $\mathcal{C}_{w_0 w_0^K}$ and $\mathrm{Fac}(P_J)$ have the same projective–injective generator. □

Let S denote the self-duality of $\mathrm{mod}(\Lambda)$ induced by the involution $a \mapsto a^*$ mapping an arrow a of $\overline{Q}$ to its opposite arrow a^*, see [34, Section 1.7]. This restricts to an anti-equivalence of categories $\mathrm{Fac}(P_J) \to \mathrm{Sub}(Q_J)$, that we shall again denote by S.

Lemma 17.3. *For every* $X \in \mathrm{nil}(\Lambda)$ *and every* $n \in N$ *we have*

$$\varphi_X\big(n^{-1}\big) = (-1)^{\dim X} \varphi_{S(X)}(n).$$

Proof. We know that N is generated by the one-parameter subgroups $x_i(t)$ attached to the simple positive roots. By Proposition 6.1 we have

$$\varphi_X\big(x_{i_1}(t_1)\cdots x_{i_k}(t_k)\big) = \sum_{\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k} \chi_c(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X}) \frac{t_1^{a_1}\cdots t_k^{a_k}}{a_1!\cdots a_k!}.$$

Now, if $n = x_{i_1}(t_1)\cdots x_{i_k}(t_k)$, we have $n^{-1} = x_{i_k}(-t_k)\cdots x_{i_1}(-t_1)$ and the result follows from the fact that $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X} \cong \mathcal{F}_{\mathbf{i}_{\mathrm{op}}^{\mathbf{a}_{\mathrm{op}}},S(X)}$, where $\mathbf{i}_{\mathrm{op}}$ and $\mathbf{a}_{\mathrm{op}}$ denote the sequences obtained by reading $\mathbf{i}$ and $\mathbf{a}$ from right to left. □

We can now prove the following:

Theorem 17.4. *Conjecture 9.6 of [37] holds.*

Proof. As before, let $w := w_0 w_0^K$, and let $\mathbf{i}$ be a reduced expression of w. The cluster algebra $\mathcal{R}(\mathcal{C}_w) = \mathcal{R}(\mathrm{Fac}(P_J))$ is isomorphic to $\mathcal{A}_J$ via the map $\varphi_X \mapsto \varphi_{S(X)}$. This comes from the fact that $S : \mathrm{Fac}(P_J) \to \mathrm{Sub}(Q_J)$ is an anti-equivalence which maps the $\mathcal{C}_w$-maximal rigid module $V_{\mathbf{i}}$ used to define the initial seed of $\mathcal{R}(\mathcal{C}_w)$ to the maximal rigid module $U_{\mathbf{j}}$ of [37, Section 9.2] used to define the initial seed of $\mathcal{A}_J$. (Here we assume that $\mathbf{j}$ is the reduced expression of $w_0^K w_0$ obtained by reading the reduced expression $\mathbf{i}$ of $w_0 w_0^K$ from right to left.) In particular the cluster variables φ_{M_k} which, by Theorem 15.1, generate $\mathcal{R}(\mathrm{Fac}(P_J)) \equiv \mathbb{C}[N(w_0 w_0^K)]$ are mapped to cluster variables $\varphi_{S(M_k)}$ of $\mathcal{A}_J$. They also form a system of generators of the polynomial algebra $\mathbb{C}[N(w_0 w_0^K)] = \mathbb{C}[N_K]$ by Lemma 17.3, because the map $n \mapsto n^{-1}$ is a biregular automorphism of N_K. Hence $\mathcal{A}_J = \mathbb{C}[N_K]$. □

Remark 17.5. The previous discussion shows that we obtain two different cluster algebra structures on $\mathbb{C}[N_K]$, coming from the two different subcategories $\mathrm{Fac}(P_J)$ and $\mathrm{Sub}(Q_J)$. When using $\mathrm{Fac}(P_J) = \mathcal{C}_{w_0 w_0^K}$, we regard $\mathbb{C}[N_K]$ as the subring of $N'(w_0 w_0^K)$-invariant functions of $\mathbb{C}[N]$ for the action of $N'(w_0 w_0^K)$ on N by *right* translations, see Section 8.1. This allows us to relate the first cluster structure to the cluster structure of the unipotent cell $\mathbb{C}[N^{w_0 w_0^K}]$, see Proposition 8.5. When using $\mathrm{Sub}(Q_J)$, we regard $\mathbb{C}[N_K]$ as the subring of $N'(w_0 w_0^K)$-invariant functions of $\mathbb{C}[N]$ for the action of $N'(w_0 w_0^K) = N(w_0^K)$ on N by *left* translations. These functions can then be "lifted" to B_K^--invariant functions on G for the action of B_K^- on G by left translations. This allows us to "lift" the second cluster structure to a cluster structure on $\mathbb{C}[B_K^- \backslash G]$, see [37, Section 10].

18. Remarks and open problems

18.1. Calculation of $M_{\mathbf{i}}(R)$

Let $\mathbf{i}$ be a reduced expression of a Weyl group element w, and let R be a $V_{\mathbf{i}}$-reachable Λ-module, see Section 3.1. Based on Theorem 3.1 we can combine Corollary 10.7 and Proposition 12.4 to determine algorithmically $M_{\mathbf{i}}(R)$. (For the definition of $M_{\mathbf{i}}(R)$ see Section 10.) Recall that the $V_{\mathbf{i}}$-reachable modules R are in 1-1 correspondence with the cluster monomials δ_R in $\mathcal{R}(\mathcal{C}_w)$.

18.2. Calculation of Euler characteristics

Let $\mathbf{i}$ be a reduced expression of a Weyl group element w, and let R be a $V_{\mathbf{i}}$-reachable Λ-module, and let $\mathbf{j} = (j_1, \dots, j_p)$. By Proposition 6.1 the Euler characteristic $\chi_c(\mathcal{F}_{\mathbf{j},R})$ is equal to the coefficient of $t_1 \cdots t_p$ in $\varphi_R(x_{j_1}(t_1) \cdots x_{j_p}(t_p))$. Using mutations, we can express algorithmically φ_R as a Laurent polynomial in the functions $\varphi_{V_1}, \dots, \varphi_{V_r}$. Now we can use the calculations from Section 9.6 to compute all the Euler characteristics $\chi_c(\mathcal{F}_{\mathbf{j},R})$.

18.3. Open orbit conjecture

It is known that the (specialized) dual canonical basis $\mathcal{B}^*$ and the dual semicanonical basis $\mathcal{S}^*$ of $\mathcal{M}^* \equiv U(\mathfrak{n})_{\mathrm{gr}}^*$ do not coincide, see [31, Section 1.5]. But one might at least hope that both bases have some interesting elements in common:

Conjecture 18.1 *(Open orbit conjecture). Let Z be an irreducible component of Λ_d, and let b_Z and ρ_Z be the associated dual canonical and dual semicanonical basis vectors of $\mathcal{M}^*$. If Z contains an open* GL_d*-orbit, then $b_Z = \rho_Z$.*

We know that each cluster monomial of the cluster algebra $\mathcal{A}(\mathcal{C}_w)$ is of the form ρ_Z, where Z contains an open GL_d-orbit. So if the conjecture is true, then all cluster monomials belong to the dual canonical basis.

18.4. Example

Finally, we would like to ask the following question. Is it possible to realize every element of the dual canonical basis of $\mathcal{M}^*$ as a δ-function? We know several examples of elements b of

$\mathcal{B}^*$ which do not belong to $\mathcal{S}^*$. In all these examples, b is however equal to δ_X for a non-generic Λ-module X. (We say that $X \in \mathrm{nil}(\Lambda)$ is *generic* if $\delta_X \in \mathcal{S}^*$.)

Let us look at an example. Let Q be the quiver $1 \leftleftarrows 2$ and let Λ be the associated preprojective algebra. For $\lambda \in \mathbb{C}^*$ we define representations $M(\lambda, 1)$ and $M(\lambda, 2)$ of Q as follows:

$$M(\lambda,1) := \mathbb{C} \underset{(\lambda)}{\overset{(1)}{\leftleftarrows}} \mathbb{C} \quad \text{and} \quad M(\lambda,2) := \mathbb{C}^2 \underset{\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)}{\leftleftarrows}} \mathbb{C}^2$$

Let $\iota : \mathrm{rep}(Q, (2,2)) \to \Lambda_{(2,2)}$ be the obvious canonical embedding. Clearly, the image of ι is an irreducible component of $\Lambda_{(2,2)}$, which we denote by Z_Q. It is not difficult to check that the set

$$\bigl\{M(\lambda,1) \oplus M(\mu,1) \bigm| \lambda, \mu \in \mathbb{C}^*\bigr\}$$

is a dense subset of Z_Q. It follows that

$$\delta_{M(\lambda,1)\oplus M(\mu,1)} = \rho_{Z_Q}$$

is an element of the dual semicanonical basis $\mathcal{S}^*$. It is easy to check that

$$\delta_{M(\lambda,2)} \neq \delta_{M(\lambda,1)\oplus M(\mu,1)}.$$

Indeed, the variety $\mathcal{F}_{\mathbf{j},X}$ of composition series of type $\mathbf{j} = (1,2,1,2)$ has Euler characteristic 2 for $X = M(\lambda,1) \oplus M(\mu,1)$ and Euler characteristic 1 for $X = M(\lambda,2)$. Furthermore, one can show that

$$\delta_{M(\lambda,2)} = b_{Z_Q}$$

belongs to the dual canonical basis $\mathcal{B}^*$ of $\mathcal{M}^*$.

Note that the functions $\delta_{M(\lambda,1)\oplus M(\mu,1)}$ and $\delta_{M(\lambda,2)}$ do not belong to any of the subalgebras $\mathcal{R}(\mathcal{C}_w)$, since $M(\lambda,1)$ and $M(\lambda,2)$ are regular representations of the quiver Q for all λ.

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