Completeness Problem of One-Dimensional Binary Scope-3 Tessellation Automata

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In this report we deal with the question of whether or not, for a given tessellation automaton, there exists a finite state pattern of the array that can never be assumed by the array no matter what sequence of global transformations is applied to a certain canonical starting pattern. We are exclusively concerned with the one-dimensional tessellation automata. We show that any finite state pattern can evolve from the starting pattern in the case where the neighborhood scope is three or more and the cell in the array has $2^{3^i} j$ states ($i, j = 0, 1, 2, ...$).

1. INTRODUCTION

The tessellation automaton is a formalization of the concept of an infinite regular array of identical finite-state machines uniformly interconnected in the sense that each machine can directly receive information by means of interconnecting wires from a finite number of neighboring machines where the spatial arrangement of these neighboring machines is the same relative to each machine in the array. Each machine can synchronously change its state at discrete time steps as a function of the state of the neighboring machines. This function can change from time step to time step, but will be identical for each machine in the array at any given time step. The simultaneous action of these local functions will define global functions which will act on the entire array changing patterns of machine states in the array to other patterns.

Such arrays have been applied in such diverse areas as pattern recognition, e.g. Unger [1, 2] Beyer [3] and Smith [4]; machine self-reproduction, e.g. von Neumann [5] and Codd [6] and Arbib [7]; and evolution theory, e.g. Barricelli [8, 9].

Among these works, Yamada and Amoroso have studied a fundamental problem concerning behavior of such automata, namely; For an arbitrarily given tessellation automaton, can all the (finite) state patterns of the array be reached by some sequence of global transformations from certain canonical starting pattern? We shall refer to this as the completeness problem for tessellation automata. Yamada and Amoroso
dealt with the problem for the one-dimensional binary scope-\(n\) tessellation automata, and they showed that any finite pattern can evolve from the starting pattern if the neighborhood scope is four or more, and that there are finite patterns that cannot evolve from the starting pattern for the scope-two case.

In this paper we investigate the completeness problem for the one-dimensional binary scope-three tessellation automata which Yamada and Amoroso have left open. And we prove that any finite pattern can evolve from the starting pattern in the scope-three case.

2. THE TESSELLATION AUTOMATA AND RESULTS OF PREVIOUS PAPERS

The abbreviation \(TA\) is used to mean a tessellation automaton. By a one-dimensional, \(q\)-state, scope-\(n\) \(TA\), we mean \(TA\) of the form:

\[(A, Z, X, T),\]

where \(A = \{0, 1, 2, \ldots, q - 1\}\) is the set of states that can be assumed by any cell (finite-state machine) in the one-dimensional array. \(Z\) is the set of integers which is used to name the cells of the one-dimensional array. \(X = ((-1), (0), (1), (2), \ldots, (n - 2))\) is called the neighborhood index of the \(TA\) and is used to define the uniform interconnection pattern. A configuration is defined as a mapping \(c: Z \rightarrow A\). \(C\) denotes the set of all configurations with respect to \(Z\) and \(A\). The image of \(i\) under \(c \in C\) is written \(c(i)\) and will be referred to as "the contents of cell \(i\) in configuration \(c\)." Given scope-\(n\) \(TA\), a mapping \(\sigma: A^n \rightarrow A\) is called a local map. A global map \(\tau: C \rightarrow C\) is defined from local map \(\sigma\) as follows.

\[c\tau = c' \text{ if and only if for any } i \in Z,\]

\[c'(i) = \sigma(c(i - 1), c(i), c(i + 1), \ldots, c(i + n - 2)).\]

Such global maps defined in this way will be called parallel maps on the set of configurations. \(T\) is the set of all possible parallel maps for a given \(A\), \(Z\) and \(X\). Usually \(0 \in A\) denotes the quiescent state. A configuration \(c\) is called finite if and only if \(c(i) = 0\) for all but finitely many cells \(i\). The set of all finite configuration for the given \(TA\) is denoted by \(\bar{C} \subseteq C\). It is easy to see that \(\bar{C}\) is closed for a given parallel map (i.e., for any \(c \in \bar{C}\), \(c\tau\) is also in \(\bar{C}\)) if and only if \(\sigma(0, \ldots, 0) = 0\) where \(\sigma\) is the local map defining \(\tau\). \(\bar{T}\) denotes the set of all the finite-configuration preserving parallel maps for the \(TA\).

Let us denote the one-dimensional, \(q\)-state, scope-\(n\) \(TA\) by \(\mathcal{A}^{(q,n)}\). For any \(TA\) \(\mathcal{A}^{(q,n)} = (A, Z, X, T)\), if there exists a sequence of configurations \(c_0, c_1, \ldots, c_m\), \(m \geq 0\), such that \(c_i\tau_i = c_{i+1}\), \(0 \leq i \leq m\), where the \(\tau\)'s are arbitrary elements of \(\bar{T}\),
then we say $A^{(q,n)}$ generates $c_n$ from $c_0$. Let us define $c_1$ and $c_2$ as shift-equivalent if and only if there exists a $k \in \mathbb{Z}$ such that for any $i \in \mathbb{Z}$, $c_1(i + k) = c_2(i)$. The equivalence classes on $C$ determined by the relation of shift-equivalence are called finite patterns, or just patterns. $[c]$ denotes the pattern containing configuration $c$. Let $c_p$ be the configuration such that $c_p(0) = 1$ and $c_p(i) = 0$ for all $i \in \mathbb{Z}$ with $i \neq 0$. Let

$$A^{(q,n)}(c_p) = \{ [c] | A^{(q,n)} \text{ generates } c \text{ from } c_p \}.$$ 

The set of all finite patterns for one-dimensional TA with respect to $A = \{0, 1, \ldots, q - 1\}$ is denoted by $P(q)$.

Yamada and Amoroso have proved the following [10].

THEOREM 1 (Yamada and Amoroso). $A^{(2, 2)}(c_p) \subseteq P(2)$.

THEOREM 2 (Yamada and Amoroso). $A^{(2, 4)}(c_p) = P(2)$.

Kubo and Kimura have proved the next theorem.

THEOREM 3 (Kubo and Kimura). $A^{(3, 3)}(c_p) = P(3)$.

In this paper, we prove $A^{(2, 3)} = P(2)$ which has been left open by Yamada and Amoroso.

3. A Pattern Decomposition Lemma

Most of the proof of $A^{(2, 3)}(c_p) = P(2)$ is devoted to establishing Lemma 17 which is closely related to the “decomposition lemma” [10, Lemma IX, 1]. In this section we establish Lemma 14. This is accomplished after a series of preliminary results.

In this section we consider one-dimensional, two-state, scope-three TA. So the set of cell’s states is $A = \{0, 1\}$ and the neighborhood index is $X = (-1, 0, 1)$. We represent patterns as follows: If $c \in C$ is such that $c(k + i) = b_i$, $1 \leq i \leq j$ for some integer $k$ and positive integer $j$, and if $c(k + i) = 0$ for all $i < 1$ and all $i > j$, then $[c]$ can be represented by

$$0 b_1 b_2 \cdots b_j 0.$$ 

The decomposition lemma in [10] is stated as follows. For any $c \in C$, there exist $c' \in C$ and $\xi \in \{\tau_{166}, \tau_{180}\}^*$ such that

$$c' \xi = c$$

$^1$ In general, $\{a_1, \ldots, a_i\}^*$ denotes the set of all the sequences of finite length composed of $a_1, \ldots, a_i$, including the sequence of zero length.
and
\[ [c'] = 0 1 1 0^{n_1} 1 1 0^{n_2} \cdots 1 1 0^{n_r} 1 0 0^{m_1} 1 0 0^{m_2} \cdots 1 0 0^{m_{s-1}} 1 \]

where \( n_1 \geq 1, \ldots, n_r \geq 1, \ r \geq 0, \ m_1 \geq 1, \ldots, m_{s-1} \geq 1, \ s \geq 0. \) In the above lemma, \([c'] = 0 1 1 0^{n_1} \cdots 1 1 0^{n_r-1} 1 1 0\) if \( s = 0. \) Lemma 17 tells us that, by choosing \( \xi \) in \( \{\tau_{76}, \tau_{166}, \tau_{180}\}^* \) appropriately, we can put \( s = 0 \) or \( s = 1 \) in the above Lemma. Parallel maps \( \tau_{76}, \tau_{166} \) and \( \tau_{180} \) are defined later.

Local maps \( \sigma_{166} \) and \( \sigma_{180} \) defining \( \tau_{166} \) and \( \tau_{180} \), respectively, are specified by the following Table I.

<table>
<thead>
<tr>
<th>( \sigma_{166} )</th>
<th>( \sigma_{180} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 0</td>
<td>0 0 0 0 0</td>
</tr>
<tr>
<td>0 0 1 1 0</td>
<td>1 0 0 0 0</td>
</tr>
<tr>
<td>0 1 0 1 1</td>
<td>1 1 0 0 1</td>
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<td>0 1 1 0 0</td>
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<td>1 0 0 0 0</td>
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<td>1 1 0 0 0</td>
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<tr>
<td>1 1 0 0 1</td>
<td>1 1 1 1 1</td>
</tr>
</tbody>
</table>

Note that the subscripts have been obtained from the table columns considered as binary numerals. It is easily verified [10] that \( \tau_{166} \) and \( \tau_{180} \) are one-to-one, onto mapping: \( C \to C. \)

It is convenient to introduce a notion of framing to determine \( c_1(c_3) \) by \( c_2(c_4) \) such that \( c_1\tau_{166} = c_3 \) \( c_2\tau_{180} = c_4 \) holds. Let \( A_1, A_2, \ldots, A_j \) be nonempty subsets of \( \{0, 1\}^+ \), where \( \{0, 1\}^+ \) denotes the set of all the sequences of finite length composed of symbols 0 and 1, excluding the sequence of zero length. A sequence of pairs of integers \( ((t_1, e_1), (t_2, e_2), \ldots, (t_k, e_k)) \), \( k \geq 1 \), is called a left framing of \( c \in C \) with respect to \( A_1, \ldots, A_j \) if and only if the following three conditions are satisfied.

(i) If the leftmost 1 (which we henceforth denote by \( 1_L \)) in \( c \) is in cell \( i \), \( t_1 = i \) holds.

(ii) \( e_l + 1 = t_{l+1} \) for \( l = 1, \ldots, k - 1 \).

(iii) For all \( l \in \{1, \ldots, k\} \), \( t_l \leq e_l \) holds and there exists \( A \in \{A_1, \ldots, A_j\} \) such that \( c(t_l) c(t_l + 1) \cdots c(e_l) \in A. \)

In a similar way, we define a right framing. That is, a sequence of pairs of integers
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\(((e_k, t_k), (e_2, t_2), (e_1, t_1)), \ k \geq 1\), is called a right framing of \(c \in C\) with respect to \(A_1, \ldots, A_j\) if and only if the following three conditions are satisfied.

(i) If the rightmost 1 (which we henceforth denote by \(1_R\)) in \(c\) is in cell \(i\),
\[t_1 = i\] holds.

(ii) \(e_i - 1 = t_{i+1}\) for \(l = 1, \ldots, k - 1\).

(iii) For all \(l \in \{1, \ldots, k\}\), \(e_i \leq t_i\) holds and there exists \(A \in \{A_1, \ldots, A_j\}\) such that \(c(e_i) \ c(e_i + 1) \cdots c(t_i) \in A\). \((t_1, e_1)\) and \((e_1, t_1)\) are called a left frame and a right frame, respectively. \(t_i\) (or \(c(t_i)\)) is called the top of left frame \((t_1, e_1)\) and of right frame \((e_1, t_1)\).

Given \(A_1, \ldots, A_j\) and \(c \in C\), there may be several left or right framings in general. Specifying \(A_1, \ldots, A_j\) and \(e_k\), we define an \(L\)-framing (\(R\)-framing) in the next definition. It is easy to verify that these framings are uniquely determined if they exist.

**DEFINITION 1.** Let
\[
A_1 = \{101^j00^i \mid j = 1, 2, \ldots, i = 0, 1, 2, \ldots\},
A_2 = \{1000^i \mid i = 0, 1, 2, \ldots\},
A_3 = \{110^i \mid i = 0, 1, 2, \ldots\}.
\]

Let \(c \in C\) and let \(1_R\) in \(c\) be in cell \(i\). \(((t_1, e_1), \ldots, (t_k, e_k))\) is the \(L\)-framing of \(c\) if and only if \(((t_1, e_1), \ldots, (t_k, e_k))\) is a left framing with respect to \(A_1, A_2\) and \(A_3\), and \(e_k = r + 2\) holds.

Let
\[
B_1 = \{0^j01^i01 \mid j = 1, 2, \ldots, i = 0, 1, 2, \ldots\},
B_2 = \{0^j00^i1 \mid i = 0, 1, 2, \ldots\},
B_3 = \{0^j11^i \mid i = 0, 1, 2, \ldots\}.
\]

Let \(1_L\) in \(c\) be in cell \(l\). \(((e_k, t_k), \ldots, (e_1, t_1))\) is the \(R\)-framing of \(c\) if and only if \(((e_k, t_k), \ldots, (e_1, t_1))\) is a left framing of \(c\) with respect to \(B_1, B_2, \) and \(B_3\), and \(e_k = l - 2\) holds. Let us denote the left frames (right frames) for the \(L\)-framing (\(R\)-framing) by \(L\)-frames (\(R\)-frames).

**EXAMPLE 1.** Let \(c\) be a configuration such that
\[
[c] = 01111100011001010110.
\]

Then the relative positions of \([c]\) and \([c_{\tau_{166}}]\) are
\[
[c] = 0011111000110010101100,
\]
\[
[c_{\tau_{166}}] = 010111000100011111000.
\]
where a symbol in \([c]\) is written directly above a symbol in \([c_{\text{180}}]\) if these symbols represent the contents of the same cell for \(c\) and \(c_{\text{180}}\). Assume that \(1_L\) in \(c_{\text{180}}\) is in cell 0. Then the \(L\)-framing of \(c_{\text{180}}\) is

\[
((0, 7), (8, 11), (12, 13), (14, 15), (16, 18)).
\]

In the sequence of \(c_{\text{180}}\), \(\cdot\) is marked above the tops of \(L\)-frames.

**Example 2.** Let \(c\) be the configuration in the Example 1. The relative positions of \([c]\) and \([c_{\text{180}}]\) are

\[
[c] = \overline{00111110001100101100},
\]

\[
[c_{\text{180}}] = \overline{0011110100001011100}.\]

Then \(R\)-framing of \(c_{\text{180}}\) is

\[
((0, 6), (7, 11), (12, 14), (15, 16), (17, 19)).
\]

In the sequence of \(c_{\text{180}}\), \(\cdot\) is marked above the tops of \(R\)-frames.

From these examples, it is easy to see that the following statements hold. Let \(c'_{\text{180}} = c\) and let \(((t_1, e_1), \ldots, (t_k, e_k))\) be the \(L\)-framing of \(c\). Then

\[
\text{if } c(t_i) \cdots c(e_i) = 1010001, \\
\quad c'(t_i) \cdots c'(e_i) = 0110100; \]

\[
\text{if } c(t_i) \cdots c(e_i) = 100001, \\
\quad c'(t_i) \cdots c'(e_i) = 011001; \]

\[
\text{if } c(t_i) \cdots c(e_i) = 11001, \\
\quad c'(t_i) \cdots c'(e_i) = 01001,
\]

where \(1 \leq l \leq k, j \geq 1, i \geq 0\). When \(c'_{\text{180}} = c\) holds, the similar relations between \(c\) and \(c''\) are established by the \(R\)-framing of \(c''\).

If \(c(i - 1) = c(j + 1) = 0\) and \(c(i) = c(i + 1) = \cdots = c(j) = 1\) for \(c \in \mathbb{C}\), then we call \(c(i) c(i + 1) \cdots c(j)\) a block of \(c\). By the length of \(c \in \mathbb{C}\), \(lg(c)\), we mean the nonnegative integer \(j - i + 1\), where \(c(i) = 1_L\) and \(c(j) = 1_R\).

**Lemma 4.** If \(c'_{\text{186}} = c_1\) for \(c_1, c_2 \in \mathbb{C}\), then the following statements hold.

(i) \(lg(c_1) = lg(c_2)\) if and only if both the lengths of the rightmost blocks (which we henceforth denote by \(B_R\)) in \(c_1\) and \(c_2\) are odd or even.

(ii) \(lg(c_1) + 1 = lg(c_2)\) if and only if the lengths of \(B_R\)'s in \(c_1\) and \(c_2\) are odd and even, respectively.
(iii) \( \lg(c_1) - 1 = \lg(c_2) \) if and only if the lengths of B_R's in \( c_1 \) and \( c_2 \) are even and odd, respectively.

Proof. Depending on the form of the rightmost L-frame of \( c_1 \), one of the following cases below (that shows relative positions of \( c_1 \) and \( c_2 \)) must hold.

Case 1.

\[
\begin{align*}
[c_2] &= 001 \overline{0} \overline{1} \overline{1} \overline{1} \cdots 1 1 \overline{0}, \\
[c_1] &= \overline{0} 1 \overline{1} \overline{1} \overline{1} \cdots 1 1 0 \overline{0}.
\end{align*}
\]

In this case, the parity of the length of B_R in \( c_1 \) is the same as that of \( c_2 \).

Case 2.

\[
\begin{align*}
[c_2] &= 001 \overline{0} \overline{1} \overline{1} \overline{0}, \\
[c_1] &= \overline{0} 1 \overline{1} \overline{0} \cdots 1 0 \overline{0}.
\end{align*}
\]

In this case, it is easily seen that the length of B_R in \( c_1 \) is odd.

Case 3.

\[
\begin{align*}
[c_2] &= 001 \overline{0} \overline{1} \overline{1} \overline{1} \cdots 0 1 \overline{0}, \\
[c_1] &= \overline{0} 1 \overline{1} \overline{1} \overline{1} \cdots 0 1 \overline{0}.
\end{align*}
\]

In this case, it is easily seen that the length of B_R in \( c_1 \) is even. Q.E.D.

The proof of the next lemma is very much like the proof of the last one and is omitted.

**Lemma 5.** If \( c_2 \tau_{180} = c_1 \) for \( c_1, c_2 \in \overline{C} \), then the following statements hold.

(i) \( \lg(c_1) = \lg(c_2) \) if and only if both the lengths of the leftmost blocks (which we henceforth denote by B_L) in \( c_1 \) and \( c_2 \) are odd or even.

(ii) \( \lg(c_1) + 1 = \lg(c_2) \) if and only if the lengths of B_L's in \( c_1 \) and \( c_2 \) are odd and even, respectively.

(iii) \( \lg(c_1) - 1 = \lg(c_2) \) if and only if the lengths of B_L's in \( c_1 \) and \( c_2 \) are even and odd, respectively.

**Lemma 6.** Let \( c_0 \in \overline{C} \) and let \( c_i = c_0 \tau_{166}^{-i} \), \( i = 1, 2, \ldots \). Then there exists \( c_j \) such that \( [c_0] = [c_j], j \in \{1, 2, \ldots \} \).

\[2\] The lines between 1 and 0 (1 and 1) indicate we are unconcerned with the symbols between 1 and 0(1 and 1). We use this convention in what follows.
Proof. From Lemma 4, \( \lg(c_i) \leq \lg(c_0) + 1 \) for any \( c_i \) \((i = 1, 2, \ldots)\). Let \( j \) be the least integer such that there exists an integer \( i \) with \([c_i] = [c_j]\) and \( 0 \leq i < j \). Let us assume \( i \neq 0 \). Then \( c_i = c_{i-1}^{(166)} \) and \( c_i = c_{j-1}^{(166)} \). Since \([c_i] = [c_j]\), we have \([c_{i-1}] = [c_{j-1}]\). This contradicts the definition of the integer \( j \). Q.E.D.

In a similar way to the proof of Lemma 6, we can verify the next lemma from Lemma 5.

**Lemma 7.** Let \( c_0 \in \mathcal{C} \) and let \( c_i = c_0^{(180)} \), \( i = 1, 2, \ldots \). Then there exists \( c_j \) such that \([c_0] = [c_j]\), \( j \in \{1, 2, \ldots\} \).

**Lemma 8.** Let \( c_0 \in \mathcal{C} \) and let \( c_1 = c_0^{\xi_1} \) for \( \xi_1 \in \{\tau_{166}, \tau_{180}\}^+ \). Then there exists \( \xi_2 \in \{\tau_{166}, \tau_{180}\}^+ \) such that \( c_2 = c_0^{\xi_2^{-1}} \) and \([c_1] = [c_2]\).

**Proof.** If \( c_1 = c_0^{\tau_{180}} \) and \( k > 1 \) is the \( j \) of Lemma 6, then \([c_0^{\tau_{180}}(1)] = [c_1]\). Similarly, if \( c_1 = c_0^{\tau_{180}} \) and \( l > 1 \) is the \( j \) of Lemma 7, then \([c_0^{\tau_{180}}(l-1)] = [c_1]\). From these facts, we can easily drive the lemma. Q.E.D.

**Definition 2.** Let \( c \in \mathcal{C} \) and let \( A = \{1 1 0 0 0 \mid j = 0, 1, 2, \ldots\} \). Let \((t_1, e_1), \ldots, (t_k, e_k)\) be the longest left framing of \( c \) with respect to \( A \), where the length of the framing \(((t_1, e_1), \ldots, (t_k, e_k))\) is defined to be \( e_k - t_1 + 1 \). Then the remainder of \( c \), \( R(c) \), is the configuration defined as follows.

\[
R(c)(i) = \begin{cases} 
  c(i) & \text{if } i < t_1 \text{ or } e_k < i, \\
  0 & \text{if } t_1 \leq i \leq e_k.
\end{cases}
\]

Let us denote the pattern \([c]\) of configuration \( c \) by \( \bar{c} \), where \( c \) is the configuration such that \( c(i) = 0 \) for all \( i \in \mathbb{Z} \).

The next lemma can easily be verified.

**Lemma 9.** Let \( c \in \mathcal{C} \) be a configuration such that \([R(c)] = \bar{c}\). Then for any \( \xi \in \{\tau_{166}, \tau_{180}\}^* \) whose length is even,

\([c] = [c\xi]\),

and

\([c] = [c\xi^{-1}]\).

**Lemma 10.** Let \( c \in \mathcal{C} \). If the length of \( B_L \) in \( R(c) \) is longer than two, there exists \( \xi \in \{\tau_{166}, \tau_{180}\}^* \) such that

\[
\lg(R(c')) \leq \lg(R(c)) - 1
\]

and

\( c' = c\xi^{-1} \).
Proof. Let 

\[ [c] = 0110^n 110^n \dots 110^n 1 \ldots 10, \]

\[ n_1 \geq 1, \ldots, n_r \geq 1. \]

We shall show that there exists a configuration \( c' \) that satisfies the conditions of the lemma in each of the following cases. Let \( l_L \) in \( R(c) \) be in cell \( i \).

Case 1. The case where the length of \( B_L \) in \( R(c) \) is longer than four.

The relative positions of \( c, c_1 = c_{166} \) and \( c_2 = c_{180} \) are

\[ [c] = 00110^n \dots 110^n 111 \ldots 110 \ldots, \]

\[ [c_1] = 0100^n \dots 10^n 101 \ldots 110 \ldots, \]

\[ [c_2] = 0110^n \dots 110^n 1101 \ldots 101 \ldots. \]

Note that the length of the block in \( c_1 \) which contains \( c_1(i + 1) \) is longer than two. Let \( l_R \) in \( c, c_1 \) and \( c_2 \) be in cell \( r, r_1 \) and \( r_2 \), respectively. Then, from Lemma 4 and inspection of the relative positions of \( l_L \) of \( c \) and \( l_L \) of \( c_1 \) we have \( r - 2 \leq r_1 \leq r \). Furthermore, \( r_1 + 1 = r_2 \) holds. Thus \( r - 1 \leq r_2 \leq r + 1 \). Hence, we have \( \lg(R(c)) - 3 \leq \lg(R(c_2)) \leq \lg(R(c)) - 1 \). Then, from Lemma 8 there exists \( c' \in \bar{C} \) with \( [c'] = [c_2] \) which satisfies the conditions of the lemma.

Case 2. The case where the length of \( B_L \) in \( R(c) \) is four. The proof of this case is essentially the same as the proof of Case 1.

Case 3. The case where the length of \( B_L \) in \( R(c) \) is three.

\[ [c_1] = c_{166}, \quad [c_2] = c_{186}, \quad [c_3] = c_{216}, \quad [c_4] = c_5_{180} \]

\[ [c_1] = 0100^n \ldots 10^n 101 \ldots 110 \ldots, \]

\[ [c_2] = 0110^n \ldots 110^n 1101 \ldots 101 \ldots. \]

It is easy to see that there exists the required configuration \( c' \in \bar{C} \) such that \( [c'] = [c_4] \) by a similar argument to that for Case 1.

Q.E.D.

Lemma 11. Let \( c \in \bar{C} \). If the length of \( B_L \) in \( R(c) \) is longer than five, then there exists \( \xi \in \{ \tau_{166}, \tau_{180} \}^* \) such that

\[ \lg(R(c)) \leq \lg(R(c)) - 2, \]
and
\[ c' = c\xi^{-1}. \]

Proof. In the case where the length of \( B_L \) in \( R(c) \) is six, the lemma can be established in the same way as Case 1 in the proof of Lemma 10. In the case where the length of \( B_L \) in \( R(c) \) is longer than six, the length of \( B_L \) in \( R(c_{166,180}) \) is longer than two. Furthermore in the same way as in the proof of Case 1 in Lemma 10 it is easy to see there exists \( \xi' \in \{\tau_{166}, \tau_{180}\}^* \) such that \( \lg (R(c')) \leq \lg (R(c)) - 1 \), \( c'' = c\xi'^{-1} \) and \( [c''] = [c_{166,180}] \). So we have the desired configuration \( c' \in \mathcal{C} \), applying again Lemma 10. Q.E.D.

Lemma 12. Let \( c \in \mathcal{C} \). If \( [R(c)] \) is of the form
\[ [R(c)] = 0101 \]
there exists \( \xi \in \{\tau_{166}, \tau_{180}\}^* \) such that
\[ \lg (R(c')) \leq \lg (R(c)) - 1 \]
and
\[ c' = c\xi^{-1}. \]

Proof. We shall show that there exists a configuration \( c' \) that satisfies the lemma in each of the following cases. Let \( 1_L \) in \( R(c) \) be in cell \( i \).

Case 1. The case where \( [R(c)] \) is of the form \( 0101 \), where \( k > 0 \) is even.

Let \( c_1 = c\tau_{166} \). Then \( c_1(i+1) = 1 \) and \( c_1(i+1) \) is the top of an \( R \)-frame of \( c_1 \). So in the same way as in the proof of Lemma 10 it follows that there exists the desired configuration \( c' \) such that \( [c'] = [c_{1}\tau_{180}] \).

Case 2. The case where \( [R(c)] \) is of the form \( 0101 \), where \( k \geq 3 \) and \( k \) is odd.

Let
\[ c_3 = c\tau_{166}^{-1}, \quad c_4 = c\tau_{180}^{-1}. \]
Then we have \( c_3(i+1) = 1 \). If \( c_3(i+1) \) is the top of an \( R \)-frame of \( c_3 \), then it is easily seen that there exists the desired configuration \( c' \) such that \( [c'] = [c_4] \). Otherwise the lemma follows from the fact that the length of \( B_L \) in \( R(c_4) \) is longer than six and \( \lg (R(c_4)) \leq \lg (R(c)) + 1 \), using Lemma 7.

Case 3. The case where \( [R(c)] \) is of the form
\[ 010100 \].
Let $c_3$ and $c_4$ be the same as Case 2. Then we have $c_3(i+1)=1$ and $c_3(i+1)$ is the top of an $R$-frame of $c_3$. Therefore, there exists the desired configuration $c'$ such that $[c']=[c_4]$.

Case 4. The case where $[R(c)]$ is of the form

$$010101\ldots$$

It is easily seen that this case can further divided into the following two cases.

Case 4-1. The case where $[R(c)]$ is of the form

$$0(10)^j0\ldots, \quad j \geq 3.$$

If $j=3$ the proof is similar to Case 3. If $j \geq 4$, then the lemma follows from the fact that

$$\lg(R(c_{166}\tau_{180})) \leq \lg(R(c)) + 1$$

and the length of $B_L$ in $R(c_{166}\tau_{180})$ is longer than five, using Lemma 11 and 12.

Case 4-2. The case where $[R(c)]$ is of the form

$$0(10)^j11\ldots, \quad j \geq 2$$

Let $c_1 = c_{166}, c_2 = c_{1}\tau_{180}$. The length of the block that contains $c(i)$ must be longer than four and must be odd. Therefore, if $1_R$ of the block is the top of an $R$-frame of $c_1$, there exists the desired configuration $c'$ such that $[c']=[c_2]$. Otherwise, since the length of $B_L$ of $R(c)$ is longer than six and $\lg(R(c_2)) \leq \lg(R(c)) + 1$ the lemma follows from Lemma 11.

Let $c_1, c_2 \in \mathcal{C}$ and let $c_1(l_1) = 1_L, c_1(r) = 1_R, c_2(l_2) = 1_L, c_2(r) = 1_R$. $c_1$ and $c_2$ are disjoint each other if and only if $r < l_2$ or $r < l_1$. When $c_1$ and $c_2$ are disjoint from each other, the union of $c_1$ and $c_2$, $c_1 \cup c_2$ is the configuration defined as follows.

$$(c_1 \cup c_2)(i) = \begin{cases} c_1(i) & \text{if } l_1 \leq i \leq r, \\ c_2(i) & \text{if } l_2 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma is easily verified.

Lemma 13. Let $c \in \mathcal{C}$ and let $c'$ be the configuration such that $c = c' \cup R(c)$. Then for any $\xi \in \{\tau_{166}\tau_{180}\}^*$,

$$(c' \cup R(c))\xi = c'\xi \cup R(c)\xi.$$
**Lemma 14.** Let \( c \in C \) and let \( c(i) = 1_L \). \([c]\) is of the form \( \overline{0} 1 0 0 \) — if and only if \( 1_L \) in \( c_{\tau_{166} \tau_{180}} \) is in cell \( i + 1 \).

**Proof.** \( 1_L \) in \( c_{\tau_{166} \tau_{180}} \) is in cell \( i + 1 \) if and only if the relative positions of \( c \), \( c_{\tau_{166}} \) and \( c_{\tau_{166} \tau_{180}} \) are

\[
[c] = \overline{0} 0 1 0 0 \\
[c_{\tau_{166}}] = \overline{0} \ 1 \ 1 \ 0 \\
[c_{\tau_{166} \tau_{180}}] = \overline{0} 0 0 1
\]

Q.E.D.

The next two lemmas are easily verified.

**Lemma 15.** Let \( c, c' \in \overline{C} \). If \( c_{\tau_{166} \tau_{180}} = c' \), then

\[
1 - 1 \leq l' \leq l + 1, \\
r - 1 \leq r' \leq r + 1,
\]

where \( c(l) = 1_L \), \( c(r) = 1_R \), \( c'(l') = 1_L \) and \( c'(r') = 1_R \).

**Lemma 16.** Let \( c \in \overline{C} \) be a configuration such that \( [c] = \overline{0} 1 0 0 \alpha_1 \alpha_2 \cdots \alpha_t \overline{0} \) and let \( c' \in \overline{C} \) be a configuration such that \( [c'] = \overline{0} \alpha_1 \cdots \alpha_t \overline{0} \), where \( \alpha_1, \ldots, \alpha_t \in \{0, 1\} \). The relative position of \( c \) and \( c_{\tau_{166} \tau_{180}} \) are

\[
[c] = \overline{0} 1 0 0 \alpha_1 \alpha_2 \cdots \alpha_t \overline{0} \\
[c_{\tau_{166} \tau_{180}}] = \overline{0} 0 1 0 0 \beta_2 \cdots \beta_t \overline{0}, \quad \beta_2, \ldots, \beta_t \in \{0, 1\},
\]

if and only if the relative positions of \( c' \) and \( c'_{\tau_{166} \tau_{180}} \) are

\[
[c'] = \overline{0} \alpha_1 \alpha_2 \cdots \alpha_t \overline{0} , \\
[c'_{\tau_{166} \tau_{180}}] = \overline{0} 0 \beta_2 \cdots \beta_t \overline{0}.
\]

**Lemma 17.** Let \( c_0 \in \overline{C} \) and let \( c_i = c_0(\tau_{166} \tau_{180})^i \), \( i = 0, 1, 2, \ldots \). If \( [c_i] \) is of the form \( \overline{0} 1 0 0 \) — for any \( i \), then

\[
[c_i] = \overline{0} 1 0 0^m_1 1 \ 0^m_2 \cdots 1 \ 0^m_s \ 1 \ 0, \quad m_1 \geq 1, \ldots, m_s \geq 1, \quad s \geq 0, \]

where \( m_1, \ldots, m_s \) are constants, not depending on \( i \).

**Proof.** Since \( [c_i] = \overline{0} 1 0 0 \) — for any \( i \),

\[
\lg(c_0) \geq \lg(c_1) \geq \cdots,
\]
from Lemmas 14 and 15. In a way similar to the proof of Lemma 6 we can verify that there exists \( e_k (k \geq 1) \) such that \([c_0] = [e_k]\) because \( \tau_{1667180} \) is one-to-one, onto mapping from \( \mathcal{C} \) to \( \bar{\mathcal{C}} \). Therefore we have
\[
\text{lg}(c_0) = \text{lg}(c_1) = \ldots.
\]
Let the second 1 in \( c_i \) from the left be in cell \( q_i \). Then from Lemmas 15 and 16 and the fact that \([c_0] = [e_k]\) for infinitely many \( k \), it is easily verified that
\[
q_{i+1} = q_i + 1 \quad \text{for} \quad i = 0, 1, 2, \ldots.
\]
Define the configuration \( c'_i \) as follows.
\[
c'_i(j) = \begin{cases} 
    e_i(j) & \text{if } j \neq i, \\
    0 & \text{if } j = i,
\end{cases}
\]
where \( e_i(i) = 1_L \). Clearly \( 1_L \) in \( c' \) is in cell \( q_i \). Furthermore,
\[
c'_i = e_i(\tau_{1667180}) \quad \text{for} \quad i = 1, 2, \ldots.
\]
Therefore, from Lemma 14,
\[
[c'_i] = \bar{0} 1 0 0 \quad \text{for} \quad i = 1, 2, \ldots.
\]
Continuing in this way, the lemma is proved. Q.E.D.

**Lemma 18.** For any \( c \in \mathcal{C} \), there exist \( c' \in \bar{\mathcal{C}} \) and \( \xi \in \{ \tau_{76}, \tau_{166}, \tau_{180} \}^* \) such that
\[
c' \xi = c,
\]
and
\[
[c'] = \bar{0} 1 1 0^{n_1} 1 1 0^{n_2} \ldots 1 1 0^{n_r} 1^m \bar{0},
\]
\[
n_1 \geq 1, \ldots, n_r \geq 1, \quad r \geq 0, \quad m = 0 \quad \text{or} \quad 1,^3
\]
where \( \tau_{78} \) is the parallel map specified in the proof below.

**Proof.** We show the algorithm to obtain the desired \( c' \) and \( \xi \) for an arbitrarily given \( c \), where \( c' \xi = c \) and \([c'] = \bar{0} 1 1 0^{n_1} \ldots 1 1 0^{n_r} 1^m \bar{0}\). Before the discussion of the algorithm, we illustrate the transformation denoted by \( \sim \). This transformation is applied to configuration \( c \) such that
\[
[c] = \bar{0} 1 1 0^{m_1} 1 1 0^{m_2} \ldots 1 1 0^{m_r} 1 0^{m_{s-1}} 1 0, \quad \text{for} \quad n \geq 1, \ldots, n_r \geq 1, \quad r \geq 0, \quad m_1 \geq 1, \ldots, m_{s-1} \geq 1, \quad s \geq 1.
\]
\( \bar{c} \) is defined as follows.

^3 If \( m = 0 \), we mean \([c'] = \bar{0} 1 1 0^{m_1} \ldots 1 1 1^{m_{s-1}} 1 \bar{0}\).
\[ \hat{c} = \begin{cases} \bar{0}110^{n_1} \cdots 110^{n_s} 111^{m_1} 100^{m_2} 111^{m_3} 100^{m_4} 1 \cdots 111^{m_{s-1}} 1 \bar{0} & \text{if } s \text{ is even}, \\ \bar{0}110^{n_1} \cdots 110^{n_s} 111^{m_1} 100^{m_2} 111^{m_3} 100^{m_4} 1 \cdots 100^{m_{s-1}} 1 \bar{0} & \text{if } s \text{ is odd}, \end{cases} \]

where \( 1_L \) in \( c \) and \( 1_L \) in \( \hat{c} \) are in the same cell. The next two examples illustrate the transformation.

**Example 3.**

\[
\begin{align*}
[c_1] &= \bar{0}110001101001000010, \\
[c_2] &= \bar{0}110001101001000010, \\
[c_3] &= \bar{0}11000110100100001000, \\
[c_4] &= \bar{0}110001101001000011110.
\end{align*}
\]

Let \( \sigma_{76} \) specified in Table II define parallel map.

<table>
<thead>
<tr>
<th>( \sigma_{76} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

Then for any configuration of the form shown by (1), we have

\[ \hat{c}\sigma_{76} = c. \]

**Step 1.** If \([R(c)]\) is of the form \( \bar{0} \) or \( \bar{0}100 \), put \( c_1 = c \). Otherwise, from Lemmas 10 and 12 there exist \( c_1 \in \hat{C} \) and \( \xi' \in \{\tau_{166}, \tau_{180}\}^* \) such that

\[ c_1 \xi' = c, \]

\[ \lg(R(c_1)) \leq \lg(R(c)), \]

and

\[ [R(c_1)] = \bar{0} \text{ or } \bar{0}1\bar{0} \text{ or } \bar{0}100. \]
Step 2. If \([R(c_1)] = \overline{0}\) or \(\overline{0}1\overline{0}\), proceed to Step 5. If \([R(c_1)]\) is of the form such that
\[
[R(c_1)] = \overline{0}100^m100^{m_2} \cdots 100^{m_{u-1}}1\overline{0}
\]
(2)
\[m_1 > 1, \ldots, m_{u-1} > 1, \quad u \geq 2,
\]
proceed to Step 3. Otherwise (i.e., \([R(c_1)]\) is of the form \(\overline{0}100\ldots\overline{0}\), but cannot be of the form shown by (2), including \(u = 1\)), proceed to Step 4.

Step 3. Put \(c = c_1\). Then proceed to Step 1. Note \(\lg(R(c)) = \lg(R(c_1))\).

Step 4. From Lemmas 14, 15, and 17, there exists \((\tau_{166}, \tau_{180}) \in \{\tau_{166}, \tau_{180}\}^*\) such that
\[
[R(c_1(\tau_{166}, \tau_{180}))] \neq \overline{0}100\ldots\overline{0},
\]
and
\[
\lg(R(c_1(\tau_{166}, \tau_{180}))) \leq \lg(R(c_1)).
\]
Put \(c = c_1(\tau_{166}, \tau_{180})\). Then, proceed to Step 1.

Step 5. Put \(c' = c_1\) and terminate the process.

Since the length of the remainder of the configuration decreases strictly in Step 1, the algorithm must terminate after a finite number of steps. \(c'\) is the desired configuration. Furthermore, although the required sequence of parallel maps \(\xi\) is not explicitly given in the algorithm, we can easily obtain it in view of Lemmas 6 and 7. Q.E.D.

4. The Proof of \(\mathcal{A}^{(2,3)}(p) = \mathcal{P}^{(2)}\) and Some Results Derived from It

Yamada and Amoroso [10] have proved the following two lemmas.

Lemma 19 (Yamada and Amoroso). For any \(n_i, 1 \leq i \leq k\), and any \(m_j, 1 \leq j \leq k - 1\),
\[
\overline{0}(1\,1)^{n_1}0^{m_1}(1\,1)^{m_2}0^{m_2} \cdots 0^{m_{k-1}}(1\,1)^{n_k} \overline{0} \quad \text{is in} \quad \mathcal{A}^{(2,3)}(p).
\]

Lemma 20 (Yamada and Amoroso). For any \(n_i, 1 \leq i \leq k\),
\[
\overline{0}1^n1^n2^01^n2^0 \cdots 01^n2^0 \quad \text{is in} \quad \mathcal{A}^{(2,3)}(p).
\]

Theorem 21. \(\mathcal{A}^{(2,3)}(p) = \mathcal{P}^{(2)}\).
Proof. Let $c$ be an arbitrary configuration in $C$ such that

$$[c] = 0110^n110^{n_2} \cdots 110^{n_r-1}110^{n_r}10.$$ 

Let $c'$ be the configuration in $C$ such that

$$[c'] = 0111111111 \cdots 11111111010.$$ 

where $1_L$ in $c$ is in cell $i$ and $1_L$ in $c'$ is in cell $i+1$. Then it is easy to verify

$$c'\tau_{42} = c,$$

where $\tau_{42}$ is defined by the local map $\sigma_{42}$ specified in Table III.

<table>
<thead>
<tr>
<th>$\sigma_{42}$</th>
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<tbody>
<tr>
<td>0 0 0 0</td>
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<td>0 0 1 1</td>
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<td>0 1 0 0</td>
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<td>0 1 1 1</td>
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<td>1 1 1 0</td>
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Thus the theorem follows from Lemmas 18, 19, and 20. Q.E.D.

Theorem 22. For any $i \geq 0, j \geq 0$,

$$\mathcal{A}^{(2j+3),3} = \mathcal{B}^{(2j+3)},$$

Proof. In general, if $\mathcal{A}^{(p,n)} = \mathcal{B}(p)$ and $\mathcal{A}^{(q,n)} = \mathcal{B}(q)$, then $\mathcal{A}^{(pq,n)} = \mathcal{B}(pq)$. This is easily verified from the fact that cells in $\mathcal{A}^{(pq,n)}$ can act as the parallel combinations of cells in $\mathcal{A}^{(p,n)}$ and $\mathcal{A}^{(q,n)}$ do. The theorem follows from this fact and Theorems 3 and 21. Q.E.D.

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REFERENCES