Topology Vol. 31, No. 3, pp. 545–555, 1992.
Printed in Great Britain

0040-9383/92 $5.00 + 0.00

THE CHARACTERISTIC MAPPING OF A FOLIATED BUNDLE

J. L. ARRAUT and N. M. DOS SANTOS

(*Received* 28 *March* 1990; *in revised form* 15 *March* 1991)

To the memory of C. Godbillon

INTRODUCTION

LET $\mathscr{F}$ denote a p-dimensional smooth foliation of a closed, orientable, connected m-manifold M and $H^1(\mathscr{F})$ the first cohomology group of the foliated manifold $(M, \mathscr{F})$. To the pair $(M, \mathscr{F})$ we associate a symmetric $(q + 1)$-linear mapping α of $H^1(\mathscr{F})$ into the real cohomology group $H^{2q+1}(M)$, $q = m - p$, called the *characteristic mapping* of the foliated manifold. In 1.3 we discuss the relation between the characteristic mapping and previous work on the Godbillon–Vey class, and especially the Godbillon operators and measures. In 1.5 we associate to each locally free smooth action A of a Lie group G on M a $(q + 1)$-linear symmetric mapping α_A of the first cohomology group $H^1(\mathscr{G})$ of the Lie algebra $\mathscr{G}$ of G into the real cohomology group $H^{2q+1}(M)$. The mapping α_A is called the *characteristic mapping* of A. In this paper we study α when M is the total space of a fiber bundle $F \to M \xrightarrow{\tau} T^p$ and $\mathscr{F}$ is a foliation of M transverse to the fibers of τ. $\mathscr{F}$ is therefore a suspension foliation of an action $\varphi: Z^p \to \mathrm{diff}(F)$. In Section 3 we identify ergodic properties of φ that imply the vanishing of α. The action φ is said to be an *E-action* if it leaves invariant a Borel probability measure μ on F and for every continuous function $h: F \to R$ the Birkhoff sums $B_n(\varphi)\cdot h$ converge pointwise to $\mu(h)$ for m-almost all $x \in F$ where m is a Lebesgue measure on F. We give some important examples of E-actions. The main theorem of the paper is

3.8. THEOREM. *Let $F \to (M, \mathscr{F}) \xrightarrow{\tau} T^p$ be the suspension foliated bundle of an action φ of Z^p on a closed connected q-manifold F. If φ is an E-action then the characteristic mapping α of $(M, \mathscr{F})$ vanishes.*

We obtain in particular the following generalization of a theorem of M. Herman, [14].

3.9 COROLLARY. *The characteristic mapping of a C^2 foliation by planes of T^3 vanishes.*

As an application of 3.8 and a previous theorem of the authors in [1] we obtain

4.1 THEOREM. *Let A be a C^2 locally free action of R^2 on a closed connected orientable 3-manifold M. If the characteristic mapping*

$$\alpha_A: R^2 \times R^2 \to H^3(M) = R$$

of A is non-degenerate, then

(i) *A has a compact orbit;*
(ii) *there is a neighbourhood V of A in the C^1 topology such that every action in V has a compact orbit.*

In case $M = T^3$ the hypothesis is in fact weaker, namely: (i) *and* (ii) *hold if $\alpha_A \neq 0$.*

1. THE CHARACTERISTIC MAPPING OF A FOLIATION

In [1] we associate to each non-singular C^r, $r \geq 2$ action A of R^p on a closed orientable connected m-manifold M a $(q+1)$-linear symmetric mapping α_A of R^p into the real cohomology group $H^{2q+1}(M)$, $q = m - p$. We called α_A the *characteristic mapping* of A. The foliated cohomology introduced by B. L. Reinhart in [18] appears naturally in the study of actions [2]. Using this cohomology we present here a generalization to foliations of the characteristic mapping. This also directly genaralizes a construction of C. Benson and G. Ratcliff for actions of simply connected nilpotent Lie groups, [5]. Our construction is a natural generalization of the Godbillon–Vey invariant of a foliation and of Duminy's Godbillon operator. For a foliation $\mathscr{F}$ of subexponential growth all residual secondary characteristic classes vanish whereas the characteristic mapping will in general not vanish. See 1.2 and [[1], 1.5] for more examples.

Let $\mathscr{F}$ be a p-dimensional foliation of M and $\Lambda(M)$ the graded algebra of all C^∞ differential forms on M. If $I(\mathscr{F}) \subset \Lambda(M)$ is the annihilating ideal of $\mathscr{F}$, then

$$I(\mathscr{F})^{q+1} = 0 \tag{1.1}$$

$q = m - p$ being the codimension of $\mathscr{F}$. Thus $\Lambda(\mathscr{F}) = \Lambda(M)/I(\mathscr{F})$ is a graded algebra, called the algebra of *differential forms along* $\mathscr{F}$.

The elements of $\Lambda^j(\mathscr{F})$ may be thought of as sections of the j-th exterior power of the dual bundle of the tangent bundle $T\mathscr{F}$ of $\mathscr{F}$. Since $dI(\mathscr{F}) \subset I(F)$ the differential $d\colon \Lambda(M) \to \Lambda(M)$ induces the *foliated differential* $d_f\colon \Lambda(\mathscr{F}) \to \Lambda(\mathscr{F})$. The kernel $Z(\mathscr{F})$ of d_f is the set of d_f-closed forms and the image $B(\mathscr{F})$ of d_f is the set of d_f-exact forms. The cohomology $H^*(\mathscr{F})$ of the differential complex $(\Lambda(\mathscr{F}), d_f)$ is the *cohomology of the foliated manifold* $(M, \mathscr{F})$. This is a natural generalization to foliations of the de Rham cohomology. Let $\pi\colon \Lambda(M) \to \Lambda(\mathscr{F})$ be the canonical projection. We say that $\xi \in \Lambda^j(M)$ is d_f-closed if $d\xi \in I(\mathscr{F})$ which is equivalent to $\pi(\xi) \in Z^j(\mathscr{F})$ and we denote by $\bar{\xi}$ the cohomology class of $\pi(\xi)$ in $H^j(\mathscr{F})$. $Z(M)$ and $B(M)$ denote, as usual, the kernel and image of $d\colon \Lambda(M) \to \Lambda(M)$, respectively. $[\xi]$ denotes the de Rham class of $\xi \in Z(M)$.

1.1 LEMMA. *Let ξ_j be d_f-closed 1-forms, $j = 1, \ldots, q+1$ and $\omega = \xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_{q+1}$. Then*

(i) *ω is a closed $(2q+1)$-form on M;*
(ii) *ω is exact if ξ_1 is closed;*
(iii) *ω is exact if some $\pi(\xi_j)$ is d_f-exact.*

Proof. (i) Since $d\xi_j \in I(\mathscr{F})$, $j = 1, \ldots, q+1$ the result follows from (1.1).

(ii) note that $\omega = -d(\xi_1 \wedge \xi_2 \wedge d\xi_3 \wedge \ldots \wedge d\xi_{q+1})$.

(iii) we may assume, after a permutation of indexes, that $\pi(\xi_1) = \pi(dh)$ for some C^∞ function $h\colon M \to R$. Thus $\xi_1 - dh \in I(\mathscr{F})$ which by (1.1) implies that $\omega = d(h\,d\xi_2 \wedge \ldots \wedge d\xi_{q+1})$, proving the lemma.

By the above lemma there exists a well defined $(q+1)$-linear symmetric mapping

$$\alpha\colon H^1(\mathscr{F}) \times \cdots \times H^1(\mathscr{F}) \to H^{2q+1}(M) \tag{1.2}$$

given by $\alpha(\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}) = [\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_{q+1}]$, called the *characteristic mapping* of the foliated manifold $(M, \mathscr{F})$.

1.2. *Example.* Let $T^2 \to T^3 \xrightarrow{\tau} T^1$, $\tau(x_1, x_2, x_3) = x_3$, be the canonical fibration of T^3 over T^1 and $\mathscr{F}$ be the foliation given by the fibers of τ. Choose any two functions f and g in

$C^\infty(T^1)$ such that $\int_{T^1} f dg \neq 0$. The 1-forms $\xi_1 = (f \circ \tau) dx_1$ and $\xi_2 = (g \circ \tau) dx_2$ are d_f-closed and

$$\alpha(\bar{\xi}, \bar{\xi}_2) = \int_{T^3} \xi_1 \wedge d\xi_2 = 4\pi^2 \int_{T^1} f dg.$$

Thus the characteristic mapping α of $(T^3, \mathscr{F})$ does not vanish.

1.3. We now describe the relation between the characteristic mapping and the Godbillon operator as defined in codimension 1 by G. Duminy in [9] and generalized by J. Heitsch and S. Hurder in [13].

Since $dI(\mathscr{F})^q \subset I(\mathscr{F})^q$ it follows that $(I(\mathscr{F})^q, d)$ is a differential complex whose cohomology is denoted by $H^*(M, \mathscr{F})$. We denote by $\langle\theta\rangle$ the cohomology class in $H^j(M, \mathscr{F})$ of a closed j-form in $I(\mathscr{F})^q$. For each j, $q \leq j \leq m-1$ there is a well defined pairing

$$H^1(\mathscr{F}) \times H^j(M, \mathscr{F}) \xrightarrow{\chi_j} H^{j+1}(M, \mathscr{F}) \tag{1.3}$$

given by $\chi_j(\bar{\xi}) \cdot \langle\theta\rangle = [\xi \wedge \theta]$. To see that χ_j is well-defined just observe that if $\pi(\xi) = \pi(d\eta)$, then $\xi \wedge \theta = d(\eta \wedge \theta)$ and if $\theta = d\omega$, $\omega \in I(\mathscr{F})^q$, then $\xi \wedge \theta = -d(\xi \wedge \omega)$. Let Ω be an integrable C^∞ q-form defining a foliation $\mathscr{F}$. Then there exists a C^∞ 1-form λ on M such that $d\Omega = \lambda \wedge \Omega$. Clearly λ is d_f-closed and defines the *Reeb class* $\bar{\lambda} \in H^1(\mathscr{F})$ of $\mathscr{F}$.

The class $\alpha(\bar{\lambda}, \ldots, \bar{\lambda}) = [\lambda \wedge (d\lambda)^q]$ in $H^{2q+1}(M)$ is the *Godbillon–Vey invariant* of $\mathscr{F}$. The *Godbillon operators* are the linear mappings

$$g: H^j(M, \mathscr{F}) \to H^{j+1}(M, \mathscr{F}) \tag{1.4}$$

given by $g(\langle\theta\rangle) = \chi_j(\bar{\lambda}) \cdot \langle\theta\rangle$, $q \leq j \leq m-1$. For $j = m-1$ we actually have for each Borel saturated set $B \subset M$ i.e., B is a union of leaves of $\mathscr{F}$ a well defined linear functional

$$g_B: H^{m-1}(M, \mathscr{F}) \to R, \quad g_B(\langle\theta\rangle) = \int_B \lambda \wedge \theta.$$

The mapping $B \to g_B$ is a vectorial measure on the σ-algebra generated by the saturated Borel sets of $(M, \mathscr{F})$ and is called the *Godbillon measure* of $\mathscr{F}$.

The characteristic mapping α is given by

$$\alpha(\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}) = \chi_{2q}(\bar{\xi}_1) \cdot \langle d\xi_2 \wedge \ldots \wedge d\xi_{q+1}\rangle. \tag{1.5}$$

1.4 *Example.* Let $A: R^p \times M \to M$ be a C^∞ locally free action, $\mathscr{F}$ its underlying foliation and $X = \{X_1, \ldots, X_p\}$ the frame of infinitesimal generators of A. Let $\xi = \{\xi_1, \ldots, \xi_p\}$ be a coframe adapted to A, i.e., ξ_i are C^∞ 1-forms on M such that $\xi_i(X_j) = \delta_{ij}$, $1 \leq i$, $j \leq p$. Since the vector fields X_i pairwise commute, the 1-forms ξ_i are d_f-closed, [1]. Thus they define p linearly independent classes $\bar{\xi} = \{\bar{\xi}_1, \ldots, \bar{\xi}_p\}$ in $H^1(\mathscr{F})$. This gives a representation $i_A: R^p \to H^1(\mathscr{F})$ defined by $i_A(x_1, \ldots, x_p) = x_1\bar{\xi}_1 + \ldots + x_p\bar{\xi}_p$. The $(q+1)$-linear symmetric mapping $\alpha_A: R^p \times \ldots \times R^p \to H^{2q+1}(M)$ given by $\alpha_A = i_A^*\alpha$ was called in [1] the *characteristic mapping* of A. It was proved in [1] that α_A does not depend on the choice of the adapted coframe ξ of A.

1.5 *Example.* More generally, let $A: G \times M \to M$ be a locally free smooth action of a Lie group G on M. Associated to A we have.a natural injection $E \to \bar{E}$ of the Lie algebra $\mathscr{G}$ of G into the Lie algebra of all smooth tangent vector fields of the underlying foliation $\mathscr{F}$ of A. This injection is given by $\bar{E}(x) = DA_x(0) \cdot E$, $x \in M$ where as usual $A_x: G \to M$,

$A_x(g) = A(g, x)$. Dually, we have a natural injection $\omega \to \omega'$ of the Lie algebra $\Lambda_R^*(\mathscr{G})$ of the invariant forms on G into the algebra $\Lambda(\mathscr{F})$ of smooth forms along $\mathscr{F}$. To see this, choose a normal bundle ν to the tangent bundle $T\mathscr{F}$ of $\mathscr{F}$. Consider the linear mapping $\mathscr{G}^* \to \Lambda^1(M)$, $\omega \to \omega^\nu$ where ω^ν is the 1-form defined by $\omega^\nu(\bar{E}) = \omega(E)$ if $E \in \mathscr{G}$ and $\nu \subset \ker \omega^\nu$. If τ is another normal bundle, then $\omega^\nu - \omega^\tau$ belongs to $I(\mathscr{F})$. Thus we have a well defined injective linear mapping $\omega \to \omega' = \pi(\omega^\nu)$ of $\mathscr{G}^*$ into $\Lambda^1(\mathscr{F})$. This mapping extends to an injective homomorphism

$$\Lambda_R^*(\mathscr{G}^*) \to \Lambda^*(\mathscr{F}) \tag{1.6}$$

which commutes with the differentials i.e. $d_f\omega' = (d\omega)'$ and depends only on the action A and induces an injective representation

$$i_A: H^*(\mathscr{G}) \to H^*(\mathscr{F}) \tag{1.7}$$

of the cohomology of the Lie algebra $\mathscr{G}$ of G into the cohomology of the foliated manifold $(M, \mathscr{F})$. The $(q + 1)$-linear symmetric mapping

$$\alpha_A: H^1(\mathscr{G}) \times \cdots \times H^1(\mathscr{G}) \to H^{2q+1}(M) \tag{1.8}$$

given by $\alpha_A = i_A^*\alpha$ is called the *characteristic mapping of A.*

2. ON THE COHOMOLOGY OF FOLIATED BUNDLES

Let B and F be closed orientable connected C^r manifolds, $r \geq 2$ and

$$\varphi: \pi_1(B) \to \mathrm{Diff}^r(F) \tag{2.1}$$

be an action.

The *foliated bundle* $F \to (M, \mathscr{F}) \xrightarrow{\tau} B$ *suspension* of φ is constructed as follows: Let $p: \tilde{B} \to B$ be the universal covering of B and $x_0 \in B$. Associated to φ there is an action

$$\Phi: \pi_1(B, x_0) \to \mathrm{Diff}^r(\tilde{B} \times F) \tag{2.2}$$

given by $\Phi_{[\gamma]}(\tilde{x}, y) = ([\gamma]\cdot\tilde{x}, \varphi_{[\gamma]}^{-1}(y))$ where $[\gamma]\cdot\tilde{x}$ denotes the image of $\tilde{x}$ by the deck transformation of $\tilde{B}$ corresponding to the homotopy class $[\gamma]$ of $\pi_1(B, x_0)$. The orbit space M of Φ is a manifold and actually we have a fiber bundle $F \to M \xrightarrow{\tau} B$. Every object of $\tilde{B} \times F$ which is invariant under Φ induces a corresponding object on M. To the natural foliation defined by the projection $\tilde{B} \times F \to F$, which is invariant under Φ, corresponds the foliation $\mathscr{F}$ on M which is transverse to the fibers of τ. We think of $\Lambda(\mathscr{F})$ as the set of differential forms along the natural foliation $\tilde{B} \times y$, $y \in F$ which are invariant under Φ. The image of τ^*: $\Lambda(B) \to \Lambda(M)$ is called the space of *basic forms.*

To each probability measure μ on F which is invariant under $\varphi: \pi_1(B, x_0) \to \mathrm{Diff}(F)$ we associate an epimorphism of differential complexes

$$P_\mu: \Lambda(\mathscr{F}) \to \Lambda(B) \tag{2.3}$$

called the *averaging operator* i.e., P_μ is a continuous surjective linear mapping which commutes with the differentials: $P_\mu \circ d_f = d \circ P_\mu$. If $h \in \Lambda^0(\mathscr{F})$, then the function

$$P_\mu(h) = \int_F h(\tilde{x}, y)\, d\mu(y) = \bar{h}(\tilde{x}) \tag{2.4}$$

is basic since it is invariant under all the deck transformations of p. If $\tau^*\omega$ is a basic form, we let $P_\mu(\tau^*\omega) = \omega$. Let $\{(\mathscr{U}_\alpha; \alpha)\}_{\alpha \in A}$ be an atlas of B and $(d\alpha_1, \ldots, d\alpha_p)$ be the coframe associated to the chart α. Let $\{\lambda_\alpha\}$ be a C^∞ partition of unit subordinate to the cover $\{\mathscr{U}_\alpha\}$. If $\xi \in \Lambda^k(\mathscr{F})$, then $\xi = \sum_\alpha(\sum_I(\lambda_\alpha \circ \tau) f_I^\alpha \tau^* d\alpha_I)$ where, as usual, $I = (i_1 < \cdots < i_k)$, $1 \le i_j \le p$ and $d\alpha_I = d\alpha_{i_1} \wedge \ldots \wedge d\alpha_{i_k}$. We define $P_\mu(\xi)$ by

$$P_\mu(\xi) = \sum_\alpha \lambda_\alpha\left(\sum_I P_\mu(f_I^\alpha) d\alpha_I\right). \tag{2.5}$$

A routine verification shows that P_μ is well defined i.e., does not depend on the choice of the atlas and the partition of unit, and commutes with the differentials.

The split short exact sequence of differential complexes

$$0 \to \ker \to \Lambda^*(\mathscr{F}) \overset{P_\mu}{\to} \Lambda^*(B) \to 0 \tag{2.6}$$

gives the split short exact cohomology sequences

$$0 \to H^j(\ker) \to H^j(\mathscr{F}) \to H^j(B) \to 0 \tag{2.7}$$

$0 \le j \le p$. Thus

2.1: $H^j(\mathscr{F}) = H^j(B) \oplus H^j(\ker)$ for $0 \le j \le p$.

Now let G be a connected Lie group and Γ be a discrete cocompact subgroup of G i.e., the right coset space $B = G/\Gamma$ is a compact manifold. Each right translation $R_z: G \to G$ induces a diffeomorphism $R_z^0: B \to B$. Let $E \in \mathscr{G}$ be a left invariant vector field on G and $a: R \times G \to G$ its flow. Thus $L_x \circ a_t = a_t \circ L_x$ and $a_t(x) = R_{a_t(e)}(x)$. The restriction of the right action $G \times G \overset{R}{\to} G$ to the 1-parameter subgroup $a_t(e)$ is the flow of E. The right action R induces a right action $G \times B \overset{R^0}{\to} B$ and the restriction to the subgroup $a_t(e)$ gives a flow on B. Let E_0 be the corresponding vector field on B. Thus we have an injective homomorphism of the Lie algebra $\mathscr{G}$ into the Lie algebra $\mathscr{X}(B)$ of all C^∞ vector fields on B. Let $\mathscr{G}_0$ be its image. Associated to a left action $\Gamma \overset{\varphi}{\to} \mathrm{Diff}(F)$ we have a left action

$$\Gamma \overset{\Phi}{\to} \mathrm{Diff}(G \times F) \tag{2.8}$$

given by $\Phi_\gamma(x, y) = (\gamma x, \varphi_\gamma^{-1}(y))$, $\gamma \in \Gamma$. The orbit space $M = G \times F/\Gamma$ is a manifold and we have a foliated bundle $F \to (M, \mathscr{F}) \overset{\tau}{\to} B$. Each right translation R_z gives an automorphism H_z of τ i.e. $H_z: (M, \mathscr{F}) \to (M, \mathscr{F})$ preserves the leaves of $\mathscr{F}$ and $\tau \circ H_z = R_z^0 \circ \tau$.

Each vector field $E \in \mathscr{G}$ may be thought as a vector field on $G \times F$ tangent to the first factor. Since it is invariant by the action Φ it induces a vector field $\bar{E}$ tangent to the leaves of $\mathscr{F}$ and in fact $\bar{E} = p_*(E)$ and $\tau_* \bar{E} = E_0$ where $p: G \times F \to M$ is the projection. Thus we have an action

$$H: G \to \mathrm{Aut}(\tau) \tag{2.9}$$

of G as automorphisms of the foliated bundle τ, called the *canonical action* of τ. The restriction of H to Γ gives an action

$$\Gamma \overset{H^*}{\to} \mathrm{Aut}(H^*(\mathscr{F})) \tag{2.10}$$

of Γ on $H^*(\mathscr{F})$.

The foliated cohomology $H^*(\mathscr{F})$ is not invariant by homotopy but it is invariant by *integrable homotopy*, a concept introduced by A. Haefliger in [12].

2.2. *Definition.* Let f and g be two mappings of the foliated manifold $(M, \mathscr{F})$ into the foliated manifold $(M', \mathscr{F}')$ sending leaves of $\mathscr{F}$ into leaves of $\mathscr{F}'$; morphisms of foliated manifolds. They are *integrably homotopic* if there is a mapping

$$\rho:(M \times R, \mathscr{F} \times R) \to (M', \mathscr{F}')$$

sending the leaves of $\mathscr{F} \times R$ into the leaves of $\mathscr{F}'$ and such that

$$\rho(x, t) = f(x) \text{ for } t \leq 0 \text{ and } \rho(x, t) = g(x) \text{ for } t \geq 1.$$

The following is well-known and a proof may be found in [16].

2.3 PROPOSITION. *Let $f, g:(M, \mathscr{F}) \to (M', \mathscr{F}')$ be two morphisms of foliated manifolds. If f is integrably homotopic to g then $f^* = g^*:H^*(\mathscr{F}) \to H^*(\mathscr{F}')$.*

2.4. LEMMA. *The action $H^*: \Gamma \to Aut(H^*(\mathscr{F}))$ is trivial i.e., $H_\gamma^* = id$ for all $\gamma \in \Gamma$.*

Proof. By 2.3 it suffices to show that each H_γ is integrably homotopic to the identity. Choose any smooth curve $c: R \to G$ such that $c(t) = \gamma$ for $t \leq 0$ and $c(t) = e$ for $t \geq 1$. The mapping

$$\rho:(M \times R, \mathscr{F} \times R) \to (M, \mathscr{F})$$

given by $\rho(x, t) = H(c(t), x)$ is the desired homotopy.

2.5 *Remark.* The representation $i_H: H^*(\mathscr{G}) \to H^*(\mathscr{F})$ associated to the canonical action $G \xrightarrow{H} \mathrm{Aut}(\tau)$ as in (1.7) is actually given by closed forms on M. In fact, each closed invariant form ω on G induces a closed form ω_0 on B and i_H sends the cohomology class of ω, in $H^*(\mathscr{G})$ into the cohomology class of $\tau^*\omega_0$ in $H^*(\mathscr{F})$ and $\tau^*\omega_0$ is closed in M. Thus to each basis of $\mathscr{G}^*$ corresponds a closed coframe of $(M, \mathscr{F})$.

2.6 PROPOSITION. *Let $F \to (M, \mathscr{F}) \xrightarrow{\tau} B$ be the suspension foliated bundle defined by the action of a discrete cocompact subgroup Γ of a connected Lie group G on a manifold F. Then the characteristic mapping vanishes on*

$$(i_H H^*(\mathscr{G})) \times H^*(\mathscr{F}) \times \ldots \times H^*(\mathscr{F}).$$

Here H: $G \to \mathrm{Aut}(\tau)$ is the canonical action of τ.

Proof. Follows from the above remark and Lemma 1.1, (ii).
Roussarie's example shows that in the above Propositon it is crucial to have the foliation transverse to a fibration over the group G modulo a cocompact lattice. We recall this example.

2.7 *Example* (Roussarie). Let $G = SL(2; R)$ and Γ be any discrete cocompact subgroup of G. The Lie algebra $\mathscr{G}$ of G is the space of matrices of zero trace and is generated by the matrices

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Clearly

$$[X, Y] = 2Y, \quad [Y, Z] = X \quad \text{and} \quad [X, Z] = -2Z.$$

Thus, the dual base $\{\xi, \eta, \zeta\}$ satisfies the relations $d\xi = -\eta \wedge \zeta$, $d\eta = -2\xi \wedge \eta$ and $d\zeta = 2\xi \wedge \zeta$. Hence ζ is an integrable 1-form on G. Let $\mathscr{G}_0$ be the corresponding Lie algebra on $M = G/\Gamma$. The isomorphism $\mathscr{G} \to \mathscr{G}_0$ gives an isomorphism $\mathscr{G}^* \to \mathscr{G}_0^*$ and we also denote by ξ, η and ζ the corresponding 1-forms on M. The foliation $\mathscr{F}$ on M given by ζ is the underlying foliation of the action of the solvable group T of triangular matrices in $SL(2; R)$ acting by right translations on M. The cohomology group $H^1(t)$ of the Lie algebra t of T is generated by the 1-form ξ which is closed in T (t is generated by X and Y). As $\xi \wedge d\xi = -4\xi \wedge \eta \wedge \zeta$ we see that $[\xi \wedge d\xi] \neq 0$. So the characteristic mapping of $\mathscr{F}$ does not vanishes on $(i_H H^*(t)) \times H^1(\mathscr{F})$. We also remark that $\mathscr{F}$ has no holonomy invariant transverse measures since T is not unimodular. This follows from a theorem of Bowen [7].

3. ERGODICITY AND THE VANISHING OF THE CHARACTERISTIC MAPPING

In this section we prove a theorem that relates the vanishing of the characteristic mapping of a foliated bundle with ergodic properties of its holonomy action.

Let, as before, G be a connected Lie group and Γ a discrete cocompact subgroup of G. To each left action $\varphi: \Gamma \to \mathrm{Diff}(F)$ there corresponds the suspension foliated bundle $F \to (M, \mathscr{F}) \xrightarrow{\tau} B$ with the homogeneous space $B = G/\Gamma$ as base-space (see (2.9)). The canonical action $H: G \to \mathrm{Aut}(\tau)$ induces an action $H^*: \Gamma \to \mathrm{Aut}(H^*(\mathscr{F}))$ as in (2.10). We restrict ourselves to the case $G = R^p$ and $\Gamma = Z^p$.In this case, as G is abelian the action H leaves invariant the closed basic forms $\tau^*\omega_0$, as remarked in 2.5.

Now, we consider ergodic averages associated to forms ξ in $\Lambda^j(\mathscr{F}), 0 \leq j \leq p$. Let $J_n = \{0, 1, \ldots, n\}$ and $U_n = J_n^p, n \in Z_+$ be the standard summing sequences of Z^p. We consider the Birkhoff sums

$$B_n(H) \cdot \xi = \frac{1}{(n+1)^p} \sum_{\gamma \in U_n} H_\gamma^*(\xi) \tag{3.1}$$

and

$$B_n(\varphi) \cdot h = \frac{1}{(n+1)^p} \sum_{\gamma \in U_n} h \circ \varphi_\gamma \tag{3.2}$$

$h \in C^0(F)$.

Choose a basis $B = \{E_1, \ldots, E_p\}$ of $\mathscr{G}$ and let $\{\omega_1, \ldots, \omega_p\}$ be the coframe of $(M, \mathscr{F})$ associated to the dual basis B^* of $\mathscr{G}^*$ as in 2.5. As G is abelian the above coframe is invariant under the action H. For each k-tuple $I = (i_1, \ldots, i_k)$ of integers $1 \leq i_1 < \cdots < i_k \leq p$ we have the k-forms $\omega_I = \omega_{i_1} \wedge \ldots \wedge \omega_{i_k}$ which give an invariant basis of the $C^\infty(M)$-module $\Lambda^k(\mathscr{F})$. We consider the c^0-norm on $\Lambda^k(\mathscr{F})$ given by

$$|\xi|_0 = \sum_I |h_I|_0 \text{ if } \xi = \sum_I h_I \omega_I \quad \text{and} \quad |h_I|_0 = \sup_{x \in M} |h_I(x)|.$$

3.1. LEMMA. *Let $H: R^p \to Aut(\tau)$ be the canonical action of the suspension foliated bundle of an action φ of Z^p into a closed orientable manifold F. Then the Birkhoff sums $B_n(H)$ are uniformly bounded i.e., $|B_n(H) \cdot \xi|_0 \leq |\xi|_0$ for every $n \in Z_+$ and $\xi \in \Lambda^j(\mathscr{F})$, $0 \leq j \leq p$.*

Proof. Since the ω_I are invariant under H, then

$$B_n(H) \cdot \xi = \sum_I (B_n(H) \cdot h_I) \omega_I. \tag{3.3}$$

Now, $|B_n(H) \cdot h_I|_0 \leq |h_I|_0$ implies that $|B_n(H) \cdot \xi|_0 \leq \sum_I |h_I|_0 = |\xi|_0$.

A Borel probability measure μ on F is *invariant* under an action φ: $\Gamma \to \text{Diff}(F)$ if $\mu(\varphi_\gamma(X)) = \mu(X)$ for every Borel set $X \subset F$. In this case we say that (F, φ, μ) is a *measured action.* If Γ has polynomial growth then every action $\varphi: \Gamma \to \text{Diff}(F)$ has an invariant measure. The ergodic property of the holonomy action φ which enables us to show the vanishing of the characteristic mapping of a foliated bundle is the following.

3.2 *Definition.* Let φ be an action of Z^p on a Borel measure space (F, μ). The action φ is an *E-action* if it has the following properties

(i) μ is invariant under φ
(ii) for every continuous function h: $F \to R$ the Birkhoff sums $B_n(\varphi) \cdot h$ converge pointwise to $\mu(h)$ for m-almost all $x \in F$, m being a Lebesgue measure on F.

We give some relevant examples of E-actions.

Examples.

3.3. An action $\varphi: Z^p \to \text{Diff}(F)$ is *uniquely ergodic* if it has a unique invariant probability measure. Unique ergodicity is equivalent to the uniform convergence of the sums $B_n(\varphi) \cdot h$ to $\mu(h)$ for every $h \in C^0(F)$, [11]. Thus a uniquely ergodic action is an *E-action.*

3.4. A measured action (F, φ, μ) is *ergodic* if every invariant Borel set have either μ-measure 0 or 1. The measure μ is *absolutely continuous with full support* if $\mu = hm$ where m is a Lebesgue measure on F and h is a non-vanishing m-a.e. L^1-function. If the action φ: $Z^p \to \text{Diff}(F)$ is ergodic and μ is absolutely continous with full support then by the Pointwise Ergodic Theorem [10] the Birkhoff sums $B_n(\varphi) \cdot h$, $h \in L^1(F)$ converge in L^1 and pointwise to $\mu(h)$ for m-almost all $x \in F$. Thus φ is an E-action.

3.5. The ergodic theory of Anosov diffeomorphisms [6] gives important examples of E-actions. In fact, if φ: $Z^p \to \text{Diff}(F)$ is generated by iterates of a transitive C^2 Anosov diffeomorphism then by a theorem of Y. Sinai [21], φ is an E-action. Among the C^2 Anosov diffeomorphism the ones that do not admit an invariant absolutely continuous measure μ form a dense open subset, see [6]. We also observe that there is a dense open subset of Anosov diffeomorphisms of the torus T^q with trivial centralizer i.e., diffeomorphisms commuting only with integer powers of themselves. This is a result of J. Palis and J. Yoccoz in [17].

3.6. An E-action may have a singular invariant measure. If φ is a C^1 diffeomorphisms of the circle S^1 with only one fixed point and no periodic points then φ is uniquely ergodic and the invariant probability measure is concentrated at the fixed point. Now if φ: $S^1 \to S^1$ has two fixed points, a sink and a source, and μ is the invariant probability measure concentrated at the sink, then the action of Z generated by φ is an E-action. Observe that this action is ergodic but not uniquely ergodic.

3.7. Lemma. *Let* φ: $Z^p \to \text{Diff}(F)$ *be an E-action and* H: $R^p \to Aut(\tau)$ *the canonical action of the suspension foliated bundle* τ *of* φ. *Then for every* $\xi \in \Lambda(\mathscr{F})$, *the Birkhoff sums* $B_n(H) \cdot \xi$ *converge both* L^1 *and pointwise to the averaged basic form* $P_\mu(\xi)$ *for v-almost all x in M where v is a Lebesgue measure in M.*

Proof. Consider the invariant measure $dx \times \mu$ an $R^p \times F$ where dx is the Lebesgue measure of R^p and let λ be the quotient measure on M. Thus λ is invariant under the action

$H: Z^p \to \mathrm{Aut}(\tau)$. As the canonical coframe of $(M, \mathscr{F})$ induced by the canonical coframe $\{dx_1, \ldots, dx_p\}$ of R^p is invariant under H then by (3.3) it is sufficient to consider the case $h \in \Lambda^0(\mathscr{F})$. By the Pointwise Ergodic Theorem, [10] the averages $B_n(H)\cdot h$ converge both L^1 and pointwise λ-a.e to an H-invariant L^1-function h^*. Now since φ is an E-action, it follows that $(B_n(H)h)(x, y) = (B_n(\varphi)h_x)(y)$, $h_x(y) = h(x, y)$ for the suspended function on $R^p \times F$ and $B_n(\varphi)h_x \to \mu(h_x)m$-a.c on F for every $x \in R^p$, thus $h^* = P_\mu(h)$ and $B_n(H)\cdot h$ converge pointwise to $P_\mu(h)\,\nu$-a.e. on M.

Now we prove the main theorem of this paper.

3.8 THEOREM. *Let $F \to (M, \mathscr{F}) \overset{\tau}{\to} T^p$ be the suspension foliated bundle of an action φ of Z^p on a closed connected q-manifold F. If φ: $Z^p \to Diff(F)$ is an E-action, then the characteristic mapping*

$$\alpha: H^1(\mathscr{F}) \times \ldots \times H^1(\mathscr{F}) \to H^{2q+1}(M)$$

vanishes.

The idea of the proof is to show that for any leafwise 1-class $\bar{\xi}$ the averaging operator $B_n(H)$ does not change the leafwise cohomology class of ξ, Lemma 2.4, so does not change the characteristic pairing $\chi: H^1(\mathscr{F}) \times H^j(M; \mathscr{F}) \to H^{j+1}(M)$ i.e., the cohomology classes of ξ and $B_n(H)\cdot\xi$ give the same operator $H^j(M; \mathscr{F}) \to H^{j+1}(M, \mathscr{F})$. This is similar to Duminy's ideas for Godbillon–Vey class, [9].

The cohomology calculation 2.1 shows that the averaged form $P_\mu(\xi)$ is a closed basic form, so by the general vanishing property of the characteristic pairing, Lemma 1.1, ii, the leafwise class of $P_\mu(\xi)$ yields a zero pairing. Since the averages $B_n(H)\cdot\xi$ converge pointwise ν-a.e. to $P_\mu(\xi)$, ν a Lebesgue measure on M, and the sequence $B_n(H)$ is uniformly bounded (Lemmas 3.1 and 3.7), the Dominated Convergence Theorem applies giving the vanishing of α. This is similar to the method of Hurder in [15].

We now give a formal proof.

Proof of 3.8. Let $\xi_1, \ldots, \xi_{q+1}$ be d_f-closed 1-forms on $(M, \mathscr{F})$ and $\eta = \xi_2 \wedge d\xi_3 \wedge \ldots \wedge d\xi_{q+1}$. Thus $d\eta$ is a closed $2q$-form whose cohomology class $\langle d\eta \rangle$ in $H^{2q}(M, \mathscr{F})$ is not necessarily trivial. As we observed in (1.5) the characteristic mapping is given by

$$\alpha(\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}) = \chi_{2q}(\bar{\xi}_1)\cdot\langle d\eta \rangle = [\xi_1 \wedge d\eta]. \tag{3.4}$$

Now, as in Lemma 2.4, the action H^*: $Z^p \to \mathrm{Aut}(H^1(\mathscr{F}))$ is trivial, so the averaging operators $B_n(H)$ do not change the leafwise class of ξ_1 and therefore

$$\alpha(\bar{\xi}_1, \ldots, \bar{\xi}_{q+1}) = [B_n(H)\cdot\xi_1 \wedge d\eta] \tag{3.5}$$

for every positive integer n. To apply the Poincaré duality theorem to conclude that $\xi_1 \wedge d\eta$ is zero in the de Rham cohomology we observe that (3.4) and (3.5) give

$$\int_M \xi_1 \wedge d\eta \wedge \omega = \int_M B_n(H)\cdot\xi_1 \wedge d(\eta \wedge \omega) \tag{3.6}$$

for every closed $(p - q - 1)$-form ω on M and every positive integer n. As φ: $Z^p \to \mathrm{Diff}(F)$ is an *E-action*, then, by Lemma 3.7, the sequence $B_n(H)\cdot\xi_1$ converges both L^1 and pointwise ν-a.e. to the averaged closed form $P_\mu(\xi)$. Now, by Lemma 3.1 the convergence is dominated, hence the Lebesgue dominated convergence theorem applies to (3.6), showing that

$$\int_M \xi_1 \wedge d\eta \wedge \omega = \int_M P_\mu(\xi_1) \wedge d\eta \wedge \omega.$$

The last integral is zero since $P_\mu(\xi_1)$ is a closed form. This shows that the characteristic mapping α vanishes.

As a consequence of Theorem 3.8 we obtain the following generalization of a theorem of M. Herman [14].

3.9 COROLLARY. *The characteristic mapping of a C^2-foliation by planes of T^3 vanishes.*

Proof. If $(T^3, \mathscr{F})$ is a C^2-foliation by planes then by a theorem of H. Rosenberg and R. Roussarie in [19] it is suspension of a uniquely ergodic action φ: $Z^2 \to \text{Diff}(S^1)$. Thus by Theorem 3.8 the characteristic mapping α: $H^1(\mathscr{F}) \times H^1(\mathscr{F}) \to H^3(M) = R$ vanishes.

3.10. There is an analogue of Theorem 3.8 for actions of finitely generated nilpotent groups [20].

3.11. We observe that the operators $B_n(H)$ do not change the leafwise cohomology class of a $d_{\mathscr{F}}$-closed form ξ but that the class of the averaged form $P_\mu(\xi)$ may change even if $B_n(H) \cdot \xi$ converge to $P_\mu(\xi)$ in the C^∞ topology. In fact, for Liouville linear foliations of T^m, the kernel of P_μ is non trivial in leafwise cohomology, see [4].

4. THE COMPACT ORBIT THEOREM

In this section, as an application of Theorem 3.8 and a previous theorem of the authors in [1] we prove a theorem that relates the non-degeneracy of the characteristic mapping of a non-singular action of R^2 on a 3-manifold M with the existence and stability of compact orbits. Let $A^r(R^2, M)$ be the space of all C^r non-singular actions of R^2 on $M, r \geq 2$.

4.1. THEOREM. *Let M be a closed orientable connected 3-manifold and $A \in A^r(R^2, M)$. If the characteristic mapping*

$$\alpha_A : R^2 \times R^2 \to H^3(M) = R$$

of A is non-degenerate, then

(i) *A has a compact orbit;*
(ii) *There exists a neighborhood $V \subset A^r(R^2, M)$ of A in the C^1 topology such that every action in V has a compact orbit.*

Actually, if $M = T^3$ then (i) and (ii) hold if $\alpha_A \neq 0$.

Proof. Assume A has no compact orbit. Then by Theorem 2 in [8] all the orbits are either planes or cylinders but there is no mixture of the two and every orbit is dense. If all the orbits are planes then M is diffeomorphic to T^3 and by Corollary 3.9 α_A vanishes. Assume every orbit is a dense cylinder. Fix an orbit L and a point x in L. Let X_1 and X_2 be the infinitesimal generators of A. There exist constains c_1 and c_2 such that the orbit of the vector field $E = c_1 X_1 + c_2 X_2$ through x will be closed with minimal period 1, and so will be the orbit of any point of L, and therefore the orbit or every point of M, since L is dense. Thus E defines a free action of S^1 on M and clearly A is a linear reparameterization of an action of the cylinder $S^1 \times R$ on M. Now by Corollary 2.2, of [3] α_A is degenerate. Actually if $M = T^3$, then $\alpha_A = 0$. Therefore A has a compact orbit. Part (ii) follows from (i) and the continuity of the characteristic mapping $A \to \alpha_A$ in the C^1 topology, [[1], 1.11].

Acknowledgements—We thank the referee for detailed comments on an earlier version of this paper, which allowed us to make a number of improvements and to M. Craizer for useful conversations on ergodic theory.

REFERENCES

1. J. L. ARRAUT and N. M. DOS SANTOS: Actions of R^p on closed manifolds, *Topology Appl.* **29** (1988), 41–54.
2. J. L. ARRAUT and N. M. DOS SANTOS: Differentiable conjugation of actions of R^p, *Bol. Soc. Brasil. Mat.* **19** (1988), 1–19.
3. J. L. ARRAUT and N. M. DOS SANTOS: Actions of cylinders, *Topology Appl.* **36** (1990) 57–71.
4. J. L. ARRAUT and N. M. DOS SANTOS: Linear foliations of T^n, *Bol. Soc. Brasil. Mat.* **21** (1991), 189–204.
5. C. BENSON and G. RATCLIFF: An invariant for unitary representations of nilpotent Lie groups, *Michigan Math. Journal.* **34** (1987), 23–30.
6. R. BOWEN: Equilibrium states and the ergodic theory of Anosov Diffeomorphisms, Springer Lect. Notes in Math. **470** (1975).
7. R. BOWEN: Weak mixing and unique ergodicity on homogeneous spaces, *Israel J. Math.* **23** (1979), 337–342.
8. G. CHATELET, H. ROSENBERG and D. WEIL: A classificaton of topological types of R^2-actions on closed orientable 3-manifolds, *IHES* **43** (1973), 261–272.
9. G. DUMINY: L'invariant de Godbillon–Vey d'un feuilletage se localise sur les feuilles ressort, preprint Lille (1982).
10. W. EMERSON: The pointwise ergodic theorem for amenable groups, *Am. J. Math.* **96** (1974), 472–487.
11. H. FURSTENBERG: Strict ergodicity and transformations of the torus *Am. J. Math.* **83** (1961), 573–601.
12. A. HAEFLIGER: Homotopy and integrability, Springer Lect. Notes in Math. **197** (1971), 133–163.
13. J. HEITSCH and S. HURDER: Secondary classes, Weil measures and the geometry of foliations, *J. Diff. Geom.* **20** (1984), 291–309.
14. M. HERMAN: The Godbillon–Vey invariant of foliations by planes of T^3, Springer Lect. Notes in Math. **597** (1977), 294–307.
15. S. HURDER: The Godbillon measure for amenable foliations, *J. Diff. Geom.* **23** (1986), 347–365.
16. A. El KACIMI-ALAOUI: Sur la cohomologie feuilletée, *Compositio Math.* **49** (1983), 195–215.
17. J. PALIS and J. C. YOCCOZ: Centralizers of Anosov diffeomorphisms on Tori, *Ann. Scient. Ec. Norm. Sup.* **22** (1989), 99–108.
18. B. L. REINHART: Harmonic integrals on almost product manifolds, *Transactions of the Amer. Math. Soc.*, **88** (1958), 243–276..
19. H. ROSENBERG and R. ROUSSARIE: Topological equivalence of Reeb foliations, *Topology* **9** (1970), 231–242.
20. N. M. DOS SANTOS: Foliated cohomology and Characteristic classes, to appear.
21. YA SINAI: Markov partitions and C-diffeomorphisms, *Func. Anal. and its Appl.* **2** (1968), 64–89.

PUC-RIO, Mathematics,
Rua Margues de Sao Vicente 225
Rio de Janeiro 22453
Brazil