Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 322 (2006) 990–1000

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On the eigenvalues of certain canonical higher-order ordinary differential operators

Albrecht Böttcher [a,*], Harold Widom [b]

[a] *Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany*

[b] *Department of Mathematics, University of California, Santa Cruz, CA 95064, USA*

Received 14 January 2005

Available online 28 October 2005

Submitted by W.L. Wendland

Abstract

We consider the operator of taking the $2p$th derivative of a function with zero boundary conditions for the function and its first $p-1$ derivatives at two distinct points. Our main result provides an asymptotic formula for the eigenvalues and resolves a question on the appearance of certain regular numbers in the eigenvalue sequences for $p=1$ and $p=3$.

Keywords: Ordinary differential operator; Eigenvalue; Asymptotic expansion

1. Introduction

Let α be a natural number. We consider the eigenvalue problem

$$(-1)^\alpha u^{(2\alpha)}(x) = \lambda u(x) \quad \text{for } x \in [0,1], \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(\alpha-1)}(0) = 0, \qquad u(1) = u'(1) = \cdots = u^{(\alpha-1)}(1) = 0. \tag{2}$$

This problem has countably many eigenvalues, which are all positive and converge to infinity. We denote the sequence of the eigenvalues by $\{\lambda_{n,\alpha}\}_{n=n_0}^{\infty}$ where n_0 will be chosen in dependence on α (the a priori choice $n_0 = 1$ will turn out to be inconvenient). Thus, the first eigenvalue

* Corresponding author.

E-mail addresses: aboettch@mathematik.tu-chemnitz.de (A. Böttcher), widom@math.ucsc.edu (H. Widom).

0022-247X/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2005.09.080

is $\lambda_{n_0,\alpha}$, the second is $\lambda_{n_0+1,\alpha}$, and so on. We also put $\mu_{n,\alpha} = \sqrt[2\alpha]{\lambda_{n,\alpha}}$. In [1] we determined the asymptotics of the minimal eigenvalue $\lambda_{\min,\alpha} = \lambda_{n_0,\alpha}$ as α goes to infinity. The result is

$$\lambda_{\min,\alpha} = \sqrt{8\pi\alpha}\left(\frac{4\alpha}{e}\right)^{2\alpha}\left(1 + O\left(\frac{1}{\sqrt{\alpha}}\right)\right). \tag{3}$$

In [1] we also observed that $\lambda_{\min,3}$ is very close to $(2\pi)^6$. We here show that $\lambda_{\min,3}$ is in fact equal to $(2\pi)^6$. It is furthermore well known that $\lambda_{\min,1} = \pi^2$. The purpose of this paper is to give an answer to the question whether these coincidences are accidents or not. We shall prove that they are due to the accidents that $\cos\pi = -1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ are nonzero rational numbers.

We remark that there is a close connection between the eigenvalues of problem (1), (2) and the eigenvalues of certain positive definite Toeplitz matrices: Parter [4] showed that for each fixed ν the appropriately scaled νth eigenvalue of a certain sequence $\{T_m\}$ of Toeplitz matrices depending on α converges to the νth eigenvalue of problem (1), (2) as $m \to \infty$. This result is another source of motivation for the investigation of the present paper.

For $\alpha = 1$, the eigenvalues of (1), (2) are known to be $\lambda_{n,1} = (n\pi)^2$ $(n \geqslant 1)$. If $\alpha = 2$, the eigenvalues are given by $\lambda_{n,2} = \mu_{n,2}^4$ where $\{\mu_{n,2}\}$ is the sequence of the positive solutions of the equation

$$\cos\mu = \frac{1}{\cosh\mu}. \tag{4}$$

This was shown in [2,3]. From (4) we infer that if n_0 is appropriately chosen, then the sequence $\{\mu_{n,2}\}_{n=n_0}^{\infty}$ has the asymptotics $\mu_{n,2} = \frac{\pi}{2} + n\pi + \delta_n$ with $\delta_n \sim 2(-1)^{n+1}e^{-\pi/2}e^{-n\pi}$ as $n \to \infty$. As usual, $x_n \sim y_n$ means that $x_n/y_n \to 1$ as $n \to \infty$. Thus, in contrast to the case $\alpha = 1$, $\{\mu_{n,2}\}_{n=n_0}^{\infty}$ does not contain an arithmetic progression. In both [2,3] it was also established that $\mu_{\min,2} = 4.7300$ and, accordingly, $\lambda_{\min,2} = 500.5467$. We may therefore take $n_0 = 1$ and write the eigenvalue sequence in the form $\{(\frac{\pi}{2} + n\pi)^4 + \xi_n\}_{n=1}^{\infty}$ with $\xi_n \sim 4\pi^3 n^3 \delta_n$.

Now let $\alpha = 3$. The general solution of Eq. (1) is

$$u(x) = \sum_{j=0}^{5} C_j \exp\bigl(\mu\varepsilon^{2j+1}x\bigr) \quad \text{with } \varepsilon = e^{\pi i/6},\ \mu = \sqrt[6]{\lambda}.$$

Consequently, the boundary conditions (2) are satisfied if and only if the determinant of the matrix

$$A_3(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \varepsilon & \varepsilon^3 & \varepsilon^5 & \varepsilon^7 & \varepsilon^9 & \varepsilon^{11} \\ \varepsilon^2 & \varepsilon^6 & \varepsilon^{10} & \varepsilon^{14} & \varepsilon^{18} & \varepsilon^{22} \\ a\omega & \omega^2 & b\omega & b\omega^{-1} & \omega^{-2} & a\omega^{-1} \\ a\omega\varepsilon & \omega^2\varepsilon^3 & b\omega\varepsilon^5 & b\omega^{-1}\varepsilon^7 & \omega^{-2}\varepsilon^9 & a\omega^{-1}\varepsilon^{11} \\ a\omega\varepsilon^2 & \omega^2\varepsilon^6 & b\omega\varepsilon^{10} & b\omega^{-1}\varepsilon^{14} & \omega^{-2}\varepsilon^{18} & a\omega^{-1}\varepsilon^{22} \end{pmatrix} \tag{5}$$

is zero, where

$$a = e^{\mu\operatorname{Re}\varepsilon} = e^{\mu\sqrt{3}/2}, \qquad b = e^{-\mu\operatorname{Re}\varepsilon} = e^{-\mu\sqrt{3}/2}, \qquad \omega = e^{i\mu\operatorname{Im}\varepsilon} = e^{i\mu/2}. \tag{6}$$

Figure 1 shows the minimum of the absolute values of the eigenvalues of the matrix $A_3(\mu)$ in dependence on μ. The unit on the horizontal axis is π. We see that the minimum of the moduli of the eigenvalues and hence the determinant of $A_3(\mu)$ is zero for $\mu_{n,3}$ sharply concentrated at the values of $n\pi$ with $n \geqslant 2$. The question is whether $\mu_{n,3}$ is exactly $n\pi$ or not. Theorem 1.1 provides the answer.

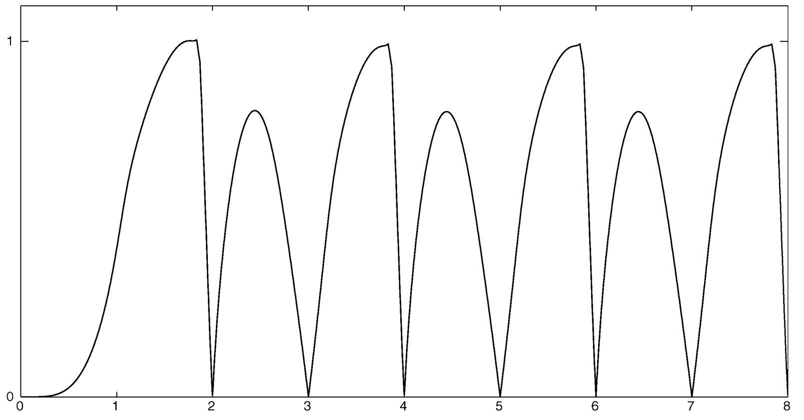

Fig. 1. The minimum of the moduli of the eigenvalues of $A_3(\mu)$ in dependence of μ.

Theorem 1.1. *For $\alpha = 3$, the eigenvalues of* (1), (2) *are* $\{\lambda_{n,3}\}_{n=2}^{\infty}$ *with*

$$\lambda_{n,3} = (n\pi)^6 \quad \text{if } n \geqslant 2 \text{ is even}$$

and

$$\lambda_{n,3} = (n\pi + \delta_n)^6 \quad \text{if } n \geqslant 3 \text{ is odd},$$

where the δ_n's are nonzero numbers satisfying $\delta_n \sim 8(-1)^{\lfloor n/2 \rfloor + 1} e^{-(\pi\sqrt{3}/2)n}$. Here $\lfloor n/2 \rfloor$ stands for the integral part of $n/2$. In particular, $\mu_{n,3} = n\pi$ if and only if n is even.

Mark Embree computed the first five roots of Eq. (9) and thus the numbers $\mu_{n,3}$ for $n = 3$, 5, 7, 9, 11 up to ten correct digits after the comma. The results are presented Table 1. The last column shows the values of $8(-1)^{\lfloor n/2 \rfloor + 1} e^{-(\pi\sqrt{3}/2)n}$.

If $\alpha = 1$ then $\{\mu_{n,1}\}_{n=1}^{\infty}$ is an arithmetic progression, and if $\alpha = 3$ then $\{\mu_{n,3}\}_{n=2}^{\infty}$ contains an arithmetic progression. The following result involves the reason for these two peculiarities. Actually it does more. It gives the asymptotics of the eigenvalues of problem (1), (2) for arbitrary α.

Theorem 1.2. *If $\alpha \geqslant 5$ is odd, one can choose n_0 so that the sequence $\{\mu_{n,\alpha}\}_{n=n_0}^{\infty}$ satisfies*

$$\mu_{n,\alpha} = n\pi + \frac{2(-1)^n}{\sin^2 \frac{\pi}{2\alpha}} e^{-n\pi \sin \frac{\pi}{\alpha}} \sin\left(n\pi \cos \frac{\pi}{\alpha}\right) + O\left(e^{-n\pi \sin \frac{2\pi}{\alpha}}\right). \tag{7}$$

If $\alpha \geqslant 4$ is even, there is an n_0 such that $\{\mu_{n,\alpha}\}_{n=n_0}^{\infty}$ can be written as

Table 1

	$\mu_{n,3}$	$8(-1)^{\lfloor n/2 \rfloor + 1} e^{-(\pi\sqrt{3}/2)n}$
$n = 3$	$9.4270555708 = 3\pi + 0.0022776101$	$+0.0022821082$
$n = 5$	$15.7079533785 = 5\pi - 0.0000098894$	-0.0000098893
$n = 7$	$21.9911486179 = 7\pi + 0.0000000428$	$+0.0000000428$
$n = 9$	$28.2743338821 = 9\pi - 0.0000000002$	-0.0000000002
$n = 11$	$34.5575191894 = 11\pi + 0.0000000000$	$+0.0000000000$

$$\mu_{n,\alpha} = \frac{\pi}{2} + n\pi + \frac{2(-1)^{n+1}}{\sin^2 \frac{\pi}{2\alpha}} e^{-(\frac{\pi}{2}+n\pi)\sin\frac{\pi}{\alpha}} \cos\left(\left(\frac{\pi}{2} + n\pi\right)\cos\frac{\pi}{\alpha}\right) + O\left(e^{-2n\pi \sin\frac{\pi}{\alpha}}\right). \tag{8}$$

The sequence $\{\mu_{n,\alpha}\}_{n=n_0}^{\infty}$ *contains an arithmetic progression if and only if* $\cos\frac{\pi}{\alpha}$ *is a nonzero rational number, that is, if and only if* $\alpha = 1$ *or* $\alpha = 3$.

We remark that (7) is also true for $\alpha = 3$, but that in this case it amounts to $\mu_{n,3} = n\pi + O(e^{-(\pi\sqrt{3}/2)n})$, which is weaker than Theorem 1.1. Formula (8) becomes valid for $\alpha = 2$ after replacing the numerator $2(-1)^{n+1}$ by $(-1)^{n+1}$. The reason for this discrepancy will be given at the end of Section 3.

In the cases $\alpha = 1, 2, 3$ the values of $\mu_{\min,\alpha}$ are π, 4.7300 ($\approx \frac{3}{2}\pi$), 2π, respectively. This suggests that the n_0 in Theorem 1.2 is the integral part of $\frac{\alpha+1}{2}$ and hence asymptotically equals $\frac{\alpha}{2}$ as $\alpha \to \infty$. However, (3) implies that $\mu_{\min,\alpha} \sim \frac{4\alpha}{e}$ and thus $n_0\pi \sim \frac{4\alpha}{e}$, which shows that actually n_0 is asymptotically equal to $\frac{4}{\pi e}\alpha = 0.4684\alpha$. It was just these tiny but significant differences that have kindled our interest in the subject in [1] and here.

The properties of problem (1), (2) depend on whether α is odd or even. We therefore study these two cases separately.

2. The odd case

There remains nothing to say on the case $\alpha = 1$. So let us begin with $\alpha = 3$. The following theorem provides us with a formula for the determinant of matrix (5) for arbitrary a, b, ω, that is, for a, b, ω that are not necessarily of the form (6).

Theorem 2.1. *Let* $\varepsilon = e^{\pi i/6}$ *and let* a,b, *and* $\omega \neq 0$ *be arbitrary complex numbers. Then the determinant of matrix* (5) *is*

$$\begin{aligned}\det A_3(\mu) &= 12ab(a+b)\left(\omega - \omega^{-1}\right) + 3\left(a^2+b^2\right)\left(\omega^{-2} - \omega^2\right) \\ &\quad + 3ab\left(\omega^4 - 8\omega^2 + 8\omega^{-2} - \omega^{-4}\right) + 12(a+b)\left(\omega - \omega^{-1}\right) \\ &= 12ab(a+b)\left(\omega - \omega^{-1}\right) - 3\left(a^2+b^2\right)\left(\omega - \omega^{-1}\right)\left(\omega + \omega^{-1}\right) \\ &\quad + 3ab\left(\omega - \omega^{-1}\right)\left(\omega + \omega^{-1}\right)\left(\omega^2 + \omega^{-2} - 8\right) + 12(a+b)\left(\omega - \omega^{-1}\right).\end{aligned}$$

Proof. Let V_{ijk} denote the determinant of the (Vandermonde) matrix that is constituted by the first three rows and the columns i, j, k of $A_3(\mu)$. We expand the determinant of $A_3(\mu)$ by its last three rows using Laplace's theorem and group the $\binom{6}{3} = 20$ terms so that we get a polynomial in a and b. What results is

$$\begin{aligned}& a^2b\left(-\omega V_{136}V_{245} + \omega^{-1}V_{146}V_{235}\right) + ab^2\left(-\omega V_{134}V_{256} + \omega^{-1}V_{346}V_{125}\right) \\ &\quad + a^2\left(\omega^2 V_{126}V_{345} - \omega^{-2}V_{156}V_{234}\right) + b^2\left(\omega^2 V_{234}V_{156} - \omega^{-2}V_{345}V_{126}\right) \\ &\quad + ab\left(-\omega^4 V_{123}V_{456} + \omega^2 V_{124}V_{356} - \omega^{-2}V_{145}V_{236} + \omega^2 V_{236}V_{145} - V_{246}V_{135}\right. \\ &\quad \left. + V_{135}V_{246} - \omega^{-2}V_{356}V_{124} + \omega^{-4}V_{456}V_{123}\right) \\ &\quad + a\left(-\omega V_{125}V_{346} + \omega^{-1}V_{256}V_{134}\right) + b\left(-\omega V_{235}V_{146} + \omega^{-1}V_{245}V_{136}\right).\end{aligned}$$

The determinants V_{ijk} are

$$V_{136} = V_{235} = V_{145} = 2\sqrt{3}\varepsilon^2, \qquad V_{245} = V_{146} = V_{236} = 2\sqrt{3}\varepsilon^4,$$
$$V_{256} = V_{346} = V_{124} = 2\sqrt{3}\varepsilon^8, \qquad V_{134} = V_{125} = V_{356} = 2\sqrt{3}\varepsilon^{10},$$
$$V_{126} = V_{234} = V_{456} = \sqrt{3}, \qquad V_{345} = V_{156} = V_{123} = -\sqrt{3},$$
$$V_{246} = -\frac{3}{2}\sqrt{3}, \qquad V_{135} = \frac{3}{2}\sqrt{3}.$$

Inserting these values in the above expression for the determinant of $A_3(\mu)$, we arrive at the asserted formula. □

Theorem 2.2. *For $\alpha = 3$, the sequence of the eigenvalues of* (1), (2) *is $\{\lambda_{n,3}\}_{n=2}^{\infty} = \{\mu_{n,3}^6\}_{n=2}^{\infty}$ where $\mu_{n,3} = n\pi$ if n is even and $\mu_{3,3}, \mu_{5,3}, \mu_{7,3}, \ldots$ are the positive solutions of the equation*

$$\cos\frac{\mu}{2} = \frac{4\cosh\frac{\mu\sqrt{3}}{2}}{\cosh\mu\sqrt{3}} + \frac{1}{\cosh\mu\sqrt{3}}\left[\cos\frac{\mu}{2}\cos\mu - 4\cos\frac{\mu}{2}\right]. \tag{9}$$

Proof. We apply Theorem 2.1 to the case where a, b, ω are given by (6). Since in that case $ab = 1$ and

$$a + b = 2\cosh\frac{\mu\sqrt{3}}{2}, \qquad a^2 + b^2 = 2\cosh\mu\sqrt{3},$$
$$\omega - \omega^{-1} = 2i\sin\frac{\mu}{2}, \qquad \omega + \omega^{-1} = 2\cos\frac{\mu}{2}, \qquad \omega^2 + \omega^{-2} = 2\cos\mu,$$

we obtain that $\det A_3(\mu)$ equals

$$24i\sin\frac{\mu}{2}\left[4\cosh\frac{\mu\sqrt{3}}{2} - \cosh\mu\sqrt{3}\cos\frac{\mu}{2} + \cos\frac{\mu}{2}\cos\mu - 4\cos\frac{\mu}{2}\right]. \tag{10}$$

The factor $\sin\frac{\mu}{2}$ produces the eigenvalues $\lambda_{2k,3} = (2k\pi)^6$ $(k = 1, 2, 3, \ldots)$. The term in the brackets of (10) is $\cosh\mu\sqrt{3}$ times

$$\frac{4\cosh\frac{\mu\sqrt{3}}{2}}{\cosh\mu\sqrt{3}} + \frac{1}{\cosh\mu\sqrt{3}}\left[\cos\frac{\mu}{2}\cos\mu - 4\cos\frac{\mu}{2}\right] - \cos\frac{\mu}{2}, \tag{11}$$

and the zeros of (11) are the solutions of (9). □

Theorem 1.1 is almost straightforward from Theorem 2.2. Figure 2 shows the graphs of the functions on the left and right of (9). We see that the first intersection occurs at approximately 3π. Thus, the zeros of (11) may be written as

$$\mu_{n,3} = n\pi + \delta_n \quad (n = 3, 5, 7, \ldots)$$

with $\delta_n \to 0$ as $n \to \infty$. The right-hand side of (9) has the asymptotics $4e^{-(\sqrt{3}\pi/2)\mu}$. It follows that

$$4e^{-(\pi\sqrt{3}/2)n} \sim \cos\frac{\mu_{n,3}}{2} = \cos\frac{n\pi + \delta_n}{2} = -\sin\frac{n\pi}{2}\sin\frac{\delta_n}{2} \sim (-1)^{\lfloor n/2\rfloor + 1}\frac{\delta_n}{2}$$

and, in particular, that $\delta_n \neq 0$ for all sufficiently large n. As (11) is definitely nonzero for $\mu = 3\pi, 5\pi, 7\pi, \ldots$, we conclude that $\delta_n \neq 0$ for *all* n.

We now turn to general odd numbers $\alpha \geqslant 5$. One can build up the analogue $A_\alpha(\mu)$ of matrix (5). The following theorem gives the first two terms of the asymptotics of $\det A_\alpha(\mu)$ as $\mu \to \infty$.

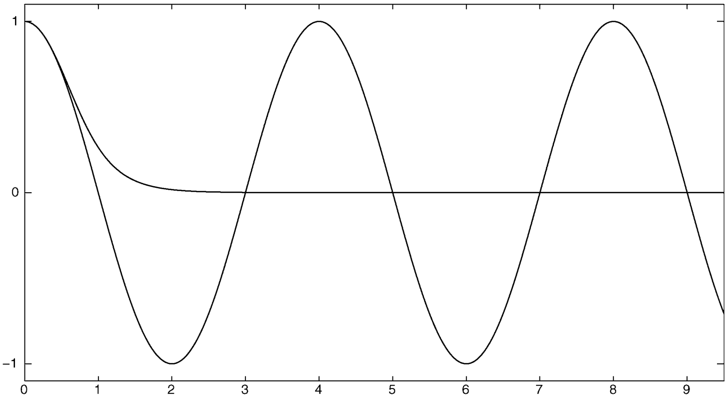

Fig. 2. The graphs of the functions on the left and right of (9), the unit on the horizontal axis being π.

Theorem 2.3. *If* $\alpha = 2k+1 \geqslant 5$, *then*

$$\det A_\alpha(\mu) = K_1 e^{\gamma_1 \mu} \sin\mu + K_2 e^{\gamma_2 \mu} \sin(s_{2k-1}\mu) + O\big(e^{\gamma_3 \mu}\big), \tag{12}$$

where K_1, K_2 *are nonzero constants satisfying* $-K_2/K_1 = 2/\sin^2 \frac{\pi}{2\alpha}$,

$$\gamma_1 = \sigma + 2c_{2k-3} + 2c_{2k-1}, \qquad \gamma_2 = \sigma + 2c_{2k-3} + c_{2k-1},$$
$$\gamma_3 = \sigma + c_{2k-3} + 2c_{2k-1},$$
$$\sigma = 2\sum_{j=1}^{k-2} c_{2j-1}, \qquad c_\ell = \cos\frac{\ell\pi}{2\alpha}, \quad s_\ell = \sin\frac{\ell\pi}{2\alpha} \quad (\ell = 1, 3, \ldots, 2k-1).$$

Proof. To avoid heavy notation and to show the essence of the matter, we restrict ourselves to the case where $\alpha = 5$. The general solution of (1) is

$$u(x) = \sum_{j=0}^{9} C_j \exp\big(\mu \varepsilon^{2j+1} x\big), \quad \varepsilon = e^{\pi i/10}, \ \mu = \sqrt[10]{\lambda}.$$

Thus, the analogue $A_5(\mu)$ of matrix (5) is a 10×10 matrix of a structure completely analogous to the one of (5). The first 5 rows are of Vandermonde type, the second row being

$$\big(\varepsilon \quad \varepsilon^3 \quad \varepsilon^5 \quad \varepsilon^7 \quad \varepsilon^9 \quad \varepsilon^{11} \quad \varepsilon^{13} \quad \varepsilon^{15} \quad \varepsilon^{17} \quad \varepsilon^{19}\big).$$

The remaining 5 rows result from the first 5 rows by multiplying each column by a factor. These factors appear without the powers of ε in the 6th row, which is

$$\big(a\omega \quad b\tau \quad \theta \quad b^{-1}\tau \quad a^{-1}\omega \quad a^{-1}\omega^{-1} \quad b^{-1}\tau^{-1} \quad \theta^{-1} \quad b\tau^{-1} \quad a\omega^{-1}\big)$$

with

$$a = e^{\mu c_1}, \qquad b = e^{\mu c_3}, \qquad \omega = e^{i\mu s_1}, \qquad \tau = e^{i\mu s_3}, \qquad \theta = e^{i\mu},$$
$$c_1 = \cos\frac{\pi}{10}, \qquad c_3 = \cos\frac{3\pi}{10}, \qquad s_1 = \sin\frac{\pi}{10}, \qquad s_3 = \sin\frac{3\pi}{10}.$$

To make things absolutely safe, we still note that the first and second columns of $A_5(\mu)$ are

$$\begin{pmatrix}1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & a\omega & a\omega\varepsilon & a\omega\varepsilon^2 & a\omega\varepsilon^3 & a\omega\varepsilon^4\end{pmatrix}^\top,$$
$$\begin{pmatrix}1 & \varepsilon^3 & \varepsilon^6 & \varepsilon^9 & \varepsilon^{12} & b\tau & b\tau\varepsilon^3 & b\tau\varepsilon^6 & b\tau\varepsilon^9 & b\tau\varepsilon^{12}\end{pmatrix}^\top.$$

We expand $\det A_5(\mu)$ by its last 5 rows using the Laplace theorem. Taking into account that

$$a^2b^2 = e^{\mu(2c_1+2c_3)}, \qquad a^2b = e^{\mu(2c_1+c_3)}, \qquad a^2 = e^{\mu\cdot 2c_1}, \qquad ab^2 = e^{\mu(c_1+2c_3)}$$

and that $c_1 + 2c_3 > 2c_1$, we see that the asymptotics of $\det A_5(\mu)$ is

$$L_1a^2b^2 + L_2a^2b + O\big(ab^2\big).$$

We are left with determining L_1 and L_2. Let $V_{j_1,\dots,j_5}$ denote the determinant of the Vandermonde matrix at the intersection of the first 5 rows and the columns $j_1,\dots,j_5$ of $A_5(\mu)$. With $X := 10$, we have

$$L_1 = -V_{1239X}V_{45678}\theta + V_{1289X}V_{34567}\theta^{-1},$$
$$L_2 = V_{1238X}V_{45679}\tau + V_{1249X}V_{35678}\tau - V_{1279X}V_{34568}\tau^{-1} - V_{1389X}V_{24567}\tau^{-1}.$$

It remains to compute the products of the form $V_MV_{M'}$ where M' denotes the set $\{1,2,\dots,10\}\setminus M$. Obviously,

$$V_M = \prod_{\substack{j,\ell\in M\\ j<\ell}}\big(\varepsilon^{2\ell-1} - \varepsilon^{2j-1}\big) = \prod_{\substack{j,\ell\in M\\ j<\ell}}\big(\varepsilon^{2\ell-1} + \varepsilon^{2j-1+10}\big)$$
$$= \prod_{\substack{j,\ell\in M\\ j<\ell}}\varrho_{\ell-j}\varepsilon^{(2\ell-1+2j-1+10)/2} = \prod_{\substack{j,\ell\in M\\ j<\ell}}\varrho_{\ell-j}\varepsilon^{\ell+j+4},$$

where

$$\varrho_{\ell-j} = \big|\varepsilon^{2\ell-1} - \varepsilon^{2j-1}\big| = 2\sin\frac{(\ell-j)\pi}{10}.$$

Letting

$$R_M = \prod_{\substack{j,\ell\in M\\ j<\ell}}\varrho_{\ell-j}\prod_{\substack{j,\ell\in M'\\ j<\ell}}\varrho_{\ell-j},$$

we get $V_MV_{M'} = R_M\varepsilon^q$ with

$$q = 4(1+2+\dots+10) + 2\binom{5}{2}\cdot 4 = 300.$$

Since $\varepsilon^{20} = 1$, it follows that $\varepsilon^{300} = 1$ and thus $V_MV_{M'} = R_M$. A direct computation shows that

$$R_{1239X} = R_{1289X} = -\varrho_1^8\varrho_2^6\varrho_3^4\varrho_4^2 =: -S,$$
$$R_{1238X} = R_{1249X} = R_{1279X} = R_{1389X} = \varrho_1^6\varrho_2^6\varrho_3^4\varrho_4^2\varrho_5^2 =: T,$$

whence

$$L_1 = S\big(\theta - \theta^{-1}\big) = 2iS\sin\mu, \qquad L_2 = 2T\big(\tau - \tau^{-1}\big) = 4iT\sin(\mu s_3).$$

In summary,

$$\det A_5(\mu) = 2iSe^{\mu(2c_1+2c_3)}\sin\mu + 4iTe^{\mu(2c_1+c_3)}\sin(\mu s_3) + O\big(e^{\mu(c_1+2c_3)}\big),$$

and since $-4iT/2iS = 2\varrho_5^2/\varrho_1^2 = 2/\sin^2\frac{\pi}{10}$, this is (12) for $\alpha = 5$.

The above arguments work equally well for general odd $\alpha \geqslant 5$. It is only the computation of the constants R_M that is more expensive and requires special care. □

We are now in a position to prove Theorem 1.2 for odd α. By Theorem 2.3, the equation $\det A_\alpha(\mu) = 0$ is of the form

$$\sin\mu = -\frac{K_2}{K_1} e^{-(\gamma_1-\gamma_2)\mu} \sin(s_{2k-1}\mu) + O\big(e^{-(\gamma_1-\gamma_3)\mu}\big). \tag{13}$$

The solutions of (13) are $n\pi + \delta_n$ with small δ_n's satisfying

$$\sin(n\pi + \delta_n) = (-1)^n \sin\delta_n = O\big(e^{-n\pi(\gamma_1-\gamma_2)}\big),$$

which implies that $\delta_n = O(e^{-n\pi(\gamma_1-\gamma_2)})$. Consequently,

$$\begin{aligned}&(-1)^n\big(\delta_n + O\big(e^{-2n\pi(\gamma_1-\gamma_2)}\big)\big)\\ &\quad = -\frac{K_2}{K_1} e^{-(n\pi+\delta_n)(\gamma_1-\gamma_2)} \sin(s_{2k-1}n\pi + s_{2k-1}\delta_n) + O\big(e^{-n\pi(\gamma_1-\gamma_3)}\big),\end{aligned}$$

and since $e^{-\delta_n(\gamma_1-\gamma_2)} = 1 + O(e^{-n\pi(\gamma_1-\gamma_2)})$ and

$$\begin{aligned}\sin(s_{2k-1}n\pi + s_{2k-1}\delta_n) &= \sin(s_{2k-1}n\pi)\cos(s_{2k-1}\delta_n) + \cos(s_{2k-1}n\pi)\sin(s_{2k-1}\delta_n)\\ &= \sin(s_{2k-1}n\pi) + O\big(e^{-n\pi(\gamma_1-\gamma_2)}\big),\end{aligned}$$

we arrive at the formula

$$(-1)^n\delta_n = -\frac{K_2}{K_1} e^{-n\pi(\gamma_1-\gamma_2)} \sin(s_{2k-1}n\pi) + O\big(e^{-n\pi\gamma}\big) \tag{14}$$

with $\gamma = \min(\gamma_1-\gamma_3, 2(\gamma_1-\gamma_2))$. Because

$$2(\gamma_1-\gamma_2) = 2c_{2k-1} = 2\sin\frac{\pi}{\alpha} > \sin\frac{2\pi}{\alpha} = c_{2k-3} = \gamma_1 - \gamma_3,$$

we see that $\gamma = \gamma_1 - \gamma_3 = \sin\frac{2\pi}{\alpha}$. Finally, taking into account that $-K_2/K_1 = 2/\sin^2\frac{\pi}{2\alpha}$ and $s_{2k-1} = \cos\frac{\pi}{\alpha}$, we obtain (7) from (14).

Now suppose $\{\mu_{n,\alpha}\}$ contains an arithmetic progression. Because $\mu_{n,\alpha} = n\pi + o(1)$ by virtue of (7), this progression must be of the form $\{n\ell\pi\}_{n=N}^\infty$ with some natural number ℓ. As $\sin(n\ell\pi) = 0$, we obtain from the second-order asymptotics of Theorem 2.3 that

$$K_2 e^{\gamma_2 n\ell\pi} \sin(s_{2k-1}n\ell\pi) + o\big(e^{\gamma_2 n\ell\pi}\big) = 0$$

for all $n \geqslant N$, which implies that $\sin(s_{2k-1}n\ell\pi) \to 0$ as $n \to \infty$. Consequently, $s_{2k-1}n\ell$ must converge to zero modulo 1. By Kronecker's theorem, this is only possible if $s_{2k-1}\ell$ and thus s_{2k-1} is a rational number. Since $s_{2k-1} = \cos\frac{\pi}{\alpha}$, we conclude that $\cos\frac{\pi}{\alpha}$ has to be rational. In Appendix we prove that this is not the case for $\alpha \geqslant 4$.

3. The even case

Let $\alpha \geqslant 4$ be an even number. Here is the analogue of Theorem 2.3.

Theorem 3.1. *If $\alpha = 2k \geqslant 4$, then*

$$\det A_\alpha(\mu) = K_1 e^{\gamma_1\mu}\cos\mu + K_2 e^{\gamma_2\mu}\cos(s_{k-1}\mu) + O\big(e^{\gamma_3\mu}\big),$$

where K_1, K_2 are nonzero constants such that $-K_2/K_1 = 2/\sin^2\frac{\pi}{2\alpha}$,

$$\gamma_1 = \sigma + 2c_{k-1}, \qquad \gamma_2 = \sigma + c_{k-1}, \qquad \gamma_3 = \sigma,$$
$$\sigma = 1 + 2\sum_{j=1}^{k-2} c_j, \qquad c_j = \cos\frac{j\pi}{\alpha}, \quad s_j = \sin\frac{j\pi}{\alpha} \quad (j = 1, 2, \dots, k-1).$$

Proof. We confine ourselves to the case $\alpha = 4$. The matrix $A_4(\mu)$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & \varepsilon^5 & \varepsilon^6 & \varepsilon^7 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^6 & \varepsilon^8 & \varepsilon^{10} & \varepsilon^{12} & \varepsilon^{14} \\ 1 & \varepsilon^4 & \varepsilon^6 & \varepsilon^9 & \varepsilon^{12} & \varepsilon^{15} & \varepsilon^{18} & \varepsilon^{21} \\ a & b\omega & \tau & b^{-1}\omega & a^{-1} & b^{-1}\omega^{-1} & \tau^{-1} & b\omega^{-1} \\ a & b\omega\varepsilon & \tau\varepsilon^2 & b^{-1}\omega\varepsilon^3 & a^{-1}\varepsilon^4 & b^{-1}\omega^{-1}\varepsilon^5 & \tau^{-1}\varepsilon^6 & b\omega^{-1}\varepsilon^7 \\ a & b\omega\varepsilon^2 & \tau\varepsilon^4 & b^{-1}\omega\varepsilon^6 & a^{-1}\varepsilon^8 & b^{-1}\omega^{-1}\varepsilon^{10} & \tau^{-1}\varepsilon^{12} & b\omega^{-1}\varepsilon^{14} \\ a & b\omega\varepsilon^4 & \tau\varepsilon^6 & b^{-1}\omega\varepsilon^9 & a^{-1}\varepsilon^{12} & b^{-1}\omega^{-1}\varepsilon^{15} & \tau^{-1}\varepsilon^{18} & b\omega^{-1}\varepsilon^{21} \end{pmatrix}$$

with

$$\varepsilon = e^{\pi i/4}, \qquad a = e^{\mu}, \qquad b = e^{\mu c_1}, \qquad \omega = e^{i\mu s_1}, \qquad \tau = e^{i\mu},$$
$$c_1 = \cos\frac{\pi}{4}, \qquad s_1 = \sin\frac{\pi}{4}.$$

Laplace expansion through the last 4 rows yields

$$\det A_4(\mu) = L_1 ab^2 + L_2 ab + O(a)$$

with

$$L_1 = V_{1238}V_{4567}\tau + V_{1278}V_{3456}\tau^{-1},$$
$$L_2 = -V_{1237}V_{4568}\omega - V_{1248}V_{3567}\omega - V_{1268}V_{3457}\omega^{-1} - V_{1378}V_{2456}\omega^{-1}. \tag{15}$$

We have

$$V_M V_{M'} = \varepsilon^6 \prod_{\substack{j,\ell\in M \\ j<\ell}} \varrho_{\ell-j} \prod_{\substack{j,\ell\in M' \\ j<\ell}} \varrho_{\ell-j} =: \varepsilon^6 R_M = -iR_M,$$

where

$$\varrho_{\ell-j} = \left|\varepsilon^{\ell-1} - \varepsilon^{j-1}\right| = 2\sin\frac{(\ell-j)\pi}{8}.$$

This gives

$$R_{1238} = R_{1278} = \varrho_1^6\varrho_2^4\varrho_3^2 =: S,$$
$$R_{1237} = R_{1248} = R_{1268} = R_{1378} = \varrho_1^4\varrho_2^4\varrho_3^2\varrho_4^2 =: T.$$

We finally obtain that

$$\det A_4(\mu) = -iSab^2\left(\tau + \tau^{-1}\right) + 2iTab\left(\omega + \omega^{-1}\right) + O(a)$$
$$= -2iSe^{\mu(1+2c_1)}\cos\mu + 4iTe^{\mu(1+c_1)}\cos(s_1\mu) + O\left(e^{\mu}\right)$$

with $-4iT/(-2iS) = 2\varrho_4^2/\varrho_1^2 = 2/\sin^2\frac{\pi}{8}$. $\square$

Armed with Theorem 3.1 we can prove Theorem 1.2 for even numbers α. The equation $\det A_\alpha(\mu) = 0$ reads

$$\cos\mu = -\frac{K_2}{K_1} e^{-(\gamma_1-\gamma_2)\mu} \cos(s_{k-1}\mu) + O\big(e^{-(\gamma_1-\gamma_3)\mu}\big)$$

and we have $\gamma_1 - \gamma_2 = c_{k-1}$ and $\gamma_1 - \gamma_3 = 2c_{k-1}$. The solutions of this equation are $\frac{\pi}{2} + n\pi + \delta_n$ with $\delta_n = O(e^{-n\pi c_{k-1}})$. It follows that

$$\begin{aligned}(-1)^{n+1}\sin\delta_n &= \cos\left(\frac{\pi}{2} + n\pi + \delta_n\right)\\ &= -\frac{K_2}{K_1} e^{-((\frac{\pi}{2}+n\pi+\delta_n)c_{k-1})} \cos\left(s_{k-1}\left(\frac{\pi}{2} + n\pi + \delta_n\right)\right) + O\big(e^{-2n\pi c_{k-1}}\big),\end{aligned}$$

and since

$$\sin\delta_n = \delta_n + O\big(e^{-2n\pi c_{k-1}}\big), \qquad e^{-\delta_n c_{k-1}} = 1 + O\big(e^{-n\pi c_{k-1}}\big),$$
$$\cos\left(s_{k-1}\left(\frac{\pi}{2} + n\pi + \delta_n\right)\right) = \cos\left(s_{k-1}\left(\frac{\pi}{2} + n\pi\right)\right) + O\big(e^{-n\pi c_{k-1}}\big),$$

we get the representation

$$(-1)^{n+1}\delta_n = -\frac{K_2}{K_1} e^{-((\frac{\pi}{2}+n\pi)c_{k-1})} \cos\left(s_{k-1}\left(\frac{\pi}{2} + n\pi\right)\right) + O\big(e^{-2n\pi c_{k-1}}\big).$$

It remains to notice that $-K_2/K_1 = 2/\sin^2\frac{\pi}{2\alpha}$, $s_{k-1} = \cos\frac{\pi}{\alpha}$, and $c_{k-1} = \sin\frac{\pi}{\alpha}$.

If $\{\mu_{n,\alpha}\}$ contains an arithmetic progression, we can argue as in Section 2 to see that there is a natural number ℓ such that

$$\frac{1}{2}s_{k-1} + \ell n s_{k-1} \to \frac{1}{2} \quad \text{modulo } 1$$

as $n \to \infty$. Kronecker's theorem implies again that $s_{k-1} = \cos\frac{\pi}{\alpha}$ must be rational.

Finally, in the case $\alpha = 2$ we have to deal with the matrix

$$A_2(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3\\ a & \tau & a^{-1} & \tau^{-1}\\ a & \tau\varepsilon & a^{-1}\varepsilon^2 & \tau^{-1}\varepsilon^3 \end{pmatrix},$$

where $\varepsilon = e^{\pi i/2}$, $a = e^\mu$, $\tau = e^{\mu i}$. The determinant of this matrix is $L_1 a + L_2 + O(a^{-1})$ with

$$L_1 = V_{12}V_{34}\tau + V_{34}V_{12}\tau^{-1}, \qquad L_2 = -V_{13}V_{24} - V_{24}V_{13}.$$

We see that the constant L_2 is the sum of two terms, which is in contrast to the case $\alpha \geqslant 4$, where the constant L_2 is the sum of four terms as in (15). This explains why for $\alpha = 2$ the numbers $\mu_{n,2}$ are $\frac{\pi}{2} + n\pi$ plus half of the subsequent term of (8). Incidentally, a straightforward computation yields $\det A_2(\mu) = 8(1 - \cos\mu\cosh\mu)$, which leads to (4).

Appendix

Here is, just for completeness, a proof of the fact that $\cos\frac{\pi}{\alpha}$ is irrational for $\alpha \geqslant 4$. Since $\cos(n\theta)$ is a polynomial with integer coefficients of $\cos\theta$, the rationality of $\cos\theta$ implies that of $\cos(n\theta)$. We are therefore left with proving that $\cos\frac{\pi}{\alpha}$ is irrational if α is 4, 6, 9 or a prime number $p \geqslant 5$.

The cases $\alpha = 4$ and $\alpha = 6$ are trivial. So let α be 9 or a prime number $p \geqslant 5$ and put $x = \cos\frac{\pi}{\alpha}$. We denote by φ the polynomial $\varphi(y) = 2y^2 - 1$ and by φ^n the nth iterate of φ, $\varphi^2(y) = \varphi(\varphi(y))$ and so on. Clearly $\varphi^n(x) = \cos\frac{2^n\pi}{\alpha}$. If $n = 6$ then $2^n = 1$ modulo 9, and if $n = p-1$ then $2^n = 1$ modulo p (Fermat). Thus, for these n we have $\varphi^n(x) = \cos\frac{(\ell\alpha+1)\pi}{\alpha} = \pm x$. Since $\varphi^n(0) = 1$ for $n \geqslant 2$ and the leading coefficient a_m of $\varphi^n(y)$ is a power of 2, $a_m = 2^k$, it follows that x satisfies an algebraic equation of the form $2^k x^m + \cdots + 1 = 0$. The only rational solutions of such an equation are of the form $x = \pm 1/2^j$. Because $1 > \cos\frac{\pi}{\alpha} \geqslant \cos\frac{\pi}{5} > \frac{1}{2}$, we arrive at the conclusion that $\cos\frac{\pi}{\alpha}$ must be irrational.

References

[1] A. Böttcher, H. Widom, From Toeplitz eigenvalues through Green's kernels to higher-order Wirtinger–Sobolev inequalities, math.FA/0412269.

[2] C.O. Horgan, A note on a class of integral inequalities, Proc. Cambridge Philos. Soc. 74 (1973) 127–131.

[3] S.V. Parter, Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations, Trans. Amer. Math. Soc. 99 (1961) 153–192.

[4] S.V. Parter, On the extreme eigenvalues of truncated Toeplitz matrices, Bull. Amer. Math. Soc. 67 (1961) 191–196.